Multi-scale methods in numerical analysis

W. N. Polyzou

Other contributors: Gerald Payne, Brian Kessler, Fatih Bulut, Tracie Michlin

The University of Iowa

U.S. DEPARTMENT OF ENERGY | Office of Science

Outline

- **Background and motivation.**
- **Fractals self-similarity and the renormalization group.**
- **Numerical analysis using the renormalization group.**
- **Applications.**

Standard numerical methods

- Designed to work with functions that with enough magnification locally look like straight lines.

- Global basis functions (orthogonal polynomials, Laguerre functions, Hermite functions) are not efficient for representing local structures, or structures on multiple scales.

$$f(x) = \sum_n f_n \phi_n(x) \qquad f_n = \int f(x) \phi_n^*(x) dx$$

Wavelets are basis functions that can overcome these difficulties

- Used for data compression in digital photography (JPEG).

- Efficient at treating images with many different scales and structures.

- A digital photograph is just a matrix of numbers. Could this same data compression method be used to efficiently solve problems in linear algebra?

- Wavelet bases result in sparse matrices. Faster algorithms can be used and they require less storage.

Challenges/questions

- **Wavelets are fractal valued functions.**
- **How do you evaluate fractal functions $\{\phi_n(x)\}$?**

$$f(x) = \sum f_n \phi_n(x)$$

- **How do you calculate integrals involving fractal functions?**

$$f_n = \int \phi_n^*(x) f(x) dx$$

- **How do you calculate derivatives of fractal functions?**

$$f''(x) = \sum f_n \phi_n''(x)$$

- **Why would you want to use them?**

What do we mean by a fractal valued function

- **Looks like a copy of itself on smaller scales.**

- **How do we change scales mathematically?**

$$Df(x) = \sqrt{2} f(2x)$$

- **Shrinks the support of the function by a factor of 2, preserving the Hilbert space norm of the function.**

- D **is called a scaling or dilatation operator**

Renormalization group equation

$$\underbrace{s(x) = D\ (\underbrace{\sum_{l=0}^{2K-1} h_l T^l s(x)}_{\text{weighted average}})}_{\text{rescale}}\ .$$

solutions $s(x)$ are fractal!

T: unit translation **D scale transformation**

$$Ts(x) = s(x-1) \qquad Ds(x) = \sqrt{2}s(2x).$$

h_l are numbers

The renormalization group equation is homogeneous
$s(x)$ a solution implies $cs(x)$ is a solution
the scale c is fixed by

$$\int dxs(x) = 1$$

Properties of $s(x)$

$$\tilde{s}(k) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx} s(x) dx \qquad \tilde{h}(k) := \sum_{l} \frac{h_l}{\sqrt{2}} e^{-ikl}.$$

$$\tilde{s}(k) = \tilde{s}\left(\frac{k}{2}\right) \tilde{h}\left(\frac{k}{2}\right)$$

$$\tilde{s}(k) = \lim_{n\to\infty} \tilde{s}(\frac{k}{2^n}) \prod_{m=1}^{n} \tilde{h}(\frac{k}{2^m}) = \tilde{s}(0) \prod_{m=1}^{\infty} \tilde{h}(\frac{k}{2^m}).$$

$$k = 0 \qquad \to \qquad 1 = \prod_{l=1}^{\infty} \tilde{h}(0) = \tilde{h}(0) = \sum_{l=0}^{2K-1} \frac{h_l}{\sqrt{2}}$$

$\sum_{l=0}^{2K-1} h_l = \sqrt{2}$: **Necessary for the renormalization group equation to have a solution**

Support of the solution $s(x)$ to the RG equation

$$s(x) = \frac{\tilde{s}(0)}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk e^{ikx} \prod_{m=1}^{\infty} \tilde{h}(\frac{k}{2^m}) =$$

$$\sqrt{2\pi}\tilde{s}(0) \prod_{m=1}^{\infty} (\sum_{l=0}^{N-1} \frac{h_l}{\sqrt{2}} \delta(x - \frac{l}{2^m}))$$

which vanishes for $x \notin [0, N-1] = [2K-1]$.

Support of $s(x) \subseteq [0, 2K-1]$ where $N = 2K$ is the number of h_l's

Support is compact!

All fractals are not created equal

Additional properties of $s(x)$ depend on the choice of h_l

$$\sum_{l=1}^{2K-1} h_l = \sqrt{2}$$

Additional conditions:

$$\int s(x-n)s(x-m)dx = \delta_{mn}$$

$$\int x^m \sum_{l=0}^{2K-1} g_l s(x-l)dx = 0 \qquad m = 0, 1, \cdots K-1$$

$$g_l := (-)^l h_{2K-l-1}$$

Conditions define Daubechies K scaling functions:

Ingrid Daubechies

Meaning of conditions:

- $s_n(x) := T^n s(x) = s(x-n)$ **are orthonormal.**

- $x^m = \sum_n c_n s_n(x)$ **pointwise for** $m \leq K$.

- $\sum_{l=1}^{2K-1} h_l = \sqrt{2}$ **necessary for existence of solution.**

- **Equations determine** h_l **up to reflection:**
  $h_l \to h'_l = h_{2k-1-l}$. **See table:**

Weight coefficients h_l for different K values

h_l	K=1	K=2	K=3
h_0	$1/\sqrt{2}$	$(1+\sqrt{3})/4\sqrt{2}$	$(1+\sqrt{10}+\sqrt{5+2\sqrt{10}})/16\sqrt{2}$
h_1	$1/\sqrt{2}$	$(3+\sqrt{3})/4\sqrt{2}$	$(5+\sqrt{10}+3\sqrt{5+2\sqrt{10}})/16\sqrt{2}$
h_2	0	$(3-\sqrt{3})/4\sqrt{2}$	$(10-2\sqrt{10}+2\sqrt{5+2\sqrt{10}})/16\sqrt{2}$
h_3	0	$(1-\sqrt{3})/4\sqrt{2}$	$(10-2\sqrt{10}-2\sqrt{5+2\sqrt{10}})/16\sqrt{2}$
h_4	0	0	$(5+\sqrt{10}-3\sqrt{5+2\sqrt{10}})/16\sqrt{2}$
h_5	0	0	$(1+\sqrt{10}-\sqrt{5+2\sqrt{10}})/16\sqrt{2}$

Basis construction ($L^2(\mathbb{R})$)

Rescale and translate fixed point, $s(x)$

$$s^k_n(x) := D^k T^n s(x) = 2^{k/2} s\left(2^k(x - 2^{-k}n)\right).$$

Support of $s^k_n(x)$ **is** $[2^{-k}n, 2^{-k}(n + 2K - 1)]$

$\mathcal{S}_k :=$ **resolution** 2^{-k} **subspace of** $L^2(\mathbb{R})$**:**

$$\mathcal{S}_k := \{f(x) | f(x) = \sum_{n=-\infty}^{\infty} c_n s^k_n(x), \quad \sum_{n=-\infty}^{\infty} |c_n|^2 < \infty\}.$$

$$\mathcal{S}_k := D^k \mathcal{S}_0$$

Renormalization group equation implies

$$\mathcal{S}_k \subset \mathcal{S}_{k+n} \qquad n > 0$$

$$\mathcal{S}_{k+1} = \mathcal{S}_k \oplus \mathcal{W}_k \qquad \mathcal{W}_k \neq \{\emptyset\}.$$

Multi-resolution decomposition of $L^2(\mathbb{R})$

$$\mathcal{S}_{k+1} = \mathcal{S}_k \oplus \mathcal{W}_k$$

Iterate

$$L^2(\mathbb{R}) = \mathcal{S}_k \oplus \mathcal{W}_k \oplus \mathcal{W}_{k+1} \oplus \mathcal{W}_{k+2} \oplus \mathcal{W}_{k+3} \oplus \cdots =$$

$$\cdots \oplus \mathcal{W}_{k-2} \oplus \mathcal{W}_{k-1} \oplus \mathcal{W}_k \oplus \mathcal{W}_{k+1} \oplus \mathcal{W}_{k+2} \oplus \cdots$$

Wavelets ($\{w_n^k(x)\}$ orthonormal basis for $\mathcal{W}_k$)

$$w(x) := D \sum_{l=0}^{2K-1} g_l T^l s(x) \qquad g_l = (-)^l h_{2K-1-l}$$

$$w_n^k(x) := D^k T^n w(x) = 2^{k/2} w\left(2^k(x - 2^{-k} n)\right).$$

Same support as $\{s_n^k(x)\}$

Comments

Orthonormal basis for $L^2(\mathbb{R})$:

$$\{s_n^k(x)\}_{n=-\infty}^{\infty} \cup \{w_n^l(x)\}_{n=-\infty, l\geq k}^{\infty}$$

All basis functions constructed from a single function $s(x)$.

The condition

$$\int x^m \sum_{l=0}^{2K-1} g_l s(x-l) = 0 \qquad m = 0, 1, \cdots K$$

that determines the h_l is equivalent to the requirement

$$\int x^m w_n^k(x) dx = 0 \qquad \forall k, n \qquad \text{and} \qquad m = 0, 1, \cdots, K-1$$

Local pointwise polynomial condition

Completeness implies

$$f(x) = \sum_n c_n s_n^k(x) + \sum_{n,l \geq k} d_{nl} w_n^l(x)$$

For $f(x) = x^m$, $m < K$ all the $d_{nl} = 0$ which means

$$x^m = \sum_n c_n s_n^k(x) \qquad m < K$$

where for any x only a finite number of the $s_n^k(x)$ are non-zero.

This requires that both sides of the equation agree at every x

- **This explains why JPEG works. All of coefficients d_n^k associated with structures that are smooth on scale 2^{-k} vanish, or are very small, resulting in an efficient representation of the data.**

- **The transformation relating $\mathcal{S}_{k+n}$ and $\mathcal{S}_k \oplus \mathcal{W}_k \oplus \cdots \oplus \mathcal{W}_{k+n}$ is a real orthogonal transformations, called the wavelet transform, that can be performed very efficiently.**

- **The wavelet transformation can be used to recover an approximation to the original function.**

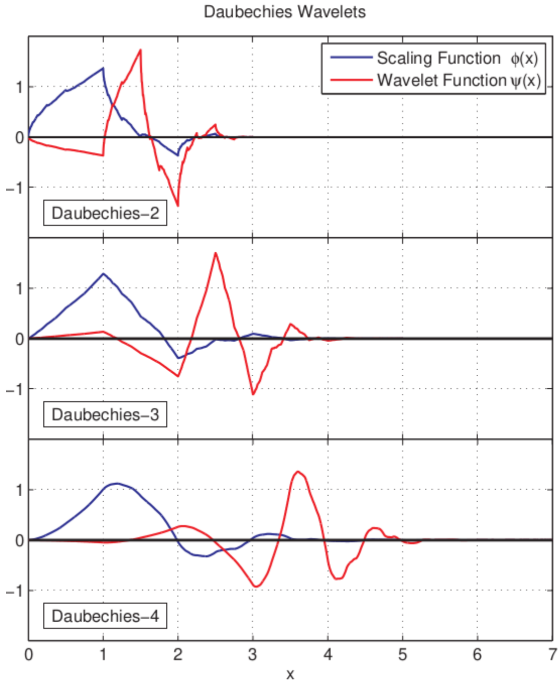
- **Basis allows natural resolution and volume truncations.**

Weird stuff!

$$1 = \underbrace{\int s(x)dx}_{\text{setting scale}} = \underbrace{\int s^2(x)dx}_{\text{normalization}}$$

- **Locally finite sums of fractal functions can be differentiable. This means that a FINITE number of these functions with complex fractal boundaries fit together like a jigsaw puzzle!**

- **Under any magnification the functions do not look like straight lines, but they are differentiable!**



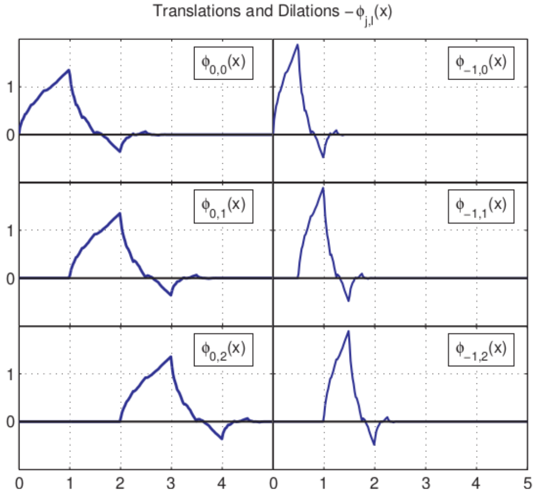
Daubechies Wavelets
Scaling Function φ(x)
Wavelet Function ψ(x)
Daubechies−2
Daubechies−3
Daubechies−4
1
0
−1
0
1
2
3
4
5
6
7
x

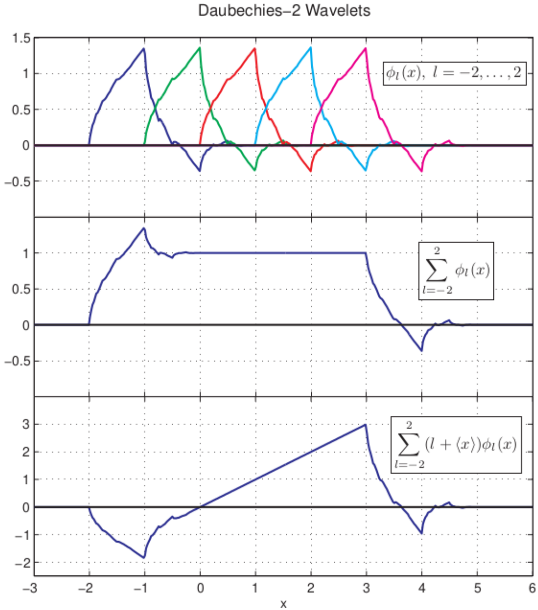
As K increases:

Support increases,

Smoothness increases,

Number of non-zero functions at a point increases

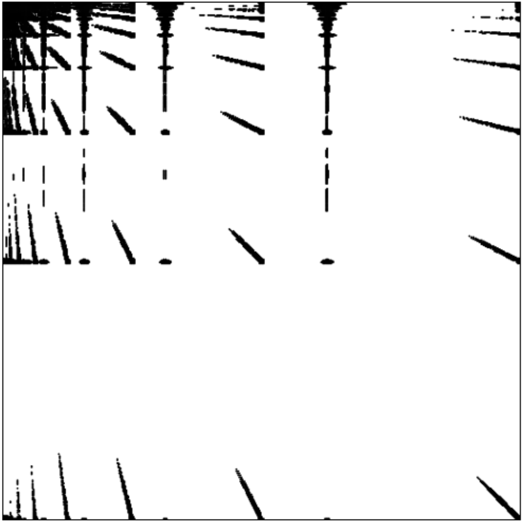

Translations and Dilations – $\phi_{j,l}(x)$
$\phi_{0,0}(x)$
$\phi_{-1,0}(x)$
$\phi_{0,1}(x)$
$\phi_{-1,1}(x)$
$\phi_{0,2}(x)$
$\phi_{-1,2}(x)$
1
0
1
0
1
0
0
1
2
3
4
0
1
2
3
4
5



Daubechies−2 Wavelets
$\phi_l(x),\ l=-2,\ldots,2$
$\sum_{l=-2}^{2}\phi_l(x)$
$\sum_{l=-2}^{2}(l+\langle x\rangle)\phi_l(x)$
x

Kernel, $K(x,y)$ of a scattering integral equation after wavelet transform

$$F(x) = D(x) + \int K(x,y)F(y)dy$$



Advantages of Daubechies wavelets

- **Basis for $L^2(\mathbb{R})$.**
- **Basis functions have compact support.**
- **Infinite number of basis functions with support in any open set.**
- **Basis functions have limited (controllable) smoothness.**
- $2^{k/2}s_n^k(x)$ **are locally finite partitions of unity.**
- **Efficient at representing problems with multiple scales.**
- **In many applications basis functions do not have to be computed.**

To do numerical analysis with fractal functions you need to be able to:

- Evaluate basis functions.
- Evaluate expansion coefficients.
- Integrals of polynomials $\times$ basis functions.
- Implement boundary conditions.
- Integrate products of basis functions $\times$ polynomials.
- Evaluate derivatives of basis functions.
- Evaluate products of basis functions, derivatives of basis functions and polynomials.
- Evaluate singular integrals.
- Evaluate integrals with moving singularities.

Standard methods cannot be used to satisfy these requirements.

The renormalization group equation is a new computational tool!

The renormalization group equation and the scale fixing condition provide a means to solve the problems on the previous slide!

Computation of fractal basis functions?

Approximation by iteration.
Pick any function $f_1(x)$ satisfying $\int f_1(x)dx = 1$.

$$f_n(x) = \sum_l h_l D T^l f_{n-1}(x)$$

$$s(x) = \lim_{n\to\infty} f_n(x)$$

Exact calculation at dyadic rationals,
use the renormalization group equation at dyadic rational points:

$$s(n) = \sqrt{2}\sum_l h_l s(2n-l) \qquad \sum_n s(n) = 1 \qquad n = 1, 2, \cdots 2K-2$$

$$s(r/2) = \sqrt{2}\sum_l h_l s(r-l) \qquad r = \frac{m}{2^k}$$

Calculations of integrals use the renormalization group equation and the scale fixing condition.

All moments of $s_n^k(x)$ and $w_n^l(x)$ can be computed exactly.

$$< x^m >_{s_n^k} := \int x^m s_n^k(x) dx = (\frac{1}{2})^{\frac{3nm}{2}} \int (x+n)^m s(x) dx$$

$$< x^m >_{w_n^k} := \int x^m w_n^k(x) dx = (\frac{1}{2})^{\frac{3nm}{2}} \int (x+n)^m w(x) dx$$

Calculating moments:

$$< x^m >_s = \int s(x) x^m dx \qquad < x^m >_w = \int w(x) x^m dx.$$

$$< x^0 >_s = (x^0, s) = \int dx s(x) = 1$$

Using the renormalization group equation:

$$< x^m >_s = (x^m, s) = (D^{-1} x^m, D^{-1} s)$$

$$= \frac{1}{\sqrt{2}} \frac{1}{2^m} \sum_l h_l (x^m, T^l s)$$

$$= \frac{1}{\sqrt{2}} \frac{1}{2^m} \sum_l h_l ((x+l)^m, s)$$

$$= \frac{1}{\sqrt{2}} \frac{1}{2^m} \sum_l h_l \sum_{k=0}^{m} \frac{m!}{k!(m-k)!} l^{m-k} < x^k >_s .$$

Using $\sum_l h_l = \sqrt{2}$, and moving the $k = m$ term to the left side of the above equation gives the recursion relation:

$$< x^m >_s = \frac{1}{2^m - 1} \frac{1}{\sqrt{2}} \sum_{k=0}^{m-1} \frac{m!}{k!(m-k)!} \left(\sum_{l=1}^{2K-1} h_l l^{m-k} \right) < x^k >_s \ .$$

Note $k < m$

$$< x >_s := \int x s(x) dx = \frac{1}{\sqrt{2}} \sum_{l=0}^{2K-1} l h_l .$$

With this definition:

$$\int (a + bx) s(x) dx = a + b < x >_s \ .$$

Boundary conditions:

$$< x^m >_{s_n:0} = \int_0^\infty x^m s_n(x) dx$$

The renormalization group equation relates these endpoint partial moments to ordinary moments.

The computation reduces to linear algebra.

Integrating functions with small support

One-point quadrature

Property of moments

$$\int x^2 s(x)dx = (\int xs(x)dx)^2$$

$$< x^2 >_s = < x >_s^2$$

$$\int p(x)s(x)dx = \int (a + bx + cx^2)s(x)dx =$$

$$a + b\langle x\rangle_s + c\langle x\rangle_s^2 = p(\langle x\rangle_s)$$

$$\langle x\rangle_s = \frac{1}{\sqrt{2}} \sum_{l=0}^{2K-1} l h_l.$$

General methods - integrals of products of basis functions:

Step 1: Increasing resolution 2^{-k}:

$$s_n^k(x) = \sum_{l=0}^{2K-1} h_l s_{2n+l}^{k+1}(x)$$

Step 2: Replacing wavelets by scaling functions:

$$w_n^k(x) = \sum_{l=0}^{2K-1} g_l s_{2n+l}^{k+1}(x)$$

Step 3: Changing scale: $2^{-k} \to 0$

$$\int s_{n_1}^k(x) \cdots s_{n_m}^k(x) dx = 2^{\frac{km}{2}-k} \int s_{n_1}^0(x) \cdots s_{n_m}^0(x) dx$$

Calculating the $k = 0$ integrals

Step 4:Use the renormalization group equation

$$\Gamma_{n_1,\cdots,n_k} := \int s^0_{n_1}(x)\cdots s^0_{n_m}(x)dx \qquad \text{use } n_1 = 0$$

Homogeneous equation (RG equation):

$$\Gamma_{n_1,\cdots n_m} = \sum 2^{m/2-1} h_{l_1}\cdots h_{l_m}\Gamma_{2n_1+l_1,\cdots,2n_m+l_m}$$

Step 5: Inhomogeneous equation: use $\sum s_n(x) = 1$:

$$\sum_{n_1} \Gamma_{n_1,\cdots n_m} = \Gamma_{n_2,\cdots n_m}$$

Step 6: Solve the finite linear system

Integrating products of scaling functions and polynomials

Homogeneous equations (RG equation)

$$I^m_{n_1\cdots n_k} := \int x^m s_{n_1}(x)\cdots s_{n_k}(x)dx =$$

$$2^{-\frac{2m+k}{2}}\sum h_{l_1}\cdots h_{l_k}\int x^m s_{2n_1+l_1}(x)\cdots s_{2n_k+l_k}(x)dx =$$

$$2^{-m-k/2}\sum h_{l_1}\cdots h_{l_k} I^m_{2n_1+l_1,\cdots 2n_k+l_k}$$

Inhomogeneous equations

$$\sum_{n_1} I^m_{n_1\cdots n_k} = I^m_{n_2\cdots n_k}$$

Solve using linear algebra and recursion on k.

A necessary condition for the solution of the RG equation to have k derivatives can be obtained by differentiating the RG equation k times, which gives

$$\frac{d^k s(x)}{dx^k}(x) = \sqrt{2}2^k \sum_l h_l \frac{d^k s(x)}{dx^k}(2x - l)$$

Letting $x = m$ and $n = 2m - l$ gives the eigenvalue equation

$$\frac{d^k s}{dx^k}(m) = \sqrt{2}2^k \sum_n h_{2m-n} \frac{d^k s}{dx^k}(n)$$

$$\sum_n H_{mn} \frac{d^k s}{dx^k}(n) = 2^{-k-\frac{1}{2}} \frac{d^k s}{dx^k}(m)$$

The matrix is $(2K - 2) \times (2K - 2)$ which limits the number of eigenvalues (again - calculus replaced by linear algebra).

Derivatives of basis functions

$$\frac{ds(x)}{dx} = 2\sum h_l D T^l \frac{ds(x)}{dx}$$

Replaces renormalization group equation

Differentiate

$$x = < x >_{s_n} + \sum_n n s_n(x)$$

to get a normalization condition

$$1 = \sum_n n \frac{ds_n(x)}{dx}$$

Example: Integral of product of derivatives $(K = 3)$

0 unless the support of s_m and s_n overlap

$$D_{mn} = D_{m-n,0} = \int \frac{ds_m(x)}{dx} \frac{ds_n(x)}{dx} dx$$

non-zero $D_{m,0}$ have exact rational values

$$D_{40} = D_{-40} = -3/560$$

$$D_{30} = D_{-30} = -4/35$$

$$D_{20} = D_{-20} = 92/105$$

$$D_{10} = D_{-10} = -356/105$$

$$D_{00} = 295/56.$$

Singular integrals

$$S_n^+ := \int \frac{s_n(x)dx}{x + i0^+}$$

Renormalization group equation

$$S_n^+ := \sqrt{2} \sum_l h_l S_{2n+l}^+$$

Treatment of singularity (partition of unity)

$$-i\pi = \sum_n \int_{-a}^{a} \frac{dx s_n(x)}{x + i0^+} = \sum_n S_{n:a}^+$$

$$S^+_{n:a} = \int_{-a}^{a} \frac{s_n(x)dx}{x + i0^+} = \frac{1}{n}\int_{-a-n}^{a-n} \frac{s(x)dx}{1 + x/n} =$$

$$\frac{1}{n}\sum_{k=0}^{\infty}(\frac{-1}{n})^k \int_{-a-n}^{a-n} x^k s(x)dx$$

RG equation couples integrals over $x = 0$ to integrals with support far from $x = 0$. The integrals far from the singularity can be approximated in terms of moments. Rigorous error bounds can be computed. The evaluation of the integrals reduce to algebra.

$$S_n^+ = \int \frac{s_n(x)dx}{x + i0^+} \qquad n = -1, -2, -3, -4$$

Table 2: Singular integrals $(K = 3)$

S_{-1+}	-0.1717835441734- i4.041140804162
S_{-2+}	-1.7516314066967+ i1.212142562305
S_{-3+}	-0.3025942645356- i0.299291822651
S_{-4+}	-0.3076858066180- i0.013302589081

Integrals with natural logs

$$L(n) := \int_0^\infty s_n(x)\ln(x)dx$$

The renormalization group equations gives

$$L(n) = \frac{1}{\sqrt{2}}\left(\sum_l h_l L(2n+l) - \ln(2)\right).$$

$L(n)$ for large n can be expressed in terms of moments

$$L(n) = \int s_n(x)\ln(x)dx = \int s(y)\ln(n(1+y/n))dy$$

$$= \ln(n) - \sum_{m=1}^{\infty} \frac{(-1)^m}{m}\frac{< x^m >_s}{n^m}.$$

Table 2: Log integrals

$K = 2$		
$n = -2$	$\int s_n(x) \ln \lvert x \rvert dx$	0.456927033732831
$n = -1$	$\int s_n(x) \ln \lvert x \rvert dx$	-1.64215549088219
$K = 3$		
$n = -4$	$\int s_n(x) \ln \lvert x \rvert dx$	1.15737952417967
$n = -3$	$\int s_n(x) \ln \lvert x \rvert dx$	0.750468355278047
$n = -2$	$\int s_n(x) \ln \lvert x \rvert dx$	0.315624303943019
$n = -1$	$\int s_n(x) \ln \lvert x \rvert dx$	-1.83646456399118

Autocorrelation function

$$A(x) := \int s(x-y)s(y)dy$$

Renormalization group equation for $A(x)$

$$A(x) = \sum_{m,n} h_m h_n A(2x - m - n)$$

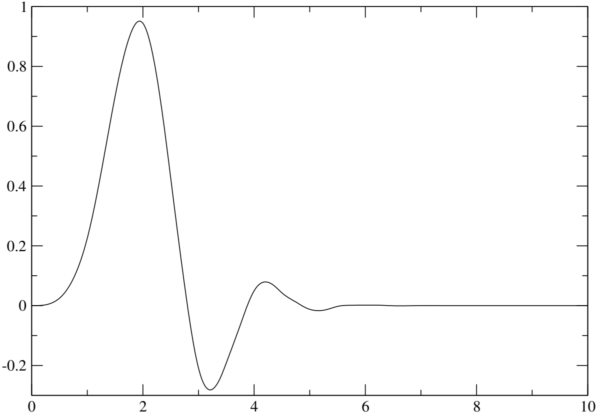
$$a_l = \frac{1}{\sqrt{2}} \sum_n h_{l-n} h_n$$

$$A(x) = \sum_{l=0}^{4K-2} a_l D T^l A(x)$$

Scale fixing for A(x)

$$\int A(x) = 1$$

Autocorrelation function A(x) (K=3)



Integrals with moving singularities

$$J_{k-m-n} := \int \frac{s_m(x)s_n(y)dxdy}{k-x-y+i0^+} = \int \frac{A(x)dx}{k-m-n-x+i0^+}$$

Renormalization group equation for $A(x)$

$$J_n = \sqrt{2}\sum_l a_l J_{2n-l}$$

$$1 = \sum_n A(x+n)$$

$$i\pi = \sum_n \int_{-a}^{a} \frac{A(x+n)dx}{-x+i0^+} = \sum_n J_{n:a}$$

Applications (my interests)

- **Solving singular integral equations for scattering and imaging.**

- **Renormalizing equations of quantum field theory.**

  Detemine how parameters of theory behave as a function of resolution and volume for fixed measured quantities. Looking for fixed point.

- **Path integral representations of quantum field theory.**

  Replaces time evolution of system with an infinite number of degrees of freedom by an infinite dimensional integral. Important for quantum computing.

Conclusions

- **Daubechies wavelets are a useful basis for problems involving multiple scales.**
- **Standard numerical methods do not work very well when applied to fractal functions.**
- **In most cases the standard numerical methods can be replaced by new methods based on the renormalization group equation.**
- **The calculations of derivatives and integrals are replaced by linear algebra.**
- **Basis provides control over resolution and volume while remaining efficient.**

Local references:

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