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The light-front vacuum and dynamics*

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Abstract. I give a quantum theoretical description of kinematically invariant vacuua on the algebra of free fields restricted to a light front and discuss the relation between the light-front Hamiltonian, P^-, the vacuum, and Poincaré invariance. This provides a quantum theoretical description of zero modes.

1 Introduction

A light-front field, $\phi(f)$, is a free field restricted to the light front $x^+ = 0$ and smeared with a Schwartz test function of the light-front variables $\tilde{x} = (x^-, \boldsymbol{x}_\perp)$. The commutator of two light-front fields is:

$$[\phi(f), \phi(g)]_- = \frac{1}{2}[(f,g)_f - (g,f)_f] \qquad (f,g)_f := \int \frac{d\tilde{p}\theta(p^+)}{p^+} \tilde{f}^*(\tilde{p})\tilde{g}(\tilde{p}) \tag{1}$$

where $\tilde{f}(\tilde{p})$ is the Fourier transform of $f(\tilde{x})$. In the absence of restrictions on the test functions, the light-front scalar product in (1) diverges logarithmically due to the $p^+ = 0$ singularity in the denominator. The commutator (1) becomes finite if the test functions are restricted to have the form $\tilde{f}(\tilde{p}) = p^+ \tilde{g}(\tilde{p})$, where $\tilde{g}(\tilde{p})$ are ordinary Schwartz test functions of the light-front momenta. This space of test functions, introduced by Schlieder and Seiler [1], is denoted by $\mathcal{S}^+$.

The light-front Fock algebra, $\mathcal{A}_f$, is generated by finite linear combinations of the form

$$A := \sum_{k=1}^{N} c_k e^{i\phi(f_k)} \tag{2}$$

where c_k are complex and $f_k(\tilde{x})$ are real Schlieder-Seiler functions. It is straightforward to show that $\mathcal{A}_f$ is an abstract $*$-algebra with following properties:

1. $\mathcal{A}_f$ is closed under kinematic Poincaré transformations.
2. $\mathcal{A}_f$ is a Weyl algebra.

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3. Rotations induce algebraic isomorphisms from $\mathcal{A}_f$ to algebras associated with different light-front orientations.

A vacuum is a positive, invariant, linear functional $E[\cdot]$ on this algebra. The properties of a vacuum can be expressed in terms of its generating functional

$$S\{f\} := E[e^{i\phi(f)}] = \langle 0|e^{i\phi(f)}|0\rangle.$$

The generating functional of a light-front vacuum must be normalized, $S\{0\} = 1$, real $S^*\{f\} = S\{-f^*\}$, continuous

$$f_n \to f \in \mathcal{S}^+ \Rightarrow S\{f_n\} \to S\{f\},$$

non-negative

$$S\{f_i - f_j\} := M_{ij} \geq 0$$

for any sequence $\{f_n\}$ of real test functions in $\mathcal{S}^+$, kinematically invariant

$$S\{f\} = S\{f'\}$$

where $f'(x) = f(\Lambda x + a)$ for any *kinematic* Poincaré transformation, and satisfy cluster properties

$$\lim_{\lambda\to\infty} S\{f + g_\lambda\} \to S\{f\}S\{g\}$$

where $g_\lambda(x) = g(x + \lambda y)$ and y is any space-like vector in the light-front hyperplane [2].

The Hilbert space representation of $\mathcal{A}_f$ associated with a given vacuum functional is defined as follows. A dense set of vectors is given by expressions of the form

$$|\psi\rangle = \sum_{n=1}^{N} c_n e^{i\phi(f_n)}|0\rangle \qquad N < \infty.$$

The inner product can be expressed in terms of the generating functional

$$\langle\xi|\psi\rangle := \sum_{mn} d_m^* c_n e^{\frac{1}{2}[\phi(g_m),\phi(f_n)]} S\{f_n - g_m\}.$$

The generating functional of the Fock representation, $S_0\{f\} := e^{-\frac{1}{4}(f,f)}$, satisfies all of the required properties. The representation of the kinematic Poincaré transformations on this Hilbert space is unitary.

The algebra $\mathcal{A}_f$ has another class of kinematically invariant vacuua. Given a Schlieder-Seiler test function $\tilde{f}(\tilde{p})$ define

$$\hat{f}(\boldsymbol{p}_\perp) = \lim_{p^+\to 0} \frac{\tilde{f}(p^+, \boldsymbol{p}_\perp)}{p^+}. \tag{3}$$

Vacuum generating functionals have the form

$$S\{f\} = S_0\{f\}s\{\hat{f}\} \qquad s\{\hat{f}\} = e^{\sum_n i^n s_n(\hat{f},\cdots,\hat{f})} \tag{4}$$

where

$$s_n(\boldsymbol{p}_{1\perp},\cdots,\boldsymbol{p}_{n\perp}) := \delta(\sum_{i=n} \boldsymbol{p}_{i\perp}) w_{tn}(\boldsymbol{p}_{1\perp},\cdots,\boldsymbol{p}_{n\perp}) \tag{5}$$

and $w_{tn}(\boldsymbol{p}_{1\perp}, \cdots, \boldsymbol{p}_{n\perp})$ are connected, two-dimensional, Euclidean invariant Schwartz distribution in $2(n-1)$ independent variables. The functional, $s\{\hat{f}\}$, is the Fourier transform of a positive measure on the cylinder sets of Schwartz distributions in two variables [3]. The generating function $S\{f\}$ will be the generating function of a vacuum functional if $s\{\hat{f}\}$ satisfies the same properties as $S\{f\}$ with respect to Schwarz functions of two variables, where the invariant subgroup is the two-dimensional Euclidean group. While positivity is a strong constraint, non-trivial examples associated with coherent states and Gaussian measures exit. The different vacuum generating functionals lead to inequivalent Hilbert space representations of the $*$ algebra.

To construct a relativistic model it is necessary to complete the Lie algebra of the Poincaré group by finding dynamical generators that are compatible with a given vacuum state. The first step is to construct a mass operator that is compatible with the vacuum. Let O be any non-negative kinematically invariant operator. The Fock representation mass operator, M_f, is one such example; others can be constructed by choosing $O = B^\dagger B$ for a kinematically invariant operator B. Let $\Pi := (I - |0\rangle\langle 0|)$.

The operator $M := \Pi O \Pi$ is non-negative, kinematically invariant, and annihilates the vacuum. It is a suitable candidate for the mass operator of a unitary representation of the Poincaré group. The associated light-front Hamiltonian P^- is:

$$P^- := \frac{\boldsymbol{P}_\perp \boldsymbol{P}_\perp + M^2}{P^+}.$$

The mass operator M can be formally expressed as a limit of elements of $\mathcal{A}_f$. To do this let $\{A_n\} \in \mathcal{A}_f$ generate an orthonormal basis for the Hilbert space representation with vacuum functional E, $E[A_n^\dagger A_m] = \delta_{mn}$. It follows that

$$M = \sum_{k,l} A_k^\dagger A_l m_{kl} \qquad \text{where}$$

$$m_{kl} := E[A_k^\dagger O A_l] - E[A_k^\dagger]E[OA_l] - E[A_k^\dagger O]E[A_l] + E[A_k^\dagger]E[O]E[A_l]$$

which explicitly exhibits M as the limit of elements of the algebra.

The last step in constructing a dynamics is to complete the Lie algebra of the Poincaré group by including rotations. The ability to complete the Lie algebra is intrinsic to the choice of kinematically invariant M. Free rotations, $U_0(R)$, act on the fields covariantly and define algebraic isomorphisms from $\mathcal{A}_f$ to light-front Fock Algebras with *different* light fronts. Given M and the kinematic observables it is possible to formulate a scattering theory. The point eigenstates of the mass operator are needed to formulate the scattering asymptotic condition. They are acceptable if under *free* rotations they satisfy

$$U_0(R)|m\rangle_{\hat{n}} = |m\rangle_{R\hat{n}} D(R)$$

where $D(R)$ is an irreducible representation of the rotation group (note the change in the orientation of the light front). Wave operators, constructed using these one-body solutions to formulate the asymptotic condition, satisfy

$$U_0(R)\Omega_{\hat{n}\pm} = \Omega_{R\hat{n}\pm} U_f(R)$$

where $\Omega_{R\hat{n}\pm}$ is the wave operator associated with the rotated light front, $R\hat{n}$ and $U_f(R)$ is the asymptotic representation of $SU(2)$. Poincaré invariance requires rotation operators that leave the vacuum invariant. If the scattering operators are asymptotically complete and *independent* of the orientation of the light-front,

$$\Omega^{\dagger}_{+\hat{n}}\Omega_{-\hat{n}} = \Omega^{\dagger}_{+R\hat{n}}\Omega_{-R\hat{n}} \Rightarrow A_R := \Omega_{+R\hat{n}}\Omega^{\dagger}_{+\hat{n}} = \Omega_{-R\hat{n}}\Omega^{\dagger}_{-\hat{n}}$$

then it follows that

$$U_{\hat{n}}(R) := U_0(R)A_{R^{-1}} = U_0(R)\Omega_{R^{-1}\hat{n}\pm}\Omega^{\dagger}_{\hat{n}\pm} = \Omega_{\hat{n}\pm}U_f(R)\Omega^{\dagger}_{\hat{n}\pm}$$

extends the Poincaré group on the Hilbert space with the given non-trivial vacuum [4]. The invariance of the S matrix ensures that A_R does not depend on the choice of asymptotic condition. Here wave operators are two-Hilbert space wave operators that necessarily include the point spectrum contributions to the mass operator. The $\hat{n}$-invariance of S can be tested in calculations, however the restrictions are non-trivial.

The above construction shows that the singularities in the light-front scalar product lead to a restriction on the test functions in the light-front Fock algebra. The resulting $*$-algebra, $\mathcal{A}_f$, has a large class of kinematically invariant vacuua that lead to inequivalent Hilbert space representations of the algebra. It is possible to find dynamical P^- or mass operators M that are positive, annihilate the vacuum and are kinematically covariant (resp. invariant). A large class of such operators exist, but a given operator can only annihilate one of the vacuum vectors. The dynamical operator P^- or M can be used to formulate a scattering theory. A sufficient condition for full Poincaré invariance is that the scattering matrix associated with different light fronts is independent the light front. This is a strong, but testable condition.

The construction provides a direct means to formulate dynamical models associated with inequivalent vacuum representations. This construction does not utilize classical equations of motion or models with a finite number of degrees of freedom, it provides a provide direct means to formulate and study models with non-trivial zero modes.

References

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