Perturbation theory of the continuous spectrum in the theory of nuclear reactions

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Abstract

Background: Nuclear reactions are complex, with a large number of possible channels. Understanding how different channels contribute to a given reaction is investigated by perturbing the continuous spectrum.

Purpose: To develop tools to investigate reaction mechanisms by identifying the contributions from each reaction channel.

Method: Cluster decomposition methods, along with the spectral theory of proper subsystem problems, is used to identify the part of the nuclear Hamiltonian responsible for scattering into each channel.

Results: The result is an expression of the nuclear Hamiltonian as a sum over all scattering channels of channel Hamiltonians. Each channel Hamiltonian is constructed from solutions of proper subsystem problems. Retaining any subset of channel Hamiltonians results in a truncated Hamiltonian where the scattering wave functions for the retained channels differ from the wave functions of the full Hamiltonian by N-body correlations. The scattering operator for the truncated Hamiltonian satisfies an optical theorem in the retained channels. Because different channel Hamiltonians do not commute, how they interact determines their contribution to the full dynamics.

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I. INTRODUCTION

The exact solution of the many-body scattering problem is an intractable numerical problem due to the large number of degrees of freedom. Even if it could be solved numerically, the solution does not determine the role of different channels and how they interact in a given reaction. The ability to examine the response to turning different channels on or off provides a means to investigate the impact of that channel to a complicated reaction. This is useful for determining which channels are most responsible for nuclear binding, decays, or a given cross section. Mathematically, this is a problem in perturbation theory of the continuous spectrum. This is not trivial from a mathematical point of view because the continuous spectrum is not stable with respect to small perturbations [1][2][3]. Formal scattering theory provides tools to study the continuous spectrum. The optical theorem provides a means to isolate contributions from different channels.

In this work, the exact many-body Hamiltonian is expressed as a linear combination of non-commuting self-adjoint operators associated with each scattering channel. These are referred to as channel Hamiltonians. Each channel Hamiltonian is constructed using only proper subsystem solutions. Time-dependent scattering theory is used to show that each channel Hamiltonian has scattering solutions in that channel. Sums of subsets of channel Hamiltonians result in a truncated Hamiltonian that satisfies an optical theorem in that subset of channels. The resulting scattering wave functions agree with the exact scattering wave functions up to, but not including, N-body correlations. Because the channel Hamiltonians do not commute, including additional channels in the sum affects the dynamics of each retained channel, but the additional channels only impact the N-body correlations in the channel wave function.

Since N-particle bound states are determined by the Hamiltonian, the expression of the full Hamiltonian as a sum of channel Hamiltonians provides a means to determine how each channel Hamiltonian contributes to the binding. It can also happen that a subset of scattering channels can support a bound state, which may become unstable as more channels are added.

This problem was formally solved in [4], but the focus was on constructing few-body models of reactions dominated by a small number of few-body channels. The interest at that time was largely academic due to computational limitations. This work provides an alternate treatment of the method that appears in [4]. It provides an alternate proof of the optical theorem and uses time-dependent scattering theory to show that the resulting approximate scattering wave functions in the retained scattering channels agree with the exact scattering wave functions up to, but not including, N-body correlations. The application to bound systems and to relativistic systems, which is relevant for hadronic reactions, is also discussed.

The representation of the Hamiltonian as a sum of channel Hamiltonians is applicable to any nuclear Hamiltonian. There are no limitations on the choice of retained scattering channels, and the method can be applied equally to theories with two-body and many-body interactions. This is relevant because Hamiltonians generated using effective field theory [5][6] or scattering equivalences [7][8][9][10] generally have many-body interactions.

The next section summarizes the time-dependent formulation of many-body scattering theory used in the subsequent sections. Cluster expansions and their relation to expansions in subsystem Hamiltonians are discussed in section three. The expansion of the Hamiltonian as a sum of channel Hamiltonians is introduced in section four. The optical theorem for any selected set of scattering channels is proved in section five. Section six discusses a formulation of the time-independent treatment of many-body scattering using the coupled equations of Bencze, Redish and Sloan, which can be applied to the sum of selected channel Hamiltonians. The treatment of identical particles is discussed in section seven. An example illustrating the structure of the dynamical equations is given in section eight. Section nine discusses the application to bound states. The treatment of relativistic reactions is discussed in section ten. A summary and conclusion is given in section eleven. The appendix has the derivation of a technical result that is used in the proof of the optical theorem.

II. SCATTERING CHANNELS

The decomposition in [4] uses spectral expansions of the Hamiltonians for all *proper subsystems* as input. These expansions involve complete sets of subsystem bound states and scattering states. This section defines what is meant by a scattering channel in the context of this work.

Let H be the Hamiltonian for a system of N particles with short-range interactions. In general, the Hamiltonian H will have both two-body and many-body interactions. The notation a denotes a partition of the N particles into n_a non-empty disjoint subsystems, labeled by a_i, and H_{a_i} is the part of H involving only the particles in the i^{th} subsystem of the partition a.

In this work, scattering channels will always be associated with the N-particle system. There is a scattering channel, α, associated with the partition a if each subsystem Hamiltonian, H_{a_i}, has a bound state or is a single particle Hamiltonian. A bound state associated with H_{a_i} is denoted by

$$|(E_i, j_i)\,\mathbf{p}_i, \mu_i\rangle \qquad \text{where} \qquad 1 \le i \le n_a.$$

In this notation, j_i is the total intrinsic angular momentum of the i^{th} bound state, μ_i is the magnetic quantum number of the i^{th} bound state, $\mathbf{p}_i$ is the total momentum of the i^{th} bound state, and

$$E_i = \frac{\mathbf{p}_i^2}{2M_i} - e_{a_i}$$

is the total kinetic energy minus the binding energy e_{a_i} of the i^{th} bound subsystem (M_i is the total mass of the i^{th} bound subsystem). In general, for a given partition a of the N particles into n_a subsystems, there may be one or more scattering channels or zero channels associated with the partition a. The notation $\mathcal{A}$ is used to denote the set of all scattering channels of the N-body system, which by convention also includes the one-body channels (N-body bound states). Except for the one-body channels, the set $\mathcal{A}$ of scattering channels is determined by the solution of *proper subsystem problems.*

The notation discussed so far can be illustrated by considering the subsystem Hamiltonians for a seven-particle system associated with the partition $a = (135)(27)(46)$. There is a scattering channel associated with this partition if each of the three subsystem Hamiltonians can form bound states:

$$a = \underbrace{(135)}_{a_1}\underbrace{(27)}_{a_2}\underbrace{(46)}_{a_3} \qquad n_a = 3 \qquad N = 7 = n_{a_1} + n_{a_2} + n_{a_3}$$

$$H_{a_1} = K_1 + K_3 + K_5 + V_{13} + V_{15} + V_{35} + V_{135}$$

$$H_{a_2} = K_2 + K_7 + V_{27}$$

$$H_{a_3} = K_4 + K_6 + V_{46}$$

$$H_{a_1}|(E_1, j_1)\,\mathbf{p}_1, \mu_1\rangle = \left(\frac{\mathbf{p}_1^2}{2(m_1 + m_3 + m_5)} - e_{135}\right)|(E_1, j_1)\,\mathbf{p}_1, \mu_1\rangle$$

where $\mathbf{p}_1 = \mathbf{k}_1 + \mathbf{k}_3 + \mathbf{k}_5$,

$$H_{a_2}|(E_2, j_2)\,\mathbf{p}_2, \mu_2\rangle = \left(\frac{\mathbf{p}_2^2}{2(m_2+m_7)} - e_{27}\right)|(E_2, j_2)\,\mathbf{p}_2, \mu_2\rangle$$

where $\mathbf{p}_2 = \mathbf{k}_2 + \mathbf{k}_7$,

$$H_{a_3}|(E_3, j_3)\,\mathbf{p}_3, \mu_3\rangle = \left(\frac{\mathbf{p}_3^2}{2(m_4+m_6)} - e_{46}\right)|(E_3, j_3)\,\mathbf{p}_3, \mu_3\rangle$$

where $\mathbf{p}_3 = \mathbf{k}_4 + \mathbf{k}_6$, $\mathbf{k}_i$ are the single-particle momenta, K_i are the single-particle kinetic energies, V_{ij} are two-body interactions, V_{135} is a three-body interaction, and e_{135}, e_{27} and e_{46} are the binding energies of the bound states.

For a given scattering channel α, there are scattering states associated with two different asymptotic conditions. The different asymptotic conditions replace the initial conditions of the scattering states with conditions that relate the scattering states in the asymptotic past $(-)$ or asymptotic future $(+)$ to states of non-interacting bound subsystems. The scattering states, $|\Psi_\alpha^{(\pm)}\rangle$, associated with the channel α are defined by strong limits:

$$\lim_{t\to\pm\infty} \||\Psi_\alpha^{(\pm)}\rangle - \sum_{\mu_1,\cdots,\mu_{n_a}} \int e^{iHt}e^{-iH_at} \otimes_{i=1}^{n_a} |(E_i, j_i)\,\mathbf{p}_i, \mu_i\rangle\, \phi_i(\mathbf{p}_i, \mu_i)\, d\mathbf{p}_i\| = 0, \tag{1}$$

where the partition Hamiltonian, H_a, is the sum of subsystem Hamiltonians

$$H_a = \sum_{i=1}^{n_a} H_{a_i} \qquad \text{with} \qquad H_{a_i}|(E_i, j_i)\,\mathbf{p}_i, \mu_i\rangle = E_i|(E_i, j_i)\,\mathbf{p}_i, \mu_i\rangle \tag{2}$$

and satisfies

$$H_a \otimes_{i=1}^{n_a} |(E_i, j_i)\,\mathbf{p}_i, \mu_i\rangle = \left(\sum_{q=1}^{n_a} E_q\right) \otimes_{i=1}^{n_a} |(E_i, j_i)\,\mathbf{p}_i, \mu_i\rangle. \tag{3}$$

The operator H_a is the part of the Hamiltonian with all of the interactions between particles in the different clusters of the partition a turned off, and $\phi_i(\mathbf{p}_i, \mu_i)$ are wave packets in the total momentum and magnetic quantum numbers of each bound subsystem in the channel α. The variables in the wave packets are the experimentally detectable degrees of freedom (momentum and spin polarization) of the bound subsystems.

The limit in (1) is a strong limit, and this means that the integral over the wave packets must be computed *before* taking the limit. If this is done in the correct order, then inserting an extra factor of $e^{\mp\epsilon t}$ and taking the limit as $\epsilon \to 0$ *after* performing the integral does not change the result. This makes it possible to define the limit using "plane wave" states where the $\epsilon \to 0$ limit can be taken at the end of the calculation *after* integrating against the wave packets. After including the factor of $e^{\mp\epsilon t}$, the channel α scattering states

$$|\Psi_\alpha^{(\pm)}\rangle = \lim_{\epsilon\to 0} \sum_{\mu_1,\cdots,\mu_{n_a}} \int |\Psi_\alpha^{(\pm)}(\mathbf{p}_1,\mu_1,\cdots,\mathbf{p}_{n_a},\mu_{n_a})\rangle \prod_{i=1}^{n_a} \phi_i(\mathbf{p}_i,\mu_i)\, d\mathbf{p}_i \tag{4}$$

can be expressed in terms of the channel α "plane wave" scattering states defined by

$$|\Psi_\alpha^{(\pm)}(\mathbf{p}_1,\mu_1,\cdots,\mathbf{p}_{n_a},\mu_{n_a})\rangle := \lim_{t\to\pm\infty} e^{iHt\mp\epsilon t} e^{-iH_a t} \otimes_{i=1}^{n_a} |(E_i,j_i)\,\mathbf{p}_i,\mu_i\rangle =$$

$$\otimes_{i=1}^{n_a}|(E_i,j_i)\,\mathbf{p}_i,\mu_i\rangle + \lim_{t\to\pm\infty}\int_0^t \frac{d}{dt}\left(e^{iHt\mp\epsilon t}e^{-iH_a t}\right)\otimes_{i=1}^{n_a}|(E_i,j_i)\,\mathbf{p}_i,\mu_i\rangle\, dt =$$

$$\otimes_{i=1}^{n_a}|(E_i,j_i)\,\mathbf{p}_i,\mu_i\rangle$$

$$+i\lim_{t\to\pm\infty}\int_0^t e^{iHt\mp\epsilon t}\,(H\pm i\epsilon - H_a)\, e^{-iH_a t}\otimes_{i=1}^{n_a}|(E_i,j_i)\,\mathbf{p}_i,\mu_i\rangle\, dt =$$

$$\otimes_{i=1}^{n_a}|(E_i,j_i)\,\mathbf{p}_i,\mu_i\rangle + (E_\alpha - H \mp i\epsilon)^{-1}\, H^a \otimes_{i=1}^{n_a}|(E_i,j_i)\,\mathbf{p}_i,\mu_i\rangle. \tag{5}$$

The operator $H^a := H - H_a$ is the sum of interactions between particles in different clusters of a, and

$$E_\alpha = \sum_{q=1}^{n_a}\left(\frac{\mathbf{p}_q^2}{2M_q} - e_{a_q}\right) \tag{6}$$

is the total energy of the system (M_q is the total mass of the q^{th} subsystem). The limit $\epsilon \to 0$ in (5) can only be taken after integrating against products of wave packets which are functions of the momenta and magnetic quantum numbers of each bound cluster.

The tensor product of the wave packets span a channel Hilbert space $\mathcal{H}_\alpha$. The operator, Φ_α, that maps the channel α Hilbert space $\mathcal{H}_\alpha$ to the N-body Hilbert space $\mathcal{H}$ is defined by

$$\Phi_\alpha|\phi_{o\alpha}\rangle := \sum_{\mu_1,\cdots,\mu_{n_a}} \int \otimes_{i=1}^{n_a}|(E_i,j_i)\,\mathbf{p}_i,\mu_i\rangle\, \phi_i(\mathbf{p}_i,\mu_i)\, d\mathbf{p}_i, \tag{7}$$

where $|\phi_{o\alpha}\rangle \in \mathcal{H}_\alpha$ represents the product of wave packets given by

$$\langle \mathbf{p}_1,\mu_1,\cdots,\mathbf{p}_{n_a},\mu_{n_a}|\phi_{o\alpha}\rangle := \prod_{q=1}^{n_a}\phi_q(\mathbf{p}_q,\mu_q). \tag{8}$$

The wave packets describe the experimentally accessible momentum and spin distributions for the reaction. The mapping, $\Phi_\alpha : \mathcal{H}_\alpha \to \mathcal{H}$, is called the channel injection operator, and it includes the internal variables of the bound state wave functions for each bound subsystem. In this two-Hilbert space notation, the channel α "plane wave" scattering states are expressed in terms of channel wave operators [11]:

$$|\Psi^{(\pm)}_{\alpha}(\mathbf{p}_1,\mu_1,\cdots,\mathbf{p}_{n_a},\mu_{n_a})\rangle = \lim_{t\to\pm\infty} e^{iHt}e^{-iH_a t}\Phi_\alpha = \Omega^{(\pm)}(a)\Phi_\alpha \tag{9}$$

where

$$\Omega^{(\pm)}(a) := \lim_{t\to\pm\infty} e^{iHt}e^{-iH_a t} \tag{10}$$

only makes sense as a strong limit applied to the normalizable vector $\Phi_\alpha|\phi_{o\alpha}\rangle$. The advantage of the notation in (9) is that it separates the part of the scattering state that depends on the partition a from the part that depends on the associated scattering channel α. The operators $\Omega^{(\pm)}(a)$ act on the N-body Hilbert space, while Φ_α acts on the degrees of freedom that can be measured in an experiment.

The probability amplitude density for a transition from an initial channel state α to a final channel state β (the scattering matrix) is

$$\langle\Psi^{(+)}_{\beta}(\mathbf{p}_1',\mu_1',\cdots,\mathbf{p}_{n_b}',\mu_{n_b}')|\Psi^{(-)}_{\alpha}(\mathbf{p}_1,\mu_1,\cdots,\mathbf{p}_{n_a},\mu_{n_a})\rangle =$$

$$\langle\beta,\mathbf{p}_1',\mu_1',\cdots,\mathbf{p}_{n_b}',\mu_{n_b}'|S_{\beta\alpha}|\alpha,\mathbf{p}_1,\mu_1,\cdots,\mathbf{p}_{n_a},\mu_{n_a}\rangle, \tag{11}$$

where the channel scattering operator, $S_{\beta\alpha} := \Phi^\dagger_\beta\Omega^{(+)\dagger}(b)\Omega^{(-)}(a)\Phi_\alpha$, is used to express the scattering matrix in terms of the non-interacting bound subsystems in the channels α and β (the channel β is associated with the partition b). The channel scattering operator, $S_{\beta\alpha}$, is a mapping from $\mathcal{H}_\alpha$ to $\mathcal{H}_\beta$. The channel Hilbert spaces are spaces of square integrable functions of the experimentally observable degrees of freedom in each scattering channel.

In a scattering process, the incoming $(-)$ states look like free bound clusters long before the collision, and the outgoing $(+)$ states look like free bound clusters long after the collision. Since there can be scattering from the channel α to the channel β, the incoming and outgoing scattering states for different channels with different asymptotic conditions $(\pm)$ are not orthogonal; however, the scattering states for different channels with the same asymptotic condition $(\pm)$ are orthogonal and complete if the bound state channels are included. This assumes that the theory is asymptotically complete, which is an assumption that the original Hamiltonian is not pathological.

While the scattering matrix is the inner product of states satisfying incoming $(-)$ and outgoing $(+)$ asymptotic conditions, it can be expressed in terms of only the incoming scattering states

$$|\Psi_\alpha^{(-)}(\mathbf{p}_1,\mu_1,\cdots,\mathbf{p}_{n_a},\mu_{n_a})\rangle.$$

This is because $\Omega^{(+)\dagger}(a)$ and $\Omega^{(-)}(a)$ both involve limits of e^{-iHt} with $t\to+\infty$. While the scattering matrix elements could also be expressed in terms of states with the (+) asymptotic condition, in both cases $t\to+\infty$ is a perferred direction of time evolution in scattering reactions.

The resulting expression for scattering matrix elements is

$$\begin{aligned}\langle\beta,\mathbf{p}_1',\mu_1',\cdots,\mathbf{p}_{n_b}',\mu_{n_b}'|S_{\beta\alpha}|\alpha,\mathbf{p}_1,\mu_1,\cdots,\mathbf{p}_{n_a},\mu_{n_a}\rangle =\\ \delta_{\beta\alpha}\prod_i\delta(\mathbf{p}_i'-\mathbf{p}_i)\delta_{\mu_i'\mu_i}\\ -2\pi i\,\delta(E_\beta'-E_\alpha)\langle\mathbf{p}_1',\mu_1',\cdots,\mathbf{p}_{n_b}',\mu_{n_b}'|\Phi_\beta^\dagger H^b|\Psi_\alpha^{(-)}(\mathbf{p}_1,\mu_1,\cdots,\mathbf{p}_{n_a},\mu_{n_a})\rangle,\end{aligned} \tag{12}$$

where $H^b:=H-H_b$ is the part of H that only has interactions between particles in different clusters of b. For short-range interactions, the operator H^b will vanish as the clusters of b are asymptotically separated. Equation (12) can be expressed in operator form using the notation in (9):

$$S_{\beta\alpha}=I\delta_{\beta\alpha}-2\pi i\,\delta(E_\beta'-E_\alpha)\Phi_\beta^\dagger H^b\Omega^{(-)}(a)\Phi_\alpha, \tag{13}$$

where

$$T^{\beta\alpha}:=\Phi_\beta^\dagger H^b\Omega^{(-)}(a)\Phi_\alpha \tag{14}$$

is the right half-shell transition matrix element. The presence of the energy conserving delta function $\delta(E_\beta'-E_\alpha)$ ensures that the scattering matrix is only defined for *on-shell* values of the energy.

Assuming that the Hamiltonian commutes with the total linear momentum, the differential cross section for scattering from a 2-cluster channel α to a general channel β can be expressed in terms of the above quantities as

$$\begin{aligned}d\sigma=\frac{(2\pi)^4}{|s\,\mathbf{v}_r|}|\langle\mathbf{p}_1',\mu_1',\cdots,\mathbf{p}_{n_b}',\mu_{n_b}'\|\Phi_\beta^\dagger H^b\Omega^{(-)}(a)\Phi_\alpha\|\mathbf{p}_1,\mu_1,\mathbf{p}_2,\mu_2\rangle|^2\\ \times\ \delta(\sum_{j=1}^{n_b}E_j'-E_1-E_2)\,\delta(\sum_{j=1}^{n_b}\mathbf{p}_j'-\mathbf{p}_1-\mathbf{p}_2)\prod_{i=1}^{n_b}d\mathbf{p}_i'.\end{aligned} \tag{15}$$

In the above expression, $\mathbf{v}_r$ is the relative velocity of the incoming pair of particles and $s=\prod_q k_q!$ is a statistical normalization factor for identical bound states in the final state, with k_q denoting the number of identical bound states of type q in the final state. In (15), the $\| \cdots \|$ indicates that a momentum conserving delta function has been factored out of the expression so that

$$\langle \mathbf{p}_1', \mu_1', \cdots, \mathbf{p}_{n_b}', \mu_{n_b}' | \Phi_\beta^\dagger H^b \Omega^{(-)}(a) \Phi_\alpha | \mathbf{p}_1, \mu_1, \mathbf{p}_2, \mu_2 \rangle =$$

$$\delta(\sum_{j=1}^{n_b} \mathbf{p}_j' - \mathbf{p}_1 - \mathbf{p}_2) \, \langle \mathbf{p}_1', \mu_1', \cdots, \mathbf{p}_{n_b}', \mu_{n_b}' \| \Phi_\beta^\dagger H^b \Omega^{(-)}(a) \Phi_\alpha \| \mathbf{p}_1, \mu_1, \mathbf{p}_2, \mu_2 \rangle. \tag{16}$$

The differential cross section in (15) contains several independent variables, but in an experiment one chooses the variables that will be measured and integrates over the remaining variables in order to eliminate the delta functions. The differential cross section is only defined for *on-shell* matrix elements.

III. CLUSTER EXPANSIONS

Cluster expansions play an important role in understanding many-body reaction mechanisms and constructing approximations [12]. For scattering, it is useful to keep track of operators that satisfy or break translational invariance of subsystems. This is because momentum-conserving delta functions are broken up by short-range interactions. In this section, cluster expansions are treated abstractly, and the abstraction provides a powerful framework for managing cluster properties. Much of this section is based on [13], see also [14] [15].

A partition a of N particles is an assignment of the N particles into distinct non-empty equivalence classes called clusters. The following notation will be used is this paper:

$\mathcal{P}_N$ is the set of all partitions for a system of N particles;

n_a is the number of equivalence classes of a;

a_i is the set of particles in the i^{th} equivalence class of a;

n_{a_i} is the number of particles in the i^{th} equivalence class of a;

$i \sim_a j$ means that particles i and j are in the same equivalence class (cluster) of a;

$0 := \{(1)(2)\cdots(N)\}$ is the unique N-cluster partition (each particle in a different class);

$1 := \{(1 \cdots N)\}$ is the unique 1-cluster partition (all particles in the same class).

It follows from the definitions that

$$\sum_{i=1}^{n_a} n_{a_i} = N, \tag{17}$$

and the number of classes n_a for a given partition a satisfies $1 \leq n_a \leq N$. In order to demonstrate the use of partitions, here is a list of all the partitions of four particles:

(1234) is the unique 1-cluster partition;

(1)(234), (2)(134), (3)(124), (4)(123), (12)(34), (13)(24), (14)(23) are all of the 2-cluster partitions;

(12)(3)(4), (13)(2)(4), (14)(2)(3), (23)(1)(4), (24)(1)(3), (34)(1)(2) are all of the 3-cluster partitions;

(1)(2)(3)(4) is the unique 4-cluster partition.

The above list exhausts all possible partitions of four particles. The order of a particle within an equivalence class (or cluster) does not matter. In this example, there are two types of 2-cluster partitions, $(ij)(kl)$ and $(i)(jkl)$, with each type involving partitions that are related by permutations, and all of the 3-cluster partitions, $(ij)(k)(l)$, are related by permutations. Distinct partitions that are related by permutations are called permutation equivalent partitions, and this is important when dealing with systems of identical particles.

In what follows, partitions will be used to label parts of operators that have no interactions between particles in different clusters (equivalence classes) of the partition. The interactions between particles in the same cluster of a partition are "turned on", while the interactions between particles in different clusters are "turned off". All of the partitions in the prior example satisfy (17), and the notation so far can be illustrated by considering the four-particle partition $a = (12)(3)(4)$:

$$a = \underbrace{(12)}_{a_1}\underbrace{(3)}_{a_2}\underbrace{(4)}_{a_3} \qquad n_a = 3 \qquad N = 4 = n_{a_1} + n_{a_2} + n_{a_3}$$

where this is the same notation that was used in the seven-particle example from the previous section.

There is a natural partial ordering on the partitions a and b given by

$$a \supseteq b, \tag{18}$$

if every particle that is in the same b-equivalence class is in the same a-equivalence class ($i \sim_b j \to i \sim_a j$).

The partial ordering for a system of particles is illustrated by the following example:

$$\left.\begin{aligned} a &= (12)(34) \\ b &= (1)(2)(34) \\ c &= (123)(4) \end{aligned}\right\} => a \supseteq b, c \not\supseteq b.$$

In this example, there is a partial ordering on partitions a and b because particles 3 and 4 are in the same cluster in both partitions (with partition a only including the additional interaction between particles 1 and 2). There is no ordering on partitions c and b because particles 3 and 4 are not in the same cluster in both partitions.

For two partitions, a and b, the union $a \cup b$ is the least upper bound of a and b with respect to the partial ordering, and the intersection $a \cap b$ is the greatest lower bound of a and b with respect to the partial ordering. The union and intersection are formally defined by

$a \cup b$: $(a \cup b) \supseteq a$, $(a \cup b) \supseteq b$, and if $c \supseteq a$, $c \supseteq b$ then $c \supseteq (a \cup b)$

$a \cap b$: $a \supseteq (a \cap b)$, $b \supseteq (a \cap b)$, and if $a \supseteq c$, $b \supseteq c$, then $(a \cap b) \supseteq c$.

The union and intersection for a system of particles is illustrated by the following example:

$$\left.\begin{aligned} a &= (123)(4567)(89) \\ b &= (1234)(567)(89) \end{aligned}\right\} => a \cup b = (1234567)(89), a \cap b = (123)(4)(567)(89).$$

This example demonstrates how the union and intersection of the partitions a and b can be used to construct new partitions that set a least upper bound and greatest lower bound, respectively, on the partial ordering of the partitions a and b. It can be seen from the definitions and the above example that every partition a satisfies $1 \supseteq a \supseteq 0$.

This structure for the partial orderings on partitions is called a partition lattice. Some important tools, that will be utilized in what follows, are the incidence function and its inverse. These functions are also called the Zeta and Möbius functions on the partition lattice, respectively. The Zeta function is defined in [13][14][15] as

$$\Delta_{a \supseteq b} := \begin{cases} 1 \text{ if } a \supseteq b \\ 0 \text{ if } a \not\supseteq b \end{cases} . \tag{19}$$

Since this is upper triangular with 1's on the diagonal, it necessarily has an inverse given by

$$\Delta^{-1}_{a\supseteq b} := \begin{cases} (-)^{n_a} \prod_{i=1}^{n_a} (-)^{n_{b_i}} (n_{b_i} - 1)! & \text{if } \; a \supseteq b \\ 0 & \text{if } \; a \not\supseteq b \end{cases}, \tag{20}$$

where n_{b_i} is the number of clusters of b in the i^{th} cluster of a. Note that both the Zeta function, $\Delta_{a\supseteq b}$, and Möbius function, $\Delta^{-1}_{a\supseteq b}$, vanish when $a \not\supseteq b$. The Zeta and Möbius functions are matrix operators that satisfy

$$\sum_{c\in\mathcal{P}_N} \Delta^{-1}_{a\supseteq c}\Delta_{c\supseteq b} = \delta_{ab} \qquad \text{and} \qquad \sum_a \delta_{ab} = 1 \qquad \text{for} \qquad a \supseteq b, \tag{21}$$

which follows from the definitions of the Zeta and Möbius functions.

Partitions can be used to classify operators on the N-particle Hilbert space. The starting assumption is that the Hamiltonian H is translationally invariant and commutes with the total momentum operator. Each cluster a_i of the N-body system represents a subsystem, and the total momentum $\mathbf{p}_{a_i}$ of the particles in the cluster a_i is

$$\mathbf{p}_{a_i} := \sum_{j\in a_i} \mathbf{k}_j. \tag{22}$$

The quantity $\mathbf{k}_j$ denotes the single-particle momenta, and the total momentum $\mathbf{p}_{a_i}$ is the generator of translations of the subsystem of particles in the cluster a_i. The operator that independently translates each cluster a_i of the partition a by a vector $\mathbf{x}_i$ is

$$T_a(\mathbf{x}_1, \cdots, \mathbf{x}_{n_a}) := e^{i\sum_{i=1}^{n_a} \mathbf{x}_i\cdot\mathbf{p}_{a_i}}. \tag{23}$$

This is a $3n_a$ parameter unitary group of translations. An operator O that commutes with $T_a(\mathbf{x}_1, \cdots, \mathbf{x}_{n_a})$, and satisfies

$$[O, T_a(\mathbf{x}_1, \cdots, \mathbf{x}_{n_a})] = 0 \tag{24}$$

for all $\mathbf{x}_i$, is called an a-invariant operator.

For any partition a of the particles into non-empty disjoint subsystems, a general operator can be expressed as the sum of an operator that commutes with $T_a(\mathbf{x}_1, \cdots, \mathbf{x}_{n_a})$ and a remainder. This is represented by the notation

$$O = O_a + O^a, \tag{25}$$

where O_a is the a-invariant part of O and $O^a := O - O_a$ is the remainder that breaks the $T_a(\mathbf{x}_1, \cdots, \mathbf{x}_{n_a})$ translational invariance. It follows from the definitions that

$$\|T_a^\dagger(\mathbf{x}_1, \cdots, \mathbf{x}_{n_a})\,(O - O_a)\,T_a(\mathbf{x}_1, \cdots, \mathbf{x}_{n_a})|\psi\rangle\| = \|O^a\,T_a(\mathbf{x}_1, \cdots, \mathbf{x}_{n_a})|\psi\rangle\| \tag{26}$$

for the state $|\psi\rangle$. For an operator O that is overall translationally invariant, O^a involves operators that only contain interactions between particles in different clusters of the partition a. If these are all short-range interactions, then as all of the clusters of a are asymptotically separated O^a should vanish. A mathematical formulation of this condition is

$$\lim_{|\mathbf{x}_i - \mathbf{x}_j| \to \infty} \|O^a\,T_a(\mathbf{x}_1, \cdots, \mathbf{x}_{n_a})|\psi\rangle\| = 0 \qquad \forall i \neq j. \tag{27}$$

A many-body operator O that can be decomposed as $O = O_a + O^a$ with O_a satisfying (24) and O^a satisfying (27) will be called a "cluster expandable operator". The types of operators considered in this work are interactions, projections on bound subsystems, resolvents of the form in (5), wave operators, and time evolution operators. For suitable short-range interactions, limits of the form (27) are expected to vanish. For cluster expandable operators, it follows from (24) and (27) that O_a can be constructed from O using

$$O_a = \lim_{|\mathbf{x}_i - \mathbf{x}_j| \to \infty} T_a^\dagger(\mathbf{x}_1, \cdots, \mathbf{x}_{n_a})\,O\,T_a(\mathbf{x}_1, \cdots, \mathbf{x}_{n_a}), \tag{28}$$

which shows that O_a can be obtained by asymptotically separating the different clusters of the partition a. This notation can be illustrated by considering the four-particle Hamiltonian associated with the partition $a = (1)(2)(34)$:

$$H = \underbrace{K_1 + K_2 + K_3 + K_4 + V_{34}}_{H_{(1)(2)(34)}} +$$

$$\underbrace{V_{12} + V_{13} + V_{14} + V_{23} + V_{24} + V_{123} + V_{124} + V_{134} + V_{234} + V_{1234}}_{H^{(1)(2)(34)}},$$

where K_i are the single-particle kinetic energies, V_{ij} are two-body interactions, V_{ijk} are three-body interactions, and V_{1234} is a four-body interaction. In this example, $H^{(1)(2)(34)}$ vanishes when all of the clusters of $a = (1)(2)(34)$ are asymptotically separated, and one is left with the a-invariant operator $H_{(1)(2)(34)}$. The interactions in $H^{(1)(2)(34)}$ all involve particles in *different* clusters of a.

For $b \supseteq a$, $T_b(\mathbf{x}_1, \cdots, \mathbf{x}_{n_b})$ is a subgroup of $T_a(\mathbf{x}_1, \cdots, \mathbf{x}_{n_a})$, and it follows that

$$O_a = \lim_{|\mathbf{x}_i - \mathbf{x}_j| \to \infty} T_b^\dagger(\mathbf{x}_1, \cdots, \mathbf{x}_{n_b})\,O_a\,T_b(\mathbf{x}_1, \cdots, \mathbf{x}_{n_b}) \qquad \text{for} \qquad b \supseteq a. \tag{29}$$

This means that O_a is invariant with respect to translations that separate the clusters of b when $b \supseteq a$. As an example, consider the partitions $a = (1)(2)(34)$ and $b = (12)(34)$ which satisfy $b \supseteq a$. The operator $H_{(1)(2)(34)}$ from the previous example is invariant with respect to translations that asymptotically separate the clusters of $b = (12)(34)$ because these translations do not separate particles 3 and 4.

On the other hand, if $b \not\supseteq a$, then O_a has the decomposition

$$O_a = (O_a)_b + (O_a)^b = O_{a \cap b} + O_a^b. \tag{30}$$

In this decomposition, $O_{a\cap b}$ is invariant with respect to both $T_a(\mathbf{x}_1, \cdots, \mathbf{x}_{n_a})$ and $T_b(\mathbf{x}_1, \cdots, \mathbf{x}_{n_b})$, and O_a^b vanishes as the clusters of the partition b are asymptotically separated. Using (27),

$$\lim_{|\mathbf{x}_i - \mathbf{x}_j| \to \infty} \| O_a^b \, T_b(\mathbf{x}_1, \cdots, \mathbf{x}_{n_b}) |\psi\rangle \| = 0, \tag{31}$$

which means that

$$O_{a\cap b} = \lim_{|\mathbf{x}_i - \mathbf{x}_j| \to \infty} T_b^\dagger(\mathbf{x}_1, \cdots, \mathbf{x}_{n_b}) \, O_a \, T_b(\mathbf{x}_1, \cdots, \mathbf{x}_{n_b}) \qquad \text{for} \qquad b \not\supseteq a. \tag{32}$$

Therefore, $O_{a\cap b}$ can be obtained from O_a by asymptotically separating the clusters of b when $b \not\supseteq a$. As an example, consider the partitions $a = (1)(2)(34)$ and $b = (123)(4)$ which satisfy $b \not\supseteq a$. The operator $H_{(1)(2)(34)}$ from the prior examples can be written as

$$H_{(1)(2)(34)} = \underbrace{K_1 + K_2 + K_3 + K_4}_{H_{(1)(2)(34)\cap(123)(4)}} + \underbrace{V_{34}}_{H_{(1)(2)(34)}^{(123)(4)}} .$$

In this example, $H_{(1)(2)(34)}^{(123)(4)}$ vanishes when the clusters of $b = (123)(4)$ are asymptotically separated, and one is left with $H_{(1)(2)(34)\cap(123)(4)} = H_{(1)(2)(3)(4)}$. It should be noted that an a-invariant operator can always be decomposed as (30), but the term O_a^b vanishes whenever $b \supseteq a$ (this is because $T_b(\mathbf{x}_1, \cdots, \mathbf{x}_{n_b})$ is a subgroup of $T_a(\mathbf{x}_1, \cdots, \mathbf{x}_{n_a})$).

For applications, it is useful to define $[O]_a$, the *a-connected part* of O, by the conditions

$$[[O]_a, T_a(\mathbf{x}_1, \cdots, \mathbf{x}_{n_a})] = 0 \qquad \text{and} \qquad ([O]_a)_b = 0 \qquad \text{for} \qquad b \not\supseteq a. \tag{33}$$

This means that the operator $[O]_a$ is invariant with respect to the translations $T_a(\mathbf{x}_1, \cdots, \mathbf{x}_{n_a})$, but it is not invariant with respect to the translations $T_b(\mathbf{x}_1, \cdots, \mathbf{x}_{n_b})$ when $b \not\supseteq a$ (these translations necessarily break up at least one of the clusters of a). For $b \supseteq a$, it follows that the b-invariant part of O is a sum of the a-connected parts of O that commute with $T_b(\mathbf{x}_1, \cdots, \mathbf{x}_{n_b})$. This means that

$$O_b = \sum_{b \supseteq a} [O]_a = \sum_{a \in \mathcal{P}_N} \Delta_{b \supseteq a} [O]_a, \tag{34}$$

where O_b is expressed in terms of the Zeta function on the partition lattice. This expression can be inverted using the Möbius function on the partition lattice

$$[O]_b = \sum_{a \in \mathcal{P}_N} \Delta^{-1}_{b \supseteq a} O_a. \tag{35}$$

It follows from (34) and (35) that the Zeta and Möbius functions on the partition lattice provide a direct relation between the operators O_b and $[O]_b$. Additionally, an operator O is said to be completely connected if $O = [O]_1$, and this means that O vanishes in the limit that any pair of particles are asymptotically separated. It follows from these expressions that

$$[O]_1 = \sum_{a \in \mathcal{P}_N} \Delta^{-1}_{1 \supseteq a} O_a = \Delta^{-1}_{1 \supseteq 1} O_1 + \sum_{a \in \mathcal{P}'_N} \Delta^{-1}_{1 \supseteq a} O_a = O + \sum_{a \in \mathcal{P}'_N} \Delta^{-1}_{1 \supseteq a} O_a, \tag{36}$$

where $\mathcal{P}'_N$ is the set of all partitions of N particles excluding the 1-cluster partition. This means O has the decomposition

$$O = [O]_1 - \sum_{a \in \mathcal{P}'_N} \Delta^{-1}_{1 \supseteq a} O_a. \tag{37}$$

It is useful to define the coefficients appearing in (37) as

$$\mathcal{C}_a := -\Delta^{-1}_{1 \supseteq a} = (-)^{n_a} (n_a - 1)! \qquad \text{with} \qquad \sum_{a \in \mathcal{P}'_N} \mathcal{C}_a = 1, \tag{38}$$

which are combinatoric factors that ensure that the decomposition has the correct overall counting. The sum in (38) follows because

$$\sum_{a \in \mathcal{P}'_N} \Delta^{-1}_{1 \supseteq a} = -\Delta^{-1}_{1 \supseteq 1} + \sum_a \Delta^{-1}_{1 \supseteq a} = -\Delta^{-1}_{1 \supseteq 1} + \sum_a \Delta^{-1}_{1 \supseteq a} \underbrace{\Delta_{a \supseteq 0}}_{=1} = -1 + 0.$$

An important consequence of the invertibility of the incidence matrix is that a general cluster expandable operator can be expressed in two equivalent ways:

$$O = \sum_{a \in \mathcal{P}_N} [O]_a = [O]_1 + \sum_{a \in \mathcal{P}'_N} \mathcal{C}_a O_a. \tag{39}$$

The first sum is the cluster expansion of O, and the second sum is the operator decomposition of O in terms of proper subsystem operators. This is a generalization of the linked cluster theorem for identical particles. The cluster expansion is a sum over the a-connected parts of O, and the operator decomposition consists of the completely connected part of O (the N-body interaction) and a linear combination of the a-invariant parts of O. For an operator like a Hamiltonian, the cluster expansion is a linear combination of interactions, while the sum over the a-invariant parts of the Hamiltonian in the operator decomposition corresponds to a linear combination of proper subsystem Hamiltonians. As an example, a three-body Hamiltonian with two-body and three-body interactions can be expressed as a cluster expansion or as a sum of proper subsystem Hamiltonians:

$$
\begin{aligned}
H = \underbrace{K_1 + K_2 + K_3}_{[H]_0=[H]_{(1)(2)(3)}} + \underbrace{V_{12}}_{[H]_{(12)(3)}} + \underbrace{V_{13}}_{[H]_{(13)(2)}} + \underbrace{V_{23}}_{[H]_{(23)(1)}} + \underbrace{V_{123}}_{[H]_1=[H]_{(123)}} = \\
-2 \underbrace{(K_1 + K_2 + K_3)}_{H_0=H_{(1)(2)(3)}=H_1+H_2+H_3} + \underbrace{K_1 + K_2 + K_3 + V_{12}}_{H_{(12)(3)}=H_{12}+H_3} \\
+ \underbrace{K_1 + K_2 + K_3 + V_{13}}_{H_{(13)(2)}=H_{13}+H_2} + \underbrace{K_1 + K_2 + K_3 + V_{23}}_{H_{(23)(1)}=H_{23}+H_1} + \underbrace{V_{123}}_{[H]_1=[H]_{(123)}}
\end{aligned}
$$

where K_i are the single-particle kinetic energies, V_{ij} are two-body interactions, V_{123} is the three-body interaction (completely connected part), and $H_{(ij)(k)}$ is a sum of subsystem Hamiltonians. The coefficients in the second and third lines are the combinatoric factors $\mathcal{C}_a$, and they ensure that the cluster expansion and operator decomposition are equal to one another. In this example, the three kinetic energy terms appear in each of the three 2-cluster partition Hamiltonians, and the (-2) in front of the 3-cluster Hamiltonian corrects for this overcounting of the three kinetic energy terms. This example demonstrates how the operator decomposition in (39) is used to express the cluster decomposition of the N-body Hamiltonian in terms of a linear combination of subsystem Hamiltonians. The decomposition in (39) is useful for identifying the parts of the Hamiltonian that are responsible for the different channel asymptotic states in a manner that treats all scattering channels democratically.

In general, if A and B are bounded operators, then

$$
\|T_a^\dagger ABT_a|\psi\rangle\| = \|T_a^\dagger(A_a + A^a)(B_a + B^a)T_a|\psi\rangle\| \leq
$$

$$
\|A_aB_a|\psi\rangle\| + \|T_a^\dagger A^aT_aB_a|\psi\rangle\| + \|A_aT_a^\dagger B^aT_a|\psi\rangle\| + \|T_a^\dagger A^aT_aT_a^\dagger B^aT_a|\psi\rangle\| \leq
$$

$$
\|A_aB_a|\psi\rangle\| + \|T_a^\dagger A^aT_aB_a|\psi\rangle\| + \|A_a\|\|T_a^\dagger B^aT_a|\psi\rangle\| + \|A^aT_a\|\|T_a^\dagger B^aT_a|\psi\rangle\| \leq
$$

$$\|A_a B_a|\psi\rangle\| + \|A^a T_a B_a|\psi\rangle\| + \|A_a\|\|B^a T_a|\psi\rangle\| + \|A^a T_a\|\|B^a T_a|\psi\rangle\|.$$

If A and B are cluster expandable operators then the last three terms in the fourth line vanish in the limit that all of the clusters of a are asymptotically separated. This means that a-invariant operators satisfy

$$(AB)_a = A_a B_a. \tag{40}$$

It follows from (37) and (38) that

$$\sum_{a\in\mathcal{P}'_N} \mathcal{C}_a A_a B_a = AB - [AB]_1 = \left(\sum_{a\in\mathcal{P}'_N} \mathcal{C}_a A_a + [A]_1\right)(B_a + B^a) - [AB]_1 =$$

$$\sum_{a\in\mathcal{P}'_N} \mathcal{C}_a A_a B_a + \sum_{a\in\mathcal{P}'_N} \mathcal{C}_a A_a B^a + [A]_1 B - [AB]_1.$$

Canceling $\sum_{a\in\mathcal{P}'_N} \mathcal{C}_a A_a B_a$ on both sides of this equation gives

$$\sum_{a\in\mathcal{P}'_N} \mathcal{C}_a A_a B^a = -[A]_1 B + [AB]_1, \tag{41}$$

where the terms on the right are connected. This means that sums of the form

$$\sum_{a\in\mathcal{P}'_N} \mathcal{C}_a A_a B^a \tag{42}$$

are either 0 or connected.

For an N-particle system, the dynamics is given by the unitary time evolution operator $U(t) = e^{-iHt}$ where H is the N-particle Hamiltonian. By turning off the interactions between particles in different clusters of the partition a, H becomes H_a, and this is the infinitesimal generator of time translation, $U_a(t)$, of non-interacting clusters of a. In general, due to the kinetic energy terms, Hamiltonians are not bounded operators, but Hunziker in [16] proved that $U(t)$ satisfies (27) for Hamiltonians with square integrable interactions (i.e., the Hamiltonian is a cluster expandable operator). This means that

$$\lim_{|\mathbf{x}_i - \mathbf{x}_j|\to\infty} \|U^a(t)\, T_a(\mathbf{x}_1, \cdots, \mathbf{x}_{n_a})|\psi\rangle\| = 0. \tag{43}$$

He also proved that, under the same assumptions, the wave operators are cluster expandable operators. In everything that follows, it will be assumed that these properties are satisfied.

IV. CHANNEL DECOMPOSITION

The set of scattering channels $\mathcal{A}$ can be separated into disjoint sets of channels; a selected set $\mathcal{A}_1$ and a remainder $\mathcal{A}_2$ [4]. This section provides the construction of a representation of a general N-particle Hamiltonian H as a sum of a Hamiltonian $H_{\mathcal{A}_1}$ that depends on the channels $\mathcal{A}_1$ and a Hamiltonian $H_{\mathcal{A}_2}$ that depends on the complementary set of channels $\mathcal{A}_2$.

For a general many-body Hamiltonian, the exact spectral decomposition has the form

$$H = \sum_{\alpha\in\mathcal{A}} |\psi_\alpha^{(-)}\rangle\langle\psi_\alpha^{(-)}|H = \sum_{\alpha\in\mathcal{A}} P_\alpha^{(-)} H \qquad \text{with} \qquad I = \sum_{\alpha\in\mathcal{A}} P_\alpha^{(-)}, \tag{44}$$

where

$$P_\alpha^{(-)} := \Omega^{(-)}(a)\Phi_\alpha\Phi_\alpha^\dagger\Omega^{(-)\dagger}(a) \tag{45}$$

is the orthogonal projection on the subspace spanned by the $(-)$ scattering states in the channel α. While there is an identical decomposition using the scattering states with the $(+)$ asymptotic condition, the $(-)$ asymptotic condition corresponds to forward time evolution. By convention, the channel sum in (44) includes the one-body channels (N-body bound states). The notation in (45) is shorthand for

$$P_\alpha^{(-)} H := \sum_{\mu_1,\cdots,\mu_{n_a}} \int d\mathbf{p}_1\cdots d\mathbf{p}_{n_a}\, |(\alpha,\mathbf{p}_1,\mu_1,\cdots,\mathbf{p}_{n_a},\mu_{n_a})^{(-)}\rangle E_\alpha \times$$
$$\langle(\alpha,\mathbf{p}_1,\mu_1,\cdots,\mathbf{p}_{n_a},\mu_{n_a})^{(-)}|,$$

where E_α is the total energy eigenvalue defined in (6). Both the Hamiltonian H and the channel projection operator $P_\alpha^{(-)}$ have cluster expansions of the form in (39):

$$H = \sum_{b\in\mathcal{P}_N} [H]_b = [H]_1 + \sum_{b\in\mathcal{P}'_N} \mathcal{C}_b H_b \qquad \text{and} \qquad P_\alpha^{(-)} = \sum_{b\supseteq a} [P_\alpha^{(-)}]_b, \tag{46}$$

where H_b is the infinitesimal generator of $U_b(t)$, the time evolution operator for the non-interacting clusters of b . The product of the Hamiltonian H and the channel projection operator $P_\alpha^{(-)}$ also has a cluster expansion. The expansions of these operators, using (40), gives

$$H = \sum_{\alpha\in\mathcal{A}} \sum_{\{b\in\mathcal{P}_N|b\supseteq a\}} [P_\alpha^{(-)} H]_b, \tag{47}$$

where $[P_\alpha^{(-)} H]_b$ is the b-connected part of $P_\alpha^{(-)} H = \Omega^{(-)}(a)\Phi_\alpha\Phi_\alpha^\dagger\Omega^{(-)\dagger}(a)H$. Using (37) and (38), (47) can be decomposed into a linear combination of its b-invariant parts and a completely connected part:

$$H = \sum_{\alpha \in \mathcal{A}} \left([P_\alpha^{(-)} H]_1 + \sum_{\{b \in \mathcal{P}'_N | b \supseteq a\}} \mathcal{C}_b (P_\alpha^{(-)})_b H_b \right), \tag{48}$$

where $[P_\alpha^{(-)} H]_1$ is the completely connected part of the product $P_\alpha^{(-)} H$ and a is the partition associated with the bound clusters of the channel α. Since $(P_\alpha^{(-)})_b$ is an orthogonal projector and commutes with H_b their product is Hermitian. For $b \not\supseteq a$, translating the clusters of b will separate particles in at least one of the bound clusters of a in the channel α. Therefore,

$$(\Phi_\alpha \Phi_\alpha^\dagger)_b = ([\Phi_\alpha \Phi_\alpha^\dagger]_a)_b = 0 \qquad \text{for} \qquad b \not\supseteq a, \tag{49}$$

which means that

$$(P_\alpha^{(-)})_b = 0 \qquad \text{for} \qquad b \not\supseteq a. \tag{50}$$

This means that the sum over the b-invariant parts of $P_\alpha^{(-)} H$ is zero when $b \not\supseteq a$.

Since the interactions between particles in different clusters of a that are in the same clusters of b are short-range, the wave operators satisfy a chain rule [3][17] that allows successive interactions to be turned on. This same result follows from the analysis in [16]. For any b satisfying $b \supseteq a$, the chain rule for wave operators gives

$$\Omega^{(-)}(a)\Phi_\alpha = \lim_{t \to -\infty} e^{iHt} e^{-iH_a t} \Phi_\alpha =$$

$$\lim_{t \to -\infty} e^{iHt} \underbrace{e^{-iH_b t} e^{iH_b t}}_{I} e^{-iH_a t} \Phi_\alpha = \lim_{t \to -\infty} e^{iHt} e^{-iH_b t} (\Omega^{(-)}(a))_b \Phi_\alpha =$$

$$\Omega^{(-)}(b)(\Omega^{(-)}(a))_b \Phi_\alpha. \tag{51}$$

Here $(\Omega^{(-)}(a))_b \Phi_\alpha$ replaces Φ_α when computing $\Omega^{(-)}(b)$. Since $\Omega^{(-)}(b)$ turns on the interactions between particles in different clusters of the partition b, then when $b \supseteq a$ one can write

$$(\Omega^{(-)}(a))_b \Phi_\alpha = ((\Omega^{(-)}(b))_b (\Omega^{(-)}(a))_b \Phi_\alpha)_b. \tag{52}$$

This shows that $(\Omega^{(-)}(b))_b$ acts like the identity on $(\Omega^{(-)}(a))_b \Phi_\alpha$, and this means that

$$(\Omega^{(-)}(a))_b \Phi_\alpha = ((\Omega^{(-)}(a))_b \Phi_\alpha)_b$$

is the b-invariant part of $\Omega^{(-)}(a)\Phi_\alpha$ for $b \supseteq a$. The important point is that this involves solutions of the scattering problem in the channel α for the Hamiltonian $H_b = \sum_{i=1}^{n_b} H_{b_i}$, which is a sum of proper subsystem Hamiltonians H_{b_i}. It also implies that every H_b for $b \supseteq a$ has channel α scattering states. The operator $(\Omega^{(-)}(a))_b \Phi_\alpha \Phi_\alpha^\dagger (\Omega^{(-)\dagger}(a))_b$ is the part of the exact spectral projection that remains after turning off the interactions between particles in different clusters of b (there are still remaining interactions between the asymptotically bound subsystems in the different clusters of a that are in the same clusters of b).

It follows that the exact projection of the Hamiltonian on the channel α subspace has the decomposition

$$P_\alpha^{(-)} H = [P_\alpha^{(-)} H]_1 + \sum_{\{b \in \mathcal{P}'_N | b \supseteq a\}} \mathcal{C}_b (P_\alpha^{(-)})_b H_b =$$

$$[P_\alpha^{(-)} H]_1 + \sum_{\{b \in \mathcal{P}'_N | b \supseteq a\}} \mathcal{C}_b (\Omega^{(-)}(a))_b \Phi_\alpha \Phi_\alpha^\dagger (\Omega^{(-)\dagger}(a))_b H_b. \tag{53}$$

Up to this point, everything is exact. The cluster properties imply that the terms $(\Omega^{(-)}(a))_b \Phi_\alpha \Phi_\alpha^\dagger (\Omega^{(-)\dagger}(a))_b H_b$ in (53), for $b \in \mathcal{P}'_N$, can be computed using only *proper subsystem solutions*. The assumed asymptotic completeness implies that the sum over all channel projectors is the identity

$$I = \sum_{\alpha \in \mathcal{A}} P_\alpha^{(-)}. \tag{54}$$

It follows that the Hamiltonian can be expressed as

$$H = [H]_1 + \sum_{b \in \mathcal{P}'_N} \mathcal{C}_b H_b =$$

$$\sum_{\alpha \in \mathcal{A}} \left([P_\alpha^{(-)} H]_1 + \sum_{\{b \in \mathcal{P}'_N | b \supseteq a\}} \mathcal{C}_b (P_\alpha^{(-)})_b H_b \right) =$$

$$\sum_{\alpha \in \mathcal{A}} \left([P_\alpha^{(-)} H]_1 + \sum_{\{b \in \mathcal{P}'_N | b \supseteq a\}} \mathcal{C}_b (\Omega^{(-)}(a))_b \Phi_\alpha \Phi_\alpha^\dagger (\Omega^{(-)\dagger}(a))_b H_b \right). \tag{55}$$

Comparing these expressions, the completely connected parts on both sides of (55) must be the same and are given by

$$[H]_1 = \sum_{\alpha \in \mathcal{A}} [P_\alpha^{(-)} H]_1. \tag{56}$$

This means that they add up to zero if H does not have an N-body interaction.

The next step is to introduce the channel decomposition, and this is the main result of [4]. This is done by decomposing the collection of channels into two disjoint sets, $\mathcal{A} = \mathcal{A}_1 \cup \mathcal{A}_2$, where $\mathcal{A}_1$ is a selected set of scattering channels and $\mathcal{A}_2$ represents the remaining scattering channels. There are no restrictions on how to choose the set $\mathcal{A}_1$.

It is useful to define the orthogonal projectors

$$P^{(-)}_{\mathcal{A}_1} := \sum_{\alpha \in \mathcal{A}_1} P^{(-)}_\alpha \qquad \text{and} \qquad P^{(-)}_{\mathcal{A}_2} := \sum_{\alpha \in \mathcal{A}_2} P^{(-)}_\alpha , \tag{57}$$

where by convention the one-body (N-body bound state) channels are in $\mathcal{A}_2$.

If follows from (54) that they satisfy

$$P^{(-)}_{\mathcal{A}_1} + P^{(-)}_{\mathcal{A}_2} = I. \tag{58}$$

This leads to the exact decomposition of the Hamiltonian given by

$$H = P^{(-)}_{\mathcal{A}_1} H + P^{(-)}_{\mathcal{A}_2} H = \sum_{\alpha \in \mathcal{A}_1} P^{(-)}_\alpha H + \sum_{\alpha \in \mathcal{A}_2} P^{(-)}_\alpha H =$$

$$\sum_{\alpha \in \mathcal{A}_1} \left([P^{(-)}_\alpha H]_1 + \sum_{\{b \in \mathcal{P}'_N | b \supseteq a\}} \mathcal{C}_b (P^{(-)}_\alpha)_b H_b \right)$$

$$+ \sum_{\alpha \in \mathcal{A}_2} \left([P^{(-)}_\alpha H]_1 + \sum_{\{b \in \mathcal{P}'_N | b \supseteq a\}} \mathcal{C}_b (P^{(-)}_\alpha)_b H_b \right) . \tag{59}$$

This decomposition is based on the the $(-)$ asymptotic condition in the scattering channel projectors. From (56), all of the completely connected parts in (59) add up to $[H]_1$, and this vanishes if there are no N-body interactions (note that the N-body bound state channels only contribute to the completely connected parts of the expression). It follows that

$$H = [H]_1 + \sum_{\alpha \in \mathcal{A}_1} \sum_{\{b \in \mathcal{P}'_N | b \supseteq a\}} \mathcal{C}_b (P^{(-)}_\alpha)_b H_b + \sum_{\alpha \in \mathcal{A}_2} \sum_{\{b \in \mathcal{P}'_N | b \supseteq a\}} \mathcal{C}_b (P^{(-)}_\alpha)_b H_b . \tag{60}$$

If there are no N-body interactions in the Hamiltonian, then the contributions from the N-body bound states cancel with the completely connected contributions from the scattering channels.

Channel truncated Hamiltonians are defined by

$$H_{\mathcal{A}_1} := \sum_{\alpha \in \mathcal{A}_1} \sum_{\{b \in \mathcal{P}'_N | b \supseteq a\}} \mathcal{C}_b (P^{(-)}_\alpha)_b H_b \tag{61}$$

and

$$H_{\mathcal{A}_2} := [H]_1 + \sum_{\alpha \in \mathcal{A}_2} \sum_{\{b \in \mathcal{P}'_N | b \supseteq a\}} \mathcal{C}_b (P^{(-)}_\alpha)_b H_b , \tag{62}$$

where the completely connected part $[H]_1$ (the N-body interaction) is included in the set $\mathcal{A}_2$. The individual channel Hamiltonians are defined by

$$H_\alpha := \sum_{\{b\in\mathcal{P}'_N|b\supseteq a\}} \mathcal{C}_b(P^{(-)}_\alpha)_b H_b. \tag{63}$$

With this definition, equations (61) and (62) have the form

$$H_{\mathcal{A}_1} := \sum_{\alpha\in\mathcal{A}_1} H_\alpha \tag{64}$$

and

$$H_{\mathcal{A}_2} := [H]_1 + \sum_{\alpha\in\mathcal{A}_2} H_\alpha. \tag{65}$$

This decomposition has the feature that both $H_{\mathcal{A}_1}$ and $H_{\mathcal{A}_2}$ are expressed in terms of solutions of *proper subsystem problems* and a possible N-body interaction. $(P^{(-)}_\alpha)_b$ is the projection on the scattering states of H_b if H_b has scattering states in the channel α. These scattering states are related to the exact channel α scattering states by "turning off" the interactions between particles in different clusters of b for $b \supseteq a$. It is always possible to add additional N-body operators to (61) provided that they are subtracted from (62).

Since both H_b and $(P^{(-)}_\alpha)_b$ are Hermitian and commute with one another, it follows that both $H_{\mathcal{A}_1}$ and $H_{\mathcal{A}_2}$ are Hermitian. Also, since

$$\sum_{\alpha\in\mathcal{A}_1} \sum_{\{b\in\mathcal{P}'_N|b\supseteq a\}} \mathcal{C}_b(P^{(-)}_\alpha)_b H_b = \sum_{\alpha\in\mathcal{A}_1} P^{(-)}_\alpha H - \sum_{\alpha\in\mathcal{A}_1} [P^{(-)}_\alpha H]_1, \tag{66}$$

$H_{\mathcal{A}_1}$ differs from the exact spectral projection of the Hamiltonian, $P^{(-)}_{\mathcal{A}_1} H$, on the channels $\mathcal{A}_1$ by the connected operator

$$W_I := [P^{(-)}_{\mathcal{A}_1} H]_1 = \sum_{\alpha\in\mathcal{A}_1} [P^{(-)}_\alpha H]_1. \tag{67}$$

It follows from (66) that

$$P^{(-)}_{\mathcal{A}_1} H = H_{\mathcal{A}_1} + W_I. \tag{68}$$

The channel α scattering states of $H_{\mathcal{A}_1}$ are defined using wave operators given by

$$\Omega^{(-)}_{\mathcal{A}_1}(a)\Phi_\alpha = \lim_{t\to-\infty} e^{iH_{\mathcal{A}_1}t} e^{-iH_a t}\Phi_\alpha. \tag{69}$$

In the appendix it is shown that the limit

$$\Omega_W^{(-)} := \lim_{t\to-\infty} e^{iH_{\mathcal{A}_1}t} e^{-iP_{\mathcal{A}_1}^{(-)}Ht} =$$

$$\lim_{t\to-\infty} e^{i\left(P_{\mathcal{A}_1}^{(-)}H-W_I\right)t} e^{-iP_{\mathcal{A}_1}^{(-)}Ht} =$$

$$\lim_{t\to-\infty} e^{iH_{\mathcal{A}_1}t} e^{-i(H_{\mathcal{A}_1}+W_I)t}, \tag{70}$$

exists when applied to $\Omega^{(-)}(a)\Phi_\alpha$, and $\Omega_W^{(-)}$, which is a connected perturbation of the identity, transforms the $(-)$ scattering states of the full Hamiltonian in the channels $\mathcal{A}_1$ to the $(-)$ scattering states of $H_{\mathcal{A}_1}$:

$$\Omega_{\mathcal{A}_1}^{(-)}(a)\Phi_\alpha = \Omega_W^{(-)}\left(\Omega^{(-)}(a)\Phi_\alpha\right). \tag{71}$$

The inverse relation

$$\left(\Omega^{(-)}(a)\Phi_\alpha\right) = \Omega_W^{(-)\dagger}\Omega_{\mathcal{A}_1}^{(-)}(a)\Phi_\alpha \tag{72}$$

can also be derived using the same methods used to derive (71) in the appendix by interchanging the roles of $\Omega^{(-)}(a)$ and $\Omega_{\mathcal{A}_1}^{(-)}(a)$. It is important to note that although $\Omega_W^{(-)}$ depends on $\mathcal{A}_1$, it is independent of the specific channel $\alpha \in \mathcal{A}_1$. While determining W_I involves solving the N-body problem, the observation that the scattering states of $H_{\mathcal{A}_1}$ are related to the scattering states of the exact Hamiltonian, projected onto the subspace of important channels, implies that the spectral resolution of both operators are related by $\Omega_W^{(-)}$. This also means that the incoming scattering eigenstates of H and $H_{\mathcal{A}_1}$ in the channels $\mathcal{A}_1$ are identical up to fully connected parts. This does not require the full N-body solution. This observation will be used in the next section to show that $H_{\mathcal{A}_1}$ satisfies an optical theorem with the channels $\alpha \in \mathcal{A}_1$.

The operator $\Omega_W^{(-)}$ also satisfies the intertwining relation:

$$e^{iH_{\mathcal{A}_1}s}\,\Omega_W^{(-)} = \Omega_W^{(-)}\,e^{iP_{\mathcal{A}_1}^{(-)}Hs}. \tag{73}$$

This follows from

$$e^{iH_{\mathcal{A}_1}s}\,\Omega_W^{(-)} = e^{iH_{\mathcal{A}_1}s}\left(\lim_{t\to-\infty} e^{i\left(P_{\mathcal{A}_1}^{(-)}H-W_I\right)t} e^{-iP_{\mathcal{A}_1}^{(-)}Ht}\right) =$$

$$e^{iH_{\mathcal{A}_1}s}\left(\lim_{t\to-\infty} e^{i\left(P_{\mathcal{A}_1}^{(-)}H-W_I\right)t} e^{-iP_{\mathcal{A}_1}^{(-)}Ht}\right) e^{-iP_{\mathcal{A}_1}^{(-)}Hs} e^{iP_{\mathcal{A}_1}^{(-)}Hs} =$$

$$\left(\lim_{(t+s)\to-\infty} e^{i\left(P_{\mathcal{A}_1}^{(-)}H-W_I\right)(s+t)} e^{-iP_{\mathcal{A}_1}^{(-)}H(t+s)}\right) e^{iP_{\mathcal{A}_1}^{(-)}Hs} = \Omega_W^{(-)}\,e^{iP_{\mathcal{A}_1}^{(-)}Hs}, \tag{74}$$

where the limit $t \to -\infty$ can be replaced with $(t+s) \to -\infty$ because the limit is the same for any fixed s.

It follows from the intertwining relation,(73), that

$$\frac{1}{2\pi} \int ds\, e^{-isx} f(x)\, e^{isH_{\mathcal{A}_1}}\, \Omega_W^{(-)} = \Omega_W^{(-)} \int ds\, e^{isP_{\mathcal{A}_1}^{(-)}H}\, \frac{1}{2\pi} e^{-isx} f(x). \tag{75}$$

This means that

$$f(H_{\mathcal{A}_1})\, \Omega_W^{(-)} = \Omega_W^{(-)}\, f(P_{\mathcal{A}_1}^{(-)} H). \tag{76}$$

For functions of the form $f(x) = \frac{1}{z-x}$, this gives

$$(E - H_{\mathcal{A}_1} + i\epsilon)^{-1}\, \Omega_W^{(-)} = \Omega_W^{(-)} \left(E - P_{\mathcal{A}_1}^{(-)} H + i\epsilon \right)^{-1} \tag{77}$$

or

$$(E - H_{\mathcal{A}_1} + i\epsilon)^{-1} = \Omega_W^{(-)} \left(E - P_{\mathcal{A}_1}^{(-)} H + i\epsilon \right)^{-1} \Omega_W^{(-)\dagger}. \tag{78}$$

Note that even though the eigenstates of $P_{\mathcal{A}_1}^{(-)} H$ are scattering states of H, the resolvent of $P_{\mathcal{A}_1}^{(-)} H$ is not the same as the resolvent of H; the two resolvents have different discontinuities across their scattering cuts. The discontinuity across the cut is responsible for the contributions to the optical theorem. This will be discussed in more detail in the next section.

V. OPTICAL THEOREM

This section will show that the transition operator for the truncated Hamiltonian, $H_{\mathcal{A}_1}$, satisfies an optical theorem with the channels $\alpha \in \mathcal{A}_1$, which shows that all of the scattered flux is in the channels $\mathcal{A}_1$.

The $\mathcal{A}_1$-transition operator for $2-2$ forward scattering in the two-body channel β, using (5) and (14), is

$$\begin{gathered} T_{\mathcal{A}_1}^{\beta\beta}(E_\beta + i\epsilon) = \Phi_\beta^\dagger\, T_{\mathcal{A}_1}^{bb}(E_\beta + i\epsilon)\, \Phi_\beta = \\ \Phi_\beta^\dagger \left(H_{\mathcal{A}_1}^b + H_{\mathcal{A}_1}^b \left(E_\beta - H_{\mathcal{A}_1} + i\epsilon \right)^{-1} H_{\mathcal{A}_1}^b \right) \Phi_\beta. \end{gathered} \tag{79}$$

Taking the difference with $i \to -i$ leads to

$$\begin{gathered} \Phi_\beta^\dagger \left(T_{\mathcal{A}_1}^{bb}(E_\beta + i\epsilon) - T_{\mathcal{A}_1}^{bb}(E_\beta - i\epsilon) \right) \Phi_\beta = \\ \Phi_\beta^\dagger H_{\mathcal{A}_1}^b \left(\frac{-2i\epsilon}{(E_\beta - H_{\mathcal{A}_1})^2 + \epsilon^2} \right) H_{\mathcal{A}_1}^b \Phi_\beta. \end{gathered} \tag{80}$$

Using (78), this becomes

$$\Phi_\beta^\dagger \left(T_{\mathcal{A}_1}^{bb}(E_\beta + i\epsilon) - T_{\mathcal{A}_1}^{bb}(E_\beta - i\epsilon) \right) \Phi_\beta =$$

$$\Phi_\beta^\dagger H_{\mathcal{A}_1}^b \Omega_W^{(-)} \left(\frac{-2i\epsilon}{\left(E_\beta - P_{\mathcal{A}_1}^{(-)} H\right)^2 + \epsilon^2} \right) \Omega_W^{(-)\dagger} H_{\mathcal{A}_1}^b \Phi_\beta,$$

and this means that

$$\lim_{\epsilon \to 0} \Phi_\beta^\dagger \left(T_{\mathcal{A}_1}^{bb}(E_\beta + i\epsilon) - T_{\mathcal{A}_1}^{bb}(E_\beta - i\epsilon) \right) \Phi_\beta =$$

$$-2\pi i \, \Phi_\beta^\dagger H_{\mathcal{A}_1}^b \Omega_W^{(-)} \, \delta(P_{\mathcal{A}_1}^{(-)} H - E_\beta) \, \Omega_W^{(-)\dagger} H_{\mathcal{A}_1}^b \Phi_\beta. \tag{81}$$

The advantage of expressing this in terms of $P_{\mathcal{A}_1}^{(-)} H$ is that the completeness relation for the exact projected Hamiltonian, *which only involves states in the chosen set of important channels* $\mathcal{A}_1$*,* can be used to evaluate the delta function so that (81) becomes

$$-2\pi i \sum_{\alpha \in \mathcal{A}_1} \Phi_\beta^\dagger H_{\mathcal{A}_1}^b \Omega_W^{(-)} \, \Omega^{(-)}(a) \Phi_\alpha \, \delta(E_\alpha - E_\beta) \, \Phi_\alpha^\dagger \, \Omega^{(-)\dagger}(a) \, \Omega_W^{(-)\dagger} H_{\mathcal{A}_1}^b \Phi_\beta. \tag{82}$$

The exact channel $\alpha \in \mathcal{A}_1$ scattering states can be expressed in terms of the channel α eigenstates of $H_{\mathcal{A}_1}$ using the relation in (71), which means that (82) becomes

$$-2\pi i \sum_{\alpha \in \mathcal{A}_1} \Phi_\beta^\dagger H_{\mathcal{A}_1}^b \Omega_{\mathcal{A}_1}^{(-)}(a) \Phi_\alpha \, \delta(E_\alpha - E_\beta) \, \Phi_\alpha^\dagger \, \Omega_{\mathcal{A}_1}^{(-)\dagger}(a) H_{\mathcal{A}_1}^b \Phi_\beta. \tag{83}$$

Taking the imaginary part of both sides gives

$$2Im \left\{ \Phi_\beta^\dagger \, T_{\mathcal{A}_1}^{bb}(E_\beta + i\epsilon) \, \Phi_\beta \right\} =$$

$$-2\pi \sum_{\alpha \in \mathcal{A}_1} \Phi_\beta^\dagger H_{\mathcal{A}_1}^b \Omega_{\mathcal{A}_1}^{(-)}(a) \Phi_\alpha \, \delta(E_\alpha - E_\beta) \, \Phi_\alpha^\dagger \, \Omega_{\mathcal{A}_1}^{(-)\dagger}(a) H_{\mathcal{A}_1}^b \Phi_\beta =$$

$$-2\pi \sum_{\alpha \in \mathcal{A}_1} \int |\Phi_\alpha^\dagger \, T_{\mathcal{A}_1}^{ab}(E_\beta + i\epsilon) \, \Phi_\beta|^2 \, \delta(\sum_i E_{\alpha_i} - E_\beta) \, d\mathbf{p}_1 \cdots d\mathbf{p}_i. \tag{84}$$

The right hand side of (84) is related to the total cross section σ_T by

$$\text{RHS} = -(2\pi) \frac{v}{(2\pi)^4} \sigma_T, \tag{85}$$

where v is the relative speed. This means that the total cross section can be expressed as

$$\sigma_T = -\frac{(2\pi)^3}{v} 2Im \left\{ \Phi_\beta^\dagger \, T_{\mathcal{A}_1}^{bb}(E_\beta + i\epsilon) \, \Phi_\beta \right\} =$$

$$-\frac{2(2\pi)^3\mu}{k}\left(\frac{-1}{(2\pi)^2\mu}\right) Im\left\{F_{\beta\beta}\right\} = \frac{4\pi}{k} Im\left\{F_{\beta\beta}\right\}, \tag{86}$$

where $F_{\beta\beta} := -(2\pi)^2\mu\,\Phi_\beta^\dagger T^{bb}_{\mathcal{A}_1}(E_\beta + i\epsilon)\,\Phi_\beta$ is the scattering amplitude for $2 \to 2$ forward scattering for the two-body channel β, k is the center-of-mass momentum, and μ is the reduced mass. Equation (86) gives the familiar form of the optical theorem:

$$\sigma_T = \frac{4\pi}{k} Im\left\{F_{\beta\beta}\right\}. \tag{87}$$

This means that all of the scattered flux is in the channels $\mathcal{A}_1$. It is important to note that, in this case, the discontinuity across the scattering cuts in the resolvent does not receive contributions from the channels in $\mathcal{A}_2$. An identical analysis also applies to the Hamiltonian $H_{\mathcal{A}_2}$ by interchanging the channels $\mathcal{A}_1$ with the channels $\mathcal{A}_2$.

VI. BENZCE-REDISH-SLOAN EQUATIONS

While the properties of $H_{\mathcal{A}_1}$ and the associated scattering theory were derived using time-dependent methods, computations normally utilize time-independent methods. For many-body reactions, differential cross sections are expressed in terms of transition operators, $T^{ba}(z)$, which are operators on the N-particle Hilbert space. They are related to the two-Hilbert space channel transition operators by

$$T^{\beta\alpha}_{\mathcal{A}_1}(E + i\epsilon) = \Phi_\beta^\dagger H^b \Omega^{(-)}(a)\Phi_\alpha = \Phi_\beta^\dagger T^{ba}(E + i\epsilon)\Phi_\alpha,$$

where $z = E + i\epsilon$ and E is the *on-shell* energy. These are solutions of linear integral equations. By manipulating the equations so that they have a compact iterated kernel, which can be uniformly approximated by a finite dimensional matrix, the solution involves solving a large linear system. The equations derived by Bencze, Redish and Sloan [18][19][20] have this property and are sufficiently flexible to be applicable to the dynamical models governed by Hamiltonians of the form $H_{\mathcal{A}_1}$. A short derivation of these equations following [21] is given below.

The transition operator for multichannel scattering in the notation of this paper is

$$T^{ba}(z) = H^a + H^b G(z) H^a, \tag{88}$$

where $G(z) = (z - H)^{-1}$ is the resolvent operator (or Green's operator) and $z = E + i\epsilon$ is the complex energy. If the completely connected part of H^b is zero (it doesn't contain any

N-body forces), then the operator decomposition of H^b is

$$H^b = \sum_{c\in\mathcal{P}'_N} \mathcal{C}_c H^b_c. \tag{89}$$

This means that the transition operator can be expressed as

$$T^{ba}(z) = H^a + \sum_{c\in\mathcal{P}'_N} \mathcal{C}_c H^b_c G(z) H^a. \tag{90}$$

Using the second resolvent identity

$$G(z) = G_c(z) + G_c(z) H^c G(z),$$

the transition operator becomes

$$T^{ba}(z) = H^a + \sum_{c\in\mathcal{P}'_N} \mathcal{C}_c H^b_c \left(G_c(z) + G_c(z) H^c G(z)\right) H^a =$$

$$H^a + \sum_{c\in\mathcal{P}'_N} \mathcal{C}_c H^b_c G_c(z) \left(H^a + H^c G(z) H^a\right) = H^a + \sum_{c\in\mathcal{P}'_N} \mathcal{C}_c H^b_c G_c(z) T^{ca}(z), \tag{91}$$

where $G_c(z) = (z - H_c)^{-1}$. Therefore, the transition operator $T^{ba}(z)$ satisfies

$$T^{ba}(z) = H^a + \sum_{c\in\mathcal{P}'_N} \mathcal{C}_c H^b_c G_c(z) T^{ca}(z), \tag{92}$$

and these are the equations derived by Bencze, Redish and Sloan. These equations are coupled integral equations. It follows from (92) that the iterated kernel,

$$\sum_{c\in\mathcal{P}'_N} \sum_{d\in\mathcal{P}'_N} \mathcal{C}_c \mathcal{C}_d H^b_c G_c(z) H^c_d G_d(z), \tag{93}$$

is connected since $\sum_{c\in\mathcal{P}'_N} \mathcal{C}_c H^b_c G_c(z) H^c_d$ is connected or zero by (42). Everything above holds if H is replaced by $H_{\mathcal{A}}$.

VII. IDENTICAL PARTICLES

For systems of identical nucleons, the Hilbert space is the antisymmetrized subspace of the N-nucleon Hilbert space, and the exchange symmetry can be used to reduce the number of coupled scattering integral equations. In this section, the method in [22] is applied to treat integral equations of the form in (92) involving identical particles.

Projectors on the symmetrized (antisymmetrized) subspace of the Hilbert space are constructed using permutation operators. $\Pi(N)$ is the group of permutations on N objects. For a given permutation $\sigma \in \Pi(N)$, P_σ is the operator acting on a N-particle basis vector $|\mathbf{k}_1, \mu_1, \cdots, \mathbf{k}_N, \mu_N\rangle$ defined by

$$P_\sigma|\mathbf{k}_1, \mu_1, \cdots, \mathbf{k}_N, \mu_N\rangle = (-)^{|\sigma|}|\mathbf{k}_{\sigma(1)}, \mu_{\sigma(1)}, \cdots, \mathbf{k}_{\sigma(N)}, \mu_{\sigma(N)}\rangle, \tag{94}$$

where $|\sigma| = 0$ for integer spin particles and even permutations of half-integer spin particles and $|\sigma| = 1$ for odd permutations of half-integer spin particles.

Permutations act on partitions by permuting the labels of particles in each cluster. The partition $b = \sigma(a)$ denotes the partition obtained from a by using σ to change the particle labels. This is illustrated by the following example

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 2 & 7 & 3 & 5 & 4 & 1 & 6 \end{pmatrix} \tag{95}$$

$$a = (124)(37)(65) \tag{96}$$

$$\sigma(a) = (275)(36)(14). \tag{97}$$

In this example, both partitions have one three-particle cluster and two two-particle clusters. Partitions related by permutations define equivalence classes of partitions (permutation equivalent partitions). The equivalence class containing partition a is denoted by $[a]$. In the above example, this is the set of all distinct seven-particle partitions with one three-particle cluster and two two-particle clusters. Any $b \in [a]$ can be expressed as $b = \sigma(a)$ for some $\sigma \in \Pi(N)$.

Permutations satisfying $\sigma(a) = a$ define a subgroup $\Pi_a(N)$ of $\Pi(N)$. The elements of $\Pi_a(N)$ include permutations that interchange particles in the same equivalence class of a and permutations that interchange different equivalence classes with the same number of particles. The subgroup $\Pi_a(N)$ has $N_a = \prod_i n_{a_i}! \prod_j k_j!$ elements, where n_{a_i} is the number of particles in the i^{th} cluster of a and k_j is the number of clusters with j particles. In the above example, interchanging 3 with 7 and 6 with 5 leaves a unchanged. Interchanging the (37) pair with the (65) pair also leaves a unchanged. The subgroup $\Pi_a(N)$ in the above example has $48 = 3!2!2!2!$ permutations.

For $\sigma' \in \Pi(N)$ satisfying $\sigma'(a) = a' \neq a$, if $\sigma \in \Pi_a(N)$ then $(\sigma'\sigma)(a) = \sigma''(a) = a'$. This means that $(\sigma')^{-1}\sigma'' \in \Pi_a(N)$ or that σ' and σ'' are in the same left coset of $\Pi_a(N)$. It follows that each partition in $[a]$ can be identified with a left coset of $\Pi_a(N)$. The number of partitions in $[a]$ is equal to the number of left cosets of $\Pi_a(N)$, which by Lagrange's theorem is $N_{[a]} = \frac{N!}{N_a}$. In the above example there are $105 = 7!/48$ partitions in the class $[(275)(36)(14)]$. As another example, the equivalence class, $[(1)(2)(34)]$, contains $\frac{4!}{2!2!} = 3!$ partitions, and these include: $(1)(2)(34)$, $(1)(3)(24)$, $(1)(4)(23)$, $(2)(3)(14)$, $(2)(4)(13)$, $(3)(4)(12)$.

Permutation operators, P_σ, are unitary since they are products of unitary transposition operators. The permutation operator P_σ replaces the cluster translation operator in eq. (23), that translates the clusters of the partition a, by the cluster translation operator that translates the clusters of the partition $\sigma(a)$:

$$P_\sigma T_a(\mathbf{x}_1, \cdots, \mathbf{x}_{n_a}) P_\sigma^\dagger = T_{\sigma(a)}(\mathbf{x}_1, \cdots, \mathbf{x}_{n_a}). \tag{98}$$

Since the operator O_a is defined using the cluster translation operator, it follows from (28) and (98) that

$$P_\sigma O_a P_\sigma^\dagger = P_\sigma O_a P_\sigma^{-1} = O_{\sigma(a)}. \tag{99}$$

Also, since $[O]_a$ and O_a^b are defined in terms of the O_a's, the transformation properties of these operators under permutations is

$$P_\sigma [O]_a P_\sigma^{-1} = [O]_{\sigma(a)} \qquad \text{and} \qquad P_\sigma O_a^b P_\sigma^{-1} = O_{\sigma(a)}^{\sigma(b)}. \tag{100}$$

Symmetrizers (antisymmetrizers) are defined by

$$R := \frac{1}{N!} \sum_{\sigma \in \Pi(N)} P_\sigma. \tag{101}$$

The symmetrizer (antisymmetrizer) R is an orthogonal projection operator on the N-particle Hilbert space that satisfies

$$R = R^2 = R^\dagger \qquad \text{and} \qquad R = P_\sigma R = R P_\sigma \qquad \forall\, \sigma \in \Pi(N). \tag{102}$$

The first relation in (102) can be obtained using

$$R^2 = \frac{1}{N!} \frac{1}{N!} \sum_{\sigma \in \Pi(N)} \left(\sum_{\sigma' \in \Pi(N)} P_\sigma P_{\sigma'} \right) = \frac{1}{N!} \frac{1}{N!} \underbrace{\sum_{\sigma \in \Pi(N)}}_{=N!} \underbrace{\left(\sum_{\sigma'' \in \Pi(N)} P_{\sigma''} \right)}_{=N!\,R} =$$

$$R = R^\dagger, \tag{103}$$

where $\sigma'' = \sigma\sigma'$ for a fixed σ. The adjoint replaces P_σ by $P_{\sigma^{-1}}$, which results in the same operator when summed over all permutations. The second relation in (102) can be obtained from the definitions of R and P_σ, which simply relabels the permutations in the sum of (101).

Symmetrizers (antisymmetrizers) constructed from permutations in the subgroup that leaves the partition a unchanged are defined by

$$R_a := \frac{1}{N_a} \sum_{\sigma \in \Pi_a(N)} P_\sigma . \tag{104}$$

These operators symmetrize the particles in each cluster of a and identical clusters of a. Therefore, for any partition a, the symmetrizer (antisymmetrizer) in (101) can be expressed as

$$R = \frac{1}{N_{[a]}} \sum_{a' \in [a]} P_{a'a} R_a = \frac{1}{N_{[a]}} \sum_{a' \in [a]} R_a P_{aa'} , \tag{105}$$

where $P_{aa'}$ is any permutation operator that transforms a to a' and $N_{[a]}$ is the number of partitions in the equivalence class $[a]$. The sums are over all partitions that can be obtained from a by permutation.

The Hilbert space for a system of N identical particles is the range of symmetrizer (antisymmetrizer), R, on the N-particle Hilbert space. The initial and final scattering states need to be projected on this subspace so that the unit normalized states are

$$|\psi\rangle \to |\psi\rangle_R := \frac{R|\psi\rangle}{\langle\psi|R^\dagger R|\psi\rangle^{1/2}} = \frac{R|\psi\rangle}{\langle\psi|R|\psi\rangle^{1/2}} . \tag{106}$$

For states $|\psi\rangle$ satisfying $R_a|\Psi\rangle = |\psi\rangle$ and $\langle\psi|P_\sigma\psi\rangle = 0$ for $\sigma(a) \neq a$, the normalization coefficient is

$$\langle\psi|R|\psi\rangle^{-\frac{1}{2}} = \sqrt{N_{[a]}} . \tag{107}$$

For a wave function $|\psi\rangle$, with $N_{[a]}$ partition equivalent channels, the normalized wave function has the familiar form

$$|\psi_{[a_o]}\rangle =$$

$$\sqrt{N_{[a]}} \sum_{a \in [a_o]} \frac{1}{N_{[a]}} P_{aa_o} |\psi_{a_o}\rangle = \sum_{a \in [a_o]} \frac{1}{\sqrt{N_{[a]}}} P_{aa_o} |\psi_{a_o}\rangle .$$

Unit normalized wave functions are needed for matrix elements of the (unitary) scattering operator to be interpreted as a probability amplitude.

For distinguishable particles, if $|\alpha\rangle$ and $|\beta\rangle$ are sharp-momentum initial and final channel states with energy E, then

$$H^a|\alpha\rangle = (H^a + H_a - E)\,|\alpha\rangle = (H - E)\,|\alpha\rangle \tag{108}$$

$$\langle\beta|H^b = \langle\beta|\left(H^b + H_b - E\right) = \langle\beta|\,(H - E)\,, \tag{109}$$

where $(H_a - E)|\alpha\rangle = (H_b - E)|\beta\rangle = 0$. On-shell sharp-momentum transition matrix elements can be expressed as

$$\begin{gathered}\langle\beta|T^{ba}(E + i\epsilon)|\alpha\rangle = \langle\beta|\left(H^a + H^b \frac{1}{E - H + i\epsilon} H^a\right)|\alpha\rangle = \\ \langle\beta|\left((H - E) + (H - E)\frac{1}{E - H + i\epsilon}(H - E)\right)|\alpha\rangle = \\ \langle\beta|\tilde{T}(E + i\epsilon)|\alpha\rangle,\end{gathered} \tag{110}$$

where $\tilde{T}(E + i\epsilon)$ is defined as

$$\tilde{T}(E + i\epsilon) := (H - E) + (H - E)\,\frac{1}{E - H + i\epsilon}\,(H - E)\,. \tag{111}$$

If the Hamiltonian satisfies $[H, R] = 0$, then $[\tilde{T}(E + i\epsilon), R] = 0$ because $\tilde{T}(E + i\epsilon)$ is a function of H. It follows that

$$\begin{gathered}\langle\beta|R\tilde{T}(E + i\epsilon)R|\alpha\rangle = \langle\beta|\tilde{T}(E + i\epsilon)R|\alpha\rangle = \\ \langle\beta|R\tilde{T}(E + i\epsilon)|\alpha\rangle = \langle\beta|R\tilde{T}(E + i\epsilon)P_\sigma|\alpha\rangle,\end{gathered} \tag{112}$$

where P_σ is an arbitrary permutation operator. In this case, the properly normalized on-shell transition operator from $|\alpha_o\rangle$ to $|\beta_o\rangle$ is

$$\begin{gathered}\frac{1}{N_{[b_o]}}\frac{\sum_{b\in[b_o]}\langle\beta_o|R_{b_o}P_{b_o b}\tilde{T}(E + i\epsilon)P_\sigma|\alpha_o\rangle}{\langle\alpha_o|R|\alpha_o\rangle^{1/2}\langle\beta_o|R|\beta_o\rangle^{1/2}} = \\ \frac{1}{N_{[b_o]}}\frac{\sum_{b\in[b_o]}\langle\beta_o|R_{b_o}P_{b_o b}\tilde{T}(E + i\epsilon)|\alpha\rangle}{\langle\alpha_o|R|\alpha_o\rangle^{1/2}\langle\beta_o|R|\beta_o\rangle^{1/2}}.\end{gathered} \tag{113}$$

In this expression, the matrix element is independent of the choice of $|\alpha\rangle = P_\sigma|\alpha_o\rangle$.

The normalization condition in (106) gives

$$\sqrt{\frac{N_{[a_o]}}{N_{[b_o]}}}\sum_{b\in[b_o]}\langle\beta_o|R_{b_o}P_{b_o}\tilde{T}(E + i\epsilon)|\alpha\rangle. \tag{114}$$

This can be expressed in terms of the operator T^{ba} by

$$\sqrt{\frac{N_{[a_o]}}{N_{[b_o]}}} \sum_{b\in[b_o]} \langle\beta_o|R_{b_o}P_{b_ob}T^{ba}(E+i\epsilon)|\alpha\rangle. \tag{115}$$

This expression is only valid on shell, but it is possible to construct equations for the operator

$$T^{[b_o]a} = \sum_{b\in[b_o]} R_{b_o}P_{b_ob}T^{ba}(E+i\epsilon). \tag{116}$$

After solving for $T^{[b_o]a}$, it can be evaluated on shell and multiplied by the numerical factors in (115).

To construct integral equations for $T^{[b_o]a}$, one uses the integral equations in (92) for $T^{ba}(E+i\epsilon)$ in (116) to get

$$T^{[b_o]a} = \sum_{b\in[b_o]} R_{b_o}P_{b_ob}\left(H^a + \sum_{c\in\mathcal{P}'_{\mathcal{N}}} \mathcal{C}_c H^b_c G_c(E+i\epsilon)T^{ca}(E+i\epsilon)\right). \tag{117}$$

To get an integral equation for $T^{[b_o]a}$, the second term on the right-hand side of (117) needs to be expressed in terms of $T^{[c_o]a}$.

The first step is to choose a permutation σ satisfying $\sigma(c_o) = c$, where $c_o \in [c]$ is an arbitrary but fixed element of $[c]$. For this permutation, let $P_{cc_o} = P_\sigma$. It follows from (99) and (100) that

$$H^b_c G_c(E+i\epsilon) = P_{cc_o}H^{\sigma^{-1}(b)}_{c_o}G_{c_o}(E+i\epsilon)P_{c_oc}, \tag{118}$$

where $P_{c_oc} = P^{-1}_{cc_o}$.

With this substitution the second term on the right side of (117) becomes

$$\sum_{b\in[b_o]} R_{b_o}P_{b_ob}\sum_{c\in\mathcal{P}'_{\mathcal{N}}} \mathcal{C}_c H^b_c G_c(E+i\epsilon)T^{ca}(E+i\epsilon) =$$

$$\sum_{b\in[b_o]} R_{b_o}P_{b_ob}\sum_{c\in\mathcal{P}'_{\mathcal{N}}} \mathcal{C}_c P_{cc_o}H^{\sigma^{-1}(b)}_{c_o}G_{c_o}(E+i\epsilon)P^{-1}_{cc_o}T^{ca}(E+i\epsilon). \tag{119}$$

The sum over partitions can be replaced by a sum over equivalence classes of partitions and a sum over partition equivalent partitions in a given equivalence class. With this decomposition, $\mathcal{C}_c = \mathcal{C}_{[c]}$ and c_o is the same for all permutation equivalent partitions.

Since $P_{b_ob} = P_{\sigma'}$ for some σ', it follows that $P_{b_ob}P_{cc_o} = P_{\sigma'\sigma} = P_{\sigma''}$. One can then define $P_{b_o\sigma^{-1}(b)} := P_{\sigma''}$. With this definition, (119) can be written as

$$\sum_{[c_o]\in\mathcal{P}'_{\mathcal{N}}} \mathcal{C}_{[c_o]}\sum_{c\in[c_o]}\sum_{b\in[b_o]} R_{b_o}P_{b_o\sigma^{-1}(b)}H^{\sigma^{-1}(b)}_{c_o}G_{c_o}(E+i\epsilon)P^{-1}_{cc_o}T^{ca}(E+i\epsilon). \tag{120}$$

Since for each $c \in [c_o]$, σ is a fixed permutation, summing over all $b \in [b_o]$ is the same as summing over all $\sigma^{-1}(b) \in [b_o]$. This is independent of the specific $c \in [c_o]$. Letting $b' = \sigma^{-1}(b)$, (120) becomes

$$\sum_{[c_o]\in\mathcal{P}'_{\mathcal{N}}} \mathcal{C}_{[c_o]} \sum_{b'\in[b_o]} R_{b_o} P_{b_o b'} H^{b'}_{c_o} G_{c_o}(E+i\epsilon) \sum_{c\in[c_o]} P_{c_o c} T^{ca}(E+i\epsilon). \tag{121}$$

Since the choice of $P_{c_o c}$ was arbitrary, it could be replaced by $P_\sigma P_{c_o c}$ for any $\sigma \in \Pi_{c_o}(N)$. Averaging over all of the permutations in $\Pi_{c_o}(N)$ gives

$$\sum_{[c_o]\in\mathcal{P}'_{\mathcal{N}}} \mathcal{C}_{[c_o]} \sum_{b'\in[b_o]} R_{b_o} P_{b_o b'} H^{b'}_{c_o} G_{c_o}(E+i\epsilon) \sum_{c\in[c_o]} R_{c_o} P_{c_o c} T^{ca}(E+i\epsilon), \tag{122}$$

which can be expressed in terms of $T^{[c_o]a}(E+i\epsilon)$ as

$$\sum_{[c_o]\in\mathcal{P}'_{\mathcal{N}}} \mathcal{C}_{[c_o]} \sum_{b'\in[b_o]} R_{b_o} P_{b_o b'} H^{b'}_{c_o} G_{c_o}(E+i\epsilon) T^{[c_o]a}(E+i\epsilon). \tag{123}$$

In this equation, the number of coupled equations is equal to the number of equivalence classes of partitions rather than the number of partitions. Combining (116), (117) and (122) gives the following integral equation for $T^{[b_o]a}(E+i\epsilon)$:

$$T^{[b_o]a}(E+i\epsilon) = \sum_{b'\in[b_o]} R_{b_o} P_{b_o b'} H^a$$
$$+ \sum_{[c_o]\in\mathcal{P}'_{\mathcal{N}}} \mathcal{C}_{[c]} \sum_{b'\in[b_o]} R_{b_o} P_{b_o b'} H^{b}_{c_o} G_{c_o}(E+i\epsilon) T^{[c_o]a}(E+i\epsilon). \tag{124}$$

After solving the system for $T^{[b_o]a_o}(E+i\epsilon)$, if it is evaluated between internally symmetrized channel states, $|\alpha_o\rangle$ and $|\beta_o\rangle$, then the matrix elements need to be multiplied by $\sqrt{\frac{N_{[a_o]}}{N_{[b_o]}}}$ to get the correct normalization. Since the normalization factors only depend on the partitions, they can be absorbed into the equations by replacing $T^{[b_o]a_o}(E+i\epsilon)$ by the symmetrized transition operator defined by

$$T^{[b_o]a_o}_{sym}(E+i\epsilon) := \sqrt{\frac{N_{[a_o]}}{N_{[b_o]}}} T^{[b_o]a_o}(E+i\epsilon). \tag{125}$$

The equations in (124) are for a general permutation-symmetric N-body Hamiltonian, and they are also valid for the truncated Hamiltonian, $H_{\mathcal{A}_1}$, provided $\mathcal{A}_1$ contains all channels related by permutations. The number of equations for truncated Hamiltonians depend on the choice of retained channels $\mathcal{A}_1$.

The differential cross section has the form

$$d\sigma = \frac{(2\pi)^4}{v}\frac{N_{[a_o]}}{N_{[b_o]}}|\langle\beta_o'|T^{[b_o]a_o}(E+i\epsilon)|\alpha_o\rangle|^2\delta(E_b'-E_a)\delta(\mathbf{P'}-\mathbf{P})\prod d\mathbf{p}_i \tag{126}$$

when it is expressed in terms of the operator $T^{[b_o]a_o}$.

VIII. THE STRUCTURE OF THE TRUNCATED EQUATIONS

The dynamical equations in (124) are abstract. This section considers the structure of the simplest approximation, where only 2-cluster channels are retained, in more detail.

In general, the truncated Hamiltonian in (61) has the form

$$H_{\mathcal{A}_1} = \sum_a \mathcal{C}_a (H_{\mathcal{A}_1})_a.$$

When $\mathcal{A}_1$ only includes 2-cluster channels, $(H_{\mathcal{A}_1})_a = 0$ unless a is a 2-cluster partition. For 2-cluster partitions, $\mathcal{C}_a = 1$ and, given partitions a and b,

$$(H_{\mathcal{A}_1})_a^b = (H_{\mathcal{A}_1})_a - (H_{\mathcal{A}_1})_{a\cap b}.$$

If $a \neq b$, then $a\cap b$ will have more than two clusters. This implies that

$$(H_{\mathcal{A}_1})_{a\cap b} = 0 \qquad \text{when} \qquad a \neq b.$$

It follows that

$$(H_{\mathcal{A}_1})_a^b = (H_{\mathcal{A}_1})_a \bar{\delta}_{ab} = [H_{\mathcal{A}_1}]_a \bar{\delta}_{ab},$$

where

$$\bar{\delta}_{ab} = 1 - \delta_{ab}.$$

The channel α eigenstates of $(H_{\mathcal{A}_1})_a$ have the form

$$|\alpha\rangle := |\alpha_1, s_1, \mu_1, \mathbf{p}_1\rangle \otimes |\alpha_2, s_2, \mu_2, \mathbf{p}_2\rangle,$$

where the α_i labels the bound subsystems. The energy of this state is

$$E_{\alpha_a} = \frac{\mathbf{p}_1^2}{2M_1} - e_1 + \frac{\mathbf{p}_2^2}{2M_2} - e_2,$$

and this is the sum of the kinetic energy minus the binding energy of each bound cluster (M_i is the total mass of each bound cluster).

The terms in the kernel and driving term of (124) become

$$(H_{\mathcal{A}_1})^b_c = \bar{\delta}_{bc}[H_{\mathcal{A}_1}]_c = \bar{\delta}_{bc} \int |\gamma_c\rangle E_{\gamma_c} d\gamma_c \langle \gamma_c| \tag{127}$$

and

$$(H_{\mathcal{A}_1})^b_c (G_{\mathcal{A}_1})_c(E+i\epsilon) = \bar{\delta}_{bc} \int |\gamma_c\rangle \frac{E_{\gamma_c} d\gamma_c}{E - E_{\gamma_c} + i\epsilon} \langle \gamma_c|, \tag{128}$$

where

$$d\gamma_c = d\mathbf{p}_{c1} d\mathbf{p}_{c2}.$$

In this case, the symmetrized equations in (124), with $\mathcal{C}_c = 1$ for 2-cluster partitions, are

$$\begin{aligned} \langle \beta_o | T^{[b_o]a_o}(E+i\epsilon)|\alpha_o\rangle &= \sum_{b\in[b_o]} \sum_{c\neq b} \int \langle \beta_o | R_{b_o} P_{b_o b} |\gamma_c\rangle E_{\gamma_c} d\gamma_c \langle \gamma_c | \alpha_o \rangle \\ + \sum_{b\in[b_o]} \sum_{c_o \neq b} \sum_{[c]} &\int \langle \beta_o | R_{b_o} P_{b_o b} |\gamma_{c_o}\rangle \frac{E_{\gamma_{c_o}} d\gamma_{c_o}}{E - E_{\gamma_{c_o}} + i\epsilon} \langle \gamma_{c_o} | T^{[c]a_o}(E+i\epsilon)|\alpha_o\rangle. \end{aligned} \tag{129}$$

If $|\beta_o\rangle$ is internally symmetrized, $|\beta_o\rangle = R_{\beta_p}|\beta_o\rangle$, then (129) becomes

$$\begin{aligned} \langle \beta_o | T^{[b_o]a_o}(E+i\epsilon)|\alpha_o\rangle &= \sum_{b\in[b_o]} \sum_{c\neq b} \int \langle \beta |\gamma_c\rangle E_{\gamma_c} d\gamma_c \langle \gamma_c | \alpha_o \rangle \\ + \sum_{b\in[b_o]} \sum_{c_o \neq b} \sum_{[c_o]} &\int \langle \beta |\gamma_{c_o}\rangle \frac{E_{\gamma_{c_o}} d\gamma_{c_o}}{E - E_{\gamma_{c_o}} + i\epsilon} \langle \gamma_{c_o} | T^{[c_o]a_o}(E+i\epsilon)|\alpha_o\rangle. \end{aligned} \tag{130}$$

These solutions need to be multiplied by $\sqrt{\frac{N_{[a_o]}}{N_{[b_o]}}}$ in order to get the properly normalized transition matrix elements.

What is needed as input are the overlap matrix elements

$$\langle \beta_o | R_{b_o} P_{b_o b} |\gamma_{c_o}\rangle$$

for all $b \in [b_o]$. Each one has an overall momentum conserving delta function with a rotationally covariant kernel that depends on one initial and one final relative momentum variable. They have the general structure

$$\int \sum_{n_b} \langle \tilde{\mathbf{p}}_{b_o}, n_{b_o}, \beta_o | R_{b_o} P_{b_o b} | \tilde{\mathbf{p}}_b, n_b, \beta\rangle d\tilde{\mathbf{p}}_b \langle \tilde{\mathbf{p}}_b, n_b, \beta | \tilde{\mathbf{p}}_{c_o}, n_{c_o}, \gamma_{c_o}\rangle,$$

where the $\tilde{\mathbf{p}}_a$ are the relative momenta between the clusters of a. For each fixed $\tilde{\mathbf{p}}_b$, the remaining independent variables are integrated out.

IX. BOUND STATES

While the main goal of this work is to investigate the role of different scattering channels on many-body reactions, the result is a truncated Hamiltonian that may also have bound states. If the original Hamiltonian has no N-body interactions, then the projection of the exact Hamiltonian on the bound states must exactly cancel with the connected part of the projection of the Hamiltonian on the scattering states (using the equation in (59)). The dominant part of the projection of the Hamiltonian on the bound states may be due to N-body contributions from a limited number of important incoming or outgoing scattering channels. If these limited scattering channels are major contributors to the connected part of the exact scattering states, then they should provide major contributions to the bound states of the system.

The decomposition in (61) can be used to determine which scattering channels are most responsible for the binding. An interesting example is the system consisting of two protons and four neutrons. This system has a bound state, ^6He, which is a halo nucleus. The expectation is that this system can be modeled as an alpha particle interacting with two loosely bound halo neutrons. The six-body bound state arises from the connected part of the complete set of scattering states. An interesting question is how much of the binding is due to the subset of $\alpha - n - n$ scattering channels. This can be investigated by searching for bound states of the truncated Hamiltonian $H_{\mathcal{A}_1}$, where $\mathcal{A}_1$ consists of all of the $\alpha - n - n$ scattering channels. The interesting thing about this system is that there are no bound states consisting of two protons and three neutrons, ^5He, and there are no bound states consisting of two neutrons. The interactions for this reaction mechanism are constructed from the $\alpha - n$ and $n - n$ scattering states where the third “particle” acts as a non-interacting spectator.

In this example, there are six equivalent scattering channels that differ by which pair of neutrons are bound in the alpha particle. These channels can be labeled by the pair of neutrons in the alpha particle:

$$\alpha_{ij}, a_{\alpha_{ij}} = (p_1 p_2 n_i n_j)(n_k)(n_l)$$

where p and n are the protons and neutrons, respectively. Here i and j label the neutrons in the alpha particle and k and l label the asymptotically free neutrons. There are 6 permutation equivalent channels and partitions: $\alpha_{12}, \alpha_{13}, \alpha_{14}, \alpha_{23}, \alpha_{24}, \alpha_{34}$. The partitions that appear in the Hamiltonian

$$H_{\mathcal{A}_1} = \sum_{a \in \mathcal{P}'_N} \mathcal{C}_a (H_{\mathcal{A}_1})_a$$

are the 3-cluster partitions, $(p_1 p_2 n_i n_j)(n_k)(n_l)$, and the 2-cluster partitions that include

$$(p_1 p_2 n_i n_j n_k)(n_l), (p_1 p_2 n_i n_j n_l)(n_k) \text{ and } (p_1 p_2 n_i n_j)(n_k n_l),$$

where there are six combinations of ij.

The following short-hand notation is used in what follows:

$$|\alpha_{ij}\rangle = |\alpha, \mathbf{p}_{ij}\rangle \otimes |\mathbf{p}_k, \mu_k\rangle \otimes |\mathbf{p}_l, \mu_l\rangle$$

$$|\alpha_{(ij)(k)^-}\rangle = |(\alpha, \mathbf{p}_{ij}, \mathbf{p}_k, \mu_k)^-\rangle \otimes |\mathbf{p}_l, \mu_l\rangle$$

$$|\alpha_{(ij)(l)^-}\rangle = |(\alpha, \mathbf{p}_{ij}, \mathbf{p}_l, \mu_l)^-\rangle \otimes |\mathbf{p}_k, \mu_k\rangle$$

$$|\alpha_{(ij)^-}\rangle = |\alpha, \mathbf{p}_{ij}\rangle \otimes |(\mathbf{p}_k, \mu_k, \mathbf{p}_l, \mu_l)^-\rangle$$

$$E_{ij} = \frac{\mathbf{p_{ij}^2}}{8m_N} + \frac{\mathbf{p}_k^2}{2m_k} + \frac{\mathbf{p}_l^2}{2m_l} - e_\alpha$$

$$d\alpha_{ij} = d\mathbf{p}_{ij} d\mathbf{p}_k d\mathbf{p}_l$$

where $\mathbf{p}_{ij} = \mathbf{p}_i + \mathbf{p}_j$ and e_α is the binding energy of the α particle. The kinetic energy and interaction terms are defined by

$$K_{ij} = \int \sum_{\mu_l, \mu_k} |\alpha_{ij}\rangle d\alpha_{ij} E_{ij} \langle \alpha_{ij}|$$

$$H_{ij,k} = \int \sum_{\mu_l, \mu_k} \left(|\alpha_{(ij)(k)^-}\rangle d\alpha_{ij} E_{ij} \langle \alpha_{(ij)(k)^-}| - |\alpha_{(ij)}\rangle d\alpha_{ij} E_{ij} \langle \alpha_{(ij)}| \right)$$

$$H_{ij,l} = \int \sum_{\mu_l, \mu_k} \left(|\alpha_{(ij)(l)^-}\rangle d\alpha_{ij} E_{ij} \langle \alpha_{(ij)(l)^-}| - |\alpha_{(ij)}\rangle d\alpha_{ij} E_{ij} \langle \alpha_{(ij)}| \right)$$

$$H_{kl} = \int \sum_{\mu_l, \mu_k} \left(|\alpha_{(ij)^-}\rangle d\alpha_{ij} E_{ij} \langle \alpha_{(ij)^-}| - |\alpha_{(ij)}\rangle d\alpha_{ij} E_{ij} \langle \alpha_{(ij)}| \right).$$

Using this notation, the Hamiltonian for the $\alpha - n - n$ channels is

$$H_{\mathcal{A}_1} = \sum_{ij} (K_{ij} + H_{ij,k} + H_{ij,l} + H_{kl}),$$

where the sum is over all six pairs of ij corresponding to 24 partitions. The input are $n-n$ and $n-\alpha$ scattering states. Mathematically, this is a coupled three-body system. This Hamiltonian can be diagonalized to determine if this reaction mechanism is sufficiently rich to support a bound state.

Using

$$(E-(H_{\mathcal{A}_1})_a)\,|\Psi\rangle = (H_{\mathcal{A}_1})^a|\Psi\rangle$$

and

$$\sum_{a\in\mathcal{P}'_N} \mathcal{C}_a = 1,$$

it follows that the six-body bound state is a solution of the generalized eigenvalue problem

$$|\Psi\rangle = \sum_a \mathcal{C}_a(E-(H_{\mathcal{A}_1})_a)^{-1}(H_{\mathcal{A}_1})^a|\Psi\rangle,$$

where the right-hand side of this equation is connected by (42). The sum only involves 2-cluster and 3-cluster partitions. Note that this particular form of the equation is known to have spurious solutions [23], so any solutions need to be checked to make sure that they also satisfy the Schrödinger equation.

X. THE RELATIVISTIC CASE

The same analysis can be applied to a relativistically invariant quantum theory, with some non-trivial differences. Relativistic invariance in a quantum theory requires that quantum observables cannot be used to distinguish inertial coordinate systems. This implies that the dynamics of the system is given by a unitary representation, $U(\Lambda, a)$, of the Poincaré group [24] (semi-direct product of the Lorentz group (Λ) and spacetime translation group (a)). Unitary transformations preserve the quantum observables: quantum probabilities, expectation values and ensemble averages.

The Poincaré group is a ten parameter group (three translations, three rotations, three rotationless boosts and time translation). The infinitesimal generators of $U(\Lambda, a)$ are the Hamiltonian, H, the linear momentum operator, $\mathbf{P}$, the angular momentum operator, $\mathbf{J}$, and the rotationless boost generator $\mathbf{K}$. These are self-adjoint operators. They satisfy the Poincaré commutation relations:

$$[P^\mu, P^\nu] = 0 \qquad [J^i, P^j] = i\epsilon^{ijk}P^k \qquad [J^i, J^j] = i\epsilon^{ijk}J^k \tag{131}$$

$$[J^i, K^j] = i\epsilon^{ijk}K^k \qquad [K^i, K^j] = -i\epsilon^{ijk}J^k \tag{132}$$

$$[K^i, P^i] = i\delta^{ij}H \qquad [K^i, H] = iP^i. \tag{133}$$

The relativistic analog of diagonalizing the Hamiltonian is to decompose $U(\Lambda, a)$ into a direct integral of irreducible representations. This is equivalent to simultaneously diagonalizing the invariant mass and the spin Casimir operators of the Lie algebra defined by

$$M^2 = H^2 - \mathbf{P}^2 \qquad \text{and} \qquad \mathbf{S}^2 = W^2/M^2, \tag{134}$$

where W^μ is the Pauli-Lubanski vector

$$W^\mu = (\mathbf{P} \cdot \mathbf{J}, H\mathbf{J} + \mathbf{P} \times \mathbf{K}). \tag{135}$$

Once the eigenvalues of M^2 and S^2 are fixed, the representation of $U(\Lambda, a)$ on the fixed M^2 and S^2 subspaces of the Hilbert space is determined by group theoretical considerations.

For this section, it is useful to define the following functions of the generators: The Newton-Wigner position operator [25] is

$$\mathbf{X} := \frac{1}{2}\{\frac{1}{H}, \mathbf{K}\} - \frac{\mathbf{P} \times (H\mathbf{J} + \mathbf{P} \times \mathbf{K})}{MH(M+H)}, \tag{136}$$

and the spin is

$$\mathbf{S} := \mathbf{J} - \mathbf{X} \times \mathbf{P}. \tag{137}$$

These operators satisfy

$$[X^i, P^j] = i\delta_{ij} \qquad [X^i, S^j] = [P^i, S^j] = 0. \tag{138}$$

Equations (134), (136) and (137) can be inverted to express the ten Poincaré generators as functions of $\{M^2, \mathbf{P}, \mathbf{X}, \mathbf{S}\}$:

$$H = \sqrt{\mathbf{P}^2 + M^2} \tag{139}$$

$$\mathbf{J} = \mathbf{X} \times \mathbf{P} + \mathbf{S} \tag{140}$$

$$\mathbf{K} = \frac{1}{2}\{H, \mathbf{X}\} - \frac{\mathbf{P} \times \mathbf{S}}{H + M}. \tag{141}$$

The Poincaré commutation relations follow from (138) and the requirement that M and $\mathbf{S}^2$ commute with these operators.

Bound states of the N-particle system are simultaneous eigenstates of the mass M, linear momentum $\mathbf{P}$, the square of the spin $\mathbf{S}^2$, and the projection of the spin on an axis, $\mathbf{S}\cdot\hat{\mathbf{z}}$, where the mass eigenvalue m_b is discrete . These states, denoted by

$$|(m_b, s)\,\mathbf{p}, \mu\rangle, \tag{142}$$

are eigenstates of H with eigenvalue

$$E_b = \sqrt{\mathbf{p}^2 + m_b^2}. \tag{143}$$

Poincaré transformations on these states leave m_b and s unchanged:

$$U(\Lambda, a)|(m_b, s)\,\mathbf{p}, \mu\rangle$$
$$= \sum_{\mu'} \int |(m_b, s)\,\mathbf{p}', \mu'\rangle \, d\mathbf{p}' \, \langle (m_b, s)\,\mathbf{p}', \mu' | U(\Lambda, a) | (m_b, s)\,\mathbf{p}, \mu\rangle, \tag{144}$$

where the matrix

$$\langle (m_b, s)\,\mathbf{p}', \mu' | U(\Lambda, a) | (m_b, s)\,\mathbf{p}, \mu\rangle$$

is a representation of a mass m_b and spin s irreducible representation of the Poincaré group. It is the Poincaré group analog [26] of the Wigner D-function for the rotation group:

$$D^s_{\mu\mu'}(R) := \langle s, \mu | U(R) | s, \mu'\rangle.$$

In a relativistic quantum theory, the cluster condition

$$\lim_{|\mathbf{x}_i - \mathbf{x}_j| \to \infty} T^\dagger(\mathbf{x}_1, \cdots, \mathbf{x}_{n_a})\, U(t)\, T(\mathbf{x}_1, \cdots, \mathbf{x}_{n_a}) = \otimes_{i=1}^{n_a} U_{a_i}(t) \tag{145}$$

is replaced by

$$\lim_{|\mathbf{x}_i - \mathbf{x}_j| \to \infty} T^\dagger(\mathbf{x}_1, \cdots, \mathbf{x}_{n_b})\, U(\Lambda, a)\, T(\mathbf{x}_1, \cdots, \mathbf{x}_{n_b}) = \otimes_{i=1}^{n_b} U_{b_i}(\Lambda, a) = U_b(\Lambda, a), \tag{146}$$

where $U_{b_i}(\Lambda, a)$ are unitary representations of the Poincaré group for the subsystem of particles in the i^{th} cluster of b and the limit is a strong limit. This condition means that it is possible to test special relativity on isolated subsystems of particles.

The new complication in the relativistic case is that interactions necessarily appear in more than one of the generators [27]. This is a consequence of the commutators

$$[K^i, P^i] = i\delta^{ij} H \tag{147}$$

that have the Hamiltonian on the right side. If H includes interaction terms, then they must also appear in the terms on the left-hand side of these commutators. This impacts (146) since the translation generators $\mathbf{P}_{a_i}$ for the i^{th} cluster of a do not commute with the corresponding boost generator $\mathbf{K}_{a_i}$.

The cluster condition in (146) will be satisfied for short-range interactions provided that each Poincaré generator G_i has a cluster expansion of the form

$$G_i = \sum_{a \in \mathcal{P}_N} [G_i]_a = [G_i]_1 + \sum_{a \in \mathcal{P}'_N} \mathcal{C}_a (G_i)_a, \tag{148}$$

where $(G_i)_a = \sum_{l=1}^{n_a} (G_i)_{a_l}$. Each $(G_i)_{a_l}$ has the form $(\tilde{G}_i)_{a_l} \otimes I$, where $(\tilde{G}_i)_{a_l}$ only acts on the Hilbert space associated with the particles in the l^{th} cluster of a and satisfies the Poincaré commutation relations for each cluster of the partition a. I is the identity on the rest of the Hilbert space. The construction of unitary representations of the Poincaré group consistent with the cluster condition in (146) is non-trivial and can be found in [28][29].

The construction in [29] is recursive on the number of particles. It uses sums of proper subsystem generators to construct the Poincaré generators $(G_i)_a$. These are used in equations (134), (136) and (137) to construct the operators $M_a, \mathbf{P}_a, \mathbf{X}_a$, and $\mathbf{S}_a$. For each partition a, a S-matrix preserving unitary transformation, $V(a)$, is constructed that transforms

$$\mathbf{P}_a, \mathbf{X}_a, \text{ and } \mathbf{S}_a \tag{149}$$

to

$$\mathbf{P}_0, \mathbf{X}_0, \text{ and } \mathbf{S}_0. \tag{150}$$

The operators with 0 subscripts have no interactions, and $V(a)$ is recursively constructed to satisfy

$$(V(a))_b = V(a \cap b). \tag{151}$$

The resulting transformed mass operators

$$V^\dagger(a)\, M_a\, V(a), \tag{152}$$

for each partition, commute with the operators in (150). If these are combined using

$$\tilde{M} = \sum_{a \in \mathcal{P}'_N} \mathcal{C}_a V^\dagger(a)\, M_a\, V(a), \tag{153}$$

then $\tilde{M}$ also commutes with the operators in (150). The generators $\tilde{G}_i$ can be constructed as functions of

$$\tilde{M}, \mathbf{P}_0, \mathbf{X}_0, \text{and } \mathbf{S}_0 \tag{154}$$

using relations (139)-(141) to express the generators in terms of the operators in (154). This gives a dynamical representation of the Poincaré Lie algebra. The problem is that if the interactions between particles in different clusters of a are turned off, then

$$\tilde{G}_i \to V^\dagger(a)\,(G_i)_a\,V(a).$$

This means that it will violate cluster properties. The violation of cluster properties typically involves interactions disappearing that should not disappear when subsystems are separated.

In order to restore cluster properties without breaking the commutation relations, one defines the Cayley transform, $K(a)$, of $V(a)$ by

$$K(a) = i(V(a) - I)(I + V(a))^{-1} \qquad \text{with} \qquad V(a) = (I - iK(a))(I + iK(a))^{-1}. \tag{155}$$

One can define the unitary operator V in terms of these Cayley transforms by

$$V := (I - i \sum_{a \in \mathcal{P}'_N} \mathcal{C}_a K(a))(I + i \sum_{a \in \mathcal{P}'_N} \mathcal{C}_a K(a))^{-1}. \tag{156}$$

This operator has the property that when the interactions between the clusters of a are turned off, it becomes $V(a)$:

$$V_a = V(a). \tag{157}$$

Since V is unitary, it follows that

$$G_i := V^\dagger \tilde{G}_i V \tag{158}$$

satisfies the Poincaré commutation relations and satisfies the cluster properties in (146)-(148). It also follows that

$$U(\Lambda, a) := V^\dagger \tilde{U}(\Lambda, a) V \tag{159}$$

satisfies the cluster properties

$$\lim_{|\mathbf{x}_i - \mathbf{x}_j| \to \infty} T^\dagger(\mathbf{x}_1, \cdots, \mathbf{x}_{n_a})\, U(\Lambda, a)\, T(\mathbf{x}_1, \cdots, \mathbf{x}_{n_a}) = \otimes_{i=1}^{n_a} U_{a_i}(\Lambda, a). \tag{160}$$

One consequence of this construction is that

$$G_i = [G_i]_1 + \sum_{a \in \mathcal{P}'_N} \mathcal{C}_a (G_i)_a, \tag{161}$$

where the connected term is generated by the operator V and is needed to restore the commutation relations.

The construction of the operators $V(a)$ in [29] uses the same methods discussed in section III. The construction outlined above can be performed by replacing

$$\{\tilde{M}, \mathbf{P}_0, \mathbf{X}_0, \mathbf{S}_0\} \tag{162}$$

by

$$\{\tilde{M}_I = V^\dagger \tilde{M} V, \mathbf{P}_0, \mathbf{X}_I = V^\dagger \mathbf{X}_0 V, \mathbf{S}_I = V^\dagger \mathbf{S}_0 V\}. \tag{163}$$

As in the non-relativistic case, the starting point is the construction of channels. Given a partition a, there is a scattering channel α if there are bound states in each cluster of the partition a. In the relativistic case, the states in (5) are replaced by

$$\otimes_{i=1}^{n_a} |(m_{b_i}, s_i)\, \mathbf{p}_i, \mu_i\rangle, \tag{164}$$

where the m_{b_i} are discrete mass eigenvalues of the bound state in the i^{th} cluster of a. Note that these states transform like (144).

Given $U(\Lambda, a)$ satisfying cluster properties, the construction in the relativistic case is identical to the construction in the non-relativistic case. The exact projection operator on the $\mathcal{A}_1$ scattering channels is

$$P_{\mathcal{A}_1} = \sum_{\alpha \in \mathcal{A}_1} \Omega^{(-)}(a) \Phi_\alpha \Phi_\alpha^\dagger \Omega^{(-)\dagger}(a), \tag{165}$$

where the wave operators are the same functions of the Hamiltonian as in the non-relativistic case. Cluster properties of $U(\Lambda, a)$ imply that the Hamiltonian and all of the generators have cluster expansions of the type discussed in section III. The wave operators satisfy the general intertwining relations

$$U(\Lambda, b)\, \Omega^{(-)}(a) = \Omega^{(-)}(a)\, U_a(\Lambda, b).$$

The projection of the exact generators on the channel subspace are

$$P_{i\mathcal{A}_1} G_i. \tag{166}$$

These projected Poincaré generators satisfy the Poincaré commutation relations because the channel projection operators are Poincaré invariant. The projected generators have cluster expansions of the general form

$$P_{i\mathcal{A}_1} G_i = [P_{i\mathcal{A}_1} G_i]_1 + \sum_{a \in \mathcal{P}'_N} \mathcal{C}_a (G_{i\mathcal{A}_1})_a. \tag{167}$$

The term $\sum_{a\in\mathcal{P}'_N} \mathcal{C}_a(G_{i\mathcal{A}_1})_a$ can be constructed from all proper subsystem problems. While it satisfies cluster properties, it does not satisfy the commutation relations. The operators $V(a)$ and V in (159) are also constructed from proper solutions of proper subsystems.

While the $(G_{i\mathcal{A}_1})_a$ for $a \in \mathcal{P}'_N$ can be constructed from solutions of proper subsystem problems, without the connected term these operators will not satisfy the Poincaré commutation relations. Connected operators $[G_{i\mathcal{A}_1}]_1$ that restore the commutation relations can be constructed directly from the $(G_{i\mathcal{A}_1})_a$ for $a \in \mathcal{P}'_N$. The construction is the same as the one used in the exact case except the subsystem mass operators are replaced by the channel projected subsystem mass operators.

Formally,

$$G_{i\mathcal{A}_1} =$$

$$:= V^\dagger(\sum_{a\in\mathcal{P}'_N} \mathcal{C}_a(V(a)\, G_{i\,\mathcal{A}_1})V^\dagger(a))_a V = [G_{i\mathcal{A}_1}]_1 + \sum_{a\in\mathcal{P}'_N} \mathcal{C}_a(G_{i\mathcal{A}_1})_a, \tag{168}$$

where the connected term is generated by the construction. The resulting selected channel generators satisfy the Poincaré commutations relations. The operator $[G_{i\mathcal{A}_1}]_1$ is not equal to the operator $[P_{i\mathcal{A}_1}G_i]_1$ in (167), which requires the full solution of the problem. The proof of the optical theorem is identical to the proof in the non-relativistic case.

These $G_{i\mathcal{A}_1}$ generators can be used to construct the corresponding unitary representation of the Poincaré group. In this construction the required part of the N-body interaction is frame dependent. Note that while the connected term is required, it is not unique. Different constructions of the operators $V(a)$ can result in different $[G_{i\mathcal{A}_1}]_1$. This is related to the fact that cluster properties only fix the dynamics up to a N-body interaction.

XI. CONCLUSIONS

The role of individual scattering channels in nuclear reactions and nuclear structure is identified. The nuclear Hamiltonian is expressed as a sum over contributions from each scattering channel, the individual contributions determine the structure of the exact scattering wave functions in that channel up to, but not including N-body correlations. Limiting the sum to a selected subset of channels results in a truncated Hamiltonian that satisfies an optical theorem in the selected subset of channels. No flux is lost to the discarded channels. Each time a channel is included in the sum that channel opens up, and it contributes to the total cross section.

Each scattering channel Hamiltonian is energy independent and impacts all of the other channels, even at energies where that channel is closed. The scattering channel Hamiltonians also determine both the bound-state spectrum and structure of the bound-state wave functions.

The scattering channel Hamiltonians are constructed by first expressing the Hamiltonian as sum of tensor products of proper subsystem Hamiltonians. This is done using the Möbius and Zeta functions of the lattice of partitions that relate the cluster expansion of the Hamiltonian to a linear combination of proper subsystem Hamiltonians.

Terms in the spectral decompositions of tensor products of proper subsystem Hamiltonians are identified with scattering channels. The channel Hamiltonians are obtained by changing the order of the sum of tensor products of proper subsystem Hamiltonians with the sum over channels. The result is the part of the full spectral expansion of the Hamiltonian that can be constructed out of proper subsystem solutions. This is the most detailed information about the N-body Hamiltonian that can be obtained without solving the full N-body problem. Since the terms in the sum involve only proper subsystems, the channel sum only includes scattering channels and since the channel expansion is not the full spectral decomposition, the channel Hamiltonians for different scattering channels do not commute.

Diagonalizing this Hamiltonian generates the N-body correlations in each scattering wave function, generates the bound state wave functions and determines bound state energies. Even though all of the scattering channels are closed at the N-body binding energy, in this representation of the Hamiltonian the bound state binding energies and wave functions are completely determined by the scattering channel contributions. If the original Hamiltonian has no explicit N-body interactions, then N-body scattering correlations generated by diagonalizing the Hamiltonian exactly cancel the bound state projections.

The representation of the Hamiltonian as a sum of channel Hamiltonians was originally derived in ([4]). There the intended application was the construction of few-body models of nuclear reactions dominated by a given set of channels, where flux is conserved and only appears in the given set of channels. In this work, time-dependent scattering theory is used to determine the contribution of different scattering channels to the full spectral expansion and the structure of bound states.

In the scattering channel representation, the Hamiltonian is a sum of many-body operators, even for Hamiltonians with only two-body interactions. The integral equations for the scattering states derived in section VI are compatible with many-body operators and can be used to solve for the transition matrix elements for any selected channel truncation of the Hamiltonian. When the retained channels include all channels related by particle exchanges the number of coupled integral equations is reduced.

The channel representation is developed by applying cluster expansions to the spectral representation of the exact Hamiltonian. The construction utilizes results from time-dependent scattering theory. The chain rule for wave operators in [3][17] relates the exact solutions to scattering solutions for Hamiltonians with interactions between different clusters solutions turned off. Hunziker's treatment of cluster properties in scattering [16] provides a framework for relating the exact and subsystem channel states without having to use cluster properties of unbounded operators. The approach in this work has the advantage that the optical theorem in the selected (or remaining) channels can be understood from the solved form of the optical theorem for the exact channel projected Hamiltonian.

In this framework, there are no restrictions on the choice of contributing channels. The key properties that are special about this decomposition include: 1) both the retained and discarded channel Hamiltonians satisfy optical theorems in complementary sets of channels, 2) the scattering wave functions for the retained set and discarded set of channels differ from the exact scattering wave functions only by N-body correlations and 3) the channel truncated Hamiltonians are energy independent. Because the result of this decomposition is a Hamiltonian, it is compatible with any computational method. And because the exact Hamiltonian can be expressed as a sum of parts with complementary sets of channels, it provides a framework for investigating the dynamics due to the excluded channels. It is also compatible with Hamiltonians that have many-body interactions that come from effective field theory [5][6] or unitary scattering equivalences [7][8][9][10]. The channel decomposition can also be applied to model bound systems when the reactions are dominated by few-body channels.

The general method can also be applied to relativistic models. The application requires Poincaré generators that satisfy cluster properties. While this is a non-trivial constraint [28][29], once it is satisfied the construction proceeds in the same way as in the non-relativistic construction; however an additional N-body operator, constructed from the truncated generators, is needed to restore the Poincaré commutation relations. The relativistic treatment is applicable to models with color confinement, where color-singlets naturally cluster and the constituent particles have relativistic energies. In this case, the internal structure of the elementary color singlets is required input.

While the structure of channel truncated Hamiltonians involves contributions from many subsystems with different signs, the combinatorial factors [13][14][15] ensure that there is no over counting and that the continuous spectrum is bounded from below. This is relevant for treating overlapping channels and systems with identical particles where it is necessary to retain all channels related to the selected channels $\mathcal{A}_1$ by exchange of identical particles. This needs to be done in a manner that avoids over counting.

A possible point of confusion is why there is no flux lost to the eliminated channels in $\mathcal{A}_2$? The primary reason is that $H_{\mathcal{A}_1}$ is a Hermitian Hamiltonian that only supports asymptotic states in the retained channels in $\mathcal{A}_1$. The proof of the optical theorem proceeds as it would with any Hamiltonian.

For a more detailed explanation note that the eigenfunction expansion of the full Hamiltonian in (44) is also equal to the expansion where (45) is replaced by

$$P_\alpha^{(+)} := \Omega^{(+)}(a)\Phi_\alpha\Phi_\alpha^\dagger\Omega^{(+)\dagger}(a). \tag{169}$$

While the full Hamiltonian is unchanged, when the channel sum is truncated to a subset of scattering channels the expansions are different; specifically they are related by time reversal. In calculating the multi-channel scattering operator, $S_{\beta\alpha}$, there is a preferred direction of time evolution which is apparent from the definition of the channel scattering operator

$$S_{\beta\alpha} = (\Omega_\beta^+)^\dagger\Omega_\alpha^- = \lim_{t\to+\infty} \Phi_\beta^\dagger e^{iH_bt}e^{-2iHt}e^{iH_at}.\Phi_\alpha. \tag{170}$$

When H is replaced by $H_{\mathcal{A}_1}$ there are two natural approximations to $S_{\beta\alpha}$. The first replaces $S_{\beta\alpha}$ in (170) by

$$S_{\beta\alpha}(\mathcal{A}_1) := \lim_{t\to+\infty} \Phi_\beta^\dagger e^{iH_bt}e^{-2iH_{\mathcal{A}_1}t}e^{iH_at}\Phi_\alpha. \tag{171}$$

In this case the dynamics is generated by a single approximate Hamiltonian and this is the approximation used in this manuscript.

A second natural approximation is to replace $e^{-2iH_{\mathcal{A}_1}t}$ in (170) by

$$e^{-iTH_{\mathcal{A}_1}T^{-1}t}e^{-iH_{\mathcal{A}_1}t} \tag{172}$$

which uses $H_{\mathcal{A}_1}$ to calculate $\Omega^{(-)}(a)$ and the time reversed $H_{\mathcal{A}_1}$ to calculate $(\Omega^{(+)}(b))^\dagger$,(here T is the time reversal operator). In this approximation flux is lost in the channels $\mathcal{A}_2$.

Both approximations converge for the same $S_{\beta\alpha}$ in the limit that all of the scattering channels are included, but the second one does not satisfy the optical theorem unless all channels are included.

In the first approximation the Hamiltonian $H_{\mathcal{A}_1}$ has asymptotic states only in the channels $\mathcal{A}_1$. Scattering states with Hamiltonain $H_{\mathcal{A}_1}$ exist for both, $(\pm)$, asymptotic conditions:

$$|\Psi^{(\pm)}_{\mathcal{A}_1\alpha}\rangle = \lim_{t\to\pm\infty} e^{iH_{\mathcal{A}_1}t}e^{-iH_a t}\Phi_\alpha|\phi_{o\alpha}\rangle, \tag{173}$$

and both limits give states with the same energy eigenvalues (see (73) and (74)). As α ranges over all channels in $\mathcal{A}_1$ they span the same subspace. The range of $H_{\mathcal{A}_1}$ contains this subspace and may include some additional discrete states, which correspond to bound states or unphysical normalizable eigenstates. The possibility that these unphysical states have eigenvalues in the continuum cannot be eliminated. States in the null space of $H_{\mathcal{A}_1}$ have eigenvalue 0.

The optical theorem arises because the resolvent of $H_{\mathcal{A}_1}$ has a discontinuity across the cut corresponding to the continuous spectrum, which is just due to the channels $\mathcal{A}_1$. It will be satisfied, except at possible discrete energies where $H_{\mathcal{A}_1}$ has a point eigenvalue in the continuum.

XII. APPENDIX

The existence of the wave operator $\Omega^{(-)}_{W_I}$, that transforms the $(-)$ scattering eigenstates of H in the channels $\mathcal{A}_1$ to the $(-)$ scattering states of $H_{\mathcal{A}_1}$, is proved in this appendix. This establishes the relations in equations (71) and (72) that are used in the proof of the optical theorem.

The starting assumption is the existence of the two types of $(-)$ wave operators in the channels $\mathcal{A}_1$:

$$\Omega^{(-)}(a)\Phi_\alpha = \lim_{t\to-\infty} e^{iHt}e^{-iH_a t}\Phi_\alpha \qquad \forall\alpha\in\mathcal{A}_1 \tag{174}$$

and

$$\Omega^{(-)}_{\mathcal{A}_1}(a)\Phi_\alpha = \lim_{t\to-\infty} e^{iH_{\mathcal{A}_1}t}e^{-iH_a t}\Phi_\alpha \qquad \forall\alpha\in\mathcal{A}_1. \tag{175}$$

The limit in the first equation is the $(-)$ wave operator for the full Hamiltonian in the channels $\mathcal{A}_1$. The limit in the second equation is the $(-)$ wave operator for the Hamiltonian $H_{\mathcal{A}_1}$ in the $(-)$ channels $\mathcal{A}_1$. The existence of the second limit is assumed, however its existence can be understood by realizing (see (52)) that (1) by turning off interactions between any two clusters containing a it agrees with doing the same to the exact wave operators, i.e. for any partition $b \supset a$ with two or more clusters

$$(\Omega^{(-)}_{\mathcal{A}_1}(a))_b \Phi_\alpha = (\Omega^{(-)}(a))_b \Phi_\alpha \tag{176}$$

and (2) by construction $H_{\mathcal{A}_1}$ has no N-body interactions, so the different intercluster interactions that appear in (176) for different choices of b are the only interactions in $H_{\mathcal{A}_1}$. This implies that all of the effective interactions in $e^{iH_{\mathcal{A}_1}t}$ are short ranged.

For $\alpha \in \mathcal{A}_1$ consider the limit

$$\lim_{t\to -\infty} \|(e^{iHt}e^{-iH_at} - e^{iP_{\mathcal{A}_1}Ht}e^{-iH_at})\Phi_\alpha|\phi_{o\alpha}\rangle\|.$$

Since $[H, P_{\mathcal{A}_1}] = 0$ this is equal to

$$\lim_{t\to -\infty} \|(I - e^{i(P_{\mathcal{A}_1}-I)Ht})e^{iHt}e^{-iH_at})\Phi_\alpha|\phi_{o\alpha}\rangle\|. \tag{177}$$

Because

$$Q_{\mathcal{A}_1} := I - P_{\mathcal{A}_1} \tag{178}$$

is the projector on the orthogonal complement of the range of $P_{\mathcal{A}_1}$, it follows that

$$I - e^{i(P_{\mathcal{A}_1}-I)Ht} = I - e^{-iQ_{\mathcal{A}_1}Ht} = (I - e^{-i(Q_{\mathcal{A}_1}Ht})Q_{\mathcal{A}_1}. \tag{179}$$

Using (179) in (177) gives

$$\begin{aligned}(177) = \lim_{t\to -\infty} \|(I - e^{i(P_{\mathcal{A}_1}-I)Ht})Q_{\mathcal{A}_1}e^{iHt}e^{-iH_at})\Phi_\alpha|\phi_{o\alpha}\rangle\| \leq \\ \lim_{t\to -\infty} 2\|Q_{\mathcal{A}_1}e^{iHt}e^{-iH_at}\Phi_\alpha|\phi_{o\alpha}\rangle\| = \\ 2\|Q_{\mathcal{A}_1}|\Psi^{(-)}_\alpha\rangle\| = 0\end{aligned} \tag{180}$$

for $\alpha \in \mathcal{A}_1$. This shows

$$\Omega^{(-)}(a)\Phi_\alpha = \lim_{t\to -\infty} e^{iP_{\mathcal{A}_1}Ht}e^{-iH_at}\Phi_\alpha. \tag{181}$$

To prove (71) consider the difference

$$\|\left(\Omega^{(-)}_{W_I}\Omega^{(-)}(a) - \Omega^{(-)}_{\mathcal{A}_1}(a)\right)\Phi_\alpha|\phi_{o\alpha}\rangle\| =$$

$$\lim_{t\to -\infty} \|\left(e^{iH_{\mathcal{A}_1}t}e^{-iHP^-_{\mathcal{A}_1}t}\Omega^{(-)}(a) - \Omega^{(-)}_{\mathcal{A}_1}(a)\right)\Phi_\alpha|\phi_{o\alpha}\rangle\|.$$

Adding 0 gives

$$= \lim_{t\to-\infty} \| \left(e^{iH_{\mathcal{A}_1}t} e^{-iHP^-_{\mathcal{A}_1}t} (\Omega^{(-)}(a) \underbrace{-e^{iHP^-_{\mathcal{A}_1}t} e^{-iH_a t} + e^{iHP^-_{\mathcal{A}_1}t} e^{-iH_a t}}_{0}) - \Omega^{(-)}_{\mathcal{A}_1}(a) \right) \times$$

$$\Phi_\alpha |\phi_{o\alpha}\rangle \|.$$

Using the triangle inequality gives

$$\le \lim_{t\to-\infty} \| e^{iH_{\mathcal{A}_1}t} e^{-iHP^-_{\mathcal{A}_1}t} (\Omega^{(-)}(a) - e^{iHP^-_{\mathcal{A}_1}t} e^{-iH_a t}) \Phi_\alpha |\phi_{o\alpha}\rangle \| +$$

$$\lim_{t\to-\infty} \| (e^{iH_{\mathcal{A}_1}t} e^{-iH_a t} - \Omega^{-}_{\mathcal{A}_1}(a)) \Phi_\alpha |\phi_{o\alpha}\rangle \|.$$

Since the operators $e^{iH_{\mathcal{A}_1}t} e^{-iHP^-_{\mathcal{A}_1}t}$ are unitary this becomes

$$= \lim_{t\to-\infty} \| (\Omega^{(-)}(a) - e^{iHP^-_{\mathcal{A}_1}t} e^{-iH_a t}) \Phi_\alpha |\phi_{o\alpha}\rangle \| +$$

$$\lim_{t\to-\infty} \| (e^{iH_{\mathcal{A}_1}t} e^{-iH_a t} - \Omega^{(-)}_{\mathcal{A}_1}(a)) \Phi_\alpha |\phi_{o\alpha}\rangle \|. \tag{182}$$

These limits vanish by assumptions (174) and (175) and equation (181). This shows that $\Omega^{(-)}_{W_I}$ exists and transforms the exact $(-)$ scattering stated in the channels $\mathcal{A}_1$ to the corresponding scattering eigenstates of $H_{\mathcal{A}_1}$:

$$\Omega^{(-)}_{W_I} \Omega^{(-)}(a) \Phi_\alpha |\phi_{o\alpha}\rangle = \Omega^{(-)}_{\mathcal{A}_1}(a) \Phi_\alpha |\phi_{o\alpha}\rangle \tag{183}$$

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