Euclidean representations of relativistic quantum theories

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This contribution discusses a Euclidean approach to relativistic quantum mechanics. The input is a collection of Euclidean covariant reflection positive distributions. The advantage of the Euclidean approach are (1) cluster properties are easily realized (2) analytic continuation is not needed and (3) the dynamics can be motivated by appealing to local quantum field theory.

There are computational advantages to using Euclidean formulations of relativistic quantum mechanics or local quantum field theory [**?**]. The challenges are treating problems like scattering, where real-time evolution plays an important role. In addition, the analytic continuation back to real time requires a detailed understanding of the analytic structure of the quantities being continued.

The Osterwalder-Schrader reconstruction theorem [**?**] provides a means for constructing a relativistic quantum field theory from a collection of Euclidean covariant Green functions. In their proof of the reconstruction theorem Osterwalder and Schrader construct the physical quantum mechanical Hilbert space *without any explicit analytic continuation*. In addition, the locality axiom is logically independent of the rest of the axioms, suggesting the possibility of using Euclidean methods to construct relativistic models of systems of a finite number of degrees of freedom that satisfy cluster properties and a spectral condition. While these properties have been known since the 1970's, they have not been exploited in applications.

The input to the reconstruction theorem is a collection of Euclidean covariant distributions that satisfy reflection positivity. These distributions are formally functional derivatives of the functional Fourier transform of a Euclidean path integral, which provides a formal connection to the dynamics. A corresponding connection to the dynamics is not as direct with the Minkowski-space Wightman axioms.

Reflection positivity implies the spectral condition. While reflection positivity is not expected to hold gauge theories, it is expected to hold if the theory is formulated in terms of gauge invariant degrees of freedom, like Wilson loops. This is the case in lattice formulations of QCD, and is sufficient for computing physical observables.

The structure of the *physical Hilbert space inner product* in terms of the Euclidean covariant distributions has the form

$$\langle \psi | \phi \rangle = \sum_{mn} \int \psi^*_m(\theta x) S_{m:n}(x,y) \phi_n(y) d^{4m}x d^{4n}y \tag{1}$$

where the space-time-spin variables are Euclidean, θ changes the sign of the Euclidean times and the wave functions have support for positive relative Euclidean times. Reflection positivity is the condition that this inner product leads to a non-negative norm:

$$\| |\psi\rangle \|^2 := \langle \psi | \psi \rangle \geq 0. \tag{2}$$

It is a condition on the distributions, $S_{n:m}$. The relativistic invariance of a quantum theory requires the existence of a dynamical unitary ray representation of the Poincaré group. The covering group of both the complex Lorentz group and the complex orthogonal group is $SL(2,\mathbb{C}) \times SL(2,\mathbb{C})$. This relation implies relations of the generators of Euclidean time translations and rotations in space- Euclidean time planes with the generators of Minkowski time evolution and Lorentz boosts [**?**]. By making the replacements

$$H_e \to H_m = iH_e \qquad J^{i0}_e \to J^{i0}_m = -iJ^{i0}_e \tag{3}$$

the Euclidean generators can be transformed into operators satisfying the Poincaré commutations relations. The generators H_e and H_m or J^{i0}_e and J^{i0}_m cannot both be self-adjoint operators on the same representation of the Hilbert space. The time reflection in the inner product (**??**) breaks Euclidean invariance and makes the Lorentz generators self-adjoint on the Hilbert space (**??**). The inner product with the time reflection cannot have a positive norm for general functions of Euclidean variables since the norm of functions that are even or odd with respect to time reflection

[*]Non-Perturbative QFT in Euclidean and Minkowski Workshop
Mini-proceedings, 10^{th}-12^{th} September, University of Coimbra, Portugal. This research supported by the US Department of Energy, Office of Science, grant number DE-SC0016457

cannot both be positive. The projection on states with positive relative times gives the required non-negative inner product, provided the Euclidean distributions are reflection positive.

A general unitary representation of the covering group of the Poincaré group of a relativistic quantum theory can be transformed into a direct integral of irreducible representations by a unitary transformation. This is the relativistic analog of diagonalizing a non-relativistic Hamiltonian. For a physically sensible theory the irreducible representations are representations with positive energy and positive or zero mass. The fact that finite dimensional representations of $SU(2)$ are entire functions of angles can be used to show the equivalence of mass m, spin s irreducible unitary representations of the Poincaré group, covariant two-point Wightman functions for particles with the same mass and spin and reflection positive Euclidean distributions that give the same inner products as the Wightman functions. The equivalent Wightman and Euclidean kernels are related by

$$\delta(p^2+m^2)\theta(p^0)D^s_{\mu\nu}(\sigma\cdot p) \leftrightarrow \frac{2}{\pi}\frac{D^s_{\mu\nu}(\sigma_e\cdot p_e)}{p_e^2+m^2}, \qquad \sigma_\mu=(I,\boldsymbol{\sigma}), \qquad \sigma_{e\mu}=(iI,\boldsymbol{\sigma}). \tag{4}$$

For systems of non-interacting particles the Euclidean distributions are irreducible representation or tensor products of irreducible representations so they are reflection positive. The main challenge is to show that reflection positivity is preserved after including interactions.

Widder's theorem [**?**] gives a representation for the kernel of the positive quadratic form

$$\int f^*(\tau)k(\tau+\tau')f(\tau') \geq 0 \tag{5}$$

of the form

$$k(\tau+\tau') = \int_c^\infty d\rho(\alpha)e^{-\alpha(\tau+\tau')} \tag{6}$$

where $d\rho(\alpha)$ is a positive measure. This can be transformed to look like a Lehmann representation in 1 dimension. This provides some insight into the structure of the reflection positive Euclidean covariant distributions, but we do not know of any general structure theorems for reflection positive Euclidean covariant distributions.

The Euclidean representation of the Hilbert space has some interesting properties. One is that four-dimensional delta functions are square integrable. This is a consequence of the structure of the kernel that includes the Euclidean time-reflection. In an irreducible representation the square of the mass operator is the Euclidean Laplacian. If analog of the Lehmann weight is polynomially bounded, polynomials in this Laplacian are complete [**?**]. This can be used to construct operators that project out single particle states that satisfy the relative time support condition. These states are needed to construct the asymptotic states in the Haag-Ruelle formulation of scattering theory, which is the field theoretic generalization of non-relativistic time-dependent scattering. It differs from the LSZ formulation in that the interpolating field can only create one-particle states out of the vacuum. The advantage is that the weak limits in LSZ scattering are replaced by strong limits.

The Haag-Ruelle S matrix has the standard form

$$S_{fi} = \Omega^\dagger_{f+}\Omega_{i-} := \lim_{t\to\infty}\langle\phi_f e^{iH_ft}\Pi^\dagger_f e^{-2iHt}\Pi_i e^{iH_ft}|\phi_i\rangle \tag{7}$$

where Π_f and Π_i are channel injection operators. The invariance principle [**?**] can be used to replace H, H_f and H_i by $-\exp(-\lambda H)$. This means that e^{-2iHt} can be replaced by $e^{2ine^{-\lambda H}}$ which, because of the compact spectrum of $-\exp(-\lambda H)$, can be uniformly approximated by a polynomial in $\exp(-\lambda H)$ [**?**]. Matrix elements of this polynomial can be computed using elementary Euclidean time translations by integer multiples of λ, which is a free parameter that sets an energy scale. For gauge theories, the interpolating field creates gauge invariant one-particle states out of the vacuum, which means that the Hamiltonian in (**??**) will not couple to non-singlet states.

The conclusion is that by taking advantage of the Euclidean representation of the dynamics the difficulties associated with analytic continuation can be avoided. In addition, because the construction gives the Hilbert-space inner product and self-adjoint Poincaré generators that satisfy cluster properties, standard methods can be used to compute any quantum mechanical observable.

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