Wavelet Notes

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February 11, 2019

Abstract

Notes on using wavelets in scattering calculations

file: usr/wavelets/reviesd-notes.tex

1 Continuous Wavelets

We begin by considering the continuous wavelet transform. The continuous wavelet transform is an alternate representation of a function, like a Fourier transform. Both continuous and discrete wavelets are built from a single function called a **mother function**. The notation, $\psi(x)$, is used to denote the mother function of a wavelet.

Wavelets are built from translations and scale transformations of the mother function. Translations and scale transformations of $\psi(x)$ are defined by:

$$\psi_{t,s}(x) := |s|^{-p}\psi(\frac{x-t}{s}). \tag{1}$$

The factor p is a parameter. The functions $\psi_{t,s}(x)$ are the wavelets associated with the mother function $\psi(x)$. The wavelet $\psi_{t,s}(x)$ has two continuous parameters. We investigate conditions on the mother function that allow one to expand any function in terms of wavelets.

To choose the parameter p note that

$$\int_{-\infty}^{\infty} \left| |s|^{-p} \psi(\frac{x-t}{s}) \right|^q dx =$$

$$|s|^{1-qp} \int_{-\infty}^{\infty} |\psi(u)|^q du. \tag{2}$$

It follows that if $p = 1/q$ the L^q-norm of ψ

$$\|M\|_q := \left(\int_{-\infty}^{\infty} |M(u)|^q du \right)^{1/q} \tag{3}$$

is preserved under scale transformations. Thus for $p = 1/q$:

$$\|\psi\|_q = \|\psi_{t,s}\|_q \qquad \text{for all} \qquad s, t. \tag{4}$$

The **continuous wavelet transform** of f is defined by taking the scalar product of f with the wavelet ψ_{ts}:

$$\hat{f}(s,t) := \int_{-\infty}^{\infty} \psi^*_{s,t}(x) f(x) dx = (\psi_{s,t}, f) \tag{5}$$

where asterik $'*'$ indicates the complex conjugate for a complex mother function. In what follows a $\hat{f}$ is used to indicate the wavelet transform of a function f.

Parseval's identity for the Fourier transform implies that the wavelet transform can be expressed in terms of the original function and the mother function or alternatively in terms of their Fourier transforms:

$$\hat{f}(s,t) = (\psi_{s,t}, f) = (\tilde{\psi}_{s,t}, \tilde{f}) \tag{6}$$

where the $\sim$ indicates the Fourier transform defined by:

$$\tilde{\psi}_{s,t}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx} \psi_{s,t}(x) dx \tag{7}$$

$$\tilde{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx} f(x) dx. \tag{8}$$

Note that Parseval's identity states $(f, f) = (\tilde{f}, \tilde{f})$, however using this with $f = g + h$ and $f = g + ih$ gives

$$(g, g) + (h, h) + (g, h) + (h, g) = (\tilde{g}, \tilde{g}) + (\tilde{h}, \tilde{h}) + (\tilde{g}, \tilde{h}) + (\tilde{h}, \tilde{g}) \tag{9}$$

and

$$(g, g) + (h, h) + i(g, h) - i(h, g) = (\tilde{g}, \tilde{g}) + (\tilde{h}, \tilde{h}) + i(\tilde{g}, \tilde{h}) - i(\tilde{h}, \tilde{g}) \tag{10}$$

which, using the identities $(g, g) = (\tilde{g}, \tilde{g})$ and $(h, h) = (\tilde{h}, \tilde{h})$, gives the solution to (9) and (10):

$$(g, h) = (\tilde{g}, \tilde{h}) \tag{11}$$

which is the form of Parseval's identity used in (6).

The Fourier transform of $\psi_{s,t}(x)$ can be expressed in terms of the Fourier transform of the mother function:

$$\tilde{\psi}_{s,t}(k) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx} |s|^{-p} \psi(\frac{x-t}{s}) dx =$$

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-iksu} e^{-ikt} |s|^{-p+1} \psi(u) du =$$

$$|s|^{1-p} e^{-ikt} \tilde{\psi}(sk). \tag{12}$$

The inner product of the Fourier transforms gives

$$\hat{f}(s, t) = (\tilde{\psi}_{s,t}, \tilde{f}) =$$

$$\int_{-\infty}^{\infty} \tilde{\psi}^*_{s,t}(k) \tilde{f}(k) dk$$

$$\int_{-\infty}^{\infty} |s|^{1-p} e^{ikt} \tilde{\psi}^*(sk) \tilde{f}(k) dk. \tag{13}$$

Multiply both sides of (13) by $e^{-ik't}$ and integrate over t to get

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ik't} (\tilde{\psi}_{s,t}, \tilde{f}) dt =$$

$$|s|^{1-p} \tilde{\psi}^*(sk') \tilde{f}(k'). \tag{14}$$

The representation of the delta function:

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i(k'-k)t}dt = \delta(k'-k). \tag{15}$$

was used to get (14).

The right had side of (14) is a product of the Fourier transform of the original function with another function. We can't divide by the function $\tilde{\psi}^*(sk')$ because it might be zero for some values of k'. Instead, the trick is to eliminate it using the variable s.

Multiply both sides of this equation by $\tilde{\psi}(sk')$ and a yet to be determined weight function $w(s)$ and integrate over s. This gives

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} w(s)ds \int_{-\infty}^{\infty} dte^{-ik't}\tilde{\psi}(sk')\hat{f}(s,t) =$$

$$\tilde{f}(k')\int_{-\infty}^{\infty} w(s)ds|s|^{1-p}\tilde{\psi}^*(sk')\tilde{\psi}(sk') = \tilde{f}(k')Y(k') \tag{16}$$

where

$$Y(k') = \int_{-\infty}^{\infty} dsw(s)|s|^{1-p}|\tilde{\psi}(sk')|^2. \tag{17}$$

In order to be able to extract the Fourier transform of the original function, it is sufficient that $Y(k')$ satisfies $0 < A \leq Y(k') \leq B < \infty$ for some numbers A and B. In this case

$$\tilde{f}(k) = \frac{1}{2\pi Y(k)}\int_{0}^{\infty} w(s)ds \int_{-\infty}^{\infty} dte^{-ikt}\tilde{\psi}(sk)\hat{f}(s,t). \tag{18}$$

It is convenient to rewrite this in terms of the wavelet basis:

$$\tilde{f}(k) = \frac{1}{2\pi Y(k)}\int_{-\infty}^{\infty} w(s)|s|^{p-1}ds \int_{-\infty}^{\infty} dt\tilde{\psi}_{s,t}(k)\hat{f}(s,t). \tag{19}$$

We define the **dual wavelet** by

$$\tilde{\psi}^{s,t}(k) = \frac{1}{2\pi Y(k)}\tilde{\psi}_{s,t}(k). \tag{20}$$

The dual wavelet is distinguished from the ordinary wavelet by having the parameters s,t appearing as superscripts rather than subscripts.

The inversion formula can be expressed in terms of the dual wavelet by

$$\tilde{f}(k) = \int_{-\infty}^{\infty} w(s)|s|^{p-1} ds \int_{-\infty}^{\infty} dt \tilde{\psi}^{s,t}(k) \hat{f}(s,t). \tag{21}$$

In order to recover the original function, take the inverse Fourier transform of this expressions:

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk e^{ikx} \tilde{f}(k) =$$

$$\int_{-\infty}^{\infty} w(s)|s|^{p-1} ds \int_{-\infty}^{\infty} dt \psi^{s,t}(x) \hat{f}(s,t) \tag{22}$$

where

$$\psi^{s,t}(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk e^{ikx} \tilde{\psi}^{s,t}(k). \tag{23}$$

In general this is a tedious procedure because the dual wavelet $\psi^{s,t}(x)$ must be computed using (20) and (23) for each value of s and t. If the dual wavelet also had a mother function, then it would only be necessary to Fourier transform the "dual mother" and then all of the other Fourier transforms could be expressed in terms of the transform of the "dual mother".

The first step in constructing a "dual mother" is to investigate the structure of the dual wavelets in x-space:

$$\psi^{s,t}(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk e^{ikx} \tilde{\psi}^{s,t}(k) =$$

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk e^{ikx} \frac{1}{2\pi Y(k)} |s|^{1-p} e^{-ikt} \tilde{\psi}(sk) =$$

$$\psi^{s,0}(x-t)$$

where

$$\psi^{s,0}(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk e^{ikx} \frac{1}{2\pi Y(k)} |s|^{1-p} \tilde{\psi}(sk).$$

This shows for a single scale the dual wavelet and its translation can be expressed in terms of a single function. This is not necessarily true for the dual wavelet and the scaled quantity.

$$\psi^{s,0}(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk e^{ikx} \frac{1}{2\pi Y(k)} |s|^{1-p} \tilde{\psi}(sk) =$$

$$\psi^{s,0}(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} due^{iu\frac{x}{s}} \frac{1}{2\pi Y(u/s)} |s|^{-p} \tilde{\psi}(u).$$

This fails to be a rescaling of a single function because of the s dependence in the quantity $Y(u)$. It follows that if *a weight function* $w(s)$ is chosen so $Y(u/s) = Y(u)$, the dual wavelet will satisfy

$$\psi^{s,0}(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} due^{iu\frac{x}{s}} \frac{1}{2\pi Y(u/s)} |s|^{-p} \tilde{\psi}(u) = |s|^{-p} \psi^{1,0}(x/s). \tag{24}$$

Note that in this case $Y(u)$ is a constant which we denote by Y. The function $\psi^{1,0}(x)$ serves as the dual mother wavelet.

To determine $w(s)$ note that

$$Y(sk) = \int_{-\infty}^{\infty} dtw(t)|t|^{1-p}|\tilde{\psi}(tsk)|^2.$$

Let $t' = st$ to get

$$Y(sk) = \int_{-\infty}^{\infty} dtw(t)|t|^{1-p}|\tilde{\psi}(tsk)|^2 =$$

$$|s|^{p-2} \int_{-\infty}^{\infty} dt' w(t'/s)|t|'^{1-p}|\tilde{\psi}(t'k)|^2.$$

This will equal $Y(k)$ provided

$$w(t') = |s|^{p-2} w(t'/s) \qquad w(s) = |s|^{p-2} w(1).$$

With this choice

$$Y(k) = Y = w(1) \int_{-\infty}^{\infty} \frac{dt}{|t|} |\tilde{\psi}(t)|^2.$$

Assuming this choice of weight the admissibility condition becomes

$$0 < A \leq Y \leq B < \infty.$$

Having computed the constant Y it is now possible to write down an explicit expression for the dual mother wavelet:

$$\psi^{s,0}(x-t) =$$

$$\frac{|s|^{-p}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} du e^{iu\frac{(x-t)}{s}} \frac{1}{2\pi Y} \tilde{\psi}(u)$$

Letting $k = u/s$

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk \frac{1}{2\pi Y} |s|^{1-p} e^{ik(x-t)} \tilde{\psi}(ks)$$

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk \frac{1}{2\pi Y} e^{ikx} \tilde{\psi}_{s,t}(k).$$

This has the form

$$\psi^{s,t}(x) = \frac{1}{2\pi} \frac{1}{Y} \psi_{s,t}(x). \tag{25}$$

Thus the inversion procedure can be summarized by the formulas:

$$f(x) = \int_{-\infty}^{\infty} |s|^{2p-3} ds \int_{-\infty}^{\infty} dt \psi^{s,t}(x) \hat{f}(s,t) \tag{26}$$

$$Y = \int_{-\infty}^{\infty} \frac{dt}{|t|} |\tilde{\psi}(t)|^2 \tag{27}$$

$$\psi^{s,t}(x) = \frac{\psi_{s,t}(x)}{2\pi Y} \tag{28}$$

$$\psi_{s,t} = |s|^{-p} \psi(\frac{x-t}{s}). \tag{29}$$

The mother function must satisfy $0 < Y < \infty$. This requires that the Fourier transform of the mother function vanishes at the origin. This is equivalent to saying that the integral of the mother function is zero. Using the representation for the wavelet transform gives a representation of a delta function:

$$\delta(x-y) =$$

$$\int_{-\infty}^{\infty} |s|^{2p-3} ds \int_{-\infty}^{\infty} dt \psi^{s,t}(x) \psi^*_{s,t}(y) =$$

$$\frac{1}{2\pi Y} \int_{-\infty}^{\infty} |s|^{2p-3} ds \int_{-\infty}^{\infty} dt \psi_{s,t}(x) \psi^*_{s,t}(y).$$

We can also use this to formulate a Parseval's identity for wavelets

$$(f,f) = \frac{1}{2\pi Y} \int_{-\infty}^{\infty} |s|^{2p-3} ds \int_{-\infty}^{\infty} dt |\hat{f}(s,t)|^2. \tag{30}$$

Consider the example of the **Mexican hat** wavelet. The mother function is

$$\psi(x) = \frac{1}{\sqrt{2\pi}}(x^2 - 1)e^{-x^2/2}.$$

To work with the Mexican hat mother function it is useful to derive general properties of Gaussian integrals:

$$\int_{-\infty}^{\infty} e^{-ax^2+bx+c}dx =$$

$$\int_{-\infty}^{\infty} e^{-a(x-\frac{b}{2a})^2+\frac{b^2}{4a}+c}dx.$$

Change variables to $y = \sqrt{a}(x - \frac{b}{2a})$ to obtain:

$$\frac{e^{\frac{b^2}{4a}+c}}{\sqrt{a}} \int_{-\infty}^{\infty} e^{-y^2}dy =$$

$$\sqrt{\frac{\pi}{a}}e^{\frac{b^2}{4a}+c}.$$

This can be used to compute the Fourier transform of the Mexican hat mother function:

$$\tilde{\psi}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx}\psi(x)dx =$$

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} (x^2 - 1)e^{-x^2/2-ikx}dx$$

To do the integral insert a parameter a which will be set to 1 at the end of the calculation:

$$(-2\frac{d}{da} - 1)\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-x^2a/2-ikx}dx =$$

$$(-2\frac{d}{da} - 1)\frac{1}{2\pi}\sqrt{\frac{2\pi}{a}}e^{-\frac{k^2}{2a}} =$$

$$(\frac{1}{a} - \frac{k^2}{a^2} - 1)\sqrt{\frac{1}{2\pi a}}e^{-\frac{k^2}{2a}}.$$

In the limit that $a \to 1$ this becomes

$$-\sqrt{\frac{1}{2\pi}}k^2e^{-\frac{k^2}{2}}.$$

Using this expression it is possible to calculate the coefficient Y

$$Y = \int_{-\infty}^{\infty} \frac{dk}{|k|} |\tilde{\psi}(k)|^2 =$$

$$\int_{-\infty}^{\infty} \frac{dk}{|k|} |\tilde{\psi}(k)|^2 =$$

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} |k|^3 dk e^{-k^2} =$$

$$\frac{1}{\pi} \int_{0}^{\infty} k^3 dk e^{-k^2}.$$

Inserting a parameter a which will eventually be set to 1 gives

$$\frac{1}{2\pi}(-\frac{d}{da}) \int_{0}^{\infty} 2k dk e^{-ak^2} =$$

$$\frac{1}{2\pi}(-\frac{d}{da})\frac{1}{a} \int_{0}^{\infty} dv e^{-v} =$$

$$\frac{1}{2\pi}.$$

This satisfies the essential inequality $0 < Y < \infty$ which ensures the admissibility of the Mexican hat mother function.

The expression for the wavelet transform and its inverse can be written as:

$$\hat{f}(s,t) = |s|^{-p} \int_{-\infty}^{\infty} dx \frac{1}{\sqrt{2\pi}}((\frac{x-t}{s})^2 - 1)e^{-(\frac{x-t}{s})^2/2} f(x) =$$

$$|s|^{1-p} \int_{-\infty}^{\infty} du \frac{1}{\sqrt{2\pi}}(u^2 - 1)e^{-u^2/2} f(su + t).$$

where $x = su + t$

The inverse is formally given by

$$f(x) = \int_{-\infty}^{\infty} |s|^{2p-3} ds \int_{-\infty}^{\infty} dt \frac{\psi_{st}(x)}{2\pi Y} \hat{f}(s,t) =$$

$$\int_{-\infty}^{\infty} |s|^{2p-3} ds \int_{-\infty}^{\infty} dt \frac{1}{\sqrt{2\pi}} |s|^{-p}((\frac{x-t}{s})^2 - 1)e^{-(\frac{x-t}{s})^2/2} \hat{f}(s,t)$$

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}|s|^{p-3}ds\int_{-\infty}^{\infty}dt((\frac{x-t}{s})^2-1)e^{-(\frac{x-t}{s})^2/2}\hat{f}(s,t)$$
$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}|s|^{p-3}ds\int_{-\infty}^{\infty}du(u^2-1)e^{-u^2/2}\hat{f}(s,su+x)$$

where $t = su + x$.

Initially we were concerned because we were representing an arbitrary function by a linear superposition of functions that all had zero integral. We could not understand how wavelets could be used to represent a function with non-zero integral.

We tested this by computing the wavelet transform and its inverse for a Gaussian function with the Mexican hat wavelet. The original Gaussian function was recovered.

The resolution of this paradox has to do with the difference between L^1 and L^2 convergence. The wavelet transform has a vanishing L^1 norm, but the L^2 norm is non-zero.

2 Scaling Functions and Wavelets

The concept of scaling functions is most easily understood using Haar wavelets (these are made out of simple box functions).

The Haar scaling function is defined by

$$\phi(x) = \begin{cases} 0 & x \leq 0 \\ 1 & 0 < x \leq 1 \\ 0 & x > 1 \end{cases} . \tag{31}$$

Note that it is normalized so

$$(\phi, \phi) := \int_{-\infty}^{\infty} \phi^*(x)\phi(x)dx = \int_0^1 \phi^*(x)dx = 1. \tag{32}$$

The operations of translation and dilatation are used extensively in the study of wavelets. Define the unit translation operator T by

$$(T\chi)(x) = \chi(x-1). \tag{33}$$

This operator moves $\chi(x)$ to the right by 1 unit. The translation operator has the following properties:

$$(T\psi, T\chi) = \int_{-\infty}^{\infty} \psi^*(x-1)\chi(x-1)dx = \tag{34}$$

changing variables $y = x - 1$ gives:

$$\int_{-\infty}^{\infty} \psi^*(y)\chi(y)dy = (\psi, \chi) \tag{35}$$

or

$$(T\psi, T\chi) = (\psi, \chi). \tag{36}$$

If A is a linear operator its **adjoint** $A^\dagger$ is defined by the relation

$$(\psi, A^\dagger \chi) = (A\psi, \chi). \tag{37}$$

It follows that

$$(\psi, T^\dagger \chi) = (T\psi, \chi) = \int_{-\infty}^{\infty} \psi^*(x-1)\chi(x)dx. \tag{38}$$

Changing variables to $y = x - 1$ gives

$$(\psi, T^\dagger \chi) = \int_{-\infty}^{\infty} \psi^*(y)\chi(y+1)dy \tag{39}$$

or

$$(T^\dagger \chi)(x) = \chi(x+1). \tag{40}$$

Since

$$(\psi, \chi) = (T\psi, T\chi) = (\psi, T^\dagger T\chi) \tag{41}$$

it follows that $T^\dagger = T^{-1}$. An operator whose adjoint is its inverse is called unitary. Unitary operators preserve inner products.

It follows from the definition of the Haar scaling function $\phi(x)$ that

$$(T^m\phi, T^n\phi) = (\phi, T^{n-m}\phi) = \int_{-\infty}^{\infty} \phi^*(x)\phi(x-n+m)dx =$$

$$\int_0^1 \phi(x-n+m)dx = \delta_{nm} \tag{42}$$

This means the functions

$$\phi_m(x) := (T^n\phi)(x) = \phi(x-n) \tag{43}$$

are orthonormal. There are an infinite number of these functions as $n : -\infty \to \infty$ in integer s.pdf.

Define $\mathcal{V}_0$ to be the subspace of the space of square integrable functions of the form

$$f(x) = \sum_{n=-\infty}^{\infty} f_n \phi_n(x) = \sum_{n=-\infty}^{\infty} f_n (T^n \phi)(x) \tag{44}$$

where the square integrability requires that the coefficients satisfy

$$\sum_{n=-\infty}^{\infty} |f_n|^2 < \infty. \tag{45}$$

For the Haar scaling functions $\mathcal{V}_0$ is the space of square integrable functions that are piecewise constant on each unit interval. Note that while there are an infinite number of functions in $\mathcal{V}_0$, it is a small subspace of the space of square integrable functions.

In addition to translations, define the linear operator D corresponding to scale transformations:

$$(D\chi)(x) = \frac{1}{\sqrt{2}}\chi(x/2). \tag{46}$$

When this is applied to the Haar scaling function it gives

$$(D\phi)(x) = \begin{cases} 0 & x \leq 0 \\ \frac{1}{\sqrt{2}} & 0 < x \leq 2 \\ 0 & x > 2 \end{cases} \tag{47}$$

This function has the same box structure, except it is twice as wide as the original scaling function and shorter by a factor of $\sqrt{2}$. Note that the normalization ensures

$$(D\psi, D\chi) = \int_{-\infty}^{\infty} \frac{1}{2}\psi^*(x/2)\chi(x/2)dx = \tag{48}$$

setting $y = x/2$,

$$\int_{-\infty}^{\infty} \frac{2}{2}\psi^*(y)\chi(y)dy = (\psi, \chi) \tag{49}$$

or

$$(D\psi, D\chi) = (\psi, \chi). \tag{50}$$

To compute the adjoint of D note that

$$(\psi, D^\dagger \chi) = (D\psi, \chi) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2}}\psi^*(x/2)\chi(x)dx. \tag{51}$$

Setting $y = x/2$

$$\int_{-\infty}^{\infty} \psi^*(y)\sqrt{2}\chi(2y)dy \tag{52}$$

implies that

$$(D^\dagger \chi)(x) = \sqrt{2}\chi(2x). \tag{53}$$

This shows that $D^\dagger = D^{-1}$ with D also unitary.

Define the functions

$$\phi_{mn}(x) = (D^m \phi_n)(x) = 2^{-m/2}\phi(2^{-m}x - n) = 2^{-m/2}\phi(2^{-m}(x - 2^m n)). \tag{54}$$

where the left index indicates the scale and the right index indicates the number of integer translations. It follows that for fixed m

$$(\phi_{mn}, \phi_{mk}) = (D^m \phi_n, D^m \phi_k) = (\phi_n, D^{m-m}\phi_k) = (\phi_n, \phi_k) = \delta_{nk} \tag{55}$$

This shows that the functions $\phi_{mn}(x)$ for fixed m are orthonormal. We define the subspace $\mathcal{V}_m$ of the square integrable functions to be those functions of the form:

$$f(x) = \sum_{n=-\infty}^{\infty} f_n \phi_{mn}(x) = \sum_{n=-\infty}^{\infty} f_n (D^m T^n \phi)(x) \tag{56}$$

where the square integrability requires that the coefficients satisfy

$$\sum_{n=-\infty}^{\infty} |f_n|^2 < \infty. \tag{57}$$

In general the scaling function $\phi(x)$ is defined as the solution of a scaling equation subject to a normalization condition. The scaling equation relates the scaled scaling function, $(D\phi)(x)$, to translates of the original scaling function. The general form of this relation is

$$(D\phi)(x) = \sum h_l T^l \phi(x) \tag{58}$$

where h_l are fixed constants, and the sum may be finite or infinite.

In general this equation cannot be solved analytically. In the Haar case we can write down the solution realizing that the scaled box is stretched over two adjacent boxes with a suitable reduction in height. It follows that:

$$D\phi(x) = \frac{1}{\sqrt{2}}\phi(x/2) = \frac{1}{\sqrt{2}}\phi(x) + \frac{1}{\sqrt{2}}T\phi(x) =$$

$$\frac{1}{\sqrt{2}}\phi(x) + \frac{1}{\sqrt{2}}\phi(x-1). \tag{59}$$

Here $h_0 = h_1 = 1/\sqrt{2}$. These coefficients are special to the Haar scaling function. The best way to think of this is that the scaling function $\phi(x)$ is the solution of the scaling equation up to normalization. The normalization is fixed by

$$\int \phi(x)dx = 1.$$

To proceed a few additional relations are useful for future computations. First note

$$DT\psi(x) = D\psi(x-1) = \frac{1}{\sqrt{2}}\psi(x/2-1) = \frac{1}{\sqrt{2}}\psi(\frac{x-2}{2}) = T^2D\psi(x) \tag{60}$$

which leads to the operator relation

$$DT = T^2D. \tag{61}$$

It follows from this equation that

$$D\phi_n(x) = DT^n\phi(x) = T^{2n}D\phi(x) = T^{2n}(h_0\phi(x) + h_1T\phi(x)) \tag{62}$$

This shows that all of the basis elements in $\mathcal{V}_1$ can be expressed in terms of basis elements in $\mathcal{V}_0$.

Specifically if $\psi(x) \in \mathcal{V}_1$ then

$$\psi(x) = \sum_{-\infty}^{\infty} d_n\phi_{1n}(x) = \tag{63}$$

$$\sum_{-\infty}^{\infty} d_nD\phi_n(x) = \sum_{-\infty}^{\infty}(d_nh_0\phi_{2n}(x) + d_nh_1\phi_{2n+1}(x)) = \tag{64}$$

$$\sum_{-\infty}^{\infty} c_n\phi_n(x) \tag{65}$$

where

$$c_{2n} = d_nh_0 \qquad c_{2n+1} = d_nh_1. \tag{66}$$

It is easy to show that

$$\sum_{-\infty}^{\infty} |c_n|^2 = \sum_{-\infty}^{\infty} |d_n|^2. \tag{67}$$

What we have shown, as a consequence of the scaling equation, is the inclusion

$$\mathcal{V}_0 \supset \mathcal{V}_1. \tag{68}$$

Similarly, it is not difficult to show the inclusions

$$\cdots \mathcal{V}_{-k} \supset \mathcal{V}_{-k+1} \supset \cdots \supset \mathcal{V}_0 \supset \cdots \mathcal{V}_k \supset \mathcal{V}_{k+1} \cdots \tag{69}$$

In this example these are all spaces of piecewise constant, square integrable functions that are constant on segments that differ by powers of 2.

First note that as $k \to -\infty$ the approximation to $f(x)$ given by

$$f_k(x) = \sum_{n=-\infty}^{\infty} f_{kn} \phi_{kn}(x) \tag{70}$$

with

$$f_{kn} = \int_{-\infty}^{\infty} \phi_{kn}^*(x) f(x) dx \tag{71}$$

is bounded by the upper and lower Riemann sums for s.pdf of width 2^{-k}. This is because the coefficients f_{kn} are just average values of the function on the appropriate sub-interval (to deal with the infinite interval it is best to first consider functions that vanish outside of finite intervals and take limits). Since the upper and lower Riemann sums converge to the same integral (when the function is integrable) it follows that

$$\int_{-\infty}^{\infty} |f_k(x) - f(x)| dx < .pdfilon \tag{72}$$

for sufficiently large $-k$. A related argument can be extended to get L^2 convergence .

Similarly, as $k \to +\infty$, the width of $\phi_{kn}(x)$ grows like 2^k while the height falls off like $2^{-k/2}$. Again, if the function vanishes outside of a bounded interval then for sufficiently large k there is only one (or two) $\phi_{kn}(x)$ that are non-vanishing where the function is non-vanishing. In the case that only one ϕ_{kl} overlaps the support of $f(x)$

$$f_k(x) \sim 2^{-k/2} \phi_{kn_0}(x) \int_{-\infty}^{\infty} f(x) dx \tag{73}$$

The integral of the square of this function $\sim 2^{-k} \to 0$ as $k \to \infty$.

Note that

$$\int_{-\infty}^{\infty} f_k(x)dx \to \int_{-\infty}^{\infty} f(x)dx \tag{74}$$

as $k \to \infty$. This shows that this integral is finite in L^1 but 0 in L^2.

Define the projection operators

$$P_k f(x) = \sum_{n=-\infty}^{\infty} f_{kn}\phi_{kn}(x) \tag{75}$$

where

$$f_{kn} = \int_{-\infty}^{\infty} \phi^*_{kn}(x) f(x)dx. \tag{76}$$

The above conditions can be stated in terms of these projectors:

$$\lim_{k\to-\infty} P_k = I \tag{77}$$

$$\lim_{k\to+\infty} P_k = 0. \tag{78}$$

We are now ready to construct wavelets. First recall the condition

$$\mathcal{V}_0 \supset \mathcal{V}_1. \tag{79}$$

Let $\mathcal{W}_1$ be the space of vectors in the space $\mathcal{V}_0$ that are orthogonal to the vectors in $\mathcal{V}_1$. We can write

$$\mathcal{V}_0 = \mathcal{V}_1 \oplus \mathcal{W}_1 \tag{80}$$

This notation means that any vector in $\mathcal{V}_0$ can be expressed as a sum of two vectors - one that is in $\mathcal{V}_1$ and one that is orthogonal to every vector in $\mathcal{V}_1$.

Note that the scaling equation implies that every vector in $\mathcal{V}_1$ can be expressed as a linear combination of vectors in $\mathcal{V}_0$ using

$$D\phi_n(x) = h_0\phi_{2n}(x) + h_1\phi_{2n+1}(x) \tag{81}$$

Clearly the functions that are orthogonal to these in $\mathcal{V}_1$ on the same interval can be expressed in terms of the functions

$$\psi_{1n}(x) := D\psi_n(x) = h_0\phi_{2n}(x) - h_1\phi_{2n+1}(x) = \frac{1}{\sqrt{2}}(\phi_{2n}(x) - \phi_{2n+1}(x)) \tag{82}$$

These are also the elements of $\mathcal{V}_0$ that satisfy

$$(D\psi_{1n}, D\phi_l) = 0. \tag{83}$$

Thus we have that $\mathcal{W}_1$ is that space of square integrable functions of the form

$$f(x) = \sum_{n=-\infty}^{\infty} f_n \psi_{1n}(x) \tag{84}$$

with

$$f(x) = \sum_{n=-\infty}^{\infty} |f_n|^2 \tag{85}$$

where we have used

$$(\psi_{1n}, \psi_{1k}) = \delta_{nk} \tag{86}$$

which follows from the definitions.

Similarly we can express $\mathcal{V}_l = \mathcal{V}_{l+1} \oplus \mathcal{W}_{l+1}$ for all values of l. For the special case $l = -1$ we define the mother wavelet as

$$\psi(x) = D^{-1}(h_0\phi(x) - h_1 T\phi(x)) = \tag{87}$$

$$h_0\sqrt{2}\phi(2t) - h_1\sqrt{2}\phi(2(t-1) = (\phi(2t) - \phi(2(t-1)) \tag{88}$$

which is manifestly orthogonal to the scaling function. Translates of this function define a basis for $\mathcal{W}_0$

$$\psi_n(x) = T^n\psi(x) = T^n D^{-1}(h_0\phi(x) - h_1 T\phi(x)) = \tag{89}$$

$$D^{-1}(h_0\phi_{2n}(x) - h_1\phi_{2n+1}(x)) \tag{90}$$

It is simple to show that

$$(\psi_n, \psi_k) = \delta_{nk} \tag{91}$$

If we decompose every space we have for any k

$$\mathcal{V}_{-k} = \mathcal{W}_{-k+1} \oplus \mathcal{V}_{-k+1} = \tag{92}$$

$$\mathcal{W}_{-k+1} \oplus \mathcal{W}_{-k+2} \oplus \mathcal{V}_{-k+2} = \tag{93}$$

$$\mathcal{W}_{-k+1} \oplus \mathcal{W}_{-k+2} \oplus \cdots \oplus \mathcal{W}_l \oplus \mathcal{V}_l \tag{94}$$

Note that unlike the $\mathcal{V}$ spaces, the $\mathcal{W}_k$ spaces are all mutually orthogonal, since if $k > l \to \mathcal{W}_k \subset \mathcal{V}_l$ which is orthogonal to $\mathcal{W}_l$ by definition.

If $f(x)$ is any square integrable function the conditions

$$\lim_{k \to -\infty} P_k = I \tag{95}$$

$$\lim_{k \to +\infty} P_k = 0 \tag{96}$$

mean that for sufficiently large k and l that $f(x)$ can be well approximated by a function in

$$\mathcal{W}_{-k+1} \oplus \mathcal{W}_{-k+2} \oplus \cdots \oplus \mathcal{W}_l \tag{97}$$

This means that the function can be approximated by a linear combination of basis functions (wavelets) from each of the spaces $\mathcal{W}_r$.

Basis functions for $\mathcal{W}_m$ are given by

$$\psi_{mn}(x) = D^m T^n \psi(x) = D^{m-1}(h_0 \phi_{2n}(x) - h_1 \phi_{2n+1}(x)) \tag{98}$$

That these are a basis with the required properties is easily shown by showing that these functions are orthogonal to $\mathcal{V}_m$ and can be used to recover the remaining vectors in $\mathcal{V}_{m-1}$.

The functions $\psi_{nl}(x)$ satisfy

$$(\psi_{nl}, \psi_{n'l'}) = \delta_{nn'} \delta_{ll'} \tag{99}$$

where the $\delta_{nn'}$ follows from the orthogonality of the spaces $\mathcal{W}_n$ and $\mathcal{W}_{n'}$ for $n \neq n'$.

The $\delta_{ll'}$ follows from the unitarity of D and

$$(\psi, T^n \psi) = \delta_{n0}. \tag{100}$$

To summarize the important s.pdf one starts with a scaling equation of the form:

$$D\phi(x) = \sum h_l T^l \phi(x) \tag{101}$$

where one is normally given only the coefficients h_l. This equation is solved to find the scaling function $\phi(x)$. This, along with translations and dilitations is used to construct the spaces $\mathcal{V}_l$. The scaling equation ensures the existence of space $\mathcal{W}_k$ that can be used to build discrete orthonormal basis. The mother wavelet is expressed in terms of the scaling function and the coefficients as

$$\psi(x) = D^{-1}(h_0 \phi(x) - h_1 T \phi(x)) \tag{102}$$

which is more complicated for a general scaling equation.

In general the coefficients h_l must satisfy constraints for the solution to the scaling equation to exist.

3 Scaling Functions II

In this section we introduce a more general treatment of scaling equations. In general the scaling function is the solution of the scaling equation

$$D\phi(x) = \sum_l h_l T^l \phi(x). \tag{103}$$

In addition to the scaling equations we demand that integer translates of the scaling functions are orthonormal

$$(\phi_n, \phi_m) = (T^n\phi, T^m\phi) = (\phi, T^{m-n}\phi) = \delta_{mn} \tag{104}$$

and the initial scale is fixed by the normalization condition

$$\int \phi(x)dx = 1 \tag{105}$$

We investigate the consequences of these equations.

Using the definitions of the operators D and T the scaling equation becomes:

$$\frac{1}{\sqrt{2}}\phi(\frac{x}{2}) = \sum h_l \phi(x-l). \tag{106}$$

It can be put in the form

$$\phi(x) = \sum_l \sqrt{2} h_l \phi(2x-l). \tag{107}$$

The sums are assumed to be from $-\infty \to \infty$. Finite sums are treated by assuming that only a finite number of the h_l's are non zero.

The claim is that if this equation has a solution, it is unique up to an overall normalization factor. To investigate this claim take the Fourier transform of both sides of equation (107) to get

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ikx}\phi(x)dx = \sum_l \sqrt{2}h_l \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ikx}\phi(2x-l)dx. \tag{108}$$

Changing variables $x \to 2x - l$ gives

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ikx}\phi(x)dx = \sum_l \frac{1}{\sqrt{2}}h_l \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-i(k/2)(x+l)}\phi(x)dx. \tag{109}$$

or

$$\tilde{\phi}(k) = \tilde{\phi}(\frac{k}{2})\tilde{h}(\frac{k}{2}) \tag{110}$$

where

$$\tilde{h}(k) = \sum_l \frac{h_l}{\sqrt{2}} e^{-ikl}. \tag{111}$$

This form of the scaling equation can be iterated n times to get:

$$\tilde{\phi}(k) = \tilde{\phi}(\frac{k}{2^n}) \prod_{m=1}^{n} \tilde{h}(\frac{k}{2^m}) \tag{112}$$

This equation holds for any n provided the Fourier transforms exist. For a finite n, an approximation can be made by a finite number of iterations of the form

$$\tilde{\phi}_n(k) = \tilde{\phi}_{n-1}(\frac{k}{2})\tilde{h}(\frac{k}{2}) \tag{113}$$

for any starting function $\tilde{\phi}_0(k)$. In the limit of large n the function $\tilde{\phi}_m(k)$ should converge to the scaling function. It is clear that this result is approximately independent of the choice of starting function; it really depends on the coefficients h_l.

If the limit exists as $n \to \infty$, and the scaling function is continuous in a neighborhood of zero, then

$$\tilde{\phi}(k) = \lim_{n\to\infty} \tilde{\phi}(\frac{k}{2^n}) \prod_{l=1}^{n} \tilde{h}(\frac{k}{2^l}) =$$

$$\tilde{\phi}(0) \prod_{l=1}^{\infty} \tilde{h}(\frac{k}{2^l}). \tag{114}$$

If the infinite product converges, then we have an expression for the scaling function, up to normalization, which is fixed by assigning a value to $\tilde{\phi}(0)$.

The coefficients h_l are not arbitrary. First note that setting $k = 0$ gives

$$1 = \prod_{l=0}^{\infty} \tilde{h}(0) \tag{115}$$

or

$$\tilde{h}(0) = 1 = \sum_l \frac{h_l}{\sqrt{2}} \tag{116}$$

or

$$\sum_l h_l = \sqrt{2}. \tag{117}$$

This condition is clearly satisfied by the Haar wavelets. This is a necessary condition on the scaling coefficients in order to have a solution to the scaling equation.

Another condition is that is needed to make a multi-resolution analysis is the orthogonality of the unit translates, $(\phi_n, \phi_m) = \delta_{nm}$. This requires

$$2\sum_{lk} h_l^* h_k \int_{-\infty}^{\infty} \phi^*(2x-2n-l)\phi(2x-2m-k)dx =$$

$$2\sum_{lk} h_l^* h_k \int_{-\infty}^{\infty} \phi^*(2x)\phi(2x-2(m-n)-(k-l))dx =$$

$$\sum_{lk} h_l^* h_k \int_{-\infty}^{\infty} \phi^*(x)\phi(x-2(m-n)-(k-l))dx$$

$$\sum_l h_l^* h_{l-2(m-n)} = \delta_{mn} \tag{118}$$

or equivalently

$$\sum_l h_{l-2m}^* h_l = \delta_{m0}. \tag{119}$$

This is trivially satisfied for the Haar wavelets.

Note that the assumption that h_0 and h_1 are the only non-zero coefficients in the scaling equation uniquely give the Haar wavelets. To see this first note conditions on the coefficients are

$$h_0 + h_1 = \sqrt{2} \tag{120}$$

$$h_0 h_0^* + h_1 h_1^* = 1. \tag{121}$$

These equations have the unique real solution $h_0 = h_1 = \frac{1}{\sqrt{2}}$ To see this write $h_k = h_{kr} + i h_{ki}$ for real h_{kr} and h_{ki}. Inserting these in the equations gives

$$h_{0r} + h_{1r} = \sqrt{2} \tag{122}$$

$$h_{0i} + h_{1i} = 0 \tag{123}$$

$$h_{0r}h_{0r} + h_{0i}h_{0i} + h_{1r}h_{1r} + h_{1i}h_{1i} = 1. \tag{124}$$

Note that since

$$\sum_m h_{n+2m} = \sum_m h_{n+2m+2k} \tag{125}$$

the m sum has two values according to whether n is even or odd:

$$h_e := \sum_m h_{2m} \qquad h_o := \sum_m h_{2m+1}. \tag{126}$$

This means that

$$\sum_{m,n} h_n^* h_{n+2m} = \sum_m \delta_{m0} = 1 =$$

$$h_e \sum_n h_{2n}^* + h_o \sum_n h_{2n+1}^* = 1 \tag{127}$$

or

$$h_e^* h_e + h_o^* h_o = 1 \tag{128}$$

We also have

$$h_e + h_o = \sqrt{2} \tag{129}$$

Assuming that the coefficients h_l are real these can be solved to get

$$h_e = h_o = \frac{1}{\sqrt{2}}. \tag{130}$$

These condition are useful checks and uniquely determine the Haar coefficients; but the essential equations are (117) and (119).

The other conditions that require some study are the ones relating the scaling function to the mother function. The mother function satisfies

$$\psi(x) = \sum_n \sqrt{2} g_n \phi(2x - n). \tag{131}$$

This and all of its translates should be orthogonal to the scaling function. In terms of the coefficients:

$$(\psi_m, \phi) = \sum_{n,l} h_l g_n^* (\phi_{n-2m}, \phi_l)$$

$$\sum_{n,l} h_l g_n^* \delta_{n-2m,l}$$

$$\sum_n h_{n-2m} g_n^* = 0 \tag{132}$$

for all m. We also need orthonormality of the translated mother function

$$(\psi_m, \psi_n) = \sum_{l,k} g_l g_k^* (\phi_{l-2m}, \phi_{k-2n})$$

$$\sum_k g_{k-2(n-m)} g_k^* = \delta_{mn} \tag{133}$$

or equivalently

$$(\psi_m, \psi) = \sum_k g_{k+2m} g_k^* = \delta_{m0} \tag{134}$$

If we choose $g_k := (-1)^k h_{l-k}$ where l is odd it follows that

$$\sum_k g_{k-2(n-m)} g_k^* = \sum_k (-)^{k-2(n-m)} h_{l-k+2(n-m)} (-)^k h_{l-k}^* =$$

$$\sum_k h_{k-2(n-m)} h_k^* = \delta_{mn} \tag{135}$$

where we have let $l - k \to k$ in the last term. It also follows that

$$\sum_n h_{n-2m} g_n^* = \sum_n h_{n-2m} (-)^n h_{l-n}^* =$$

$$\sum_n h_{l-n'} (-)^{l-n'-2m} h_{n'-2m}^* = (-)^l \sum_n h_{l-n'} (-)^{n'} h_{n'-2m}^* =$$

$$(-)^l \sum_n g_{n'} h_{n'-2m}^* \tag{136}$$

Since l is odd, if $h_l = h_l^*$ the sum is equal to its negative which shows that it vanishes. [this appears to only work for real scaling coefficients]. The choice of l is arbitrary - it simply involves moving the origin of the mother. Since the mother is orthogonal to the translates of all of the father wavelets, it does not matter where the origin is placed.

This shows that the coefficients h_l, satisfying

$$\sum_l h_l = \sqrt{2}. \tag{137}$$

$$\sum_l h_{l-2m} h_l^* = \delta_{m0} \tag{138}$$

$$g_k := (-1)^k h_{l-k} \qquad l \qquad \text{odd} \tag{139}$$

gives a multi-resolution analysis, scaling function, and a mother function using

$$\phi(x) = \frac{\tilde{\phi}(0)}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ikx} \prod_{l=1}^{\infty} \tilde{h}(\frac{k}{2^l}) dk =$$

$$\frac{\tilde{\phi}(0)}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ikx} \prod_{l=1}^{\infty} \sum_{n_l} \frac{h_{n_l}}{\sqrt{2}} e^{-ikn_l/2^l}.$$

$$\tilde{\phi}(0)\sqrt{2\pi} e^{ikx} \sum_{n_1} \cdots \sum_{n_m} \cdots \prod_{l=1}^{\infty} \frac{h_{n_l}}{\sqrt{2}} \delta(x - \sum_{l=1}^{\infty} n_l/2^l).$$

This is not a very useful representation for computation, however it indicates that if a scaling function has a finite number of coefficients h_l then the scaling function has support on

$$[0, (N-1)(\frac{1}{2} + \frac{1}{4} + \frac{1}{8} \cdots)] = [0, N-1]$$

where $N = 2K$ is the number of scaling coefficients.

An alternative is to compute the scaling function exactly on a dense set of points. This construction also starts from the scaling equation:

$$\phi(x) = \sum_l \sqrt{2} h_l \phi(2x - l) \tag{140}$$

Let $x = n$ to get

$$\phi(n) = \sum_l \sqrt{2} h_l \phi(2n - l) \tag{141}$$

Let $k = 2n - l$ gives

$$\phi(n) = \sum_k \sqrt{2} h_{2n-k} \phi(k) \tag{142}$$

Which gives the eigenvalue equation

$$\phi(n) = \sum_m H_{nm} \phi(m) \tag{143}$$

where

$$H_{nm} = \sqrt{2} h_{2n-m} \tag{144}$$

Eigenvectors of this equation with eigenvalue 1 are solutions of the scaling function at integer points - up to normalization. Eigenvectors with eigenvalues other than 1 can be tossed out.

Rather than solve the eigenvalue problems, one of the equations can be replaced by the condition

$$\sum_n \phi(n) = 1 \tag{145}$$

which follows from the assumption that $\int \psi(x)dx = 0$. The support condition implies that only a finite number of the $\phi(n)$ are non-zero. This condition is independent of the orthonormality condition.

For the case of the $N = 4(K = 2)$ Daubechies wavelets these equations are

$$\begin{aligned}
\phi(0) &= \sqrt{2}h_0\phi(0) \\
\phi(1) &= \sqrt{2}(h_0\phi(2) + h_1\phi(1) + h_2\phi(0)) \\
\phi(2) &= \sqrt{2}(h_1\phi(3) + h_2\phi(2) + h_3\phi(1)) \\
\phi(3) &= \sqrt{2}h_3\phi(3) \\
1 &= \phi(0) + \phi(1) + \phi(2) + \phi(3)
\end{aligned}$$

The first and fourth equation give $\phi(0) = \phi(3) = 0$ (or $h_0 = h_1 = 1/\sqrt{2}$ which is the Haar solution). This also follows from the continuity of the wavelets, since 0 and 3 are the boundaries of the support. The second and third equations are eigenvalue equations

$$\begin{pmatrix} \phi(1) \\ \phi(2) \end{pmatrix} = \begin{pmatrix} \sqrt{2}h_1 & \sqrt{2}h_0 \\ \sqrt{2}h_2 & \sqrt{2}h_3 \end{pmatrix} \begin{pmatrix} \phi(1) \\ \phi(2) \end{pmatrix} \tag{146}$$

Instead of solving the eigenvalue problem for an eigenvector with eigenvalue 1, we use

$$\phi(1) + \phi(2) = 1 \tag{147}$$

with

$$\phi(1) = \sqrt{2}(h_0\phi(2) + h_1\phi(1))$$

to get

$$\phi(1) = \sqrt{2}(h_0(1 - \phi(1)) + h_1\phi(1))$$

which can be solved for

$$\phi(1) = \frac{\sqrt{2}h_0}{1 + \sqrt{2}(h_0 - h_1)} \tag{148}$$

and

$$\phi(2) = \frac{1-\sqrt{2}h_1}{1+\sqrt{2}(h_0-h_1)} \tag{149}$$

This gives exact values of the scaling function at integer points. It also manifestly satisfies $\sum_n \phi(n) = 1$. In this case there are only two non-zero terms.

In order to construct construct the scaling function at the point x the first step is to make a dyadic approximation to x. Let m be an integer that defines a dyadic resolution. This means that we want the dyadic approximation to satisfy the inequality $|x - x_{approx}| < 2^{-m}$. For any m it is possible to find an integer n such that

$$\frac{n}{2^m} \leq x < \frac{n+1}{2^m}. \tag{150}$$

Writing this as

$$n \leq 2^m x < n_1 \tag{151}$$

immediately gives

$$n := [2^m x] \tag{152}$$

where [] means greatest integer $\leq 2^m x$, which can be constructed by simply assigning a floating point number to an integer variable.

Since the scaling function is continuous, for any $.pdfilon > 0$ we can find a large enough m so

$$|\phi(x) - \phi(\frac{n}{2^m})| < .pdfilon$$

In what follows we evaluate $\phi(\frac{n}{2^m})$ exactly. Let $x = \frac{n}{2^m}$. We also assume that $0 < n < 4 \times 2^m$, otherwise $\phi(x) = 0$ by the support condition. In order to evaluate $\phi(x)$ note that the scaling equation gives:

$$\phi(x) = \phi(\frac{n}{2^m}) = \sqrt{2}D\phi(\frac{n}{2^{m-1}}) =$$

$$\sum_l \sqrt{2}h_{l_1}T^{l_1}\phi(\frac{n}{2^{m-1}}) = \sum_l \sqrt{2}h_{l_1}\phi(\frac{n}{2^{m-1}} - l_1) = \sum_l \sqrt{2}h_{l_1}\phi(\frac{n-2^{m-1}l}{2^{m-1}}) \tag{153}$$

Repeating this process gives

$$\phi(x) = \sum_{l_1,l_2} \sqrt{2}^2 h_{l_1}h_{l_2}\phi(\frac{n-2^{m-1}l_1-2^{m-2}l_2}{2^{m-2}}) \tag{154}$$

Using the scaling equation m times gives

$$\phi(x) = \sum_{l_1,l_2\cdots l_m} \sqrt{2}^m h_{l_1} h_{l_2} \cdots h_m \phi(n - 2^{m-1}l_1 - 2^{m-2}l_2 - \cdots 2l_{m-1} - l_m) \quad (155)$$

In this case the last expression is evaluated at integer values which gives (for the Daubechies N=4 case):

$$\phi(x) =$$
$$\sum_{l_1,l_2\cdots l_m} c_{l_1} c_{l_2} \cdots c_m \times$$
$$[\delta_{n-2^{m-1}l_1-2^{m-2}l_2-\cdots 2l_{m-1}-l_m,1} \frac{\sqrt{2}h_0}{1+\sqrt{2}(h_0-h_1)} +$$
$$\delta_{n-2^{m-1}l_1-2^{m-2}l_2-\cdots 2l_{m-1}-l_m,2} \frac{1-\sqrt{2}h_1}{1+\sqrt{2}(h_0-h_1)}] \quad (156)$$

where $c_k := \sqrt{2}h_k$

To implement this in a program, given n and m use the following s.pdf:

Let $N = 4^m$, let $\sum_2 = 0$. $\sum_2$ will eventually become $\phi(x)$.

Loop on $k : 0 \leq k < N$. Here k is a single integer that contains all of the l values

$$k = l_1 + 4l_2 + 4^2 l_3 + \cdots + 4^{m-1} l_m$$

To extract $l_1, \cdots, l_m$ from k first initialize:

$M = k$; $d = 2^{m-1}$; $\sum = 0$.

Next loop m times: $n = 1, \cdots, m$

$l_n = M - 4[M/4]$; $m_n = l_n d$; $d \to d/2$; $\sum \to \sum + m_n$; $M \to (M - l_n)/4$. Here $m_n = 2^{m-n} l_n$.

Repeat m-times (until all l_n are extracted). The result is

$$\sum = 2^{m-1}l_1 - 2^{m-2}l_2 - \cdots 2l_{m-1} - l_m$$

if $\sum -n-1=0$ then compute

$$a_1 = c(l_1)c(l_2)\cdots c(l_m)\phi(1)$$

add $\sum_2 = \sum_2 + a_1$

if $\sum -n-2=0$ then compute

$$a_1 = c(l_1)c(l_2)\cdots c(l_m)\phi(2)$$

add $\sum_2 = \sum_2 + a_1$

Integer translates can be constructed using

$$\phi_m(n) = \phi(n-m). \tag{157}$$

The scaling equation gives

$$\phi(m/2) = \sum_l \sqrt{2} h_l \phi(m-l) = \sum_n \sqrt{2} h_{m-n} \phi(n). \tag{158}$$

Translates of the scaled scaling function are

$$\phi_n(m/2) = \phi(m/2-n) = \phi(\frac{m-2n}{2}) = \sum_k \sqrt{2} h_{m-2n-k} \phi(k) =$$

$$\sum_l \sqrt{2} h_l \phi_{2n+l}(m) \tag{159}$$

This procedure can be repeated inductively to obtain

$$\phi_l(\frac{k}{2^n}) = 2^{\frac{n}{2}} \sum_{m_1 \cdots m_n} h_{m_1} \cdots h_{m_n} \phi_{2^n l + 2^{n-1} m_1 + 2^{n-2} m_2 \cdots + 2 m_{n-1} + m_n}(k) =$$

$$2^{\frac{n}{2}} \sum_{m_1 \cdots m_n} h_{m_1} \cdots h_{m_n} \phi(k - 2^n l - 2^{n-1} m_1 - 2^{n-2} m_2 \cdots - 2 m_{n-1} - m_n). \tag{160}$$

Note that in any of the sums the only non-zero contributions occur when the argument of $\phi(\cdot)$ is 1 or 2. This equation gives exact values of the scaling function at points $x = \frac{n}{2^k}$. These are dense and if the scaling function is continuous this method can be used to approximate the scaling function an any point.

This has the advantage that the function is computed exactly at many points - with iterative methods it is computed approximately at one point.

The general form of the equation is

$$\sum_{n=1}^{2N-1} \phi(n) = 1$$

$$\phi(0) = \sqrt{2}h_0\phi(0)$$
$$\phi(1) = \sqrt{2}(h_0\phi(2) + h_1\phi(1) + h_2\phi(0))$$
$$\phi(2) = \sqrt{2}(h_0\phi(4) + h_1\phi(3) + h_2\phi(2) + h_3\phi(1) + h_4\phi(0))$$
$$\vdots$$
$$\phi(2N-2) = \sqrt{2}(h_{2N-1}\phi(2N-3) + h_{2N-2}\phi(2N-2) + h_{2N-3}\phi(2N-1))$$
$$\phi(2N-1) = \sqrt{2}h_{2N-1}\phi(2N-1).$$

4 Daubechies Wavelets

The Daubechies wavelets have two special properties. First is that there are a finite number of non-zero coefficients h_i. This gives them a compact support. The second feature is that the first N moments of the wavelets are zero.

The constraint on the moments has interesting consequences. First we note that

$$\int \psi(x)x^l dx = 0 \qquad l = 0 \cdots N-1. \tag{161}$$

From this we conclude

$$\int \psi_{0m}(x)x^l dx = \int \psi(x-m)x^l dx = \int \psi(y)(y+m)^l dy =$$
$$\sum_{k=0}^{l} \frac{l!}{k!(l-k)!} m^{l-k} \int \psi(x)x^k dx = 0. \tag{162}$$

Similarly

$$\int D\psi(x)x^l dx = \frac{1}{\sqrt{2}} \int \psi(x/2)x^l dx = 2^{l+1/2} \int \psi(y)y^l dy = 0. \tag{163}$$

It straight forward to proceed inductively to show that

$$\int \psi_{nk}(x)x^l = 0 \qquad l = 0 \cdots N-1. \tag{164}$$

This means that every element of the Daubechies$_N waveletbasisisorthogonaltoallpolynomialsof$

If we consider instead the orthonormal basis consisting of

$$\{T^n\phi(x), D^mT^n\psi(x) : -m \geq 0\} \tag{165}$$

we have

$$\int \phi_m(x)x^l dx \neq 0 \qquad l = 0 \cdots N-1 \tag{166}$$

Although the polynomials are not square integrable; we can multiply a polynomial by a box function $b(x)$ which is 1 between x_- and x_+ and zero elsewhere. This product is square integrable and is equal to the polynomial on the interval $[x_-, x_+]$. It follows that

$$p(x)b(x) = \sum_{mn} c_{mn}\psi_{mn}(x) = \sum_{mn} d_n\phi_n(x) + \sum_n \sum_{m\leq 0} c_{mn}\psi_{mn}(x) \tag{167}$$

where

$$c_{mn} = \int_{x_-}^{x_+} \psi_{mn}(x)p(x)dx \tag{168}$$

$$d_n = \int_{x_-}^{x_+} \phi_n(x)p(x)dx \tag{169}$$

The moment condition means that the coefficients $c_{mn} = 0$ whenever the support of the wavelet is completely contained inside of the box. Thus in the first expression the non-zero coefficients arise from end point contributions and to many small contributions from wavelets with support that are much larger than the box.

In the second expression the wavelets with support larger than the box do not appear. The endpoint contributions only affect the answer within a distance equal to the support of the wavelet from the endpoints of the box. Inside this distance the only nonzero coefficient are due to the translates of the scaling functions. There are a finite number of these coefficients, and in this region they provide an exact representation of the polynomial. Specifically let

$$I(x) = b(x)p(x) - \sum_n d_n\phi_n(x) + \sum_n \sum_{m\leq 0} c_{mn}\psi_{mn}(x) \tag{170}$$

then we have

$$0 = \|I\|^2 = \int_{x_-}^{x_-+\Delta} I(x)^2 dx + \int_{x_+-\Delta}^{x_+} I(x)^2 dx +$$

$$\int_{x_-+\Delta}^{x_+-\Delta} |p(x) - \sum_n d_n \phi_n(x)|^2 dx. \tag{171}$$

Since all three terms are non-negative we conclude that

$$\int_{x_-+\Delta}^{x_+-\Delta} |p(x) - \sum_n d_n \phi_n(x)|^2 dx = 0. \tag{172}$$

Since Δ is fixed by the choice of the wavelet and $x_\pm$ is arbitrary we have

$$\int_a^b |p(x) - \sum_n d_n \phi_n(x)|^2 dx = 0 \tag{173}$$

for any interval $[a, b]$. Since $p(x)$ and $\phi(x)$ are continuous (we did not prove this for $\phi(x)$ - but that is the claim in the literature) and the sum of translates is finite it follows that

$$p(x) = \sum_n d_n \phi_n(x) \tag{174}$$

pointwise on every finite interval. This establishes the desired result.

Note that expansion in the wavelet basis gives all coefficients zero. This is not a contradiction because none of the polynomials are square integrable. This is reminiscent of the initial problem that we had in representing functions that did not average to zero. The key point is that once one puts a box around a function, wavelets with very large support (large m) lead to many small contributions.

What is interesting is that even though the Daubechies wavelets do not have order N smoothness, there are linear combinations of the scaling function that can exactly represent polynomials locally.

These properties are the key to the normalization coefficients for the connection coefficients Γ. The key formulas are

$$x^l = \sum_m c_{lm} \phi(x - m) \tag{175}$$

where

$$c_{lm} = \int \phi(x-m) x^l dx = \sum_{k=0}^{l} \frac{l!}{(l-k)!k!} m^{l-k} \int \phi(y) y^k dx =$$

$$\sum_{k=0}^{l} \frac{l!}{(l-k)!k!} m^{l-k} c_{k0}. \tag{176}$$

Starting with $l = 0$ these equations give

$$c_{0m} = c_{00} \tag{177}$$

$$c_{1m} = mc_{00} + c_{10} \tag{178}$$

This can be continued to express all of the translates in terms of c_{k0} for $k = 0, \cdots, l$.

Note that the normalization condition gives $c_{00} = 1$

Next note that

$$1 = \sum_m c_{0m} \phi(x - m) \tag{179}$$

$$l! = \sum_m c_{lm} \frac{d^l}{dx^l} \phi(x - m). \tag{180}$$

Multiplying by $\phi(x)$ and integrating gives

$$c_{00} = \frac{1}{l!} \sum_m c_{lm} \int \phi(x) \frac{d^l}{dx^l} \phi(x - m). \tag{181}$$

5 Daubechies Scaling Coefficients

Begin by defining the two formal polynomials:

$$P(T) = \frac{1}{\sqrt{2}} \sum_{l=0}^{M} h_l T^l \tag{182}$$

and

$$Q(T) = \frac{1}{\sqrt{2}} \sum_{l=0}^{M} g_l T^l = \frac{1}{\sqrt{2}} \sum_{l=0}^{M} (-)^l h^*_{M-l} T^l \qquad N \qquad \text{odd} \tag{183}$$

First we argue that M must be odd. By contradiction assume that $M = 2K$ is even and $h_M, h_0 \neq 0$. The orthogonality condition requires

$$\sum_{l=0}^{2K} h^*_l h_{l+2K} = h^*_0 h_M = \delta_{K0} \tag{184}$$

This vanishes if $K \neq 0$, requiring either h_0 or $h_M = 0$, contradicting the assumption that M is even. It follows that M is odd or $M = 2K - 1$.

Inserting this in the expressions for the polynomials:

$$P(T) = \frac{1}{\sqrt{2}} \sum_{l=0}^{2K-1} h_l T^l \tag{185}$$

and

$$Q(T) = \frac{1}{\sqrt{2}} \sum_{l=0}^{2K-1} (-)^l h^*_{2K-1-l} T^l \tag{186}$$

It follows that if we let $m = 2K - 1 - l$

$$[Q(-T)]^* = \frac{1}{\sqrt{2}} \sum_{m=0}^{2K-1} h_m (T^*)^{2K-1-m} \tag{187}$$

This can be expressed as

$$[Q(-T)]^* = (T^*)^{2K-1} P((T^*)^{-1}) \tag{188}$$

Properties of these polynomials are used to determine the coefficients h_l. First note that for complex $-T = z = e^{i\omega}$ on the unit circle, $z = (z^*)^{-1}$. Thus

$$Q^*(e^{i\omega}) = -e^{-i(2K-1)\omega} P(e^{i(\omega+\pi)}) \tag{189}$$

Note $P(1) = \sum_{l=0}^{2K-1} \frac{1}{\sqrt{2}} h_l = 1 = -Q(-1)$. Next consider the orthonormality condition, for $z = e^{i2\pi\omega}$.

$$|P(z)|^2 + |P(-z)|^2 = \tag{190}$$

$$\frac{1}{2} \sum_{l,l'=0}^{2K-1} (h^*_l h_{l'} e^{i2\pi\omega(l'-l)} + e^{i\pi(l'-l)} h^*_l h_{l'} e^{i2\pi\omega(l'-l)}) = \tag{191}$$

$$\frac{1}{2} \sum_{l,l'=0}^{2K-1} h^*_l h_{l'} (e^{i2\pi\omega(l'-l)} + e^{i\pi(l'-l)} e^{i2\pi\omega(l'-l)}) = \tag{192}$$

$$\frac{1}{2} \sum_{l,l'=0}^{2K-1} h^*_l h_{l'} (e^{i\pi(l'-l)(2\omega)} + e^{i\pi(l'-l)(2\omega+1)}) \tag{193}$$

shifting $l' - l = k$ gives

$$\frac{1}{2}\sum_{l}^{2K-1}\sum_{k=-l}^{2K-1-l} h_l^* h_{k+l}(e^{i\pi k(2\omega)} + e^{i\pi k 2\omega}e^{i\pi k}) \tag{194}$$

In this form it is manifestly obvious that the coefficient of the k =odd terms vanish. Thus let $k \to 2n$:

$$\sum_{l}^{2K-1}\sum_{n=-l/2}^{K-1/2-l/2} h_l^* h_{l+2n} e^{i4\pi n\omega} \tag{195}$$

where the n sum is over successive integers between $-l/2$ and $N - 1/2$.

The condition that this is 1 for all ω gives

$$\sum_{l}^{2N-1} h_l^* h_{l+2n} = \quad constant \quad \times \delta_{n0} \tag{196}$$

or

$$\sum_{l}^{2N-1} h_l^* h_l = 1 \tag{197}$$

Thus we have

$$P(1) = 1; \qquad |P(1)|^2 + |P(-1)|^2 = 1 \tag{198}$$

Consistency of these two equations requires that $P(-1) = 0$,

The Daubechies wavelets have the property that $P(z)$ has a high order zero at $z = -1$:

$$P(z) = \left(\frac{1+z}{2}\right)^{N+1} W(z) \tag{199}$$

The normalization is chosen so that the polynomial $W(1) = 1$. The problem is reduced to finding polynomials that have this property.

One technique for finding $W(z)$ utilizes trigonometric polynomials. The problem is to find polynomials $P(z)$ with the property that

$$|P(z)|^2 + |P(-z)|^2 = 1 \tag{200}$$

Let $z = e^{-i2\pi\omega}$ which gives

$$\frac{1+z}{2} = e^{-i\pi\omega}\cos(\pi\omega) \tag{201}$$

$$\frac{1-z}{2} = ie^{-i\pi\omega}\sin(\pi\omega) \tag{202}$$

In terms of these equations

$$\cos(\pi\omega) = \frac{z^{1/2} + z^{*1/2}}{2} \tag{203}$$

$$\sin(\pi\omega) = \frac{z^{*1/2} - z^{1/2}}{2i} \tag{204}$$

This gives

$$1 = \left(\cos^2(\pi\omega) + \sin^2(\pi\omega)\right)^{2K-1} = \tag{205}$$

$$\sum_{n=0}^{2K-1} \frac{(2K-1)!}{n!(2K-1-n)!} \cos^{2n}(\pi\omega)\sin^{4N-2-2n}(\pi\omega) = \tag{206}$$

$$\sum_{n=0}^{2K-1} \frac{(2K-1)!}{n!(2K-1-n)!} \left(\frac{z^{1/2}+z^{*1/2}}{2}\right)^{2n} \left(\frac{z^{*1/2}-z^{1/2}}{2i}\right)^{4K-2-2n} = \tag{207}$$

$$\sum_{n=0}^{2K-1} \frac{(2K-1)!}{n!(2K-1-n)!} \left(\frac{z+1}{2}\right)^{2n} \left(\frac{1-z}{2i}\right)^{4K-2-2n} z^{*(2K-1)} = \tag{208}$$

It is useful to express the first K-terms in terms of cosines and sines:

$$1 = \sum_{n=0}^{2K-1} \frac{(2K-1)!}{n!(2K-1-n)!} \cos^{2n}(\pi\omega)(1-\cos^2(\pi\omega))^{2K-1-n} = \tag{209}$$

This sum has the following properties:

- Has 2K terms.
- The last K terms in this sum have the desired zero at -1 with the correct multiplicity.
- Let P_+ be the sum of the last K terms and P_- be the sum of the first K terms.
- Both sums are non-negative.

Note that

$$P_+ = \sum_{n=K}^{2K-1} \frac{(2K-1)!}{n!(2K-1-n)!} \cos^{2n}(\pi\omega) \sin^{4K-2-2n}(\pi\omega) = \tag{210}$$

setting $m = 2K-1-n$ gives

$$\sum_{m=0}^{K-1} \frac{(2K-1)!}{m!(2K-1-m)!} \cos^{4K-2-2m}(\pi\omega) \sin^{2m}(\pi\omega) = \tag{211}$$

$$\cos^{2K}(\pi\omega) \sum_{m=0}^{K-1} \frac{(2K-1)!}{m!(2K-1-m)!} \left(1 - \sin^2(\pi\omega)\right)^{K-1-m} \sin^{2m}(\pi\omega) \tag{212}$$

Let

$$W(x) = \sum_{m=0}^{K-1} \frac{(2K-1)!}{m!(2K-1-m)!} \left(1 - x^2\right)^{K-1-m} x^{2m} \tag{213}$$

Note that

$$P_+ = \left|\frac{1+z}{2}\right|^{2N} W(\sin(\pi\omega)). \tag{214}$$

Since W is non-negative function of $\sqrt{z}$ and $\sqrt{z^*}$, it has a non-negative square root ,$R(\sqrt{z}, \sqrt{z^*})$.

Claim (proof needs to be supplied - see page 172 of ten lectures) $W = |R(z)|^2$ where $R(z)$ is a polynomial. If this is true then

$$P(z) = \left(\frac{1+z}{2}\right)^N R(z) \tag{215}$$

has all of the desired properties.

Case $K = 2$. In this case

$$W(x) = \frac{3!}{0!3!}(1-x^2)x^0 + \frac{3!}{1!2!}(1-x^2)^0 x^2 = 1 + 2x^2 = 1 - \frac{1}{2}(\sqrt{z^*} - \sqrt{z})^2 = \tag{216}$$

$$2 - \frac{1}{2}(z + z^*). \tag{217}$$

The coefficients of the polynomial R should be real for real scaling coefficients. Try Let $R(z) = a + bz$; $|R(z)|^2 = a^2 + b^2 + ab(z + z^*)$. Equating coefficients gives

$$a^2 + b^2 = 2 \qquad 2ab = -1 \tag{218}$$

These are consistent with

$$(a-b)^2 = 3; \qquad (a+b)^2 = 1 \tag{219}$$

$$a-b = \pm\sqrt{3} \qquad a+b = \pm 1 \tag{220}$$

or

$$a = \pm(\frac{1}{2} \pm \frac{\sqrt{3}}{2}) \qquad b = \pm(\frac{1}{2} \mp \frac{\sqrt{3}}{2}) \tag{221}$$

The normalization $R(1) = 1$ gives

$$R(z) = (\frac{1}{2} \pm \frac{\sqrt{3}}{2}) + (\frac{1}{2} \mp \frac{\sqrt{3}}{2})z \tag{222}$$

and finally

$$P(z) = \left(\frac{1+z}{2}\right)^2 ((\frac{1}{2} \pm \frac{\sqrt{3}}{2}) + (\frac{1}{2} \mp \frac{\sqrt{3}}{2})z) \tag{223}$$

from which on can read off the $N = 2$ Daubechies coefficients. The two sign choices are related by the symmetry $z \to z^{-1}$ followed by multiplication by a homogeneous polynomial to preserve the polynomial nature of the function.

6 Moments and Quadrature Rules

Moments of scaling functions and mother functions are defined by

$$< x^m >_\phi = \int \phi(x) x^m dx \qquad < x^m >_\psi = \int \psi(x) x^m dx. \tag{224}$$

Normally these are integrated over the real line. For compactly supported wavelets this is equivalent to integrating over the support of the wavelet.

A polynomial quadrature rule is a collection of N points $\{x_i\}$ and weights $\{w_i\}$ with the property

$$< x^m >_\phi = \int \phi(x) x^m dx = \sum_{i=1}^{N} x_i^m w_i \tag{225}$$

which hold for $0 \leq m \leq 2N-1$. By linearity this means that

$$\int \phi(x) P(x) dx = \sum_{i=1}^{N} P(x_i) w_i \tag{226}$$

is exact for all polynomials of degree up to 2N-1.

In order to construct a quadrature rule we need to first compute the moments, and from these we can compute the points and weights.

The moments can be constructed recursively from the normalization condition

$$< x^0 >_\phi = (x^0, \phi) = \int dx \phi(x) = 1 \tag{227}$$

using

$$< x^m >_\phi = (x^m, \phi) = (D x^m, D\phi) =$$
$$\frac{1}{\sqrt{2}} \frac{1}{2^m} \sum_l h_l (x^m, T^l \phi) =$$
$$\frac{1}{\sqrt{2}} \frac{1}{2^m} \sum_l h_l ((x+l)^m, \phi)$$
$$\frac{1}{\sqrt{2}} \frac{1}{2^m} \sum_l h_l \sum_{k=0}^{m} \frac{m!}{k!(m-k)!} l^{m-k} < x^k >_\phi$$

Using $\sum_l h_l = \sqrt{2}$, and moving the $k = m$ term to the left side of the equation gives the recursion relation:

$$< x^m >_\phi = \frac{1}{2^m - 1} \sum_{k=0}^{m-1} \frac{m!}{k!(m-k)!} (\sum_{l=1}^{2N+1} h_l l^{m-k}) < x^k >_\phi$$

Note that the right hand side involves moments with $k < m$. Similarly we have for the mother function

$$< x^m >_\psi = \frac{1}{2^m} \sum_{k=0}^{m} \frac{m!}{k!(m-k)!} (\sum_{l=0}^{2N-1} \frac{g_l}{\sqrt{2}} l^{m-k}) < x^k >_\phi$$

Since the scaling equation relates the mother function to the scaling function there is no need to take the $k = m$ term to the left of the equation; it is known from the first recursion.

This gives a recursive method for generating all non-negative moments of the scaling and mother functions from the normalization integral on the scaling function.

Note the moments for $\phi_{kl} = D^k T^l \phi$ and $\psi_{kl} = D^k T^l \psi$ can be computed from these moments using the unitarity of the D and T operators

$$< x^m >_{\phi_{kl}} = (x^m, D^k T^l \phi) = (T^{-l} D^{-k} x^m, \phi) =$$

$$2^{k(m+1/2)}\sum_{n=0}^{m}\frac{m!}{n!(m-n)!}l^{m-n} < x^n >_\phi$$

and

$$< x^m >_{\psi_{kl}} = (x^m, D^k T^l \psi) = (T^{-l}D^{-k}x^m, \phi) =$$
$$2^{k(m+1/2)}\sum_{n=0}^{m}\frac{m!}{n!(m-n)!}l^{m-n} < x^n >_\psi$$

Next consider moments on the half line

$$< x^m >_{\phi_k[0:\infty]} := \int_0^\infty x^m \phi(x-k)dx.$$

Note that scale transformations are unitary on the half interval. This gives

$$< x^m >_{\phi_k[0:\infty]} := \int_0^\infty Dx^m DT^k\phi(x)dx = \frac{1}{2^{m+1/2}}\sum_l h_l \int_0^\infty x^m \phi(x-2k-l)dx =$$

$$\frac{1}{2^{m+1/2}}\sum_l h_l < x^m >_{\phi_{2k+l}[0:\infty]}$$

In order to understand how to compute these moments note that the support of $\phi(x-k)$ is on $[k, 2N-1+k]$. Consider the scaling functions that have partial overlap with the half interval. These correspond to $k = -1, \cdots, -2N+2$. For each value of k between -1 and $-2N+2$ there is a linear equation relating the k-th partial moment to the other partial moments. In each equation there are three type of moments: moments that are zero because the shifted support is not in the half interval, partial moments, and full moments because the support is shifted entirely into the right-half interval. This gives a linear system for the non-vanishing partial moments in terms of the full moments.

Specifically the sums above become

$$< x^m >_{\phi_k[0:\infty]} =$$

$$\frac{1}{2^{m+1/2}}\sum_{-1-2k\le l:0\le -2N+2-2k}^{2N-1} h_l < x^m >_{\phi_{2k+l}[0:\infty]} +$$
$$\frac{1}{2^{m+1/2}}\sum_{-1-2k<l\wedge l>-2N+2-2k}^{2N-1} h_l < x^m >_{\phi_{2k+l}}$$

Normally it is useful to absorb the $\sqrt{2}$ in the coefficient $h_l \to c_l := h_l/\sqrt{2}$:

$$\frac{1}{2^m} \sum_{-1-2k \leq l:0 \leq -2N+2-2k}^{2N-1} c_l < x^m >_{\phi_{2k+l}[0:\infty]} +$$

$$\frac{1}{2^m} \sum_{-1-2k < l \wedge l > -2N+2-2k}^{2N-1} c_l < x^m >_{\phi_{2k+l}}$$

We illustrate the equation for the case that $N = 2$:

$$< x^m >_{\phi_{-2}[0:\infty]} =$$

$$\frac{1}{2^{m+1/2}} (h_2 < x^m >_{\phi_{-2}[0:\infty]} + h_3 < x^m >_{\phi_{-1}[0:\infty]})$$

$$< x^m >_{\phi_{-1}[0:\infty]} =$$

$$\frac{1}{2^{m+1/2}} (h_0 < x^m >_{\phi_{-2}[0:\infty]} + h_1 < x^m >_{\phi_{-1}[0:\infty]} + h_2 < x^m >_{\phi_0} + h_3 < x^m >_{\phi \cdot 1}).$$

This gives two linear equations for the partial moments, $< x^m >_{\phi_{-2}[0:\infty]}$, $< x^m >_{\phi_{-1}[0:\infty]}$ in terms of the full moments $< x^m >_{\phi_0}$ and $< x^m >_{\phi_1}$

Note that having solved for the partial moments for the scaling function it is possible to get partial for the mother function using

$$< x^m >_{\psi_k[0:\infty]} := \frac{1}{2^{m+1/2}} \sum_l g_l \int_0^\infty x^m \phi(x - 2k - l) dx =$$

$$\frac{1}{2^{m+1/2}} \sum_l g_l < x^m >_{\phi_{2k+l}[0:\infty]}$$

which expresses the partial scaling coefficients for the mother function directly terms of moments and partial moments of the scaling function.

Similarly we can solve for the partial moments of ϕ_{mn} and ψ_{mn} using the fact the dilatation operator is unitary on the half interval:

$$< x^m >_{\phi_{nl}[0:\infty]} = \int_0^\infty x^m, D^n T^l \phi(x) dx =$$

$$2^{n(m+1/2)} < x^m >_{\phi_l[0:\infty]}$$

$$< x^m >_{\psi_{nl}[0:\infty]} = \int_0^\infty x^m, D^n T^l \psi(x) dx =$$

$$2^{n(m+1/2)} < x^m >_{\psi_l[0:\infty]}$$

Given a collection of $2N$ moments we can construct quadrature points and weights using the Payne-Klink method. If $\{x_i\}$ are the quadrature points define the polynomial

$$P(x) = \prod_{i=1}^{N} (x - x_i) = \sum_{n=0}^{N} p_n x^n$$

where $p_N = 1$ and the other p_n's are unknown. Define

$$Q_m(x) = x^m P(x)$$

for $m = 1, \cdots N - 1$. By construction, for each m and x_i , $Q_m(x_i) = 0$ because $P(x_i) = 0$.

If we require that the points and weights reproduce $2N$ moments exactly then it follows that

$$\int \phi(x) Q_m(x) dx = \sum_{i=1}^{N} Q_m(x_i) w_i = 0 \tag{228}$$

because $Q_m(x_i) = 0$. We also have

$$\int \phi(x) Q_m(x) dx = \sum_{n=0}^{N} p_n < x^{n+m} >_\phi$$

setting this equal to zero for each value m from $m = 0$ to $m = N - 1$ gives N linear equations for the N unknowns $p_0 \cdots p_{N-1}$':

$$\sum_{n=0}^{N} p_n < x^{n+m} >_\phi = 0 \qquad m = 1 \cdots N;\ p_N = 1$$

Solving this linear system for the coefficients p_n , using $p_N = 1$, give the polynomial $P(x)$.

Given $P(x)$ the next step is to find the roots. The N roots of $P(x)$ are the quadrature points x_i. The weights are determined from the remaining N moments by solving the linear system

$$< x^n >_\phi = \sum_{i=1}^{N} x_i^n w_i \qquad n = 0, \cdots, N - 1$$

for the weights, w_i.

This shows how to construct the quadrature points and weights from the moments. For the case of a half-interval the partial moments, $< x^m >_{\phi_l[0:\infty]}$, need to be used near 0.

In general one should check that the points are real and lie in the support of the scaling function. This is not an obvious consequence of the definitions. When this fails to occur it is best to simply assign real quadrature points that lie on the support of the scaling function. In doing this some accuracy is sacrificed, but it is easy to go to a higher order.

These points and weights can be transformed to father or mother function using unitarity. To see this consider a set of points and weights $\{x_i, w_i\}$ that satisfy

$$(x^m, \phi) = \int x^m \phi(x) dx = \sum x_i^m w_i$$

we have

$$(x^m, \phi_{nk}) = (x^m, D^n T^k \phi) 2^{n(m+1/2)} (x^m, T^k \phi,) = 2^{n(m+1/2)}((x+k)^m, \phi) =$$

$$\sum 2^{nm+n/2} w_l (x_l + k)^m = \sum (2^{n/2} w_l)(2^n(x_l + k))^m$$

If we define the transformed points and weights by

$$w_l' = 2^{n/2} w_l \qquad x_l' = 2^n (x_l + k)$$

we get

$$(x^m, \phi_{nk}) = \sum_l w_l' (x_l')^m .$$

These involve simple transformations of the original points and weights.

To use these to solve integral equations first consider the non-singular equation

$$f(x) = g(x) + \int K(x, y) f(y) dy$$

Let

$$f(x) \approx \sum_n f_n \phi_n(x)$$

where $\phi_{sn}(x)$ are translates of the scaling function on a sufficiently fine scale s.

$$\sum_n f_n \phi_{sn}(x) \approx g(x) + \sum_n \int K(x, y) f_n \phi_{sn}(y) dy$$

Using orthonormality of the father on the same scale gives equation for the coefficients:

$$f_m = \sum_l g(x_{lm}) w_{lm} + \sum_n \sum_{l,k} w_{lm} K(x_{lm}, x_{kn}) w_{kn} f_n$$

or

$$\sum_n (\delta_{mn} - \sum_{l,k} w_{lm} K(x_{lm}, x_{kn}) w_{kn}) f_n = \sum_l g(x_{lm}) w_{lm}$$

Note that no integrals need to be evaluated, except using the local quadrature rules. In addition the points and weights only have to be calculated for the father on one scale - the rest can be obtained by simple transformations.

To solve this the last step is to use the wavelet transform on the indices mn. This should give a sparse linear system that can be used to solve for f_n.

The system in question has two complications. First this integral is over a half line. The second is that the kernel has a principal value singularity.

The endpoint near $x = 0$ can be treated using special quadratures for the functions on the half interval. If there is an endpoint the δ_{mn} needs to be replaced

$$\int_0^\infty \phi_m(x) \phi_n(x) dx = N_{mn} = N_{nm}$$

which is not a kronecker delta for m, n with support containing 0. Note that these integrals can be evaluated using the same methods that were used to calculate moments on the half interval. We simply use the scaling equations and the orthonormality when the support of both terms are in the half interval. Specifically

$$N_{ij}^{ab} = \int_a^b \phi_i(x) \phi_j(x) dx =$$

$$\int_{a-i}^{b-i} \phi(x) \phi(x + i - j) dx =$$

$$\sum_{l,l'} h_l h_l' \int_{a-i}^{b-i} \phi(2x - l) \phi(2x + 2i - 2j - l') 2dx =$$

$$\sum_{l,l'} h_l h_l' \int_{2(a-i)-l}^{2(b-i)-l} \phi(x) \phi(x + 2i - 2j + l - l') dx =$$

$$\sum_{l,l'} h_l h_l' N_{0,-2i+2j-l+l}^{(2(a-i)-l)(2(b-i)-l)},$$

When both functions have support inside the inverval this is a Kronecker delta. These equations relate these elements to the elements where the support overlaps an upper or lower endpoint. These formulas simplify if $a = \infty$ or $b = -\infty$. The final relations are

$$N^{(a-i)(b-i)}_{0(j-i)} =$$

$$\sum_{l,l'} h_l h'_l N^{(2(a-i)-l)(2(b-i)-l)\prime}_{0,-2i+2j-l+l}$$

Note that $N_{0,k} = 1$ if $k = 0$, $N_{0,k} = 0$ if $k > 0$ or $k \leq -(2N-1)$ and non-trivial for $-(2N-2) \leq k < 0$. This gives a linear system for the overlap coefficients, N_{ij}. For scaling functions that overlap 0 the equation becomes:

$$\sum_n (N_{0,n-m} - \sum_{l,k} \tilde{w}_{lm} K(\tilde{x}_{lm}, x_{kn}) w_{kn}) f_n = \sum_l g(\tilde{x}_{lm}) \tilde{w}_{lm}$$

where the indicates that for m satisfying $2N-2 \leq m < 0$ the quadrature points and weights need to be replaced by the ones for the half interval.

Mapping techniques should still be valuable for reducing the coupling for functions with support near the origin. For exmaple

$$y = x\frac{2+x}{2-x}$$

mover the problem to $[-2, 2]$ with the singularity at the origin. What remains is a mechanism for treating an integrable singularity. This can be accomplished using mapping techniques. The first step it to use a mapping to place the singularity at the origin. After mapping the relevant integrals are

$$I_m(n) := \int \frac{D^m T^n \phi(x)}{x} dx$$

Using unitarity of D gives

$$I_m(n) := \int D(D^m T^n \phi(x)) D\frac{1}{x} dx =$$

$$\frac{2}{\sqrt{2}} \int \frac{D^{m+1} T^n \phi(x)}{x} dx =$$

$$\sqrt{2}\int \frac{D^m T^{2n} D\phi(x)}{x}dx =$$

$$\sum_{l=0}^{2K-1} \sqrt{2}h_l I_m(2n+l)$$

The equations

$$I_m(n) = \sum_{l=0}^{2K-1} \sqrt{2}h_l I_m(2n+l)$$

give linear relations connecting the integrals with singularities to the integrals with no singularities. The singular terms are

$$\phi_m(-1), \phi_m(-2), \cdots \phi_m(-2K+2)$$

The endpoint terms, $\phi_m(0)$ and $\phi_m(-2K+1)$ are not singular because $\phi_m(x)$ must be continuous at the endpoints.

As we found these equations are ill conditioned and can be supplemented by

$$0 = \sum_n \mathcal{P}\int_{-k}^{k} \frac{\phi_{mn}(x)}{x}dx$$

Which has the form

$$0 = \sum_n I^k_m(n)$$

where the integrals are over partial intervals near the endpoints. The linear relations relate the singular integrals to the non-singular ones. For points far enough away from the singularity and the endpoints we can express the integrals in terms of the moments:

$$I_m(n) = \int \frac{D^m T^n \phi(x)}{x}dx = \int T^n\phi(x)D^{-m}\frac{1}{x}dx = 2^{\frac{m}{2}}\int \phi(x)T^{-n}\frac{1}{x}dx =$$

$$2^{\frac{m}{2}}\int \phi(x)\frac{1}{x+n}dx = 2^{\frac{m}{2}}\frac{1}{n}\sum_{l=0}^{\infty}\left(-\frac{1}{n}\right)^l \langle x^l\rangle_\phi$$

For large enough n this converges fast. Similar methods can be used near the endpoints.

For different asymptotic conditions note

$$\int_{-m}^{m}\frac{dx}{x\pm i0^+} = \int_{-\infty}^{\infty}\frac{dx}{x\pm i0^+} - \int_{-\infty}^{-m}\frac{dx}{x\pm i0^+}\int_{m}^{\infty}\frac{dx}{x\pm i0^+} =$$

$$\int_{-\infty}^{\infty} \frac{dx}{x \pm i0^+}$$

While this does not exist as a function, as a distribution is has the well known value

$$\int_{-\infty}^{\infty} \frac{dx}{x \pm i0^+} = \mp i2\pi\delta(x)$$

Using this with the wavelet expansion

$$\int \sum_n \phi(x)\delta(x)dx = \sum_n \phi_n(0) = 1$$

With additional work one can show that required asymptotic conditions can be achieved by replacing the supplementary condition by

$$\mp 2\pi i = \sum_n I_m^k(n)$$

The result of this is exact expressions for

$$\int \frac{\phi_n(x)}{x} \tag{229}$$

In the region of the singularity. For these basis functions

$$\int \frac{K(x:y)}{y}\phi_n(y)dy = \int_{a_n}^{b_n} \frac{K(x:y) - K(x,0)}{y}\phi_n(y)dy +$$

$$K(x,0)\int_{a_n}^{b_n} \frac{\phi_n(y)}{y}dy$$

where $\phi_n(x)$ has support on $[a_n, b_n]$

$$\int_{a_n}^{b_n} \frac{K(x:y)}{y}\phi_n(y)dy = \int_{a_n}^{b_n} \frac{K(x:y) - K(x,0)}{y}\phi_n(y)dy +$$

$$K(x,0)\int_{a_n}^{b_n} \frac{\phi_n(y)}{y}dy =$$

$$\sum_l \frac{K(x:x_l) - K(x,0)}{x_l} w_l K(x,0) I_0(n)$$

For the case of $K = 1$ the equations for $I_m(-1)$ and $I_m(-2)$ are

$$I_m(-2) = c_0 I_m(-4) + c_1 I_m(-3) + c_2 I_m(-2) + c_3 I_m(-1)$$

$$I_m(-1) = c_0 I_m(-2) + c_1 I_m(-1) c_2 I_m(0) + c_3 I_m(1)$$

where $c_k = \sqrt{2} h_k$. These need to be supplemented by the equation:

$$0 = \sum_n I_m$$

The accuracy near the singularity can be improved using

$$\int \frac{x^m \phi_n}{x-k} = \int \frac{x^m - k^m \phi_n}{x-k} + k^m \int \frac{\phi_n}{x-k} = \sum l = 0^{m-1} \int \frac{x^l \phi_n}{x-k} + k^m \int \frac{\phi_n}{x-k}$$

This can be replaced by an iterative scheme:

$$\int \frac{x^{m-1}(x-k)\phi_n}{x-k} = \int x^{m-1} \phi_n =$$

$$\int \frac{x^m \phi_n}{x-k} - k \int \frac{x^{m-1} \phi_n}{x-k}$$

7 Derivatives and Differential Equations

In order to use use wavelets for differential equations is necessary to be able to compute derivatives.

A typical wavelet basis consists of a the scaling function and its translates at a fixed resolution m:

$$\{\phi_{mn}(x)\}_{n=-\infty}^{\infty} : \phi_{mn}(x) = D^m T^n \phi(x) = 2^{-m/2} \phi(2^m x - n) \tag{230}$$

and the wavelets for all resolutions k less than or equal to m and their translates

$$\{M_{kn}(x)\}_{n=-\infty, k=-\infty}^{\infty, m} : M_{mn}(x) = D^k T^n M(x) = 2^{-k/2} M(2^k x - n) \tag{231}$$

Given a function $f(x)$ represented as

$$f(x) = \sum_n f^s_{mn} \phi_{mn}(x) + \sum_{kn} f^m_{kn} M_{kn}(x) \tag{232}$$

The r-th derivative of this $f(x)$ can be represented in the following forms

$$f^{(r)}(x) = \sum_n f^s_{mn} \phi^{(r)}_{mn}(x) + \sum_{kn} f^m_{kn} M^{(r)}_{kn}(x) \tag{233}$$

or

$$f^{(r)}(x) = \sum_n f^{s(r)}_{mn} \phi_{mn}(x) + \sum_{kn} f^{m(r)}_{kn} M_{kn}(x). \tag{234}$$

The coefficients in these two expansions can be related by expanding the derivatives of the basis functions in terms of the basis:

$$\phi^{(r)}_{mn}(x) = \sum_{n'} \phi_{mn'}(x) \Gamma^{1(r)}_{mn';mn} + \sum_{k'n'} M_{k'n'}(x) \Gamma^{2(r)}_{k'n';mn} \tag{235}$$

and

$$M^{(r)}_{kn}(x) = \sum_{n'} \phi_{mn'}(x) \Gamma^{3(r)}_{mn';kn} + \sum_{k'n'} M_{k'n'}(x) \Gamma^{4(r)}_{k'n';kn} \tag{236}$$

In terms of these quantities

$$f^{s(r)}_{mn} = \sum_{n'} \Gamma^{1(r)}_{mn;mn'} f^s_{mn'} + \sum_{n'k'} \Gamma^{3(r)}_{mn;k'n'} f^m_{k'n'} \tag{237}$$

$$f^{m(r)}_{kn} = \sum_{n'} \Gamma^{2(r)}_{kn;mn'} f^s_{mn'} + \sum_{n'k'} \Gamma^{4(r)}_{kn;k'n'} f^m_{k'n'} \tag{238}$$

The expansion coefficients are the overlap matrices

$$\Gamma^{1(r)}_{mn;m'n'} := (\phi_{mn}, \phi^{(r)}_{m'n'}) \tag{239}$$

$$\Gamma^{2(r)}_{kn;m'n'} := (M_{kn}, \phi^{(r)}_{m'n'}) \tag{240}$$

$$\Gamma^{3(r)}_{mn;k'n'} := (\phi_{mn}, M^{(r)}_{k'n'}) \tag{241}$$

$$\Gamma^{4(r)}_{kn;k'n'} := (M_{mn}, M^{(r)}_{k'n'}) \tag{242}$$

The scaling equation mean that all of these coefficients can be determined from a subset of the coefficients. In order to exhibit the key relations it is useful to use operators:

$$Df(x) = \frac{1}{\sqrt{2}} f(\frac{x}{2}) \tag{243}$$

$$Tf(x) = f(x-1) \tag{244}$$

$$\Delta f(x) = \frac{df}{dx}(x) \tag{245}$$

Direct computation shows

$$\Delta D = \frac{1}{2} D \Delta \tag{246}$$

$$DT = T^2 D \tag{247}$$

$$\Delta T = T\Delta \tag{248}$$

$$\Delta^\dagger = -\Delta \qquad T^\dagger = T^{-1} \qquad D^\dagger = D^{-1} \tag{249}$$

We also have the scaling equations:

$$D\phi = \sum_l h_l T^l \phi \tag{250}$$

$$DM = \sum_l g_l T^l \phi \tag{251}$$

Using the operator relations above

$$D\Delta^r \phi = 2^r \sum_l h_l T^l \Delta^r \phi \tag{252}$$

$$D\Delta^r M = 2^r \sum_l g_l T^l \Delta^r \phi \tag{253}$$

In terms of these operators

$$\Gamma^{1(r)}_{m'n';mn} := (D^{m'} T^{n'} \phi, \Delta^r D^m T^n \phi) \tag{254}$$

$$\Gamma^{2(r)}_{m'n';kn} := (D^{m'} T^{n'} M, \Delta^r D^m T^n \phi) \tag{255}$$

$$\Gamma^{3(r)}_{k'n';mn} := (D^{k'} T^{n'} \phi, \Delta^r D^m T^n M) \tag{256}$$

$$\Gamma^{4(r)}_{k'n';kn} := (D^{k'} T^{n'} M, \Delta^r D^m T^n M) \tag{257}$$

In order to evaluate these coefficients the following s.pdf are used:

1. Move all of the factors of D to a single side of the equation. Choose the side where the power of D is positive.
2. Move the D's through all derivatives.
3. Use the scaling equations to eliminate all of the D's.
4. Move all of the T's to the left side of the scalar product.

Using these all of the Γ's can be expressed in terms of

$$\Gamma^{1(r)}_{0n;00} := (T_n \phi, \Delta^r \phi) \tag{258}$$

$$\Gamma^{2(r)}_{0n;00} := (T^n M, \Delta^r \phi) \tag{259}$$

$$\Gamma^{3(r)}_{0n;00} := (T^n\phi, \Delta^r M,) \tag{260}$$

$$\Gamma^{4(r)}_{0n;00} := (T^n M, \Delta^r M) \tag{261}$$

These quantities satisfy scaling equations. To see this consider

$$\Gamma^{1(r)}_{0n;00} := (T^n\phi, \Delta^r\phi) =$$

$$(DT^n\phi, D\Delta^r\phi) = 2^r(T^{2n}D\phi, \Delta^r D\phi) =$$

$$\sum_{ll'} h_l^* h_{l'} 2^r (T^{2n+l}\phi, T^{l'}\Delta^r\phi) =$$

$$\sum_{ll'} h_l^* h_{l'} 2^r (T^{2n+l-l'}\phi, \Delta^r\phi) =$$

$$\sum_{ll'} h_l^* h_{l'} 2^r \Gamma^{1(r)}_{0,2n+l-l';00} =$$

$$\sum_k (\sum_{l'} h^*_{k-2n+l'} h_{l'}) 2^r \Gamma^{1(r)}_{0k;00}. \tag{262}$$

This gives an eigenvalue equation for the vector $\Gamma^{1(r)}_{0n;00}$. The solution is the eigenvector with eigenvalue 1. The normalization can be fixed by computing $\Gamma^{1(r)}_{00;00} := (\phi, \Delta^r\phi)$ directly

Define $H^{(r)}_{nk}$ by

$$H^{(r)}_{nk} := \sum_{l'} h^*_{k+l'-2n} h_{l'} 2^r \tag{263}$$

With this definition the eigenvalue problem takes on the form:

$$\Gamma^{1(r)}_{0n;00} = \sum_k H^{(r)}_{nk} \Gamma^{1(r)}_{0k;00} \tag{264}$$

Note that this can be written as

$$2^{-r}\Gamma^{1(r)}_{0n;00} = \sum_k H^0_{nk} \Gamma^{1(r)}_{0k;00} \tag{265}$$

This treats all allowed derivatives with a single equation - the eigenvector with eigenvalue 2^{-r} is the coefficient for the r^{th} derivative.

These quantities can be used to determine all of the other quantities using

$$\Gamma^{2(r)}_{0n;00} := (T^n M, \Delta^r\phi) =$$

$$\sum_{ll'} g^*_{l'} h_l 2^r \Gamma^{1(r)}_{0,2n+l'-l;00} =$$

$$\sum_m (\sum_l g^*_{l'}) h_{2n+l-m}) 2^r \Gamma^{1(r)}_{0'm;00} \tag{266}$$

$$\Gamma^{3(r)}_{0n;00} := (T^n \phi, \Delta^r M) =$$

$$\sum_{ll'} h^*_{l'} g_l 2^r \Gamma^{1(r)}_{0,2n+l'-l;00} =$$

$$\sum_m (\sum_l h^*_l g_{2n+l-m}) 2^r \Gamma^{1(r)}_{0m;00} \tag{267}$$

$$\Gamma^{4(r)}_{0n;00} := (T^n M, \Delta^r M,) =$$

$$\sum_{ll'} g^*_l g_{l'} 2^r \Gamma^{1(r)}_{0,2n+l'-l;00} =$$

$$\sum_m (\sum_l g^*_l g_{2n+l-m}) 2^r \Gamma^{1(r)}_{0m;0} \tag{268}$$

This shows that all of the expansion coefficients for any number of derivatives can be constructed from the solutions of a single eigenvalue problem.

There are a number of stepping formulas:

$$\Gamma^{1(r)}_{0m;0n} = \Gamma^{1(r)}_{0m-n;00} \tag{269}$$

$$\Gamma^{1(r)}_{0m+k;0,n+k} = \Gamma^{1(r)}_{0m;0n} \tag{270}$$

$$\Gamma^{1(r)}_{kn;tm} = \sum_l h^*_l \Gamma^{1(r)}_{k-1,2n+l;tm} \tag{271}$$

$$\Gamma^{1(r)}_{kn;tm} = 2^r \sum_l h_l \Gamma^{1(r)}_{k+1,n;t,2m+l} \tag{272}$$

$$\Gamma^{1(r)}_{kn;tm} = \sum_l h_l \Gamma^{1(r)}_{kn;t-1,2m+l} \tag{273}$$

$$\Gamma^{1(r)}_{kn;tm} = 2 \sum_l h^*_l \Gamma^{1(r)}_{k,2n+1;t+1,2m+l} \tag{274}$$

which can be used to reduce the number of dilitations to zero.

8 Moments and Wavelets

Consider the equations for the moments of the scaling function and the mother function:

$$\langle x^m \rangle_\phi := \int x^m \phi(x) dx \tag{275}$$

$$\langle x^m \rangle_m := \int x^m m(x) dx \tag{276}$$

To develop relations among the moments use the unitarity of the dilitation operator:

$$\langle x^m \rangle_\phi = \int x^m \phi(x) dx =$$

$$\int D x^m D\phi(x) dx = \frac{1}{2^{m+1/2}} \int x^m D\phi(x) dx \tag{277}$$

Using the scaling equation gives

$$\frac{1}{2^{m+1/2}} \sum_l \int x^m h_l \phi(x-l) dx =$$

$$\frac{1}{2^{m+1/2}} \sum_l \int (l+y)^m h_l \phi(y) dy =$$

Nest use the binomial theorem to get:

$$\frac{1}{2^{m+1/2}} \sum_{k=0}^{m} \sum_l h_l \frac{m!}{k!(m-k)!} l^{m-k} \int y^k \phi(y) dy =$$

$$\frac{1}{2^{m+1/2}} \sum_{k=0}^{m} \sum_l h_l \frac{m!}{k!(m-k)!} l^{m-k} \langle x^k \rangle_\phi \tag{278}$$

This gives a set of eigenvalue equations for $\langle x^k \rangle_\phi$ The moments of the mother function have similar properties with h_l replaced by g_l: :

$$\langle x^m \rangle_m = \int x^m m(x) dx =$$

$$\int D x^m D m(x) dx =$$

$$\frac{1}{2^{m+1/2}} \sum_{k=0}^{m} \sum_l g_l \frac{m!}{k!(m-k)!} l^{m-k} \langle x^k \rangle_\phi \tag{279}$$

The nice thing about these equations is that they are lower triangular. Specifically for the $\langle x^k \rangle_\phi$ equations we have for $m = 0$:

$$\langle x^0 \rangle_\phi = \frac{1}{2^{1/2}} \sum_l h_l \langle x^0 \rangle_\phi \tag{280}$$

which is the constraint $\sum_l h_l = \sqrt{2}$. For $m = 1$ this equation is

$$\langle x^1 \rangle_\phi = \frac{1}{2^{3/2}} \sum_l h_l \left(\frac{1!}{0!1!} l^1 \langle x^0 \rangle_\phi + \frac{1!}{1!0!} l^0 \langle x^1 \rangle_\phi \right) \tag{281}$$

Using the first equation gives

$$\langle x^1 \rangle_\phi = \frac{1}{\sqrt{2}} \sum_l l h_l \langle x^0 \rangle_\phi \tag{282}$$

This gives $\langle x^1 \rangle_\phi$ in terms of $\langle x^0 \rangle_\phi$

For $m = 2$ is equation becomes

$$\langle x^2 \rangle_\phi =$$

$$\frac{1}{2^{5/2}} \sum_l h_l \left(\frac{2!}{0!2!} l^2 \langle x^0 \rangle_\phi + \frac{2!}{1!1!} l^1 \langle x^1 \rangle_\phi + \frac{2!}{2!0!} l^0 + \langle x^2 \rangle_\phi \right) =$$

$$\frac{1}{2^{5/2}} \left(\sum_l h_l l^2 \langle x^0 \rangle_\phi + \sqrt{2} (\sum l h_l)^2 \langle x^0 \rangle_\phi + \sqrt{2} \langle x^2 \rangle_\phi \right)$$

which can be solved for $\langle x^2 \rangle_\phi$ in terms of $\langle x^0 \rangle_\phi$

The general form of the recursion relation is

$$\langle x^m \rangle_\phi =$$

$$\frac{m!}{2^{m+1} - 2} \sum_{k=0}^{m-1} \sum_l \frac{h_l l^{m-k}}{k!(m-k)!} \langle x^k \rangle_\phi \tag{283}$$

A similar recursion can be derived for moments of the mother function:

$$\langle x^m \rangle_m =$$

$$\frac{m!}{2^{m+1}}\sum_{k=0}^{m-1}\sum_l \frac{g_l l^{m-k}}{k!(m-k)!}\langle x^k\rangle_\phi \tag{284}$$

There are all fixed in terms of the value of the integral $\int \phi(x)dx$.

It follows from this relation that the condition that the first p moments of the mother function vanishing are

$$\sum_l l^k g_l = 0 \quad k = 0, 1, \cdots, p \tag{285}$$

These equations come from looking at the coefficients of the moments of the scaling function. They give all of the moments in terms of the h_l's and

$$\langle x^0\rangle_\phi = \int \phi(x)dx \tag{286}$$

These equations, along with the normalization equation and the orthonormality equation determine the h_l's for the Daubechies wavelets. The general equations are

$$\sum_l h_l h_{l-2k} = \delta_{k0} \qquad k = 0, 1, \cdots \tag{287}$$

$$\sum_l h_l = \sqrt{2} \tag{288}$$

$$\sum_l l^k (-)^l h_{1-l} = 0 \quad k = 0, 1, \cdots, p \tag{289}$$

where 1 can be replaced by any odd integer.y

Moments of all of the other functions can be computed using

$$\langle x^m\rangle_{\phi_{rs}} = \int x^m D^r T^s \phi(x)dx =$$

$$\int (T^{-s}D^{-r}x^m)\phi(x)dx = 2^{\frac{r}{2}+mr}\int (x+s)^m \phi(x)dx =$$

$$2^{\frac{r}{2}+mr}\sum_{k=0}^{m}\frac{m!}{k!(m-k)!}s^{m-k}\langle x^r\rangle_\phi$$

and

$$\langle x^m\rangle_{m_{rs}} = \int x^m D^r T^s m(x)dx =$$

$$\int (T^{-s}D^{-r}x^m)m(x)dx = 2^{\frac{r}{2}+mr}\int (x+s)^m m(x)dx =$$

$$2^{\frac{r}{2}+mr}\sum_{k=0}^{m}\frac{m!}{k!(m-k)!}s^{m-k}\langle x^r\rangle_m$$

We can use these to determine the normalization of the $\Gamma's$: Consider

$$\frac{d\phi}{dx}(x) = \sum_l \phi_{ml}(x)\Gamma^{1(1)}_{ml;00} + \sum_{kl} m_{kl}(x)\Gamma^{2}_{kl;00} \tag{290}$$

Multiply by x to get

$$\langle x^0\rangle_\phi = \int \phi(x)dx = -\int x\frac{d\phi}{dx}(x)dx =$$

$$\sum_l \langle x^1\rangle_{\phi_{ml}}\Gamma^{1(1)}_{ml;00} + \sum_{kl}\langle x^1\rangle_{m_{kl}}\Gamma^{2(1)}_{kl;00}$$

Given that we know the Γ's up to an overall normalization, and all of the moments in terms of $\langle x^0\rangle$, this equation fixes the normalization of the $\Gamma^{(1)}$'s in terms of $\langle x^0\rangle$.

The Γ's corresponding to the higher derivatives can be derived using

$$\int x^l\frac{d^l\phi}{dx^l}(x)dx = (-)^l l!\langle x^0\rangle_\phi =$$

$$\sum_s \langle x^l\rangle_{\phi_{ms}}\Gamma^{1(l)}_{ms;00} + \sum_{ks}\langle x^l\rangle_{m_{ks}}\Gamma^{2(l)}_{ks;00} \tag{291}$$

which expresses the Γ's for the higher derivatives in terms of higher moments.

9 Normalization issues

The orthonormality of the wavelets means that the scaling function $\phi(x)$ satisfies

$$\int \phi^2(x)dx = 1. \tag{292}$$

Because this is preserved under the continuous scale transformation

$$\phi(x) \to \phi'(x) := \frac{1}{\sqrt{s}}\phi(\frac{x}{s}) \tag{293}$$

by choosing s we can simultaneously fix

$$\int \phi'(x)dx = \sqrt{s}\int \phi(x)dx \tag{294}$$

Choosing s so $\int \phi'(x)dx = 1$ fixes a starting scale. In this way we can simultaneously require

$$\int \phi^2(x)dx = \int \phi(x)dx = 1 \tag{295}$$

Given the conditions

$$\int \phi^2(x)dx = \int \phi(x)dx = 1 \tag{296}$$

we can compute all of the necessary normalization coefficients.

First we calculate the expansion of 1. In this case the assumption that the integral of the mother function gives zero implies that 1 can be expressed in terms of translates of the scaling function:

$$1 = \sum c_n^0 \phi(x-n) \tag{297}$$

Multiplying by $\phi_m(x) = \phi(x-m)$ and using the orthonormality and $\int \phi(x)dx = 1$ gives

$$1 = c_m = \int \phi_m(x)dx = \int \phi(x)dx = \beta_0 \tag{298}$$

Thus we conclude that

$$1 = \sum_n \phi_n(x) \tag{299}$$

Next we show that we can compute all of the moments

$$\beta_m := \int x^m \phi(x)dx \tag{300}$$

in terms of the h_l's and β_0. To see this note

$$\beta_m := \int x^m \phi(x)dx = \int D x^m D\phi(x)dx =$$

$$\frac{1}{\sqrt{2}} \int \left(\frac{x}{2}\right)^m \sum_l h_l \phi(x-l)dx =$$

$$2^{-m-\frac{1}{2}} \sum_l h_l \int (x+l)^m \phi(x)dx =$$

$$2^{-m-\frac{1}{2}} \sum_l h_l \sum_{k=0}^{m} \frac{m!}{k!(m-k)!} l^{m-k} \beta_k \tag{301}$$

Putting the $k = m$ term on the left and using $\sum_l h_l = \sqrt{2}$ gives

$$\beta_m = \frac{1}{2^m - 1} \frac{1}{\sqrt{2}} \sum_l h_l \sum_{k=0}^{m-1} \frac{m!}{k!(m-k)!} l^{m-k} \beta_k \tag{302}$$

This give an explicit expression for the m-th moment in terms of $\beta_0 = 1, \beta_1, \cdots, \beta_{m-1}$ and the scaling coefficients. This determines all of the moments.

Note that these equations follow directly from the scaling equations and the normalization condition that gives β_0. No special properties of the mother function have been used.

Given the moments we show that they can be used to compute the expansion coefficients for all monomial x^l that vanish when integrated against the mother function.

To see this note

$$x^l = \sum_k c_k^l \phi_k(x) \tag{303}$$

Using orthonormality gives

$$c_m^l = \int \phi_m(x) x^l dx = \int \phi(x)(x+m)^l dx$$

$$\sum_{k=0}^{l} \frac{l!}{k!(l-k)!} m^{l-k} \beta_k$$

which expresses these expansion coefficients in terms of the previously computed moments.

We can use these to get inhomogeneous equations for the Γs:

$$x^l = \sum_k c_k^l \phi_k(x) \tag{304}$$

If we differentiate this l times, multiply by $\phi(x)$, and integrate we get

$$l! \int \phi(x) dx = \sum_k c_k^l \int \phi(x) \frac{d^l \phi_k}{dx^l}(x-m) dx =$$

$$\sum_k c_k^l \int \phi(x+k) \frac{d^l \phi}{dx^l}(x) dx$$

which becomes

$$1 = \frac{1}{l!} \sum_k c^l_k \Gamma^l_{0-k;00} \tag{305}$$

This gives the needed inhomogeneous equation to determine the Γs for any allowed derivative.

Thus we have shown how $\beta_0 = 1$ determines all of the necessary normalizations.

10 Integral equations

Consider an integral equation of one of the two forms

$$f(x) = \int K_\lambda(x,y) f(y) dy \tag{306}$$

where λ is a parameter like an eigenvalue or

$$f(x) = g(x) + \int K(x,y) f(y) dy \tag{307}$$

In the first equation there will not be solutions unless λ takes on certain values. Each solution is associated with a specific value of λ. This can be considered like a matrix equation, where the matrix does not have an inverse for certain values of λ.

In the second equation there is no parameter. The function $g(x)$ is given this looks like a system of linear equations.

These equations can be solved using Galerkin or collocation methods. For the Galerkin method the solution can be expanded in terms of an orthonormal basis (wavelets):

$$f(x) = \sum_n \phi_n(x) f_n \tag{308}$$

where the f_n are unknown expansion coefficients. For the case of wavelets the index n is replaced by two indices.

Inserting this solution into either of the above equations gives

$$\sum_n \phi_n(x) f_n = \int K_\lambda(x,y) \sum_n \phi_n(y) f_n dy \tag{309}$$

or

$$\sum_n \phi_n(x) f_n = g(x) + \int K(x,y) \sum_n \phi_n(y) f_n dy \tag{310}$$

Assuming that both expression are well behaved it is possible to change the order of the sum and the integral to obtain:

$$\sum_n \phi_n(x) f_n = \sum_n \int K_\lambda(x,y)\phi_n(y)dy f_n \tag{311}$$

$$\sum_n \phi_n(x) f_n = g(x) + \sum_n \int K(x,y)\phi_n(y)dy f_n \tag{312}$$

This is not yet an equation. In applications the infinite sum has to be replaced by a finite sum. The equation then becomes approximate.

For the Galerkin method the equation is required to hold on the subspace generated by the first N basis functions. The equations for the coefficients are obtained by integrating against $\phi^*_m(x)$ for $m = 1, \cdots, N$. Using the orthonormality of the basis functions gives:

$$f_m = \sum_{n=1}^N \int \phi^*_m(x) K_\lambda(x,y)\phi_n(y)dy f_n \tag{313}$$

$$f_m = \int \phi^*_m(x) g(x) dx + \sum_{n=1}^N \int \phi^*_m(x) K(x,y)\phi_n(y)dy f_n \tag{314}$$

If we define

$$K_{\lambda mn} := \int \phi^*_m(x) K_\lambda(x,y)\phi_n(y)dxdy \tag{315}$$

$$K_{mn} := \int \phi^*_m(x) K(x-y)\phi_n(y)dxdy \tag{316}$$

$$g_m := \int \phi^*_m(x) g(x) dx \tag{317}$$

these equations become

$$f_m = \sum_{n=1}^N K_{\lambda mn} f_n \qquad m \in \{1, 2, \cdots N\} \tag{318}$$

$$f_m = g_m + \sum_{n=1}^N K_{mn} f_n \qquad m \in \{1, 2, \cdots N\} \tag{319}$$

These equation are linear algebraic equations for the coefficients f_n. The approximate solution can be expressed in terms of solution for these coefficients as:

$$f(x) \sim \sum_{n=1}^N \phi_n(x) f_n \tag{320}$$

This can be improved using the equation

$$f(x) \sim \sum_{n=1}^{N} \int K_\lambda(x,y) \sum_n \phi_n(y) dy f_n \tag{321}$$

or

$$f(x) \sim g(x0 + \sum_{n=1}^{N} \int K(x,y) \sum_n \phi_n(y) dy f_n \tag{322}$$

These equations can also be approached using the collocation method. For the collocation method the number of basis functions is also truncated. Rather than projecting on the subspace generated by the first N basis functions the equations are required to be exact at N points $\{x_i\}_{i=1}^N$. This gives a different set of linear algebraic equations:

$$\sum_n \phi_n(x_m) f_n = \sum_{n=1}^{N} \int K_\lambda(x_m,y)\phi_n(y) dy f_n \qquad m \in \{1, \cdots N\} \tag{323}$$

$$\sum_n \phi_n(x_m) f_n = g(x_m) + \sum_{n=1}^{N} \int K(x_m,y)\phi_n(y) dy f_n \qquad m \in \{1, \cdots N\} \tag{324}$$

In this case the equations have the structure

$$\sum_{n=1}^{N} \phi_{mn} f_n = \sum_{n=1}^{N} K_{\lambda mn} f_n \tag{325}$$

$$\sum_{n=1}^{N} \phi_{mn} f_n = g_m + \sum_{n=1}^{N} K_{mn} f_n \tag{326}$$

where

$$\phi_{mn} := \phi_n(x_m) \tag{327}$$

$$K_{\lambda mn} := \int K_\lambda(x_m,y)\phi_n(y) dy \tag{328}$$

$$K_{mn} := \int K(x_m,y)\phi_n(y) dy \tag{329}$$

$$g_m := g(x_m) \tag{330}$$

These equation give approximate values of the f_n, and an approximate solution of the form

$$f(x) \sim \sum_{n=1}^{N} \phi_n(x) f_n \tag{331}$$

This can be improved by the same interpolation method that was used in the Galerkin method.

In both cases the use of wavelet method will allow for efficient computation of the basis functions. The advantage of the integral equation method is that there there are no problems with integrating functions, like the Haar functions, that have discontinuous derivatives.

In order to obtain any additional benefit our of the wavelet basis the kernel $K(x, y)$ needs to have additional properties. In many cases of practical interest the kernel is translationally invariant. This means the

$$K(x, y) = K(x - y). \tag{332}$$

I consider the Galerkin case, where the these benefits are of most value. For a translationally invariant kernel the Galerkin method involves computing matrix elements of the general form:

$$\int \phi^*_{mn}(x) K(x - y) \phi_{kl}(y) dx dy \tag{333}$$

where here I have introduced indices on the scaling function. Similar equations are needed for the mother function. Define

$$c_n := \int \phi^*(x) K(x - y + n) \phi(y) dx dy \tag{334}$$

where $\phi(x)$ is the scaling function. Note that

$$K_{mn:kl} := \int (D^m T^n \phi)^*(x) K(x - y)(D^k T^l \phi)(y) dx dy \tag{335}$$

Using the scaling equations we show that it is possible to reduce the values of m and k respectively:

$$K_{mn:kl} := \int (D^m T^n \phi)^*(x) K(x - y)(D^k T^l \phi)(y) dx dy = \tag{336}$$

$$\sum_r h_r \int (D^m T^n \phi)^*(x) K(x - y)(D^{k-1} T^{2l+r} \phi)(y) dx dy = \sum_r h_r K_{mn:k-1,2l+r} \tag{337}$$

$$\sum_r h_r^* \int (D^{m-1}T^{2n+r}\phi)^*(x)K(x-y)(D^kT^l\phi)(y)dxdy = \sum_r h_f^* K_{m-1,2n+r:kl} \tag{338}$$

These equations show that it is possible to successively reduce the value of m and k. Once these are reduced to 0 what remains is

$$K_{0n:0l} := \int (T^n\phi)^*(x)K(x-y)(T^l\phi)(y)dxdy = \int \phi^*(x)K(x+n-y-l)\phi(y)dxdy = c_{n-l} \tag{339}$$

Similar results can be obtained for the mother functions. In that case the h_m's are replaced by the g_m's.

The problem with this is that it does not cover the case of negative values of m and k. We conclude that given a small enough base scale, translational invariance can be used to generate matrix elements of the kernel on all larger scales, in terms of translates on the base scale.

In order to deal with negative values of m and k the Kernel must have additional properties with respect to scale transformations. Unfortunately, unlike translations, most integral equation do not have scale invariant kernels. Roughly speaking, parameters with physical dimensions appearing in equations break scale invariance. The Maxwell's equation are an exception. They to not have a natural distance scale.

We say that a kernel has scale dimension s if

$$2^{-sn}K(2^n(x-y)) = K(x-y) \tag{340}$$

For example

$$K(x-y) := \frac{1}{|x-y|^{1/2}} \tag{341}$$

has scale dimension $s = -1/2$.

For these kernels direct integration shows

$$(f, KDg) := \int f^*(x)K(x-y)(Dg)(y)dxdy = \tag{342}$$

$$\int f^*(x)K(x-y)(Dg)(y)dxdy = \tag{343}$$

$$\int f^*(x)K(x-y)\frac{1}{\sqrt{2}}g(y/2)dxdy = \tag{344}$$

$$\frac{2}{\sqrt{2}}\int f^*(x)K(x-2u)g(u)dxdu = \tag{345}$$

$$\frac{4}{\sqrt{2}}\int f^*(2v)K(2(v-u))g(u)dvdu = \tag{346}$$

$$2\int (D^\dagger f^*)(v)K(2(v-u))g(u)dvdu = \tag{347}$$

$$2^{1+s}\int (D^\dagger f^*)(v)K(v-u)g(u)dvdu = \tag{348}$$

$$2^{1+s}(D^\dagger f, Kg) \tag{349}$$

Similarly it is possible to show

$$(Df, Kg) = 2^{s+1}(f, KD^\dagger g) \tag{350}$$

and

$$(Df, KDg) = 2^{s+1}(f, Kg) \tag{351}$$

$$(D^\dagger f, KD^\dagger g) = 2^{-(s+1)}(f, Kg) \tag{352}$$

These equations allow reductions of the form

$$(D^{-n}T^m\phi, KD^{-k}T^l\phi) = \tag{353}$$

assuming $n > k$ this becomes

$$2^{m(s+1)}(T^m\phi, KD^{n-k}T^l\phi) = \tag{354}$$

$$2^{m(s+1)}(\phi, KT^{2^{n-k}l-m}D^{n-k}\phi) \tag{355}$$

In this form the scaling equations can be used to eliminate powers of D. Similar relations can be derived fort he case $k > n$ and $k = n$.

We conclude that if the kernel is scale and translationally invariant, the matrix elements for the Galerkin method can be expressed in terms of matrix elements of translates of the scaling function and mother function.

11 Pyramid Method

Consider a function $\chi(x)$ that is periodic on $[0, L]$.

Divide this interval into 2^N subintervals of length $\Delta := L/2^N$

Let $\phi(x)$ be a Daubechies scaling function. Let

$$\psi(x) = \frac{1}{\sqrt{\Delta}}\phi(\frac{x}{\Delta}) \tag{356}$$

This has the following properties:

$$\int \psi(x)\psi(x-n\Delta) = \frac{1}{\Delta}\phi(\frac{x}{\Delta})\phi(\frac{x}{\Delta}-n)dx \int \phi(y)\phi(y-n)dy = \delta_{n0} \tag{357}$$

$$\sum_n \phi(x-n\Delta) = \sum_n \frac{1}{\sqrt{\Delta}}\psi(\frac{x}{\Delta}-n) = \frac{1}{\sqrt{\Delta}} \tag{358}$$

which follow from the orthogonality of the translated scaling function and the moment condition of the father wavelet (we have assumed both orthonormality of the translated wavelets and $\sum_n(\phi(x-n) = 1$ - if there are not compatible we need to readjust the constant).

We define

$$\psi_n(x) = \psi(x-n\Delta) \tag{359}$$

$$\psi_{kn}(x) = D^k\psi_n(x) = 2^{-k/2}\psi_n(\frac{x}{2^k}) \tag{360}$$

as well as the corresponding expressions for the mother function

$$u(x) = \frac{1}{\sqrt{\Delta}}m(\frac{x}{\Delta}) \tag{361}$$

$$u_n(x) = u(x-n\Delta) \tag{362}$$

$$u_{kn}(x) = D^k u_n(x) = 2^{-k/2}u(\frac{x}{2^k}) \tag{363}$$

The scaling equations for $\psi(x)$ and $u(x)$ are

$$D\psi(x) = \sum_n h_n\psi(x-n\Delta) \tag{364}$$

$$Du(x) = \sum_n g_n\psi(x-n\Delta) \tag{365}$$

Approximate $\chi(x)$ at resolution 2^N on $[0,L]$ by

$$\chi_{2^N}(x) := \sum_{n=1}^{2^N} c_n\psi(x-n\Delta) \tag{366}$$

where

$$c_n = \int \psi(x-n\Delta)\chi(x)dx \tag{367}$$

Here we assume that the integrals extend past $[0, L]$ by extending the functions so they are periodic. The coefficients c_n are an approximate representation of the function. Because

$$\chi(x) = \chi(x) \sum_n \sqrt{\Delta}\phi(x - n\Delta) \tag{368}$$

we expect that the sum of the c_n for the basis functions that are non-vanishing at x should be approximately the value of χ at x times $\sqrt{\Delta}$.

The approximation $\chi_{2^N}(x)$ is finest resolution approximation of the exact function. This is the V_0 representation of the function. This can be decomposed into a pair of vectors of length 2^{N-1} corresponding to the V_1 and W_1 subspace.

On these subspaces the expansions are

$$\chi_{2^N}(x) = \sum c_{1n} D\psi_n(x) + \sum d_{1n} Du_n(x) \tag{369}$$

Using the scaling relations gives

$$\chi_{2^N}(x) = \sum_n c_{1n} \sum_m h_m \psi(x - 2n\Delta - m\Delta) + \sum d_{1n} \sum_m g_m \psi(x - 2n\Delta - m\Delta) \tag{370}$$

Integrating against $\phi(x - k\Delta)$ gives

$$c_k = \sum_n (c_{1n} \sum_m h_{k-2n} + d_{1n} \sum_m g_{k-2n}) \tag{371}$$

This can be expressed as a matrix equation:

$$\begin{pmatrix} c_1 \\ \vdots \\ c_2^N \end{pmatrix} = \begin{pmatrix} h_{1-2} \cdots h_{1-2^{2N}} & g_{1-2} \cdots g_{1-2^{2N}} \\ \cdots & \cdots \\ h_{2^N-2} \cdots h_{2^N-2^{2N}} & g_{2^N-2} \cdots g_{2^N-2^{2N}} \end{pmatrix} \begin{pmatrix} c_{11} \\ \vdots \\ d_{12^{N-1}} \end{pmatrix} \tag{372}$$

This procedure can be repeated on the c_1 part, using h and g on next coarser scale. In this case the next matrix is smaller by a factor of four. Repeating the procedure on every level gives

$$\{c_N, d_N, d_{N-1}, d_{N-2}, \cdots d_1\} \tag{373}$$

The relations connecting these with c_0 can be put in the form of matrix. Since both bases are real and orthonormal this is necessarily are real orthogonal matrix. This is the wavelet transform.

The rows and columns are clearly manipulated by similarity transformations at each level.

12 Multigrid Methods

To understand the standard procedure consider a linear equation of the form:

$$Lu = f \tag{374}$$

where L is a linear operator and f is known. Let n denote a wavelet level with higher n denoting a finer grid.
Step 1: Start with a coarse grid - call it the $n = 1$ grid. Project L and f on the basis functions spanning this grid. This gives

$$L^1 u^1 = f^1 \tag{375}$$

In terms of the scaling functions we have

$$L^1_{mn} = \int \phi_m(x) L \phi_n(x) dx \tag{376}$$

$$f^1_m = \int \phi_m(x) f(x) dx \tag{377}$$

Solve this equation exactly for u^1. This gives a first approximation on the next level. In the case of wavelets this is the smooth part of the solution.
Step 2: The next step is to go to the next level. Define L^2 and f^2 by

$$L^2_{mn} = \int D^{-1}\phi_m(x) L D^{-1}\phi_n(x) dx \tag{378}$$

$$f^2_m = \int D^{-1}\phi_m(x) f(x) dx \tag{379}$$

We can map the course solution to the fine grid in two different ways. The first is simply to expand

$$\tilde{u}^2 = \sum_m (\tilde{u}^1, D^{-1}T^m\phi)\phi_{-1,m}(x) \tag{380}$$

The expansion coefficients $(\tilde{u}^1, D^{-1}T^m\phi)$ cam be expressed in terms of the scaling coefficients:

$$(\tilde{u}^1, D^{-1}T^m\phi) = \sum_n (D\phi, T^{m-2n}\phi) u_n = \sum_n (D\phi, T^{m-2n}\phi) u_n = \sum_n h_{m-2n} u_n \tag{381}$$

which gives

$$\tilde{u}^2 = \sum_{m,n} \phi_{-1,m}(x) h_{m-2n} \tag{382}$$

While this expresses the approximation in terms of the basis on the next level, it makes more sense to use the mother functions on the previous level, since they deal with the high frequency information. In this case we have the equivalent expression:

$$\tilde{u}^2 = \sum_m u_m \phi_{0,m}(x) + \sum_m v_m M_{0,m}(x) \tag{383}$$

where all of the coefficients $v_m = 0$.

On the next level we include the mothers. We note that the projected equation on the next level is

$$L^2 u^2 = f^2 \tag{384}$$

If we use the approximate solution $\tilde{2}$, which is exact on the previous level, we get

$$d_1^2 = L^2 \tilde{u}^2 - f^2 \tag{385}$$

which is called the defect. This measure how much the coarse grid solution fails to satisfy the equation at the next finest level.

We can also compute the error

$$v^2 = u^2 - \tilde{u}^2 \tag{386}$$

Knowing v^2 is equivalent to knowing u^2. We also have

$$L^2 v^2 = L^2(u^2 - \tilde{u}^2) = -d_1^2 \tag{387}$$

which gives an equation for the error in terms of the defect.

The idea is to map this up to the coarse level and solve for an approximate $\tilde{v}_1^2$. Projecting on the coarse level does not help, because d_2 is orthogonal to the coarse subspace; however, we can use the mothers - which is a space of the same size and project this equation on the mother subspace. This gives a correction; $\tilde{v}^2$ that is orthogonal to $\tilde{u}^2$

Define a new $\tilde{u}_2^2$ by

$$\tilde{u}_2^2 = \tilde{u}_1^2 + \tilde{v}_1^2 \tag{388}$$

This has a part on the coarse father space and a part on the coarse mother space.

Compute a new defect

$$d_2^2 = L^2 \tilde{u}_2^2 - f^2 \tag{389}$$

this has parts on the coarse father and mother subspaces.

We have the relation

$$L^2 \tilde{v}_2^2 = d_2^2 \tag{390}$$

We seek an approximation on the father space by projection - this gives a correction $\tilde{v}_2^2$ on the father space, and a new approximate solution

$$\tilde{u}_3^2 = \tilde{u}_2^2 + \tilde{v}_2^2 = \tilde{u}_1^2 + \tilde{v}_1^2 + \tilde{v}_2^2 \tag{391}$$

This process can be repeated, alternating between the mother and father space until a solution of the desired accuracy is obtained.

Step 3. The next step uses this solution to go to the next level. The above process needs to be used for each solution on the previous level.

The advantage is that if the coupling between the different scales is small the convergence should be fast.

13 Quadrature Methods

The problem is to compute the overlap coefficients

$$\nu_{jk} := (\phi_{jk}, f) = (D^j T^k \phi, f) \tag{392}$$

and

$$\mu_{jk} := (\psi_{jk}, f) = (D^j T^k \psi, f) \tag{393}$$

The scaling equations relate coefficients on one scale to the coefficients on a finer or coarser scale.

We use the basic scaling equations:

$$D\phi = \sum_l h_l T^l \phi \tag{394}$$

and

$$D\psi = \sum_l g_l T^l \phi \tag{395}$$

and the commutation relations

$$DT = T^2 D \tag{396}$$

In addition, the multiresolution analysis implies the inverse relations:

$$\phi_l = \sum_k (h_{l-2k} DT^k \phi + g_{l-2k} DT^k \psi) \tag{397}$$

(these following from completeness and orthogonality)

The scaling equations give

$$\nu_{jk} = (D^j T^k \phi, f) = (D^{j-1} T^{2k} D\phi, f) =$$
$$\sum_l h_l \nu_{j-1,2k+l} = \sum_m h_{m-2k} \nu_{j-1,m} \tag{398}$$

and similarly

$$\mu_{jk} = (D^j T^k \psi, f) = (D^{j-1} T^{2k} D\psi, f) =$$
$$\sum_l g_l \nu_{j-1,2k+l} = \sum_m g_{m-2k} \nu_{j-1,m} \tag{399}$$

The inverse relations give

$$\nu_{j-i,l} = \sum_k (h_{l-2k} \nu_{jk} + g_{l-2k} \mu_{jk}). \tag{400}$$

To understand the one-point quadrature rule note

$$\int f(x)\phi(x)dx = \nu \tag{401}$$

Define

$$x_1 := \int x\phi(x)dx = M_1 \tag{402}$$

Then

$$\int (a + bx)\phi(x)dx = a + bM_1 = a + bx_1 \tag{403}$$

For orthogonal wavelets note

$$k_m := \int x\phi(x)\phi(x-m) = \int (y+m)\phi(y+m)\phi(y) =$$
$$\int x\phi(x+m)\phi(x) = k_{-m}$$

It follows that

$$\sum mk_m = \sum(-m)k_m = 0 \tag{404}$$

Since

$$\sum m\phi(x-m) = x - M_1 \tag{405}$$

we have

$$0 = \sum \int mx\phi(x)\phi(x-m) = \sum s\phi(x)x(x-M_1) = M_2 - M_1^2 \tag{406}$$

This means that

$$\int \phi(x)(a+bx+cx^2) = a + bM_1 + cM_2 = a + bx_1 + cx_1^2 \tag{407}$$

or that for $x_1 = M_1$ the one-point rule integrates polynomials of degree 2 exactly.

Coiflets have the property that

$$\int x^k \phi(x)dx = 0 \tag{408}$$

for $k = 1, 2, \cdots N$. In this case for $x_1 = M_1 = 0$ we have

$$\int \phi(x) \sum_{n=1}^{N} c_n x^n = \sum_{n=1}^{N} c_n x_1^n = c_0 \tag{409}$$

which integrates polynomials of degree N exactly.

For orthogonal wavelets the error will be of the order 2^{-3n} where 2^{-n} is the finest scale wavelet.

The most useful alternative is to use a multipoint formula. These are are not limited in terms of errors and still allow wavelets with small support.

Choose wavelets that have compact support on $[0, L]$. In general we need to calculate

$$I_{mn}[f] = \int \phi_{mn}(x)f(x)dx \tag{410}$$

where $\phi(x)$ is a scaling function. We also need

$$\hat{I}_{mn}[f] = \int \psi_{mn}(x)f(x)dx \tag{411}$$

where $\psi_{mn}(x)$ is a wavelet.

First observe that if we work with wavelets on the finest scale the scaling equations map to coarser scales. Thus we have

$$I_{mn}[f] = \int D^m T^n \phi(x) f(x) dx =$$

$$\sum_l h_l \int D^{m-1} T^{2n+l} \phi(x) f(x) dx =$$

$$\sum_l h_l I_{m-1,2n+l}[f]$$

and

$$\hat{I}_{mn}[f] = \int \psi_{mn}(x) f(x) dx =$$

$$\sum_l g_l I_{m-1,2n+l}[f]$$

This can be repeated by reapplying the scaling relations $m-1$ more times to express these integrals in terms of the integrals

$$I_{0l}[f]$$

which are translates of the scaling function on the finest scale. This is done using the wavelet transform.

We seek a quadrature rule of the form

$$I[f] = \sum_{n=1}^{N} w_n f(x_n) \tag{412}$$

where the points x_n are in the interval $[0, L]$ and the formula is exact for polynomials

$$I[x^m] = \sum_{n=1}^{N} w_n x_n^m \tag{413}$$

We also note that

$$I_{0,1}[f] = I[T^{-1} f] \tag{414}$$

This means that

$$I_{0,1}[x^m] = I[(x+1)^m] = \sum_{n=1}^{N} w_n (x_n + 1)^m \tag{415}$$

or more generally

$$I_{0,k}[x^m] = I[(x+k)^m] = \sum_{n=1}^{N} w_n (x_n + k)^m \tag{416}$$

which will also be exact. Thus, for a general function we have the approximations:

$$I_{0,k}[f] = \sum_{n=1}^{N} w_n f(x_n + k) \tag{417}$$

Thus, having points and weights for the scaling function provide a means to compute all overlap integrals on the finest scale. The scaling equation for the scaling function and wavelets allow one to get to the overlap integrals for the wavelet basis.

Two problems remain - they are the computation of the moments of the scaling function and the computation of the quadrature weights.

For stability it is useful to replace the system

$$I[x^m] = \sum_{n=1}^{N} w_n x_n^m \tag{418}$$

by

$$I[P_m] = \sum_{n=1}^{N} w_n P_m(x_n) \tag{419}$$

where P_n is a system of real orthogonal polynomials on the support of the scaling function. For a scaling function with support in $[0, L]$ we assume

$$\int_0^L P_m(x) P_n(x) w(x) dx = \delta_{mn} \tag{420}$$

For a polynomial naturally supported on $y \in [-1, 1]$ let

$$y = \frac{2}{L} x - 1 \tag{421}$$

If $T_n(y)$ are orthogonal on $[-1, 1]$; i.e.

$$\int_{-1}^{1} T_m(y) T_n(y) s(y) dy = \delta_{mn} \tag{422}$$

then

$$P_n(x) = T_m(y(x))\sqrt{\frac{dy}{dx}} = \sqrt{\frac{2}{L}}T_m(\frac{2}{L}x - 1) \tag{423}$$

with weight

$$w(x) = s(y(x)) = s(\frac{2}{L}x - 1) \tag{424}$$

The main reason for including the weight is that the Chebyshev polynomials have a non-trivial weight.

Because of the structure of the wavelets it is useful to pick N equally spaced quadrature points on $[0, L]$.

$$x_n = (n-1)2^s + \tau \tag{425}$$

which go from

$$x_1 = \tau \qquad \text{to} \qquad x_N = (N-1)2^s + \tau < L \tag{426}$$

While τ can be used as an adjustable parameter to increase the order of the quadrature, it seem like a better strategy is to simply increase N. In this case the problem is to solve the linear system

$$I[P_m] = \sum_{n=1}^{N} w_n P_m(x_n) \tag{427}$$

for the weights w_n. The relevant approximate quadrature is then

$$I[f] \sim \sum_{n=1}^{N} w_n f(x_n) \tag{428}$$

which is exact for polynomials of degree $\leq N - 1$.

In order to solve these equations we need expression for the moments $I[P_m]$. To do this we use scaling and unitarity. We need the following two sets of coefficients

$$DP_n(x) = \frac{1}{\sqrt{2}}P_n(\frac{x}{2}) = \sum_{m=0}^{n} d_{nm}P_m(x) \tag{429}$$

$$T^{-l}P_n(x) = P_n(x+l) = \sum_{m=0}^{n} t^l_{nm}P_m(x) \tag{430}$$

We can get exact expressions for the matrices d_{mn} and t^l_{mn} using the appropriate gauss quadrature formula:

$$\int_0^L P_n(x)P_m(x)w(x)dx = \delta_{mn} = \sum_{i=1}^{K} P_n(u_i)P_m(u_i)w_{ui} \tag{431}$$

where $K > n/2, m/2$. Multiplying the above equations by $P_k(x)w(x)$ and using the quadrature rule gives:

$$d_{nk} = \sum_{i=1}^{K} w_{ui}\frac{1}{\sqrt{2}}P_n(\frac{u_i}{2})P_k(u_i) \tag{432}$$

$$t^l_{nk} = \sum_{i=1}^{K} w_{ui}P_n((u_i + l))P_k(u_i) \tag{433}$$

To compute the moments assuming $P_0(x) = c$ we have

$$I[P_0] = \int_0^L \phi(x)P_0(x)dx = c \tag{434}$$

Note

$$I[P_n] = \int_0^L \phi(x)P_n(x)dx =$$

$$\int_0^L D\phi(x)DP_n(x)dx =$$

$$\sum_l h_l \sum_m d_{nm} \int_0^L T^l\phi(x)P_m(x)dx =$$

$$\sum_l h_l \sum_m d_{nm} \int_0^L \phi(x)T^{-l}P_m(x)dx =$$

$$\sum_l h_l \sum_{mk} d_{nm}t^l_{mk}I[P_k]$$

We can separate off the $k = n$ term and write

$$I[P_n] = \frac{\sum_l h_l \sum_{mk}^{k<n} d_{nm}t^l_{mk}I[P_k]}{1 - \sum_l h_l \sum_{mk} d_{nm}t^l_{mn}} \tag{435}$$

Which can be used to recursively generate the required moments in terms of $I[P_0] = c$.

The advantage of the Chebyshev polynomials are that the points and weights are known analytically. We have the formulas

$$\int_{-1}^{l} f(x)\frac{dx}{\sqrt{1-x^2}} \approx \sum_{n=1}^{N} \frac{\pi}{N} f(\cos(\frac{(2n-1)\pi}{2N})) \tag{436}$$

which are exact for $f(x)$ a polynomial of degree $2N-1$. To get the u_i and w_{ui} these expressions have to be transformed from $[-1,1]$ to $[0,L]$:

$$u_k = \frac{L}{2}(\cos(\frac{(2n-1)\pi}{2N} + 1) \tag{437}$$

and

$$w_{uk} = \frac{2\pi}{LN} \tag{438}$$