A Euclidean formulation of relativistic quantum mechanics

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In this paper we discuss a formulation of relativistic quantum mechanics that uses model Euclidean Green functions or their generating functional as input. This formalism has a close relation to quantum field theory, but as a theory of linear operators on a Hilbert space, it has the advantages of quantum mechanics. One interesting feature of this approach is that matrix elements of operators in normalizable states on the physical Hilbert space can be calculated directly using the Euclidean Green functions without performing an analytic continuation. The formalism is summarized in this paper. We discuss the motivation, advantages and difficulties in using this formalism. We discuss how to compute bound states, scattering cross sections, and finite Poincaré transformations without using analytic continuation. A toy model is used to demonstrate how matrix elements of $e^{-\beta H}$ in normalizable states can be used to construct sharp-momentum transition matrix elements.

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I. MOTIVATION

In this paper we investigate a framework for constructing relativistic quantum mechanical models of few-degree-of-freedom systems that are inspired by an underlying quantum field theory. Our interest is in few-body physics at the few-GeV energy scale. Poincaré invariance is an important symmetry at this scale since the energies are comparable to the mass scale of the hadrons. Most models of systems at these energies are motivated by quantum field theory, but their connection with quantum mechanical models of a finite number of degrees of freedom is not straightforward. The advantage of a quantum mechanical model of a finite number of degrees of freedom is that the theory is linear and it can be in-principle solved, even for strongly interacting systems, with mathematically controlled errors. Bound systems of particles present no special problems; they are just point-spectrum eigenstates of the mass operator (rest energy). The construction of a unitary multichannel scattering operator, S, can be performed using the same methods that are used in non-relativistic models.

One approach that has been successfully used to formulate realistic Poincaré invariant quantum mechanical models of few-hadron systems is Poincaré invariant quantum mechanics[1]. In this approach a dynamical unitary representation of the Poincaré group is constructed on a few-particle Hilbert space. This approach has been successfully applied to treat a number of few-hadron or few-quark problems. The virtue of this approach is that it is possible construct quantum mechanical models with a finite number of degrees of freedom that have a unitary representation of the Poincaré group, satisfy a spectral condition, and for fixed numbers of particles satisfy cluster separability. These are essentially all of the axioms of quantum field theory, except microscopic locality, which cannot be tested experimentally and requires an infinite number of degrees of freedom. One of the disadvantages of this approach is that the models do not have a straightforward relation to a Lagrangian field theory. This makes it difficult to use a field theory like QCD to improve or constrain the models.

Because of the difficulties discussed in the previous paragraph it is desirable to explore alternate formulations of relativistic quantum mechanics that have a more direct relation to Lagrangian field theory while preserving the structure of the underlying quantum theory. The alternative that we pursue in this work is to formulate models that are motivated by the standard reconstruction of a quantum theory from the field theory. The Euclidean formulation of quantum field theory is convenient for this purpose because (1) it has a direct relation to the action or Lagrangian through either formal path integrals or the Dyson expansion and (2) it is possible to use the Euclidean Green functions or their generating functional to directly construct the physical Hilbert space using the Euclidean reconstruction theorem. In this paper we argue that using this framework it is possible to compute all interesting quantum mechanical observables without explicit analytic continuation.

The proposed models are constructed by restricting the number of degrees of freedom. In passing to a few-body model all of the axioms of field theory cannot be preserved. One of the attractive features of the Euclidean reconstruction theorem is that locality is an independent requirement that can be relaxed without violating the other axioms[2]. A feature of this approach is that the dynamics is introduced directly by the model Euclidean Green functions or generating functional, rather than in a model Hamiltonian or Lagrangian. One of the challenges of this approach is to find a robust framework for modeling Euclidean Green functions or generating functionals with the required properties. An advantage is that standard Euclidean field theoretic methods can be used to motivate the structure of the models.

The construction of the physical Hilbert space of the field theory from a Euclidean generating functional is discussed in the next section. The generating functional for a scalar field is used to illustrate the basic construction. The construction of the Poincaré Lie algebra is discussed in the third section. The resulting generators are self-adjoint operators on the physical Hilbert space. The construction of one-particle states and the computation of finite Poincaré transformations on these states is discussed in section four. The one-particle states are used to construct scattering states in section five. The computation of finite Poincaré transformations on the scattering states is also discussed in this section. Everything discussed in sections 2-5 uses only Euclidean generating functionals and Euclidean test functions. Section six discusses how the results of sections 2-5 are expressed in terms of Euclidean Green functions. This is more practical for formulating models. Models of a finite number of degrees of freedom are discussed in section seven by considering the structure of models of relativistic nucleon-nucleon scattering. A numerical test of the proposed method to compute scattering observables from matrix elements of $e^{-\beta H}$ in normalizable states is given in section eight using an exactly solvable non-relativistic model. The results suggest that these methods can be used to compute cross sections at the few-GeV energy scale. Our conclusions and the outlook for this approach are discussed in the last section.

II. QUANTUM MECHANICS - HILBERT SPACE REPRESENTATION

To illustrate the construction of the model quantum theory we assume that we are given an Euclidean generating functional $Z[f]$ associated with a scalar field. For the purpose of illustration we assume that $Z[f]$ has all of the properties that are expected of the generating functional of a scalar field theory. These properties include Euclidean invariance, reflection positivity, reality, and cluster separability.

The generating functional relates the Lagrangian of the field theory to the Hilbert space formulation of the theory. In Lagrangian field theory the generating functional is formally the functional Fourier transform of an Euclidean path-integral measure with action $A[\phi]$

$$Z[f] = \frac{\int D[\phi] e^{-A[\phi]+i\phi(f)}}{\int D[\phi'] e^{-A[\phi']}} = \sum_n \frac{(i)^n}{n!} S_n \underbrace{(f, \cdots, f)}_{n \text{ times}} = \exp(\sum_n \frac{(i)^n}{n!} S_n^c \underbrace{(f, \cdots, f)}_{n \text{ times}}) \tag{2.1}$$

where $f = f(\tau, \mathbf{x})$ is a Schwartz test function in the Euclidean space-time variables and S_n is the n-point Euclidean Green function, and S_n^c is the connected n-point Euclidean Green function. In this approach the generating functional $Z[f]$ is the dynamical input that replaces the Lagrangian or Hamiltonian. A connection with an underlying Lagrangian or path integral is not required and will not be generally assumed, however this connection provides an important source of phenomenology, which is a primary motivation for developing this formalism.

In what follows Euclidean space-time coordinates are denoted by $\mathsf{x} = (\tau, \mathbf{x})$. Euclidean invariance of the generating functional means that

$$Z[f_{E,a}] = Z[f]. \tag{2.2}$$

for

$$f_{E,a}(\mathsf{x}) := f\left(E^{-1}(\mathsf{x} - \mathsf{a})\right), \tag{2.3}$$

where E is an $O(4)$ rotation and a is an Euclidean space-time displacement.

The construction of the physical Hilbert space from Euclidean Green functions was given by Osterwalder and Schrader [2][3]. A simpler construction in terms of the generating functional was given by Fröhlich[4]. We use Fröhlich's approach to illustrate the construction using the generating functional for a "scalar field". To construct the physical Hilbert space Osterwalder and Schrader select a Euclidean time axis and restrict the space of test functions, $f(\tau, \mathbf{x})$, to real-valued Schwartz functions of four Euclidean space-time variables that vanish for negative Euclidean times;

$$\mathcal{S}_+ = \{f(\tau, \mathbf{x}) \in \mathcal{S} | f(\tau, \mathbf{x}) = 0 \quad \tau < 0\}. \tag{2.4}$$

We call elements of $\mathcal{S}_+$ positive-time test functions. In what follows all test functions will be assumed to be positive-time test functions unless stated otherwise.

Osterwalder and Schrader introduced the Euclidean time-reflection operator Θ defined by

$$\Theta f(\tau, \mathbf{x}) = f(-\tau, \mathbf{x}). \tag{2.5}$$

The generating functional, $Z[f]$, is reflection positive if for any finite sequence of real positive-time test functions, $\{f_1 \cdots f_n\} \in \mathcal{S}_+$, the $n \times n$ matrix,

$$M_{ij} := Z[f_i - \Theta f_j] \geq 0, \tag{2.6}$$

is non-negative. This condition is not automatic. It holds for generating functionals for free fields and for some lattice truncations of interacting theories. In general it is a requirement on physically acceptable models. One of the challenges in implementing this formalism is to find a robust class of reflection positive generating functionals. In what follows we assume this condition is satisfied; in applications it may only be satisfied approximately or when restricted to a subspace.

In Fröhlich's construction a dense set of normalizable vectors in the physical Hilbert space is represented by complex wave-functionals of the form

$$B(\phi) = \sum_{j=1}^{n_b} b_j e^{i\phi(f_j)} \tag{2.7}$$

where b_i are complex numbers and $f_j(\tau, \mathbf{x})$ are real positive-time test functions. The argument ϕ can be thought of as an abstract integration variable. This interpretation is motivated by the path integral representation of the generating functional, (2.1).

The physical scalar product of two such wave-functionals, $B(\phi)$, and

$$C(\phi) = \sum_{k=1}^{n_c} c_k e^{i\phi(g_k)} \tag{2.8}$$

is given directly in terms of the generating functional by

$$\langle B|C\rangle := \sum_{j,k}^{n_b,n_c} b_j^* c_k Z[g_k - \Theta f_j]. \tag{2.9}$$

Reflection positivity, (2.6), is equivalent to the statement that vectors have positive length

$$\langle B|B\rangle \geq 0. \tag{2.10}$$

The physical Hilbert space is obtained by identifying wave functionals whose difference has zero norm and completed by adding convergent sequences of wave functionals.

The inner product (2.9) is the physical quantum mechanical scalar product, even though the input only involves the Euclidean generating functional and positive-time Euclidean test functions! No analytic continuation is used. An explicit illustration of this relationship in given in section six for free scalar particles in eq. (6.4) and free spin 1/2 particles in eq. (6.6).

While the computation of the exact generating functional is equivalent to solving the field theory, models of generating functionals are easily constructed. For example, consider the representation of the generating functional in terms of connected Euclidean Green functions. If we use the linked cluster theorem to isolate the contribution to the generating functional from the connected n-point Green function (see 2.1),

$$Z_n^c[f] := e^{\frac{(i^n)}{n!} S_n^c(f,\cdots,f)}, \tag{2.11}$$

then the full generating functional is the product

$$Z[f] = \prod_n Z_n^c[f]. \tag{2.12}$$

It follows that the matrix that gives the quantum mechanical scalar product, (2.6), has the decomposition

$$M_{ij} := Z[f_i - \theta f_j] = \prod_n Z_n^c[f_i - \theta f_j] = \prod_n M_{ij}^n \tag{2.13}$$

which is an infinite Schur product of the matrices M_{ij}^n. A sufficient condition for positivity of M_{ij} is that each M_{ij}^n in the Schur product is positive (this is because the Schur product can be expressed as the restriction of the tensor product of positive operators to the diagonal subspace).

Thus, one strategy for constructing models is to use the representation (2.13) to build models starting with the free field two-point function. $Z_2^c[f]$ is reflection positive if it is the generating functional for a free field. Including a reflection positive $Z_4^c[f]$ in the Schur product (2.13) would give a generating functional for an interacting many-body theory with two-body interactions, where the dynamical input is a model connected Euclidean 4-point function,

$$Z_{model}[f] = Z_2^c[f] Z_4^c[f]. \tag{2.14}$$

Reflection positivity of M_{ij}^4 means that

$$Z_4^c[f_i - \theta f_j] = e^{\frac{i^4}{4!} S_4^c(f_i - \theta f_j, f_i - \theta f_j, f_i - \theta f_j, f_i - \theta f_j)} \tag{2.15}$$

is a positive matrix for any finite sequence of positive-time test functions. The model can be extended by including additional factors, $Z_n^c[f]$.

III. RELATIVISTIC INVARIANCE

Poincaré invariance of a quantum theory implies the existence of a unitary representation of the Poincaré group on the physical Hilbert space [5]. Equivalently there should be a set of ten infinitesimal generators of the Poincaré group represented by self-adjoint operators satisfying the Poincaré commutation relations.

The relation between the complex orthogonal group and the complex Lorentz group is relevant for constructing Poincaré generators. To understand the connection between $O(4)$ and the Lorentz group consider the matrices

$$X = \begin{pmatrix} t+z & x-iy \\ x+iy & t-z \end{pmatrix} \qquad \mathsf{X} = \begin{pmatrix} i\tau+z & x-iy \\ x+iy & i\tau-z \end{pmatrix}. \tag{3.1}$$

A simple calculation shows that $\det(X) = t^2 - \mathbf{x}^2$ and $\det(\mathsf{X}) = -(\tau^2 + \mathbf{x}^2)$, which are $(-)$ the Minkowski and Euclidean invariant (distances)^2 respectively. Both determinants are preserved under the linear transformations

$$X \to X' = AXB^t \qquad \mathsf{X} \to \mathsf{X}' = A\mathsf{X}B^t \qquad \det(A) = \det(B) = 1 \tag{3.2}$$

where A and B are complex matrices with unit determinant. In general the pair (A, B) defines a complex Lorentz or complex $O(4)$ transformation. Both (A, B) and $(-A, -B)$ correspond to the same linear transformation of the coordinates. In general, the transformed coordinates may become complex, but the determinant remains real and unchanged. If $B = A^*$ then the transformation $X' = AXB^t$ is a real Lorentz transformation; if $A, B \in SU(2)$ then the transformation $\mathsf{X}' = A\mathsf{X}B^t$ is a real $O(4)$ transformation.

When $A, B \in SU(2)$ the transformation $X' = AXB^t$ is a six parameter subgroup of the complex Lorentz group. While this represents a real Euclidean transformation on the Euclidean Hilbert space (without the Euclidean time reversal Θ), it defines a complex ten-parameter subgroup of the Lorentz group on the physical Hilbert space. It is possible to extract the ten Poincaré generators on the physical Hilbert space by considering the infinitesimal forms of these complex Lorentz transformations.

In order to get a unitary representation of the real Poincaré group on the physical Hilbert space the generators must be self-adjoint. It turns out that the Θ in the physical scalar product breaks Euclidean invariance in just the right way to ensure that all of the Poincaré generators are self-adjoint.

We begin by defining real Euclidean transformations on wave functionals by

$$\left(T(E, \mathsf{a})B\right)(\phi) = \sum_{j=1}^{n_b} b_j e^{i\phi(f_{j,E,a})} \qquad f_{j,E,\mathsf{a}}(\mathsf{x}) := f_j\left(E^{-1}(\mathsf{x} - \mathsf{a})\right). \tag{3.3}$$

These transformations leave the generating functional invariant, $Z[f] = Z[f_{E,\mathsf{a}}]$. In general they will not preserve the positive-time constraint.

Before we use these transformations to construct Poincaré generators on the physical Hilbert space, it is useful to note that the wave functionals can also be considered as multiplication operators. For example the operator $B(\phi)$ acting on the wave functional $C(\phi)$ is the wave functional $D(\phi)$ defined by

$$D(\phi) := B(\phi)C(\phi) = \sum_{j=1}^{n_b}\sum_{k=1}^{n_c} b_j c_k e^{i\phi(f_j + g_k)}. \tag{3.4}$$

These algebraic properties will be used when we formulate the scattering asymptotic conditions.

Next we consider the real Euclidean transformations, $T(E,\mathsf{a})$, as complex Poincaré transformations on the physical Hilbert space. It is useful to treat pure rotations, space translations, Euclidean time translations, and Euclidean rotations in planes containing the time axis separately. The Euclidean time reversal Θ operator does not commute with the last two transformations.

We define action of the Poincaré generators on the wave functionals considered as operators by:

$$[\mathbf{P}, B(\phi)] = -i\frac{\partial}{\partial \mathbf{a}}(T(I,(0,a))BT(I,(0,-a)))(\phi)_{|\mathbf{a}=0} := -i\sum_{j=1}^{n_b} b_j \frac{\partial}{\partial \mathbf{a}} e^{i\phi(f_{j,I,(0,\mathbf{a})})}_{|\mathbf{a}=0}, \tag{3.5}$$

$$[\mathbf{J}\cdot\hat{\mathbf{n}}, B(\phi)] = -i\frac{\partial}{\partial \xi}(T(R(\hat{\mathbf{n}},\xi),0)BT(R(\hat{\mathbf{n}},-\xi),0))(\phi)_{|\xi=0} := -i\sum_{j=1}^{n_b} b_j \frac{\partial}{\partial \xi} e^{i\phi(f_{j,(R(\hat{\mathbf{n}},\xi),0)})}_{|\xi=0} \tag{3.6}$$

where $R(\hat{\mathbf{n}},\xi)$ is an ordinary rotation about the $\hat{\mathbf{n}}$ axis by an angle ξ. For the Hamiltonian we require that $\beta > 0$ in $T(I,(\beta,a))$ to preserve the positive-time support condition:

$$[H, B(\phi)] = -\frac{\partial}{\partial \beta}(T(I,(\beta,0))BT(I,(-\beta,0)))(\phi)_{|\beta=0} := -\sum_{j=1}^{n_b} b_j \frac{\partial}{\partial \beta} e^{i\phi(f_{j,I,(\beta,0)})}_{|\beta=0}. \tag{3.7}$$

For the boost generators we first restrict the support of the test functions f_j in the wave functionals to a cone symmetric about the positive Euclidean time axis that makes an angle $0 \leq \chi < \pi/2$ with the Euclidean time axis

$$\mathcal{S}_{\chi,+} := \{f \in \mathcal{S}_+ \mid f(\tau,\mathbf{x}) = 0, \quad \tan^{-1}(\frac{\tau}{|\mathbf{x}|}) \geq \chi\}. \tag{3.8}$$

This ensures that the support condition is preserved for sufficiently small rotations. On these wave functionals we consider the Euclidean rotation $T(R_e(\hat{\mathbf{n}},\xi),0)$ in the $\hat{\mathbf{n}} - \tau$ plane through angle $\xi < \pi/2 - \chi$:

$$f_{j,\xi,\hat{\mathbf{n}}}(\tau,\mathbf{x}) := f_j(\tau',\mathbf{x}') \qquad f_f \in \mathcal{S}_{\chi,+}, \tag{3.9}$$

with

$$\tau' = \tau\cos(\xi) - x_{\hat{\mathbf{n}}}\sin(\xi) \qquad x'_{\hat{\mathbf{n}}} = x_{\hat{\mathbf{n}}}\cos(\xi) + \tau\sin(\xi). \tag{3.10}$$

The restrictions on the parameters ξ and χ ensure that initial and final vectors are in the physical Hilbert space. On these vectors the rotationless boost generator is defined by

$$[\mathbf{K}\cdot\hat{\mathbf{n}}, B(\phi)] = -\frac{\partial}{\partial \xi}(T(R_e(\hat{\mathbf{n}},\xi),0)BT(R_e(\hat{\mathbf{n}},-\xi),0))(\phi)_{|\xi=0} := -\sum_{j=1}^{n_b} b_j \frac{\partial}{\partial \xi} e^{i\phi(f_{j,(R_e(\hat{\mathbf{n}},\xi),0)})}_{|\xi=0} \tag{3.11}$$

where $R_e(\hat{\mathbf{n}},\xi)$ is the Euclidean space-time rotation (3.10). Note the absence of the i in the expressions for H and $\mathbf{K}$. This is compensated for by the Θ that appears in the physical scalar product.

Direct calculations show that the ten operators $H, \mathbf{P}, \mathbf{J}, \mathbf{K}$ satisfy the Poincaré commutation relations and are formally Hermitian on the physical Hilbert space. Self-adjointness of $H, \mathbf{P}, \mathbf{J}$ follows because these operators are generators of either one parameter unitary groups or a contractive Hermitian semigroup. The contractive nature of time Euclidean time evolution is proved using reflection positivity and positivity properties of the generating functional[6]. This also ensures that Hamiltonian satisfies the spectral condition:

$$H \geq 0. \tag{3.12}$$

Self-adjointness of $\mathbf{K}$ can be established by verifying that rotations in Euclidean space-time planes define local symmetric semigroups on the model Hilbert space [7][8][9].

Matrix elements of the Poincaré generators in normalizable states can be expressed directly in terms of the generating functional

$$\langle B|H|C\rangle = -\frac{\partial}{\partial \beta}\left(\sum_{j=1}^{N_b}\sum_{k=1}^{N_c} b_j^* c_k Z[g_{k,I,(\beta,0)} - \Theta f_j]\right)_{\beta=0}, \tag{3.13}$$

$$\langle B|\mathbf{P}|C\rangle = -i\frac{\partial}{\partial \mathbf{a}}\left(\sum_{j=1}^{n_b}\sum_{k=1}^{n_c} b_j^* c_k Z[g_{k,I,(0,\mathbf{a})} - \Theta f_j]\right)_{\mathbf{a}=0}, \tag{3.14}$$

$$\langle B|\hat{\mathbf{n}}\cdot\mathbf{J}|C\rangle = -i\frac{\partial}{\partial \xi}\left(\sum_{j=1}^{b_b}\sum_{k=1}^{n_c} c_j^* d_k Z[g_{k,R(\hat{\mathbf{n}},\xi),0} - \Theta f_j]\right)_{\xi=0}, \tag{3.15}$$

$$\langle B|\hat{\mathbf{n}}\cdot\mathbf{K}|C\rangle = -\frac{\partial}{\partial \xi}\left(\sum_{j=1}^{b_b}\sum_{k=1}^{n_c} c_j^* d_k Z[g_{k,R_e(\hat{\mathbf{n}},\xi),0} - \Theta f_j]\right)_{\xi=0}. \tag{3.16}$$

The formal Hermiticity of the generators defined above can be deduced from these expressions. For example

$$\langle B|H^\dagger|C\rangle = \langle C|H|B\rangle^* =$$

$$-\frac{\partial}{\partial \beta}\left(\sum_{j=1}^{N_b}\sum_{k=1}^{N_c} b_j^* c_k Z[-f_{k,I,(\beta,0)} + \Theta g_j]\right)_{\beta=0} =$$

$$-\frac{\partial}{\partial \beta}\left(\sum_{j=1}^{N_b}\sum_{k=1}^{N_c} b_j^* c_k Z[-f_k + \Theta g_{j,I,(\beta,0)}]\right)_{\beta=0} =$$

$$-\frac{\partial}{\partial \beta}\left(\sum_{j=1}^{N_b}\sum_{k=1}^{N_c} b_j^* c_k Z[-\Theta f_k + g_{j,I,(\beta,0)}]\right)_{\beta=0} =$$

$$\left(\sum_{j=1}^{N_b}\sum_{k=1}^{N_c} b_j^* c_k Z[\Theta g_{j,I,(\beta,0)} - f_k]\right)_{\beta=0} = \langle B|H|C\rangle \tag{3.17}$$

where we have used reality, $Z^*[f] = Z[-f]$, Euclidean invariance, and properties of Θ. Hermiticity of the rotationless boost generators follows using the same argument. The Euclidean time-reversal operator, Θ, plays the role of the missing factor of i when integrating by parts.

The commutation relations can be verified by explicit computation, however they also follow as a direct consequence of the relation between complex $O(4)$ and the complex Lorentz group.

Matrix elements of $e^{-\beta H}$ can also be directly computed in terms of the generating functional:

$$\langle B|e^{-\beta H}|C\rangle = \sum_{j=1}^{N_b}\sum_{k=1}^{N_c} b_j^* c_k Z[g_{k,I(\beta,0)} - \Theta f_j]. \tag{3.18}$$

These matrix elements will be used to compute scattering cross sections. They only involve elementary quadratures.

Matrix elements of the mass Casimir operator can be expressed in terms of the Poincaré generators

$$M^2 = H^2 - \mathbf{P}^2 \tag{3.19}$$

$$\langle B|M^2|C\rangle := \left(\frac{\partial^2}{\partial \beta^2} + \frac{\partial^2}{\partial \mathbf{a}^2}\right)\sum_{j=1}^{N_b}\sum_{k=1}^{N_c} b_j^* c_k Z[g_{k,I(\beta,\mathbf{a})} - \Theta f_j]_{|\beta=\mathbf{a}=0}. \tag{3.20}$$

Finally we note that the real Euclidean transformations, $T(E,\mathsf{a})$ can be formally expressed in terms of the Poincaré generators on the physical Hilbert space by

$$T(I,(\beta,\mathbf{a})) = e^{-\beta H + i\mathbf{a}\cdot\mathbf{P}} \qquad T(R(\hat{\mathbf{n}},\psi),0) = e^{i\mathbf{J}\cdot\hat{\mathbf{n}}\psi} \qquad T(R_e(\hat{\mathbf{n}},\psi),0) = e^{\mathbf{K}\cdot\hat{\mathbf{n}}\psi}. \tag{3.21}$$

Thus Euclidean time evolution and rotations in Euclidean space-time planes look like imaginary time evolution and Lorentz transformation with imaginary rapidities.

The operators defined in (3.5,3.6,3.7 and 3.11) are self-adjoint operators on the physical Hilbert space that satisfy the Poincaré commutation relations. Formally they can be exponentiated to give a unitary representation of the Poincaré group on the physical Hilbert space, but, as we will see in the next two sections, this exponentiation is never needed.

The expressions for the matrix elements of all of the Poincaré generators in normalizable states (3.13-3.16) are directly expressed in terms of the Euclidean generating functional and Euclidean test functions. Analytic continuation is not used.

IV. PARTICLES

Given the dense set of wave functionals of the form (2.7) and the physical scalar product (2.9) the Gram-Schmidt method can be formally used to construct a complete orthonormal set of wave functionals $B_n(\phi)$:

$$\langle B_n|B_m\rangle = \delta_{mn}. \tag{4.1}$$

Since the orthonormal wave functionals are complete, normalizable one-particle states are linear combinations of these orthonormal wave functionals with square summable coefficients:

$$\Psi_\lambda(\phi) = \sum_n b_n B_n(\phi) \qquad \sum_n |b_n|^2 < \infty \tag{4.2}$$

that are eigenstates of the mass square Casimir operator (3.19) of the Poincaré group with eigenvalue λ^2 in the point spectrum:

$$\sum_n \langle B_m|M^2|B_n\rangle b_n = \sum_n \langle B_m|(H^2-\mathbf{P}^2)|B_n\rangle b_n = \lambda^2\delta_{mn}b_n. \tag{4.3}$$

These normalizable states are infinitely degenerate because there is an associated wave packet in the particle's momentum and spin. For suitable wave packets these normalizable eigenstates can be decomposed into simultaneous eigenstates of mass and linear momentum using translations and Fourier transforms:

$$\Psi_{\lambda,\mathbf{p}}(\phi) = \int \frac{d\mathbf{a}}{(2\pi)^{3/2}} e^{-i\mathbf{p}\cdot\mathbf{a}} \left(T(0,\mathbf{a})\Psi_\lambda\right)(\phi). \tag{4.4}$$

These wave functionals can be given a plane-wave normalization

$$\langle\Psi_{\lambda,\mathbf{p}'}|\Psi_{\lambda,\mathbf{p}}\rangle = \delta(\mathbf{p}'-\mathbf{p}). \tag{4.5}$$

The simultaneous eigenstates of mass and linear momentum can be further decomposed into eigenstates of spin, and z-component of spin using

$$\Psi_{\lambda,j,\mathbf{p},\mu}(\phi) = \sum_{\nu=-j}^{j} \int_{S(U2)} dR \left(T(R,0)\Psi_{\lambda,R^{-1}\mathbf{p}}\right)(\phi) D^{j*}_{\mu\nu}(R) \tag{4.6}$$

where the integral is over $SU(2)$ and dR is the $SU(2)$ Haar measure. This projection gives the canonical spin. This is the spin measured in the rest frame when the particle is transformed to the rest frame by a rotationless Lorentz transformation. Different projections can be used to get states of different helicities or light-front spins. The integral in equation (4.6) vanishes if there are no states of mass λ and spin j.

The normalization of the states can be chosen so

$$\langle\Psi_{\lambda,j',\mathbf{p}',\mu'}|\Psi_{\lambda,j,\mathbf{p},\mu}\rangle = \delta(\mathbf{p}'-\mathbf{p})\delta_{j'j}\delta_{\mu'\mu}. \tag{4.7}$$

The state $|\Psi_{\lambda,j,\mathbf{p},\mu}\rangle$ is a single-particle state if λ is in the discrete spectrum of M.

Since we started from a linear combination of wave functionals, the single-particle state is formally represented by a single-particle wave functional

$$\Psi_{\lambda,j,\mathbf{p},\mu}(\phi). \tag{4.8}$$

In general it is not trivial to compute finite Poincaré transformations in terms of the generators, however if the one-particle state is a non-degenerate state (i.e. the theory has no other particles with the same mass and spin) then this state necessarily transforms irreducibly with respect to the dynamical unitary representation of the Poincaré group introduced in the previous section. It follows that

$$\langle B|U(\Lambda,a)|\Psi_{\lambda,j,\mathbf{p},\mu}\rangle =$$

$$\langle B|\psi_{\lambda,\mathbf{p}',\mu'}\rangle\sqrt{\frac{\omega_\lambda(\mathbf{p}')}{\omega_\lambda(\mathbf{p})}}e^{-i\omega_\lambda(\mathbf{p}')a^0+i\mathbf{p}'\cdot\mathbf{a}}D^j_{\mu'\mu}[\Lambda_c^{-1}(\frac{\mathbf{p}'}{\lambda})\Lambda\Lambda_c(\frac{\mathbf{p}}{\lambda})] \tag{4.9}$$

where

$$(\mathbf{p}')^j = \Lambda^j{}_0\omega_\lambda(\mathbf{p}) + \Lambda^j{}_k\mathbf{p}^k \qquad \omega_\lambda(\mathbf{p}) = \sqrt{\lambda^2+\mathbf{p}^2} \tag{4.10}$$

and $\Lambda_c(\frac{\mathbf{p}}{\lambda})$ is a rotationless Lorentz boost that transforms to a frame where a particle of mass λ is at rest to one where it has linear momentum $\mathbf{p}$.

Normalizable single-particle states have the form

$$\Psi_{\lambda,j,f}(\phi) = \int d\mathbf{p}\sum_{\mu=-j}^{j} f(\mu,\mathbf{p})\Psi_{\lambda,j',\mathbf{p}',\mu'}(\phi) \tag{4.11}$$

$$\langle\Psi_{\lambda,j,g}|\Psi_{\lambda,j,f}\rangle = \int d\mathbf{p}\sum_{\mu=-j}^{j} g^*(\mu,\mathbf{p})f(\mu,\mathbf{p}) \tag{4.12}$$

where the integral is over the 3-momentum, $\mathbf{p}$. In this formalism there is no distinction between elementary particles and bound states. They describe particles because they have discrete mass eigenvalues.

What is interesting is that it is possible to construct the Hilbert space, Poincaré generators, find single-particle states and perform finite Poincaré transformations on single-particle states using only the Euclidean generating functional, and positive-time test functions without performing any analytic continuation.

V. SCATTERING THEORY

In a quantum theory scattering states are solutions of the Schrödinger equation that evolve into asymptotically separated non-interacting single-particle states or bound states. In quantum field theory there is no free dynamics on the physical Hilbert space, so with the exception of one-particle states, there are no states of non-interacting particles on the physical Hilbert space. However, because of cluster properties, there are states that look like states of asymptotically separated particles. These states evolve like systems of free particles until the particles get close enough to interact.

One natural framework to formulate scattering asymptotic conditions that is applicable in both quantum mechanics and quantum field theory is the two-Hilbert space formulation of scattering [10]. In this framework a separate many-particle Hilbert space of non-interacting particles is introduced. This space is used to label the states of the asymptotically stable particles. There is a mapping from this asymptotic space to the physical Hilbert space that adds the correct description of the internal structure of the particles on the physical Hilbert space when the particles are asymptotically separated.

In the asymptotic Hilbert space composite particles are treated like elementary particles with a given mass, spin, and momentum distribution. The internal structure of the composite particle is contained in the mapping to the physical Hilbert space. In field theories all particles have internal structure due to their self-interactions. In the non-relativistic case the mapping from the asymptotic Hilbert space to the physical Hilbert space has the form

$$\prod_i |(\lambda_i,j_i)\mathbf{p}_i,\mu_i\rangle \tag{5.1}$$

where the product is a symmetrized tensor product of possibly composite particle states with a given momentum and spin. Normalizable states in the physical Hilbert space are obtained when this mapping is integrated over square integrable functions of the momenta and magnetic quantum numbers of each asymptotically stable particle. In this way composite particles are treated as elementary particles with mass λ_i and spin j_i in the asymptotic Hilbert space, while the mapping adds the internal structure of the asymptotically separated bound states in the physical Hilbert space.

In quantum field theory there are two standard approaches to scattering. The most practical is the LSZ treatment of scattering, which formulates the scattering asymptotic conditions using interpolating fields that create states from the vacuum that have the same quantum numbers as single-particle states. It has the advantage that the asymptotic conditions can be formulated without solving the one-body problem. The price for this advantage is that weak limits must be used to calculate scattering matrix elements. The second approach is Haag-Ruelle scattering [11][12] which uses non-local fields that create only one-particle states from the vacuum. Haag-Ruelle scattering is the natural generalization of standard quantum mechanical scattering in the field theory setting and it has a natural two-Hilbert space formulation[13][14]. Haag-Ruelle scattering states are defined by strong limits, just like in the non-relativistic case. Haag-Ruelle scattering is not commonly used in applications because it requires the solution of the one-body problem on the physical Hilbert space as input. In this work the one-body solutions discussed in the previous section are used to formulate the Haag-Ruelle asymptotic conditions.

We begin with a summary of the two-Hilbert space formulation of Haag-Ruelle scattering in Minkowski field theory. For simplicity we consider a scalar field theory with a single-particle state of mass λ. To construct the mapping from the asymptotic Hilbert space to the physical Hilbert space the Fourier transform of the field, $\tilde{\phi}(p)$, is multiplied by a smooth function $\rho(p^2)$ of the square of the Minkowski four momentum that is one when $p^2 = -\lambda^2$ (the mass of the asymptotic particle) and identically vanishes when $-p^2$ is in the rest of spectrum of M^2.

The product of the Fourier transform of the field and the smooth invariant function, $\tilde{\phi}_\rho(p) := \rho(p^2)\tilde{\phi}(p)$, is then Fourier transformed back to configuration space. The resulting field $\phi_\rho(x)$ is covariant, but no longer local. It has the property that when it is applied to the physical vacuum it creates only the single-particle eigenstate of the mass operator with mass λ. Because of the multiplication by $\rho(p^2)$, $\phi_\rho(x)$ is a well-behaved operator-valued function of time when it is smeared over a test function in three space variables.

The part of $\phi_\rho(x)$ that asymptotically looks like a creation operator is extracted by taking the linear combination of ϕ and $\dot{\phi}$ below:

$$A(f,t) := -i \int \phi_\rho(x) \overleftrightarrow{\partial_0} f(x) d\mathbf{x} \tag{5.2}$$

where $f(x)$ is a smooth, mass λ, positive-energy solution of the Klein-Gordon equation. Here smooth means that the Fourier transform of the $t = 0$ solution is a smooth function with compact support in the three momentum. Haag and Ruelle show that the N-particle scattering states in the physical Hilbert space exist and are given by the strong limits

$$|\Psi_\pm(f_1, \cdots f_N)\rangle = \lim_{t\to\pm\infty} A(f_N, t) \cdots A(f_1, t)|0\rangle. \tag{5.3}$$

Next we express the limit (5.3) in a two Hilbert space framework that can also be used in the wave functional representation of the physical Hilbert space. First we write $A(f, t)$ defined in (5.2) by expressing the time derivative of the field using the commutator with the Hamiltonian and the time derivative of the Klein-Gordon solution by an energy factor:

$$A(f,t)|0\rangle =$$

$$-\frac{i}{(2\pi)^{3/2}} \int d\mathbf{x}\, d\mathbf{p} e^{iHt} \left(i[H, \phi_\rho(0, \mathbf{x})] - i\omega_\lambda(\mathbf{p})\phi_\rho(0, \mathbf{x})\right) e^{-iHt}|0\rangle e^{-i\omega_\lambda(\mathbf{p})t + i\mathbf{p}\cdot\mathbf{x}} \tilde{f}(\mathbf{p}) \tag{5.4}$$

where $\tilde{f}(\mathbf{p})$ is a test function in the three momentum. Integrating over $\mathbf{x}$ gives a partial Fourier transform of the field so (5.2) becomes

$$A(f,t)|0\rangle = e^{iHt} \int \left([H, \phi_\rho(0, \mathbf{p})] - \omega_\lambda(\mathbf{p})\phi_\rho(0, \mathbf{p})\right)|0\rangle \tilde{f}(\mathbf{p}) e^{-i\omega_\lambda(\mathbf{p})t} d\mathbf{p}. \tag{5.5}$$

The time-dependence is in the quantities e^{iHt} and $e^{-i\omega_\lambda(\mathbf{p})t}$, where the second factor gives the time dependence of the positive-energy solution the Klein-Gordan equation. This is expressed as an operator that acts on the wave packet $\tilde{f}(\mathbf{p})$ of a free particle of mass λ. It follows that (5.2) can be interpreted as a mapping from a dense subset of the Hilbert space of square integrable functions, $f(\mathbf{p})$, to the physical Hilbert space

$$A(f,t)|0\rangle = e^{iHt} A_1 e^{-iH_0 t}|f\rangle \tag{5.6}$$

where $H_0 = \omega_\lambda(\mathbf{p})$ is the energy of the asymptotic particle and

$$A_1(\mathbf{p}) := ([H, \phi_\rho(0,\mathbf{p})] - \omega_\lambda(\mathbf{p})\phi_\rho(0,\mathbf{p}))\,. \tag{5.7}$$

By repeating this analysis N-times the products that appear in the Haag-Ruelle formula,

$$\prod A(f_1,t)\cdots A(f_N,t)|0\rangle, \tag{5.8}$$

can be expressed as mappings from an N-particle subspace of the Fock space of non-interacting particles of mass λ to the physical Hilbert space:

$$\prod A(f_1,t)\cdots A(f_N,t)|0\rangle = e^{iHt}\int A_N(\mathbf{p}_1,\cdots,\mathbf{p}_N)e^{-i(\sum\omega_\lambda(\mathbf{p}_i))t}f_1(\mathbf{p}_1)\cdots f_N(\mathbf{p}_N)d\mathbf{p}_1\cdots d\mathbf{p}_N = \tag{5.9}$$

$$e^{iHt}\int A_N(\mathbf{p}_1,\cdots,\mathbf{p}_N)e^{-iH_0 t}f_1(\mathbf{p}_1)\cdots f_N(\mathbf{p}_N)d\mathbf{p}_1\cdots d\mathbf{p}_N \tag{5.10}$$

where

$$A_N(\mathbf{p}_1,\cdots,\mathbf{p}_N) := \prod A_1(\mathbf{p_i})|0\rangle. \tag{5.11}$$

Equation (5.9) has the form

$$e^{iHt}A_N e^{-iH_0 t}|\mathbf{f}\rangle \tag{5.12}$$

where $H_0 = \sum\omega_\lambda(\mathbf{p}_i)$ is the Hamiltonian for N non-interacting particles of mass λ.

In this notation the Haag-Ruelle theorem, (5.3), has the two-Hilbert space form:

$$\lim_{t\to\pm\infty}\||\Psi_\pm(\mathbf{f})\rangle - e^{iHt}A_N e^{-iH_0 t}|\mathbf{f}\rangle\| = 0. \tag{5.13}$$

Following what is done in standard quantum mechanical multichannel scattering theory, wave operators are defined by

$$\Omega_{N\pm}|\mathbf{f}\rangle := \lim_{t\to\pm\infty} e^{iHt}A_N e^{-iH_0 t}|\mathbf{f}\rangle = |\Psi_\pm(\mathbf{f})\rangle. \tag{5.14}$$

In the field theory case [12], these wave operators satisfy relativistic intertwining relations

$$U(\Lambda,a)\Omega_{N\pm} = \Omega_{N\pm}\left(\otimes U_i(\Lambda,a)\right) \tag{5.15}$$

that relate the dynamical representation of the Poincaré group with the tensor product of n single-particle irreducible representations on the n-particle sector of the asymptotic Fock space.

The scattering states can be expressed using the representation of the physical Hilbert space in terms of wave functionals in section 2. The relevant observation is that

$$\tilde{\phi}_\rho(p)|0\rangle \tag{5.16}$$

is a single-particle state of linear momentum $\mathbf{p}$ and mass λ. The wave functionals (2.7) of section 2 are vectors in the physical Hilbert space, even though they are expressed in terms of Euclidean test functions and the Euclidean generating functional. The wave functional,

$$\Psi_{\lambda,j',\mathbf{p}',\mu'}(\phi), \tag{5.17}$$

defined in the previous section, creates a single-particle state of linear momentum $\mathbf{p}$ and mass λ.

Thus, if we make the replacements

$$\tilde{\phi}_\rho(p) \to \Psi_{\lambda,j',\mathbf{p}',\mu'}(\phi), \tag{5.18}$$

in the two-Hilbert space Haag-Ruelle injection operator, A_N (5.11), then it becomes the wave functional

$$A_N(\mathbf{p}_1,\mu_1,\cdots\mathbf{p}_N,\mu_N)(\phi) := \prod\left([H,\Psi_{\lambda_i,j_i',\mathbf{p}_i',\mu_i'}] - \omega_{\lambda_i}(\mathbf{p}_i)\Psi_{\lambda_i,j_i',\mathbf{p}_i',\mu_i'}\right)(\phi) \tag{5.19}$$

where this functional allows for the possibility of scattering of composite particles with arbitrary spin.

The proof of the Haag-Ruelle theorem does not apply to models; however for models the existence of channel wave operators can be established directly by a generalization of Cook's method[19], which gives the following sufficient condition for the existence of N-particle wave operators:

$$\int_0^\infty \|(HA_N - A_N H_0)e^{\pm iH_0 t}|\mathbf{f}\rangle\| < \infty \tag{5.20}$$

while Lorentz invariance, (5.19), follows if

$$\lim_{t\to\infty} \|(\mathbf{K}A_N - A_N \mathbf{K}_0)e^{\pm iH_0 t}|\mathbf{f}\rangle\| = 0. \tag{5.21}$$

Expression (5.19) leads to the following formal expression for S-matrix elements between normalizable states:

$$S_{fi} = \langle\Psi_+|\Psi_-\rangle =$$

$$\lim_{t\to\infty}\int f_1^*(\mathbf{p}_1,\mu_1)\cdots f_M^*(\mathbf{p}_M,\mu_M)e^{i\sum\omega_{\lambda_i}(\mathbf{p}_i)t}\times$$

$$\langle A_M^\dagger(\mathbf{p}_1,\mu_1,\cdots,\mathbf{p}_M,\mu_M)|e^{-2iHt}|A_N(\mathbf{p}_1',\mu_1',\cdots,\mathbf{p}_N',\mu_N')\rangle\times$$

$$e^{i\sum\omega_{\lambda_i}(\mathbf{p}_i')t}f_1'(\mathbf{p}_1',\mu_1')\cdots f_N'(\mathbf{p}_N',\mu_N')\prod_{ij}d\mathbf{p}_i d\mathbf{p}_j' \tag{5.22}$$

where the scalar product is expressed in terms of the Euclidean generating functional. This expression has the form

$$S_{fi} = \lim_{t\to\infty}\langle B(t)|e^{-2iHt}|C(t)\rangle. \tag{5.23}$$

While eq. (5.23) involves operators that are defined in the wave functional representation, the real time evolution operator, e^{-2iHt}, is difficult to calculate in this representation.

Fortunately this quantity can be replaced by a more easily computable quantity using the Kato-Birman invariance principle [15] [16][13][14] which identifies the limits

$$\Omega_\pm|\mathbf{f}\rangle := \lim_{t\to\pm\infty} e^{iHt}A_N e^{-iH_0 t}|\mathbf{f}\rangle = \lim_{t\to\pm\infty} e^{ig(H)t}A_N e^{-ig(H_0)t}|\mathbf{f}\rangle \tag{5.24}$$

for $g(x)$ in a suitable class of admissible functions, provided both limits exist. The content of this result is that in the large-time limit the surviving terms correspond to situations where both exponents oscillate in phase, which requires that both the dynamical and asymptotic energies are the same. Replacing H and H_0 by functions of H resp. H_0 does not change this result provided the function is increasing, with suitable smoothness. A useful choice for $g(x)$ that is in the class of admissible functions is

$$g(x) = -e^{-\beta x} \qquad \beta > 0. \tag{5.25}$$

For this choice the expressions for the wave operator becomes

$$\Omega_\pm|\mathbf{f}\rangle := \lim_{n\to\pm\infty} e^{-ine^{-\beta H}}A_N e^{ine^{-\beta H_0}}|\mathbf{f}\rangle \tag{5.26}$$

where the time parameter t has been replaced by a dimensionless integer n.

This means that the expression (5.23) for the S matrix elements can be replaced by

$$S_{fi} = \langle\Psi_+|\Psi_-\rangle = \lim_{n\to\infty}\int f_1^*(\mathbf{p}_1,\mu_1)\cdots f_k^*(\mathbf{p}_M,\mu_M)e^{-ine^{-\beta(\sum\omega_{\lambda_i}(\mathbf{p}_i)}}\times$$

$$\times\langle A_M^\dagger(\mathbf{p}_1,\mu_1,\cdots,\mathbf{p}_M,\mu_M)|e^{2ine^{-\beta H}}|A_N(\mathbf{p}_1',\mu_1',\cdots,\mathbf{p}_N',\mu_N')\rangle$$

$$\times e^{-ine^{i\sum\omega_{\lambda_i}(\mathbf{p}_i')}}f_1'(\mathbf{p}_1',\mu_1')\cdots f_M'(\mathbf{p}_M',\mu_M')\prod_{ij}d\mathbf{p}_i d\mathbf{p}_j'. \tag{5.27}$$

The virtue of this expression is that for large *fixed* n, e^{2inx} can be *uniformly* approximated by a polynomial for $x \in [0,1]$

$$|e^{2inx} - P_{n,\epsilon}(x)| < \epsilon \quad \forall x \in [0,1]. \tag{5.28}$$

Because the spectrum of $e^{-\beta H}$ is in the interval $[0,1]$ and the approximation is uniform the operator satisfies the same inequality

$$|||e^{2ine^{-\beta H}} - P_{n,\epsilon}(e^{-\beta H})||| < \epsilon \tag{5.29}$$

where the norm on the left is the uniform or operator norm and $P_{n,\epsilon}(x)$ and ϵ are the polynomial and error that appear in equation (5.28).

Using the polynomial approximation in (5.27) gives

$$S_{fi} = \lim_{n\to\infty} \sum_m d_m(n)\langle B(n)|e^{-\beta m H}|C(n)\rangle. \tag{5.30}$$

where $d_m(n)$ are the coefficients of the polynomial in (5.29). This is useful because matrix elements of powers of $e^{-m\beta H}$ between wave functionals $B(n)[\phi]$ and $C(n)[\phi]$ can be expressed directly in terms of the generating functional using (3.18).

While time-dependent methods are not traditionally used in scattering calculations, they have been used successfully in non-relativistic few-body calculations [17]. The advantage of the above formalism is that the entire calculation can be performed using only Euclidean methods.

In order to calculate sharp-momentum transition matrix elements it is necessary to use narrow wave packets. If the transition matrix is sufficiently smooth as a function of momentum, it will factor out of the S-matrix element, allowing one to define scattering observables that do not depend on the details of the wave packet. This is an assumption in the standard formulation relating time-dependent and time-independent scattering [18]. For sharp initial and final wave packets the on-shell transition matrix elements can be approximated by:

$$\langle \mathbf{p}_1, \mu_1, \cdots \mathbf{p}_N, \mu_N|T|\mathbf{p}_1, \mu_1, \mathbf{p}_2, \mu_2\rangle$$

$$\approx \frac{\langle B|S|C\rangle - \langle B|C\rangle}{-2\pi i\langle B|\delta^4(p_f - p_i)|C\rangle}. \tag{5.31}$$

After the wave packets are fixed the limit $n \to \infty$ in (5.27) can be investigated. For a large enough n the term in the limit, (5.27), which has the form

$$\langle B(n)|e^{-2ine^{-\beta H}}|C(n)\rangle, \tag{5.32}$$

will be a good approximation to $\langle B|S|C\rangle$ in (5.31) .

For this value of n, $e^{-2ine^{-\beta H}}$ can then be uniformly approximated by a polynomial in $e^{-\beta H}$ which can be evaluated using Euclidean methods.

$$\langle B(n)|e^{-2ine^{-\beta H}}|C(n)\rangle \approx \sum_m d_m(n)\langle B(n)|e^{-m\beta H}|C(n)\rangle. \tag{5.33}$$

Combining these three approximations gives an approximation to sharp-momentum transition matrix elements using matrix elements of $e^{-n\beta H}$ in normalizable states as input.

Once the scattering states are known their Poincaré transformation properties are determined by equation (5.15) and the transformation properties (4.9) of the single-particle states.

It is useful summarize the steps needed to calculate transition-matrix elements.

1. Solve the one-body problem. These are eigenstates of the mass-square operator with discrete eigenvalues: $\Psi(\phi)$.
2. Use translational and rotational covariance to construct $\Psi_{\lambda,j,\mathbf{p},\mu}(\phi)$.
3. Choose a sufficiently narrow set of single asymptotic-particle wave packets $f(\mathbf{p}_i, \mu_i)$. The width must be sufficiently narrow to factor the transition matrix elements out of the S-matrix elements.

4. Use the one-body solutions to construct the two-Hilbert space mappings

$$A_n(\mathbf{f}) := \prod_i \int \left([H, \Psi_{\lambda,j,\mathbf{p}_i,\mu_i}] - \omega_\lambda(\mathbf{p}_i)\Psi_{\lambda,j,\mathbf{p}_i,\mu_i}\right) f(\mathbf{p}_i,\mu_i) d\mathbf{p}_i e^{ine^{-\beta(\sum \omega_\lambda(\mathbf{p}_i)}}. \tag{5.34}$$

5. Pick a large enough n.

6. Make a polynomial approximation to e^{2inx} for $x \in [0,1]$

$$e^{2inx} \approx P_{2n,\epsilon}(x). \tag{5.35}$$

7. Calculate

$$S_{fi} = \langle A_n(\mathbf{f}')|P_{2n,\epsilon}(e^{-\beta H})|A_n(\mathbf{f}')\rangle \tag{5.36}$$

8. Approximate $\langle \mathbf{p}_1, \mu_1, \cdots \mathbf{p}_m, \mu_m|T|\mathbf{p}_1, \mu_1, \mathbf{p}_2, \mu_2\rangle$ using (5.31).

The result can be expressed directly in terms of the Euclidean generating functional using (3.18).

The discussion above assumes that the starting point is an Euclidean generating functional. For models the requirement that all of these approximations converge are model assumptions that restrict properties of the model generating functional or Green functions. These are reasonable requirements, since the model Green functions are modeled after the field theoretic Green functions, which are expected to have these properties.

The usual difficulties of realizing the Poincaré symmetry are replaced by the requirement of finding reflection positive Euclidean invariant Green functions or generating functionals. It is interesting that if the Green function is given perturbatively, the perturbative Green function defines a different model. The scattering matrix constructed in this model will not be perturbative, and may even be unitary.

For the simple kinds of two-body models discussed in section VII it is easy to see that the generalized Cook condition (5.20) translates into regularity properties of the connected four-point function. This is because $\|(HA_2 - A_2H_0)e^{-iH_0t}|\mathbf{f}\rangle\|$ is linear in the four-point Euclidean Green function and vanishes when the Green function is replaced by the free four-point Euclidean Green function. What controls the convergence of the integral in (5.20) is the difference between the full and free four-point Euclidean Green functions, which is the connected Euclidean four-point function.

Finally we note that even though the calculation of the scattering observables are based on Euclidean quantities, the mechanism that leads to the convergence of the wave operators is in-phase oscillations, not an exponential fall-off. The scattering states involve strong limits and replace interpolating operators by operators that create single-particle states out of the vacuum. The factor β that appears in the application of the invariance principle serves as a parameter the sets the energy scale of the reaction.

As a result, these calculations are not subject to some of the difficulties encountered in scattering calculations based on a Euclidean lattice discretization[20], however it is necessary to be able to accurately compute matrix elements of $e^{-\beta H}$, which cannot be done easily on a lattice.

VI. GREEN FUNCTION REPRESENTATION

While the Euclidean generating functionals provide a concise and elegant description of the theory as well as a consistent treatment of the few and many-body problems, the direct Green function approach of Osterwalder and Schrader may be more appropriate for constructing phenomenological few-body models.

In the Green function approach the representation of the physical Hilbert space in terms of Euclidean wave functionals is replaced by sequences of positive-time support functions of the form:

$$B(\phi) \to \langle \mathsf{x}|\mathbf{f}\rangle := \begin{pmatrix} f_0 \\ f_1(\mathsf{x}_{11}) \\ f_2(\mathsf{x}_{21}, \mathsf{x}_{22}) \\ \vdots \end{pmatrix} \tag{6.1}$$

with an inner product that is expressed in terms of multi-point Euclidean Green functions.

$$\langle \mathbf{g}|\mathbf{f}\rangle := \sum_{mn} \int d\mathsf{x}_1 \cdots d\mathsf{x}_m d\mathsf{y}_n \cdots d\mathsf{y}_1 g_m^*(\Theta\mathsf{x}_1 \cdots \Theta\mathsf{x}_m) S_{m+n}(\mathsf{x}_1, \cdots, \mathsf{x}_m, \mathsf{y}_n \cdots, \mathsf{y}_1) f_n(\mathsf{y}_1, \cdots, \mathsf{y}_n). \tag{6.2}$$

In the Green function representation the support of $f_n(\mathsf{y}_1, \cdots, \mathsf{y}_n)$ is for $0 < \mathsf{y}_1^0 < \mathsf{y}_2^0 < \cdots$. Note that the order of the support of the Euclidean times is identical to the order of the fields in the corresponding Minkowski Wightman function. Reflection positivity is the condition

$$\langle \mathbf{f} | \mathbf{f} \rangle \geq 0 \tag{6.3}$$

and $\langle \mathbf{g} | \mathbf{f} \rangle$ is the physical quantum mechanical scalar product.

The relation between the Euclidean and Minkowski scalar products is illustrated for the case of a free Euclidean two-point function, one for spin zero and one for spin 1/2:

$$\langle \mathbf{f} | \mathbf{f} \rangle = \int f(x) S_2(\Theta x, y) f(y) d^4x d^4y =$$

$$= \frac{1}{(2\pi)^4} \int d^4\mathsf{x} d^4\mathsf{y} d^4\mathsf{p} f(\mathsf{x}) \frac{e^{i\mathsf{p}\cdot(\theta\mathsf{x}-\mathsf{y})}}{\mathsf{p}^2 + m^2} f(\mathsf{y})$$

$$= \frac{1}{(2\pi)^4} \int d^4\mathsf{x} d^4\mathsf{y} d^4\mathsf{p} f(\mathsf{x}) \frac{e^{-i\mathsf{p}_0\cdot(\mathsf{x}_0+\mathsf{y}_0)+i\vec{\mathsf{p}}\cdot(\vec{\mathsf{x}}-\vec{\mathsf{y}})}}{(\mathsf{p}^0 + i\omega_m(\vec{\mathsf{p}}))(\mathsf{p}^0 - i\omega_m(\vec{\mathsf{p}}))} f(\mathsf{y})$$

$$= \int d^3\mathsf{p} \frac{|g(\vec{\mathsf{p}})|^2}{2\omega_m(\vec{\mathsf{p}})} \geq 0 \tag{6.4}$$

where

$$g(\vec{\mathsf{p}}) := \frac{1}{(2\pi)^{3/2}} \int d^4y f(\mathsf{y}) e^{-\omega_m(\vec{\mathsf{p}})y_0 - i\vec{\mathsf{p}}\cdot\vec{\mathsf{y}}}. \tag{6.5}$$

For spin 1/2:

$$\langle \mathbf{f} | \mathbf{f} \rangle = \int f(x) \gamma^0 S_2(\Theta x, y) f(y) d^4x d^4y =$$

$$= \frac{1}{(2\pi)^4} \int d^4\mathsf{x} d^4\mathsf{y} d^4\mathsf{p} f(\mathsf{x})\, e^{i\mathsf{p}\cdot(\theta\mathsf{x}-\mathsf{y})} \gamma^0 \frac{m - p\cdot\gamma_e}{\mathsf{p}^2 + m^2} f(\mathsf{y})$$

$$= \int g^\dagger(\vec{\mathsf{p}}) \frac{\Lambda_+(p)}{(2\pi)^3} g(\vec{\mathsf{p}}) d^3\mathsf{p} \tag{6.6}$$

where

$$\Lambda_+(p) := \frac{\omega_m(\vec{p}) + \gamma^0\vec{\gamma}\cdot\vec{p} - m\gamma^0}{2\omega_m(\vec{p})}. \tag{6.7}$$

We see clearly that in both case the Euclidean integrals with the Euclidean time reversal are identical to the corresponding Minkowski scalar products. The equality shows that analytic continuation is not required to compute the physical scalar product in the Euclidean representation.

In the Green-function representation the formulas for the Poincaré generators are replaced by

$$\langle \mathsf{x} | H | \mathbf{f} \rangle := \{0, \frac{\partial}{\partial \mathsf{x}_{11}^0} f_1(\mathsf{x}_{11}), \left(\frac{\partial}{\partial \mathsf{x}_{21}^0} + \frac{\partial}{\partial \mathsf{x}_{22}^0} \right) f_2(\mathsf{x}_{21}, \mathsf{x}_{22}), \cdots\} \tag{6.8}$$

$$\langle \mathsf{x} | \mathbf{P} | \mathbf{f} \rangle := \{0, -i\frac{\partial}{\partial \vec{\mathsf{x}_{11}}} f_1(\mathsf{x}_{11}), -i\left(\frac{\partial}{\partial \vec{\mathsf{x}_{21}}} + \frac{\partial}{\partial \vec{\mathsf{x}_{22}}} \right) f_2(\mathsf{x}_{21}, \mathsf{x}_{22}), \cdots\} \tag{6.9}$$

$$\langle \mathsf{x}|\mathbf{J}|\mathbf{f}\rangle := \{0, -i\vec{\mathsf{x}}_{11}\times \frac{\partial}{\partial \vec{\mathsf{x}}_{11}} f_1(\mathsf{x}_{11}), -i\left(\vec{\mathsf{x}}_{21}\times\frac{\partial}{\partial\vec{\mathsf{x}}_{21}} + \vec{\mathsf{x}}_{22}\times\frac{\partial}{\partial\vec{\mathsf{x}}_{22}}\right) f_2(\mathsf{x}_{21},\mathsf{x}_{22}),\cdots\} \tag{6.10}$$

$$\langle \mathsf{x}|\mathbf{K}|\mathbf{f}\rangle := \{0, \left(\vec{\mathsf{x}}_{11}\frac{\partial}{\partial \mathsf{x}^0_{11}} - \mathsf{x}^0_{11}\frac{\partial}{\partial\vec{\mathsf{x}}_{11}}\right) f_1(\mathsf{x}_{11}),$$

$$\left(\vec{\mathsf{x}}_{21}\frac{\partial}{\partial \mathsf{x}^0_{21}} - \mathsf{x}^0_{21}\frac{\partial}{\partial\vec{\mathsf{x}}_{21}} + \vec{\mathsf{x}}_{22}\frac{\partial}{\partial \mathsf{x}^0_{22}} - \mathsf{x}^0_{22}\frac{\partial}{\partial\vec{\mathsf{x}}_{22}}\right) f_2(\mathsf{x}_{21},\mathsf{x}_{22}),\cdots\}. \tag{6.11}$$

For particles with spin these expression are modified as follows

$$\mathbf{J}: \qquad \left(-i\vec{\mathsf{x}}_{11}\times\frac{\partial}{\partial\vec{\mathsf{x}}_{11}}\right) \to \left(-i\vec{\mathsf{x}}_{11}\times\frac{\partial}{\partial\vec{\mathsf{x}}_{11}} + \vec{\Sigma}\right) \tag{6.12}$$

$$\mathbf{K}: \qquad \left(\vec{\mathsf{x}}_{11}\frac{\partial}{\partial \mathsf{x}^0_{11}} - \mathsf{x}^0_{11}\frac{\partial}{\partial\vec{\mathsf{x}}_{11}}\right) \to \left(\vec{\mathsf{x}}_{11}\frac{\partial}{\partial \mathsf{x}^0_{11}} - \mathsf{x}^0_{11}\frac{\partial}{\partial\vec{\mathsf{x}}_{11}} + \vec{\mathcal{B}}\right) \tag{6.13}$$

where

$$\vec{\Sigma} = i\vec{\nabla}_\phi S(e^{\frac{-i}{2}\vec{\sigma}\cdot\vec{\phi}}, e^{\frac{i}{2}\vec{\sigma}^t\cdot\vec{\phi}})_{aa'} \tag{6.14}$$

and

$$\vec{\mathcal{B}} = \vec{\nabla}_\rho S(e^{\frac{-i}{2}\vec{\sigma}\cdot\vec{\rho}}, e^{\frac{-i}{2}\vec{\sigma}^t\cdot\vec{\rho}})_{aa'}. \tag{6.15}$$

Here $S(R_1, R_2)$ is a finite dimensional representations of $SU(2)\times SU(2)$ associated with the type of field. It is constructed by expressing the finite dimensional representation of the Lorentz group $S(\Lambda)$ in terms of $SL(2,C)$ matrices $S(A, A^*)$ and subsequently replacing A and A^* by independent unitary matrices, A and B.

For the case of fermions the Euclidean time reversal operator also includes a factor γ^0 for each final particle.

The formulas summarized in this section are discussed in more detail in [21].

VII. FEW-BODY MODELS

A typical application where relativistic few-body methods are used is elastic nucleon-nucleon scattering. This is normally treated using the inhomogeneous Bethe-Salpeter equation. We outline the formulation of this problem in the Euclidean quantum mechanical representation.

We consider a model Green function of the form

$$\Theta S \to \begin{pmatrix} 0 & 0 & 0 & \cdots\cdots \\ 0 & S_2(\Theta\mathsf{x}_{11},\mathsf{x}_{12}) & 0 & \\ 0 & 0 & S_4(\Theta\mathsf{x}_{21},\Theta\mathsf{x}_{22},\mathsf{x}_{23},\mathsf{x}_{24}) & 0 \\ \vdots & 0 & 0 & \ddots \end{pmatrix}. \tag{7.1}$$

For this model we assume that only S_2 and S_4 are non-zero. Furthermore we assume that S_2 and S_4 are related by cluster properties:

$$S_4 = S_2S_2 + S_c = S_0 + S_c. \tag{7.2}$$

The Euclidean Bethe Salpeter kernel is defined by

$$S_4^{-1} - S_0^{-1} = -K. \tag{7.3}$$

The structure of S_2 is determined by covariance up to an unknown Lehmann weight. If the weight is a delta function in the mass then this is a free field Euclidean Green function. In this case the one-body solutions that are needed to formulate the scattering problem are trivial. If the Lehmann weight also includes some continuous spectrum then it is necessary to solve a one-body problem to formulate the scattering asymptotic condition. To do this we take an orthonormal set of positive-time test functions and use the Gram-Schmidt method to construct an orthonormal set

$$\langle f_n | f_m \rangle = \int d\mathsf{x} d\mathsf{y} f_n^*(\Theta \mathsf{x}) \gamma^0 S_2(\mathsf{x}-\mathsf{y}) f_m(\mathsf{y}) = \delta_{mn}. \tag{7.4}$$

Because the invariant Minkowski Green function is defined with a Dirac conjugate field rather than a Hilbert space adjoint, the γ^0 needs to be eliminated from S_2 to get the continuation to the Wightman function that serves as the kernel of the Hilbert space scalar product. This is achieved by including γ^0 as the spinor part of the Θ operator.

In this basis one-body solutions have the form

$$|\lambda\rangle = \sum_n c_n |\mathbf{f}_n\rangle \tag{7.5}$$

where c_n and λ are determined solving the eigenvalue problem for discrete λ^2:

$$\lambda^2 c_n = \sum_m (\frac{\partial^2}{\partial \tau^2} + \frac{\partial^2}{\partial \mathbf{a}^2})(f_n \gamma^0 \Theta S_2 f_{m,I,(\tau,\mathbf{a})})_{\tau=\mathbf{a}=0} c_m. \tag{7.6}$$

In this case the Euclidean two-point Green function has the form

$$S_2(\mathsf{x}-\mathsf{y}) := \frac{1}{(2\pi)^4} \int d^4\mathsf{p} \rho(m) dm \frac{m - p\cdot\gamma_e}{\mathsf{p}^2+m^2} e^{i\mathsf{p}\cdot(\mathsf{x}-\mathsf{y})} \tag{7.7}$$

where

$$i\gamma_{0e} = \beta = \gamma^0 = -\gamma_0; \qquad \gamma_e^i = \gamma^i. \tag{7.8}$$

and

$$\rho(m) = \rho_m \delta(m-\lambda) + \rho_c(m). \tag{7.9}$$

The matrix elements have the form

$$(f, \Theta\gamma^0 S_2 f)$$

$$= \frac{1}{(2\pi)^4} \int d^4\mathsf{x} d^4\mathsf{y} d^4\mathsf{p} dm f(\mathsf{x})\, e^{i\mathsf{p}\cdot(\Theta\mathsf{x}-\mathsf{y})} \gamma^0 \frac{m - p\cdot\gamma_e}{\mathsf{p}^2+m^2} \rho(m) f(\mathsf{y})$$

$$= \int g^\dagger(\mathbf{p}, m) \frac{\Lambda_+(\mathbf{p},m)}{(2\pi)^3} \rho(m) g(\vec{\mathsf{p}}, m) d\mathbf{p} dm \tag{7.10}$$

where

$$\Lambda_+(p) := \frac{\omega_m(\mathbf{p}) + \gamma^0 \vec{\gamma}\cdot\mathbf{p} - m\gamma^0}{2\omega_m(\mathbf{p})} \tag{7.11}$$

is the positive-energy Dirac projector and

$$g(\mathbf{p}, m) := \int d^4\mathsf{x}\, e^{-\omega_m(\mathbf{p})\mathsf{x}_0 - i\mathsf{p}\cdot\mathsf{x}} f(\mathsf{x}). \tag{7.12}$$

Single particle eigenstates of mass, linear momentum and spin are constructed from the mass eigenstates (7.6)

$$\psi_\lambda(\mathsf{x}) = \sum_n c_n f_n(\mathsf{x}) \tag{7.13}$$

using

$$\psi_{\lambda,j,\mathbf{p},\mu}(\mathsf{x}) = \sum_n c_n \frac{1}{(2\pi)^{3/2}} \int \sum_\nu \mathbf{f}_n(\tau, R^{-1}\mathbf{x} - \mathbf{a}) e^{-iR^{-1}\mathbf{p}\cdot\mathbf{x}} d\mathbf{a} D^{j*}_{\mu\nu}(R) dR \tag{7.14}$$

The Haag-Ruelle operators $A(\mathbf{p},\mu)$ are

$$A(\mathbf{p},\mu)[\mathsf{x}] = \frac{1}{(2\pi)^{3/2}} \int d\mathbf{p}[-\frac{\partial}{\partial\beta} - \omega_\lambda(\mathbf{p})]\psi_{\lambda,j,\mathbf{p},\mu}[\tau-\beta, \mathbf{x}-\vec{a}]_{|\beta=\mathbf{a}=0}. \tag{7.15}$$

The scattering asymptotic states of interest are two-body states. S-matrix elements in normalizable states can be computed using the methods discussed in section 5. Equation (5.33) is replaced by:

$$\langle g|S|f\rangle =$$

$$\lim_{n\to\infty}\sum_m d_m(n)\int g(\mathbf{p}_1',\mu_1')g(\mathbf{p}_2',\mu_2')e^{ine^{-\beta(\omega_\lambda(\mathbf{p}_1')+\omega_\lambda(\mathbf{p}_2'))}} A^\dagger(\mathbf{p}_1',\mu_1')[\Theta\mathsf{x}_2']A^\dagger(\mathbf{p}_2',\mu_2')[\Theta\mathsf{x}_1']\gamma_1^0\gamma_2^0$$

$$\times S_4(\mathsf{x}_2',\mathsf{x}_2';\mathsf{x}_1,\mathsf{x}_2)A(\mathbf{p}_1,\mu_1)[\mathsf{x}_1-2m\beta]A(\mathbf{p}_2,\mu_2)[\mathsf{x}_2-2m\beta]e^{ine^{-\beta(\omega_\lambda(\mathbf{p}_1')+\omega_\lambda(\mathbf{p}_2'))}}\times$$

$$g(\mathbf{p}_2,\mu_2)g(\mathbf{p}_1,\mu_1)d\mathbf{p}_1 d\mathbf{p}_2 d\mathbf{p}_1' d\mathbf{p}_2' d\mathsf{x}_1 d\mathsf{x}_2 d\mathsf{x}_1' d\mathsf{x}_2'. \tag{7.16}$$

Here the reflection positivity is limited to requiring that $\gamma_1^0\gamma_2^0\Theta S_4$ is non-negative on products of positive-time test functions.

This illustrates how the approximations discussed above can be implemented in a few-body setting. We note that even if S_4 is calculated perturbatively, the resulting approximate S matrix will be formally unitary. This is similar to what is observed in non-relativistic scattering theory when the K-matrix is treated perturbatively

VIII. SCATTERING TEST

The scattering computations outlined above and in section V are based on the convergence of a sequence of three approximations that are performed in a prescribed order. While they should in principle converge for suitable model Green functions, that does not imply that the approximations can be sufficiently well-controlled to give converged predictions for reactions at the few-GeV energy scale of interest. Since to the best of our knowledge this approach to scattering, i.e. computing sharp-momentum transition matrix elements using matrix elements of $e^{-n\beta H}$ in normalizable states as input, has not even been tested in non-relativistic models, we discuss the implementation of this method in an exactly solvable non-relativistic model. This has the advantage that all of the approximations can be compared to exact results, and the accuracy of the proposed method can be precisely determined. We consider a non-relativistic Hamiltonian with a separable potential that has the range and strength of a nucleon-nucleon interaction. The range is fixed by the pion mass while the strength is adjusted to bind two nucleons with the deuteron binding energy.

The the interaction is taken as a Yamaguchi interaction, with Hamiltonian

$$H = \mathbf{k}^2/m - |g\rangle\lambda\langle g| \qquad \langle\mathbf{k}|g\rangle = g(\mathbf{k}) = \frac{1}{m_\pi^2+\mathbf{k}^2}. \tag{8.1}$$

The transition matrix elements are

$$\langle\mathbf{k}|t|\mathbf{k}'\rangle = g(\mathbf{k})\tau(\frac{k'^2}{m_n}+i0^+)g(\mathbf{k}') \tag{8.2}$$

with

$$\tau(\frac{k'^2}{m_n}+i0^+) = -\frac{\lambda}{1+\frac{m_n\lambda\pi^2}{m_\pi}\frac{1}{(im_\pi+k')^2}} \tag{8.3}$$

where $m_n = .94$GeV, $m_\pi = .14$GeV, $e_b = -2.24$MeV and the coupling constant is determined from these parameters by

$$\lambda = \frac{m_\pi}{m_n\pi^2}(m_\pi+\sqrt{-m_n e_b})^2. \tag{8.4}$$

To test the approximations we calculate sharp-momentum transition matrix elements using matrix elements of $e^{-\beta H}$ evaluated between normalizable states using the methods outlined in section five. While the eigenstates of this model can be computed exactly, we had to use the spectral expansion of the Hamiltonian to compute matrix elements of $e^{-\beta H}$. While this is a complicated construction in the non-relativistic case, it is replaced by a quadrature (3.18) in the Euclidean case.

The first approximation is to extract sharp-momentum transition matrix elements using sufficiently narrow wave packets in equation (5.31). For this model, exact expressions are available for both the S operator and the transition matrix. For our test problem we choose Gaussian wave packets in the relative momenta of the form

$$\phi(k) = Ne^{-\alpha(k-k_0)^2} = Ne^{-(k-k_0)^2/k_w^2} \tag{8.5}$$

where N is a normalization constant and k_0 is the on-shell momentum. We do not choose a particular direction because the interaction is pure s-wave, and the on-shell transition matrix elements at a given energy are given by a single complex number.

Sharp on-shell transition matrix elements computed exactly and approximately from S-matrix elements using (5.31) are compared as a function of the width, k_w of the wave packets. The wave packet widths were determined by the requirement that the approximate transition matrix elements agree with the exact transition matrix elements to an accuracy of less that 0.1%. The results are shown in Table 1 as a function of the relative momentum. The first column of Table 1 shows the on-shell momentum k_0 (center of the Gaussian). The second column shows the value of α used to get the error shown in the fourth column. All of the errors in column four round up to .1%. The third column lists $k_w := 1/\sqrt{\alpha}$ for each value of k_0 and the last column is the dimensionless ratio k_w/k_0. The values of α in column 3 are used in all of the calculations in this section.

Table 1

k_0 [GeV]	α [GeV^{-2}]	k_w [GeV]	% error	k_w/k_0
0.05	325000	0.00175412	0.1	0.035
0.1	105000	0.00308607	0.1	0.030
0.2	26000	0.00620174	0.1	0.031
0.3	10500	0.009759	0.1	0.032
0.4	5100	0.0140028	0.1	0.035
0.5	3000	0.0182574	0.1	0.036
0.6	2000	0.0223607	0.1	0.037
0.7	1350	0.0272166	0.1	0.038
0.8	1000	0.0316228	0.1	0.039
0.9	750	0.0365148	0.1	0.040
1.0	600	0.0408248	0.1	0.040
1.1	475	0.0458831	0.1	0.041
1.2	400	0.05	0.1	0.041
1.3	330	0.0550482	0.1	0.042
1.4	290	0.058722	0.1	0.041
1.5	250	0.0632456	0.1	0.042
1.6	210	0.0690066	0.1	0.043
1.7	190	0.0725476	0.1	0.042
1.8	170	0.0766965	0.1	0.042
1.9	150	0.0816497	0.1	0.042
2.0	135	0.0860663	0.1	0.043

The last column of Table 1 suggests that a 0.1% error will generally be obtained if this width is less than 3% of the on-shell momentum. This property holds over a wide range of momenta in this model. This is a simple model transition operator; one may anticipate narrower wave packets are needed for more realistic models. This approximation can be improved by decreasing the width of the wave packet; it is the largest source of error in the calculations. In any realistic calculation the wave packet width does not have to be smaller than the experimental momentum resolution.

The second step is to approximate S-matrix elements in these Gaussian wave packets using equation (5.27). It is important to first pick the wave packets because the n value needed for convergence depends on the width of the wave packet. The approximate quantities are

$$\langle\psi'|S_n|\psi\rangle := \int d\mathbf{k}d\mathbf{k}'\psi'(\mathbf{k})e^{-ine^{-\beta k^2/m}}\langle\mathbf{k}|e^{2ine^{-\beta H}}|\mathbf{k}'\rangle e^{-ine^{-\beta k'^2/m}}\psi(\mathbf{k}'). \tag{8.6}$$

In Tables 2-8 these quantities are computed using the spectral expansion for H. In these calculations the bound-state contribution is not included because it vanishes in the large n limit. Tables 2-8 show the real and imaginary parts of matrix elements $\langle\psi'|(S_n - I)|\psi\rangle$ for different values of n for $k_0 = 50, 100, 200, 500, 1000, 1500, 2000$ MeV. The exact value is given at the bottom of each table. Table 9 shows the values of β, the product $k_0 \times \beta$ and the n-values used in our final calculations. Table 9 suggests that β should be chosen so $k_0 \times \beta$ is of order unity. Except for the $k_0 = 50$MeV case, $n = 250$ or more gives errors for the n-limits that are smaller than the errors made in the factorization approximation, (5.31).

The n dependence of the real and imaginary part of matrix elements of $\langle\psi'|(S - I)|\psi\rangle$, computed using (5.27), are plotted as a function of n for different values of k_0 in figures 1-10. Figures 11 and 12 show how fast the neglected bound state contribution to the spectral expansion falls off with n for $k_0 = 1$ GeV.

Table 2: $k_0 = 50$[MeV], $\alpha = 325000$[GeV^{-2}]

n	Re $\langle\phi\|(S_n - I)\|\phi\rangle$	Im $\langle\phi\|(S_n - I)\|\phi\rangle$
50	-7.62976513315350e-1	-1.52406978178214e-1
100	-1.33113144491104e+0	-2.75546806155677e-1
150	-1.67324498184421e+0	-3.50392186517949e-1
200	-1.83391449191883e+0	-3.86136590065981e-1
250	-1.89273779093641e+0	-3.99389077283171e-1
300	-1.90951807884485e+0	-4.03228425149703e-1
350	-1.91324926322936e+0	-4.04093826445785e-1
400	-1.91389545311571e+0	-4.04245969269804e-1
450	-1.91398265491004e+0	-4.04266741575048e-1
500	-1.91399177708580e+0	-4.04268944214980e-1
550	-1.91399249452470e+0	-4.04269131167755e-1
600	-1.91399253056843e+0	-4.04269144047622e-1
650	-1.91399252976074e+0	-4.04269145893613e-1
ex	-1.91399253060872e+0	-4.04269147714400e-1

Table 3: $k_0 = 100$[MeV], $\alpha = 105000$[GeV^{-2}]

n	Re $\langle\phi\|(S_n - I)\|\phi\rangle$	Im $\langle\phi\|(S_n - I)\|\phi\rangle$
50	-8.73395186664514e-1	4.95616337213744e-1
100	-1.34576615227520e+0	7.59199494502660e-1
150	-1.49091126760062e+0	8.39700869213905e-1
200	-1.51566533604846e+0	8.53352070852317e-1
250	-1.51799902547669e+0	8.54631681615040e-1
300	-1.51811943431498e+0	8.54697554653043e-1
350	-1.51812278309620e+0	8.54699376334219e-1
400	-1.51812288480017e+0	8.54699423334768e-1
450	-1.51812290596551e+0	8.54699435479409e-1
500	-1.51812290955424e+0	8.54699438102000e-1
550	-1.51812290968227e+0	8.54699438639971e-1
600	-1.51812288857871e+0	8.54699427676778e-1
650	-1.51812275938123e+0	8.54699356334623e-1
ex	-1.51812291315971e+0	8.54699438329052e-1

Table 4: $k_0 = 200$[MeV], $\alpha = 26000$[Gev^{-2}]

n	Re $\langle\phi\|(S_n - I)\|\phi\rangle$	Im $\langle\phi\|(S_n - I)\|\phi\rangle$
50	-2.08408481834932e-1	4.56550768265380e-1
100	-3.11945696198071e-1	6.85279276148059e-1
150	-3.38623394392403e-1	7.44641333032490e-1
200	-3.42127100784575e-1	7.52475859236454e-1
250	-3.42359208499266e-1	7.52997722504423e-1
300	-3.42366821259122e-1	7.53015087494957e-1
350	-3.42366956571344e-1	7.53015382645461e-1
400	-3.42366963936415e-1	7.53015400508140e-1
450	-3.42366965122021e-1	7.53015404697748e-1
500	-3.42366965247355e-1	7.53015405389574e-1
550	-3.42366962667700e-1	7.53015400608940e-1
600	-3.42366938358008e-1	7.53015348812648e-1
650	-3.42366884413759e-1	7.53015247619997e-1
ex	-3.42366967477707e-1	7.53015410457076e-1

Table 5: $k_0 = 500$[MeV], $\alpha = 3000$[Gev^{-2}]

n	Re $\langle\phi\|(S_n - I)\|\phi\rangle$	Im $\langle\phi\|(S_n - I)\|\phi\rangle$
50	-5.93330385580271e-3	9.65738963854834e-2
100	-7.12681692349637e-3	1.18415504225673e-1
150	-7.18129565909453e-3	1.19491569949089e-1
200	-7.18179706798405e-3	1.19502423190014e-1
250	-7.18179794641838e-3	1.19502444469971e-1
300	-7.18179794670870e-3	1.19502444477783e-1
350	-7.18179794671048e-3	1.19502444477784e-1
400	-7.18179794671081e-3	1.19502444477784e-1
ex	-7.18179797016073e-3	1.19502444795275e-1

Table 6: $k_0 = 1$[GeV], $\alpha = 600$[Gev^{-2}]

n	Re $\langle\phi\|(S_n - I)\|\phi\rangle$	Im $\langle\phi\|(S_n - I)\|\phi\rangle$
50	-1.47024820732811e-4	1.55922557816223e-2
100	-1.62726188649875e-4	1.79713868930101e-2
150	-1.62967714125273e-4	1.80219591282450e-2
200	-1.62968113934903e-4	1.80220978916775e-2
250	-1.62968113982642e-4	1.80220979403700e-2
300	-1.62968113982642e-4	1.80220979403721e-2
350	-1.62968113982753e-4	1.80220979403720e-2
400	-1.62968113982975e-4	1.80220979403718e-2
ex	-1.62968113982742e-4	1.80220979403858e-2

Table 7: $k_0 = 1.5$[GeV], $\alpha = 250$[GeV^{-2}]

n	Re $\langle\phi\|(S_n - I)\|\phi\rangle$	Im $\langle\phi\|(S_n - I)\|\phi\rangle$
50	-1.40242356887477e-5	4.66175982621713e-3
100	-1.54430995201738e-5	5.52235009412764e-3
150	-1.54679181726403e-5	5.55112445776045e-3
200	-1.54679281958447e-5	5.55130663695727e-3
250	-1.54679280390813e-5	5.55130688968718e-3
300	-1.54679280390813e-5	5.55130688978369e-3
350	-1.54679280394143e-5	5.55130688978374e-3
400	-1.54679280388592e-5	5.55130688978377e-3
ex	-1.54679280242191e-5	5.55130688386830e-3

Table 8: $k_0 = 2.0$[GeV], $\alpha = 135$[GeV^{-2}]

n	Re $\langle\phi\|(S_n - I)\|\phi\rangle$	Im $\langle\phi\|(S_n - I)\|\phi\rangle$
50	-2.60094316473225e-6	1.94120750171791e-3
100	-2.82916859895010e-6	2.35553585404449e-3
150	-2.83171624670953e-6	2.37471383801820e-3
200	-2.83165946257657e-6	2.37492460997990e-3
250	-2.83165905312632e-6	2.37492527186858e-3
300	-2.83165905257121e-6	2.37492527262432e-3
350	-2.83165905190508e-6	2.37492527262493e-3
400	-2.83165905234917e-6	2.37492527262540e-3
ex	-2.83165905227843e-6	2.37492527259701e-3

Table 9: Parameters

k0 [GeV]	β[GeV^{-1}]	$k0 \times \beta$	n
0.05	80.0	4.0	630
0.1	40.0	4.0	450
0.2	10.0	2.0	470
0.3	5.0	1.5	330
0.4	4.0	1.6	235
0.5	3.0	1.5	205
0.6	2.5	1.5	225
0.7	1.6	1.2	200
0.8	1.4	1.12	200
0.9	1.05	.945	190
1.0	1.0	1.0	200
1.1	0.95	1.045	200
1.2	0.9	1.08	200
1.3	0.85	1.105	200
1.4	0.7	0.98	200
1.5	0.63	0.945	200
1.6	0.57	0.912	200
1.7	0.5	0.85	200
1.8	0.45	0.81	200
1.9	0.42	0.798	200
2.0	0.4	0.8	200

The third approximation is the polynomial approximation to $e^{i2ne^{-\beta H}}$. In this application we use the freedom to shift the zero of the energy of the free and interacting Hamiltonians by a constant (the binding energy) so the spectrum of $e^{-\beta H}$ is strictly between zero and one. Then it is only necessary to find a polynomial approximation to e^{inx} for $x \in [0,1]$.

The polynomial approximation is made using the Chebyshev expansion:

$$f(y) \approx P_N(y) := \frac{1}{2}c_0 T_0(y) + \sum_{k=1}^{N} c_k T_k(y) \tag{8.7}$$

$$c_j = \frac{2}{N+1}\sum_{k=1}^{N} f(\cos(\frac{2k-1}{N+1}\frac{\pi}{2}))\cos(j\frac{2k-1}{N+1}\frac{\pi}{2}). \tag{8.8}$$

The Chebyshev expansion is designed for functions on the interval $[-1,1]$. To make full use of the information in this approximation we approximate $e^{-2inx} = f(2x-1)$ by the polynomial approximation to $f(y)$. This gives the polynomial

$$f(2e^{-\beta H'}-1) \approx \frac{1}{2}c_0 T_0(2e^{-\beta H'}-1) + \sum_{k=1}^{N} c_k T_k(2e^{-\beta H'}-1) = P_N(e^{-\beta H'}) \tag{8.9}$$

where $H' = H + e_b$. It follows that

$$|e^{2inx} - P_N(x)| < 2\frac{n^{N+1}}{(N+1)!} \qquad |||e^{2ine^{-\beta H'}} - P_N(e^{-\beta H'})||| < 2\frac{n^{N+1}}{(N+1)!} \tag{8.10}$$

Chebyshev polynomials are known to be good approximations to the best uniform polynomial approximation [22]. Table 10 shows polynomial approximations to e^{inx} for different values of $x \in [0,1]$, and n. The errors are between 10^{-11} and 10^{-13} for the degree of the polynomial only slightly above n.

Table 10: Polynomial convergence

x	n	deg	poly error %
0.1	200	150	7.939e+00
0.1	200	200	3.276e+00
0.1	200	250	1.925e-11
0.1	200	300	4.903e-13
0.1	630	580	3.573e+00
0.1	630	630	2.069e+00
0.1	630	680	5.015e-08
0.1	630	700	7.456e-11
0.5	200	150	1.973e-13
0.5	200	200	1.627e-13
0.5	200	250	3.266e-13
0.5	630	580	1.430e-14
0.5	630	630	3.460e-13
0.5	630	680	9.330e-13
0.9	200	150	7.939e+00
0.9	200	200	3.276e+00
0.9	200	250	1.950e-11
0.9	200	300	9.828e-13
0.9	630	580	3.573e+00
0.9	630	630	2.069e+00
0.9	630	680	5.015e-08
0.9	630	700	7.230e-11

Table 10 suggests that a polynomial with degree 10-20% larger than n is needed for convergence to one part in 10^{12} (10^{-10}%).

All three approximations are combined to get the approximations to the on-shell transition operator for incident momenta between 50 MeV and 2 GeV. These results are displayed in table 11. The errors are all better than .1%. The only significant source of error is the approximate factorization of the sharp momentum matrix element from the S matrix elements. This can be made as small as desired by choosing sufficiently narrow wave packets, although there is no need to improve accuracy to better than experimental resolution.

Table 11: Final calculation

$k0$	Real T	Im T	% error
0.05	2.18499e-1	-1.03160e+0	0.0982
0.1	-2.30337e-1	-4.09325e-1	0.0956
0.2	-1.01512e-1	-4.61420e-2	0.0981
0.3	-3.46973e-2	-6.97209e-3	0.0966
0.4	-1.39007e-2	-1.44974e-3	0.0997
0.5	-6.44255e-3	-3.86459e-4	0.0986
0.6	-3.34091e-3	-1.24434e-4	0.0952
0.7	-1.88847e-3	-4.63489e-5	0.0977
0.8	-1.14188e-3	-1.93605e-5	0.0965
0.9	-7.28609e-4	-8.86653e-6	0.0982
1.0	-4.85708e-4	-4.37769e-6	0.0967
1.1	-3.35731e-4	-2.30067e-6	0.0987
1.2	-2.39235e-4	-1.27439e-6	0.0968
1.3	-1.74947e-4	-7.38285e-7	0.0985
1.4	-1.30818e-4	-4.44560e-7	0.0955
1.5	-9.97346e-5	-2.76849e-7	0.0956
1.6	-7.73390e-5	-1.77573e-7	0.0992
1.7	-6.08794e-5	-1.16909e-7	0.0964
1.8	-4.85672e-5	-7.87802e-8	0.0956
1.9	-3.92110e-5	-5.42037e-8	0.0967
2.0	-3.20000e-5	-3.80004e-8	0.0966

These tests suggest that this method can be applied to compute scattering observables at the few GeV scale.

IX. SUMMARY AND CONCLUSION

In this paper we introduced a formulation of relativistic quantum mechanics that uses model Euclidean Green functions or generating functionals as dynamical input. The motivation for this approach is to formulate few-body models at the few GeV scale. The models are quantum mechanical which means that they are formulated in terms of linear operators on a model Hilbert space. They can be treated using standard quantum mechanical methods. The advantages of this framework over some conventional treatments of relativistic quantum mechanics is that there is a formal relation to Euclidean Lagrangian field theory. Specifically, the quantum theory discussed in this paper becomes the quantum formulation of the field theory when the model Green functions are replaced by the Green functions of the field theory. A second advantage of this formalism is that all calculations can be performed in the Euclidean domain, without using analytic continuation, however the quantities computed are ordinary Minkowski scalar products of normalizable vectors.

The structure of the physical Hilbert space was discussed in section one in terms of an Euclidean generating functional for a scalar field theory. Methods to model generating functionals in terms of connected Euclidean Green functions were also discussed. Representations in terms of Euclidean Green functions were given in section 6. Examples illustrating the equivalence of the Hilbert space inner product expressed in terms of Euclidean Green functions and the standard Minkowski inner product were given in equations (6.4) and (6.5).

The Poincaré Lie algebra is realized by interpreting the real Euclidean group as a complex subgroup of the complex Lorentz group on the physical Hilbert space. Self-adjoint generators that satisfy the Poincaré commutation relations are extracted by considering infinitesimal Euclidean transformations which become infinitesimal complex Lorentz transformations on the physical Hilbert space. Expressions for the generators were given both in terms of the generating functional in section three and directly in terms of the Green functions in section six.

In section four the generators introduced in sections three and six are used to construct a mass operator whose point eigenstates correspond to particles. The translation and rotation operators introduced in sections three and six are used to construct operators that create single-particle states of a given sharp momentum and spin out of normalizable mass eigenstates. These states necessarily transform irreducibly under the Poincaré group if they are non-degenerate.

The operators that create single-particle states are used to construct two-Hilbert space injection operators that define the scattering asymptotic conditions in the Haag-Ruelle formulation of scattering. The existence of the strong limits that define the wave operators can be checked using a generalization of the standard Cook condition [19] used in non-relativistic quantum mechanics. The two-Hilbert space wave operators are computed using time-dependent methods with the Kato-Birman invariance principle to reduce the computation to the evaluation matrix elements of polynomials in $e^{-\beta H}$. The feasibility of this method for computing scattering observables was established using an exactly solvable non-relativistic test model. This model demonstrated that it is possible to perform accurate calculations over a wide range of energies using this method. Guidelines for choosing the "time" n, temperature β, and the degree of the polynomial were established in the context of this model. Both the "time" limit and polynomial approximations were shown to be accurate to about ten significant figures. Even though the input to the scattering theory involves Euclidean quantities, scattering emerges because the scattering states asymptotically oscillate in phase with free-particle states as the "time-parameter", n, gets large. The limits do not involve the exponential fall-off that is used in lattice calculations. In addition, the use of one-body solutions in the formulation of the scattering asymptotic condition implies that the time limits are strong limits. These two properties imply that calculations can be performed at relativistic energies without the complications that arise in alternative formulations of scattering involving Euclidean quantities [20]. Since the output of these calculations are wave operators, the relativistic intertwining properties provide a mechanism for performing finite Poincaré transformations on all scattering states.

The framework presented in this work is preliminary. A practical implementation that can be used to treat realistic systems clearly requires more development. For applications to realistic systems it is apparent that model Green functions or generating functionals will need a significant amount of phenomenological input. In this regard this framework is no different than realistic non-relativistic nuclear models or quasipotential models, where the interactions or kernels for realistic systems require significant phenomenological input. In this framework the phenomenological input is needed to construct few-point connected Green functions, rather than potentials or quasipotential kernels. The use of models of Green functions avoids the difficulties encountered in calculating the Green functions from a field theory. Formal representations of the Green functions [23] may be useful in making such models.

This approach has a number of potential advantages. Cluster properties, which provide a key relation between few and many-body models is much easier to implement than it is in direct interaction formulations of relativistic quantum mechanics or quasipotential models. The ability to calculate scattering observables directly from Euclidean Green functions is also useful. Treating reactions that do not conserve particle number presents no special problems in this formalism.

However, there is no free lunch. The requirements of reflection positivity, Euclidean invariance, along with some additional technical requirements (ergodicity and analyticity) are essentially all of the axioms of quantum field theory for generating functionals. The difficulties in finding non-trivial theories satisfying these axioms are well known. This problem is not special to this framework; it is a problem in all implementations of field theory. In the context of this formalism, since Euclidean invariance and cluster properties are easy to realize, one might anticipate that it will be difficult to find a robust class of models that satisfy full reflection positivity. This is an important question for future research. While reflection positivity of free-field generating functionals is easy to establish, reflection positivity is not stable with respect to small Euclidean invariant perturbations [21]. Reflection positivity provides the quantum mechanical scalar product and ensures the spectral condition. This suggest that in models one may have to deal with violations of the spectral condition or negative norm states. For few-body problems one works on a subspace of the Hilbert space of the full theory that is relevant for a given energy scale. One might anticipate that good models will satisfy reflection positivity on this subspace. This possibility requires additional investigation.

In the absence of the positive Euclidean-time-support restriction on the test functions, the group of Euclidean motions in the inner product (2.9) with the Euclidean time reversal still has a representation of a complex subgroup of the Poincaré group that acts on a space with an indefinite metric. This suggests the possibility that for some models there may be invariant subspaces with positive norm[24] and positive energy that are distinct from the ones created using the positive-time test functions. These possibilities, may provide an alternative to full reflection positivity, also require additional investigation.

While the model formally becomes the field theory in the limit that the model generating functional becomes the full generating functional, truncations that eliminate higher-order connected Green functions from the exact generating functional are not mathematical approximations. In practice, the construction of model generating functionals involve some fitting to data and the model generating functionals only include contributions from a finite set of connected Green functions. To obtain a quantitative relation between these quantities to the corresponding field theoretic quantities it is necessary to first replace the field-theoretic generating functional by an effective generating functional that only has contributions from the connected Green functions that appear in the model. There is an additional freedom due to the fact that the model is only expected to be valid on an finite energy interval. Finally these is an additional freedom associated with "field redefinitions". A formal development of these connections is beyond the scope of this paper.

While this framework is strongly motivated by the structure of the a field theory, most of the properties of the a given model can be tested without appealing to properties of the field theory. For example, the existence and invariance of scattering matrix elements can be checked using (5.20, 5.21), while existence and self-adjointness of Poincaré generators can checked by verifying that rotations in Euclidean space-time planes are local symmetric semigroups [7][8][9].

Another important open question that requires more investigation is the implementation of this strategy in QCD, where the consistency of reflection positivity with confinement and gauge invariance introduces additional constraints on the models. Some formulations of Lattice QCD satisfy reflection positivity in the presence of these constraints, which provides some hope that these constraints are also compatible in models. For the type of calculations discussed, i.e one-particle states and scattering states, it is sufficient that reflection positivity holds on gauge invariant subspaces, since one-particle states are gauge invariant and the two-Hilbert space injection operator has range in the subspace of gauge invariant states. This limited form of reflection positivity is all that is needed to implement the computational strategy discussed in this paper.

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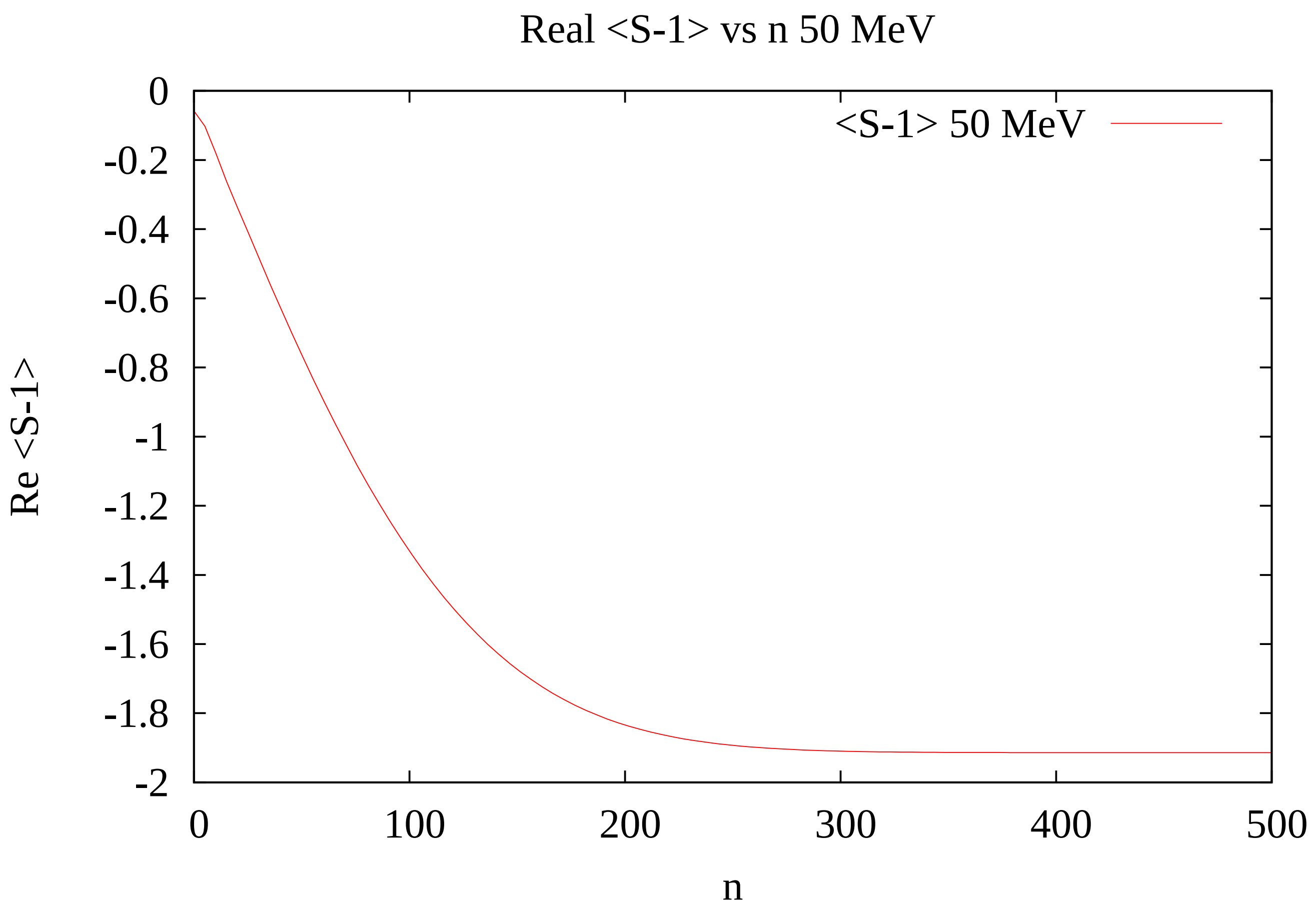

FIG. 1: Re $(\langle\phi|(S-1)|\phi\rangle)$ vs n 50[MeV] $\alpha = 325000[\mathrm{GeV}^{-2}]$

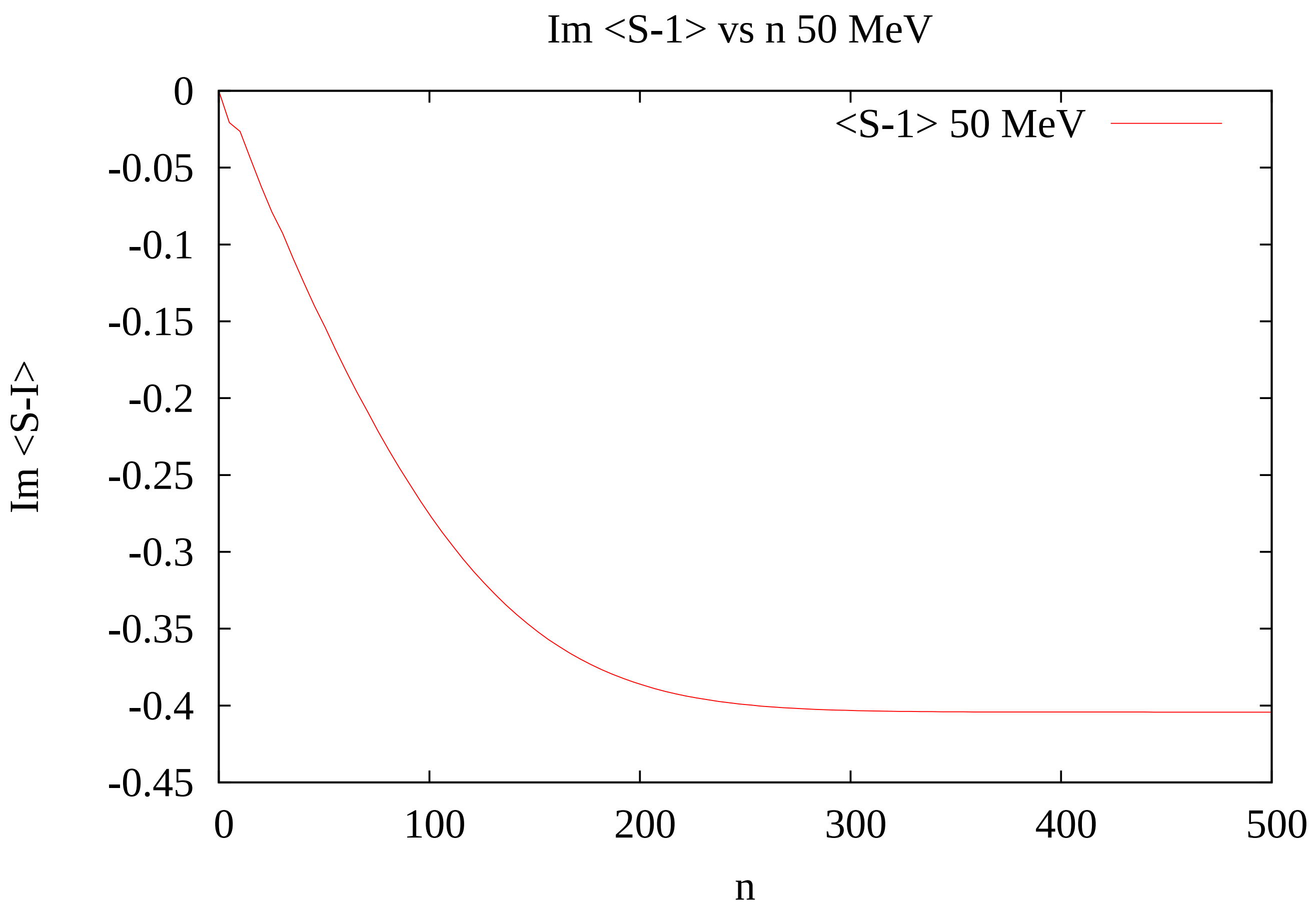

FIG. 2: Im $(\langle\phi|(S-1)|\phi\rangle)$ vs n 50[MeV] $\alpha = 325000$[GeV^{-2}]

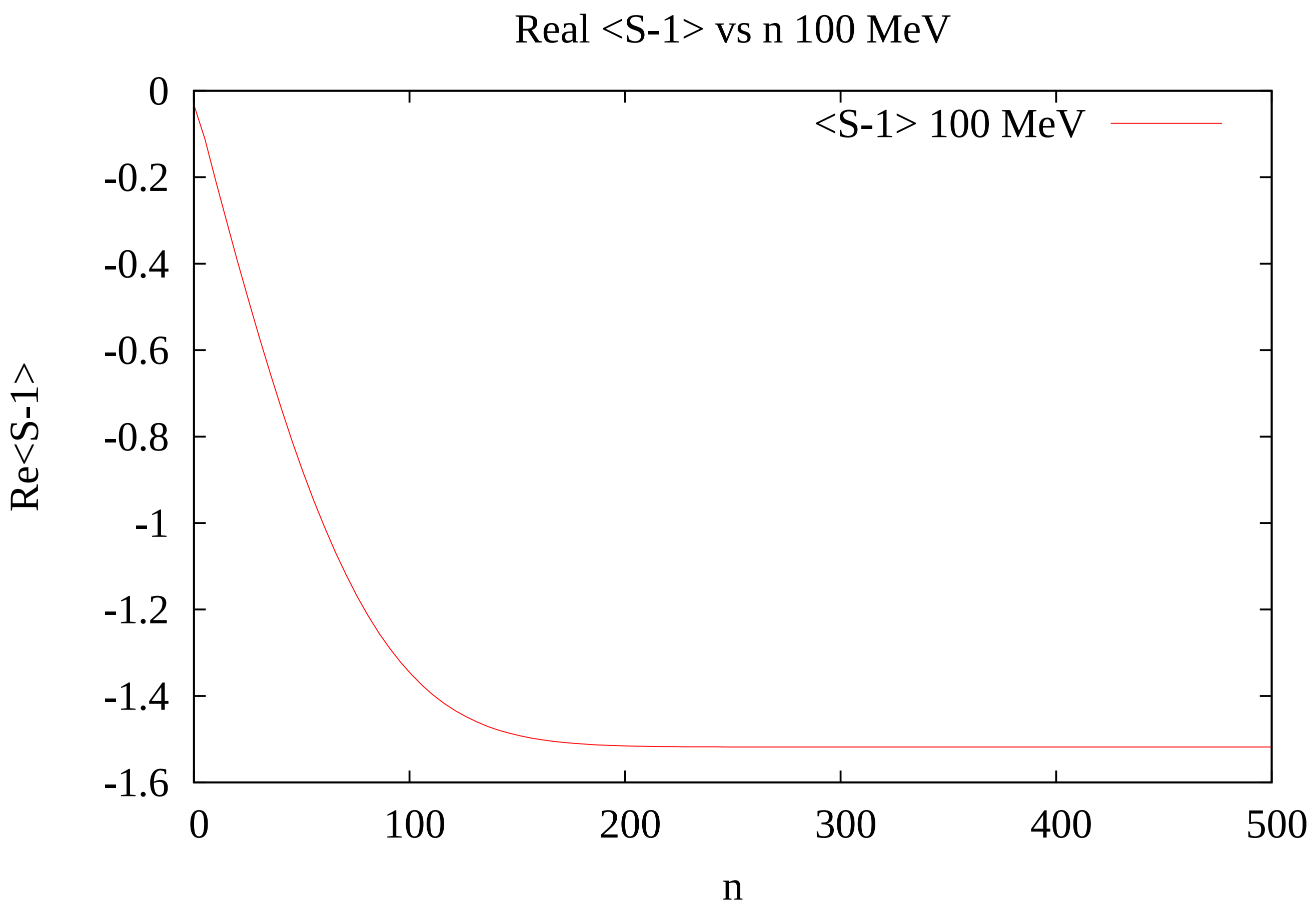

FIG. 3: Re $(\langle\phi|(S-1)|\phi\rangle)$ vs n 100[MeV] $\alpha = 105000[\mathrm{GeV}^{-2}]$

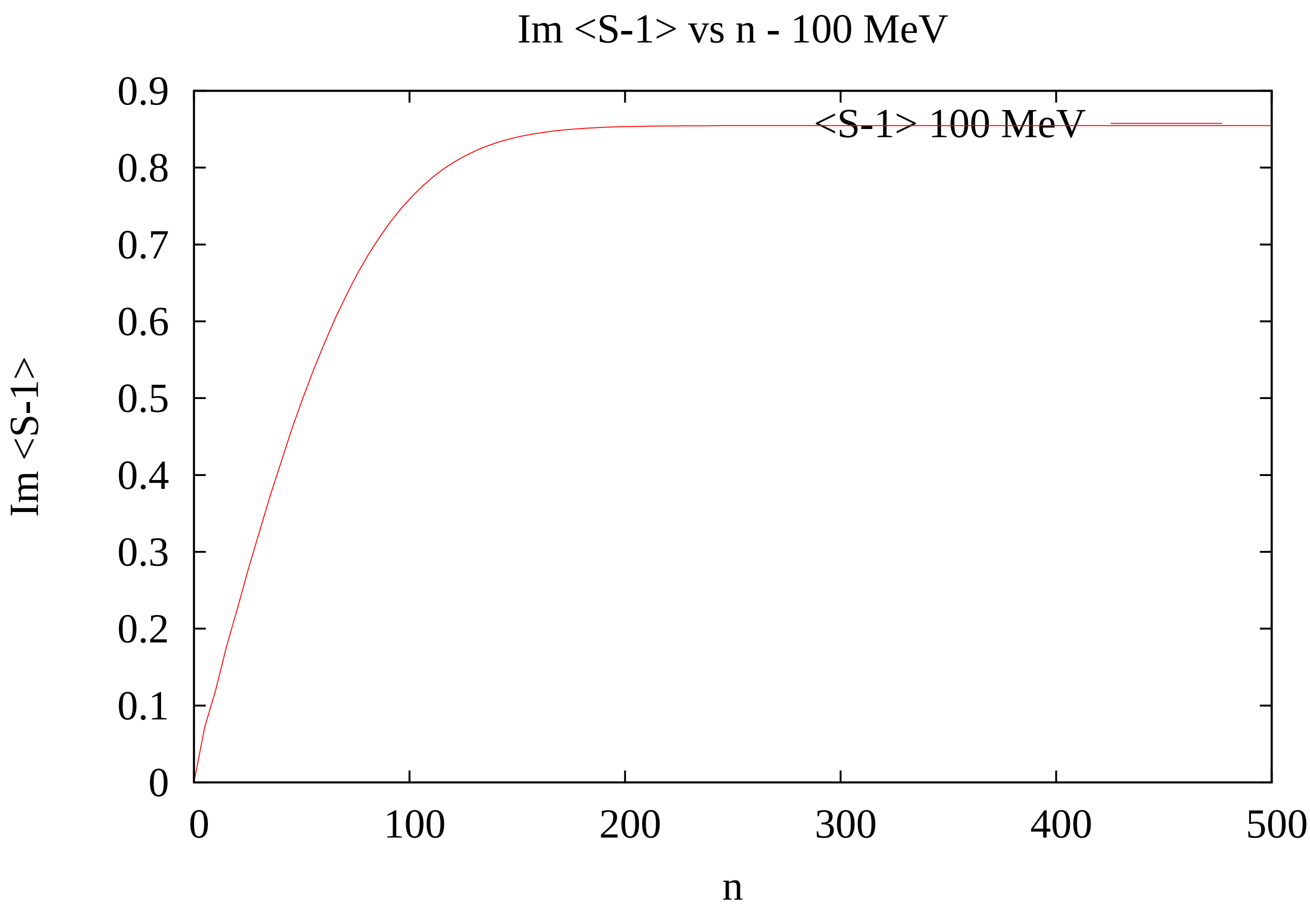

FIG. 4: Im $(\langle\phi|(S-1)|\phi\rangle)$ vs n 100[MeV] $\alpha = 105000[\mathrm{GeV}^{-2}]$

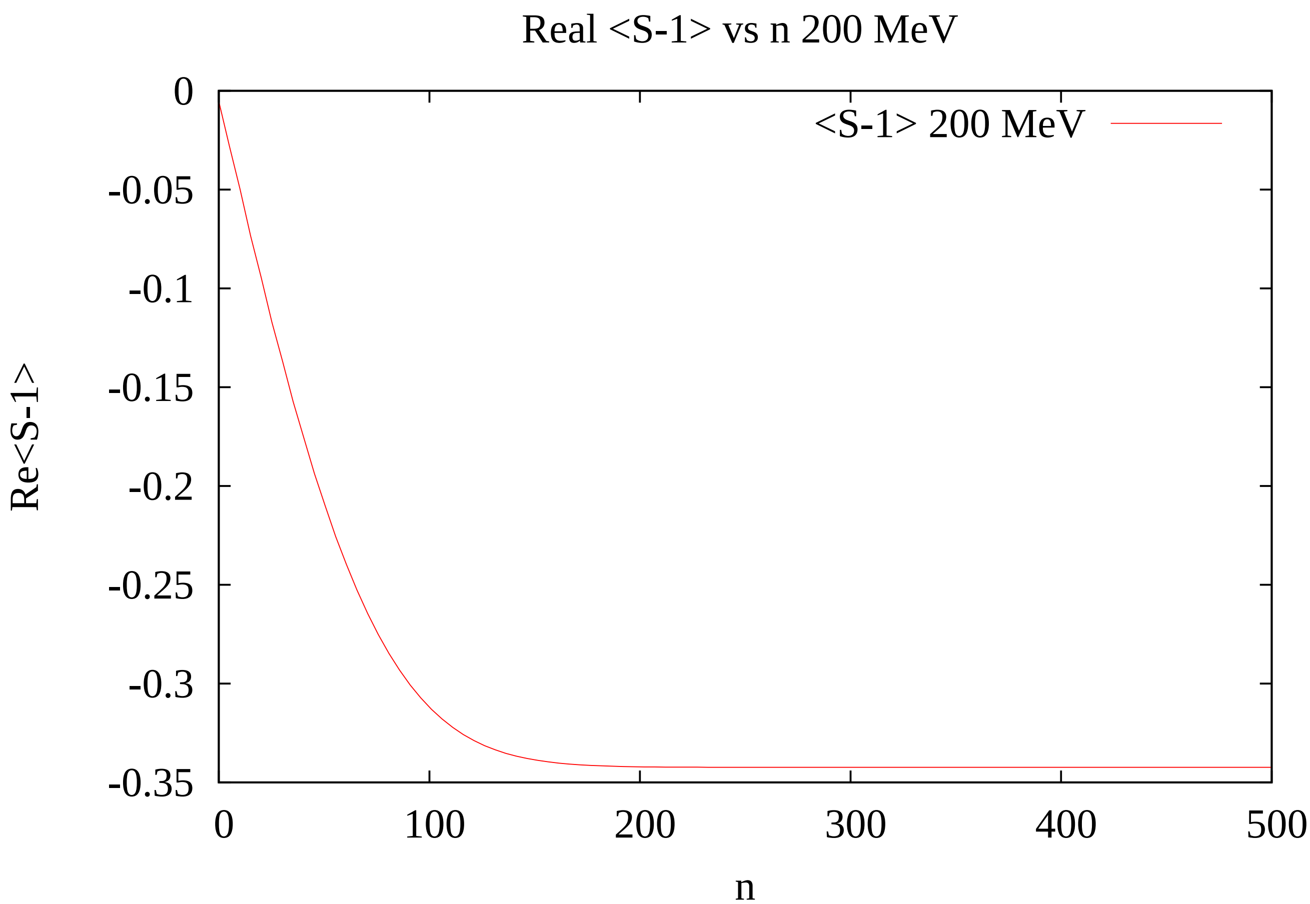

FIG. 5: Re ($\langle\phi|(S-1)|\phi\rangle$) vs n 200[MeV] $\alpha = 26000[\mathrm{GeV}^{-2}]$

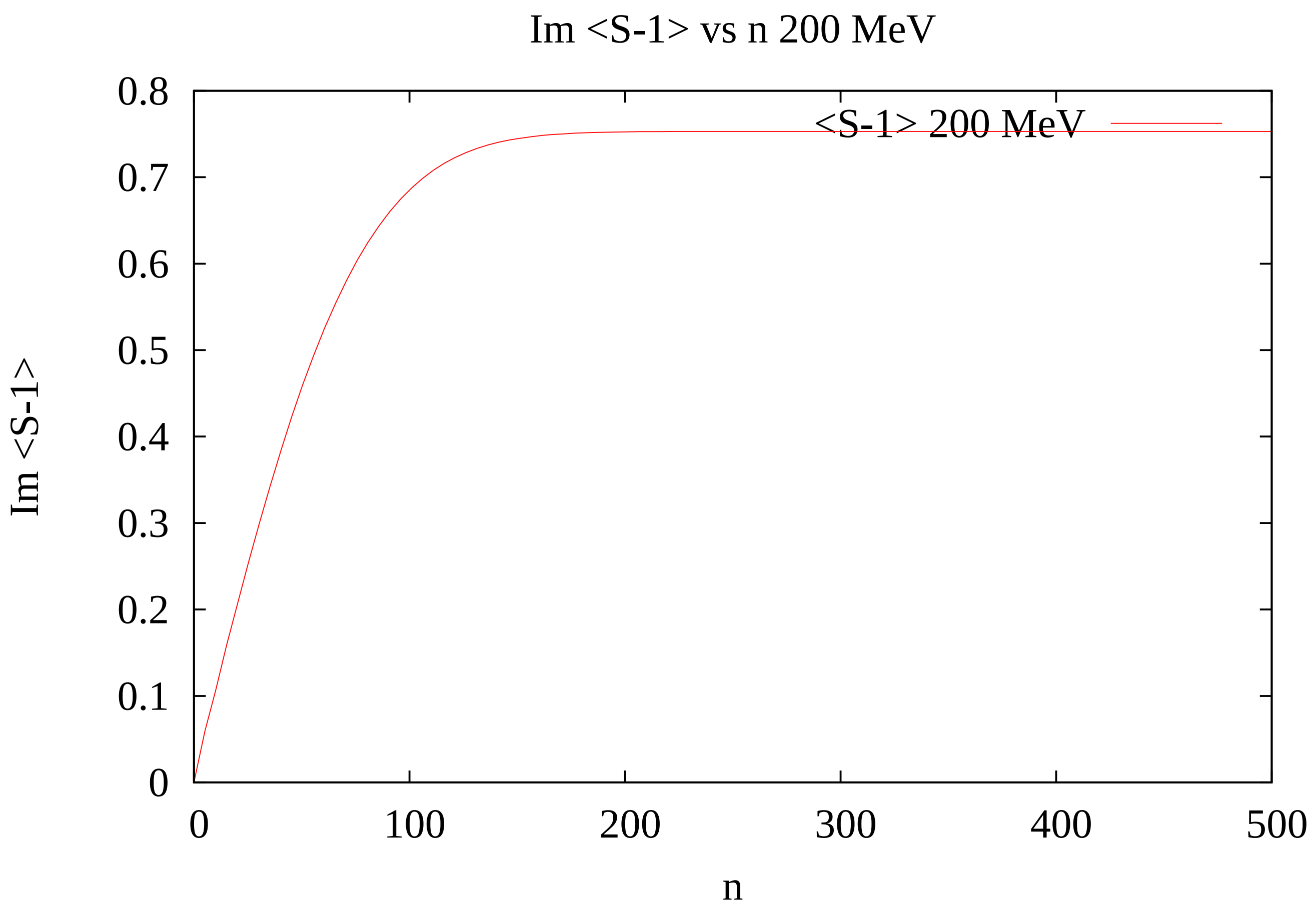

FIG. 6: Im ($\langle\phi|(S-1)|\phi\rangle$) vs n 200[MeV] $\alpha = 32000[\mathrm{GeV}^{-2}]$

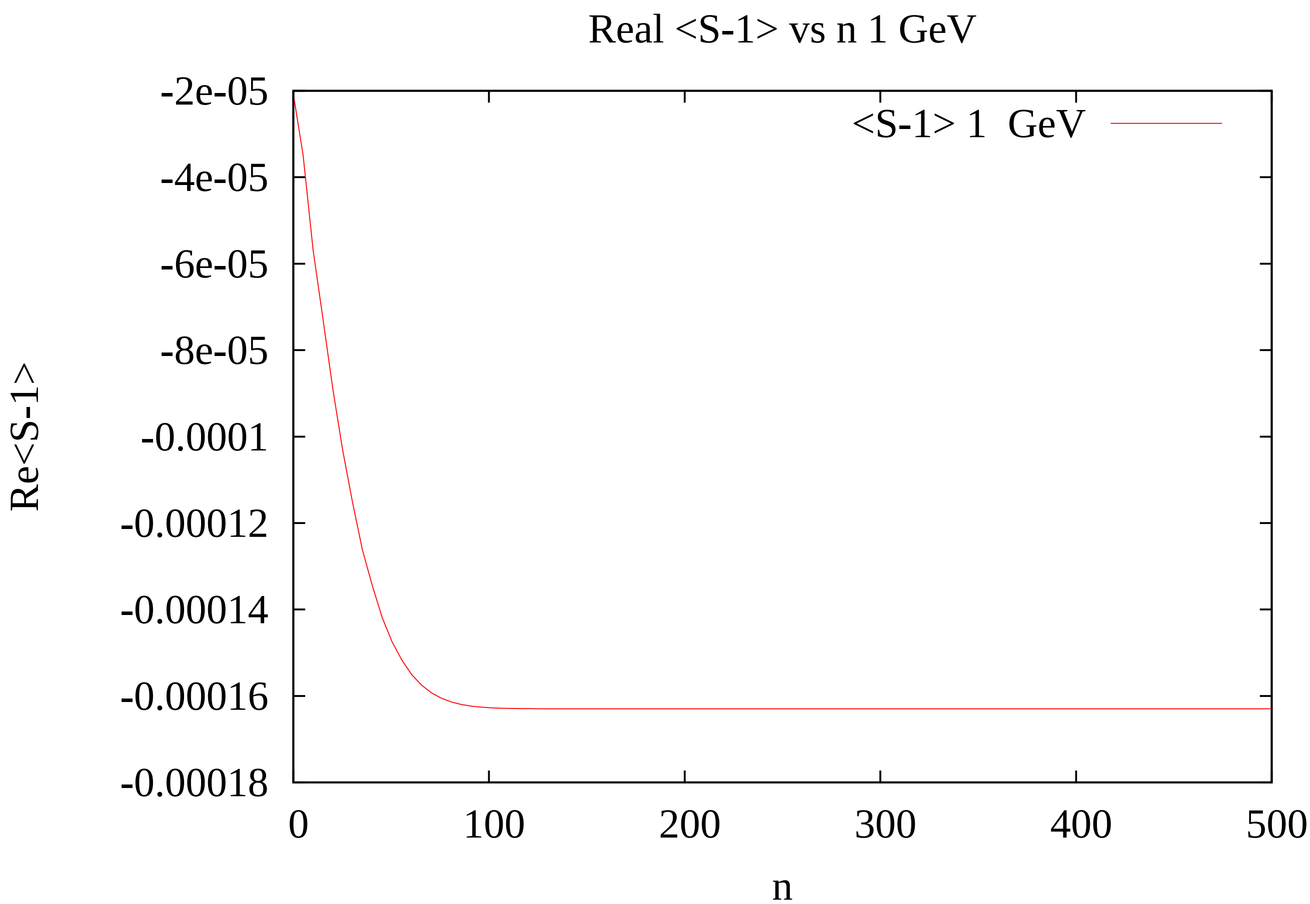

FIG. 7: Re $(\langle\phi|(S-1)|\phi\rangle)$ vs n 1[GeV] $\alpha = 600[\mathrm{GeV}^{-2}]$

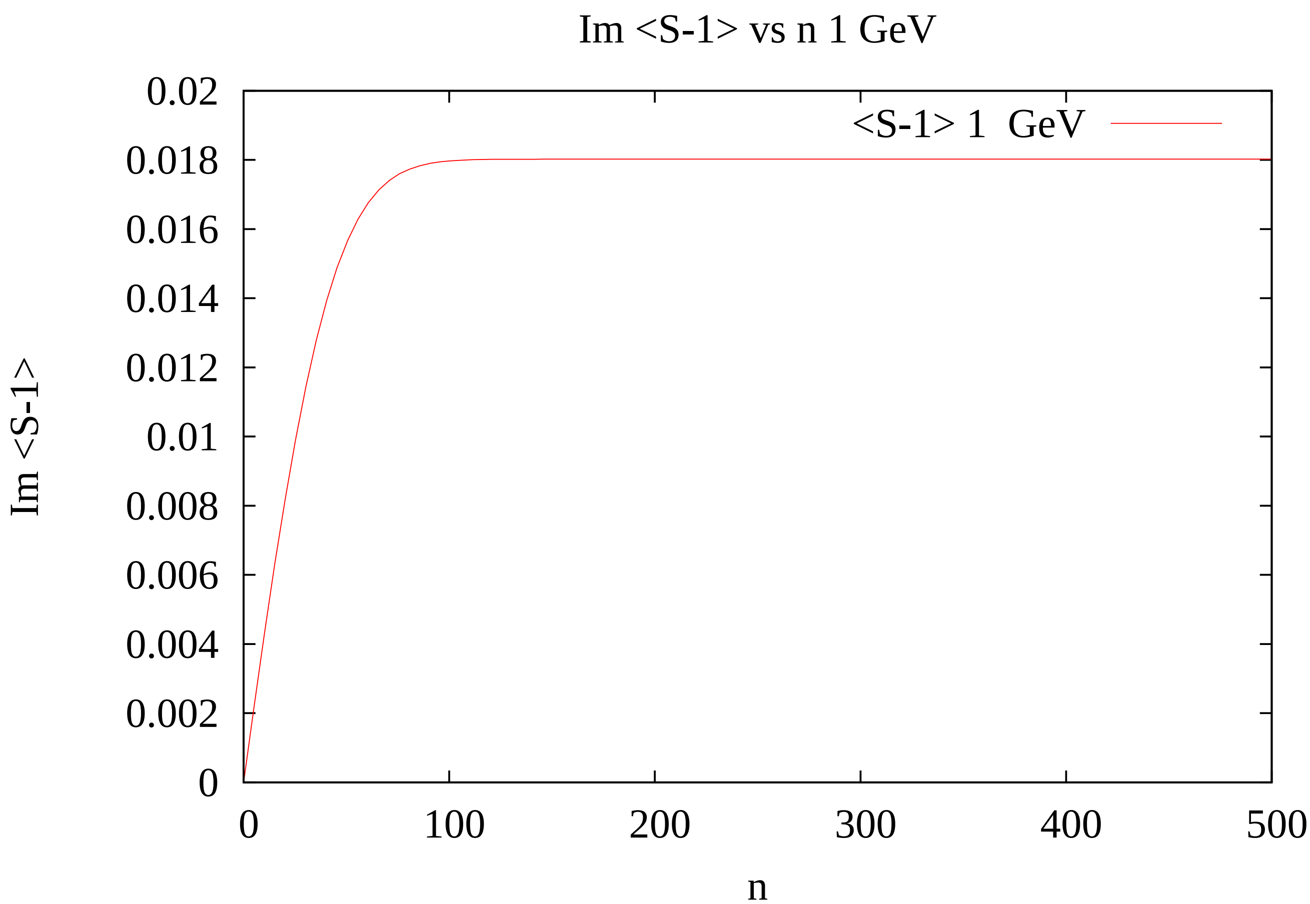

FIG. 8: Im $(\langle\phi|(S-1)|\phi\rangle)$ vs n 1[GeV] $\alpha = 600[\mathrm{GeV}^{-2}]$

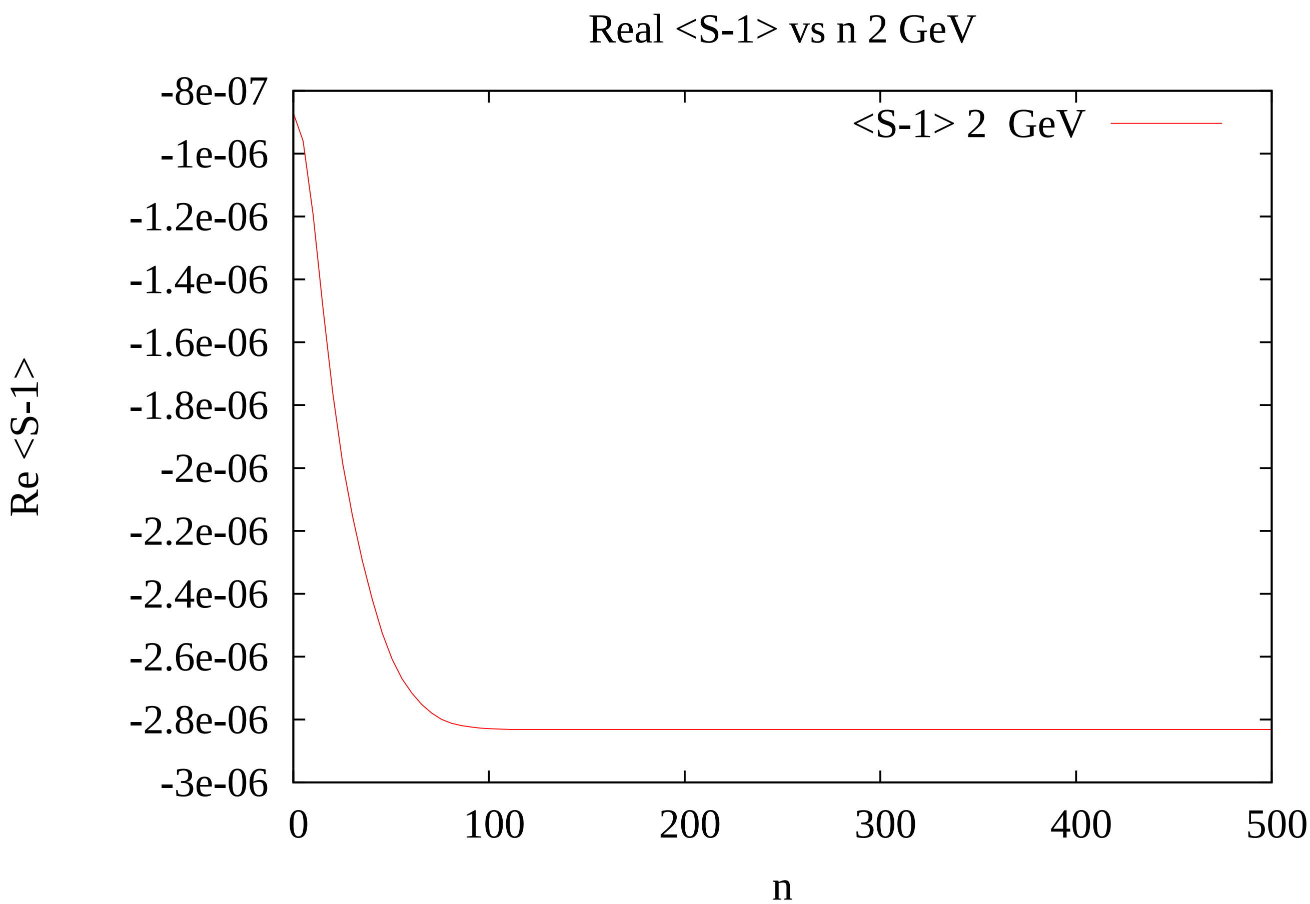

FIG. 9: $\mathrm{Re}(\langle\phi|(S-1)|\phi\rangle)$ vs n 2[GeV] $\alpha = 250[\mathrm{GeV}^{-2}]$

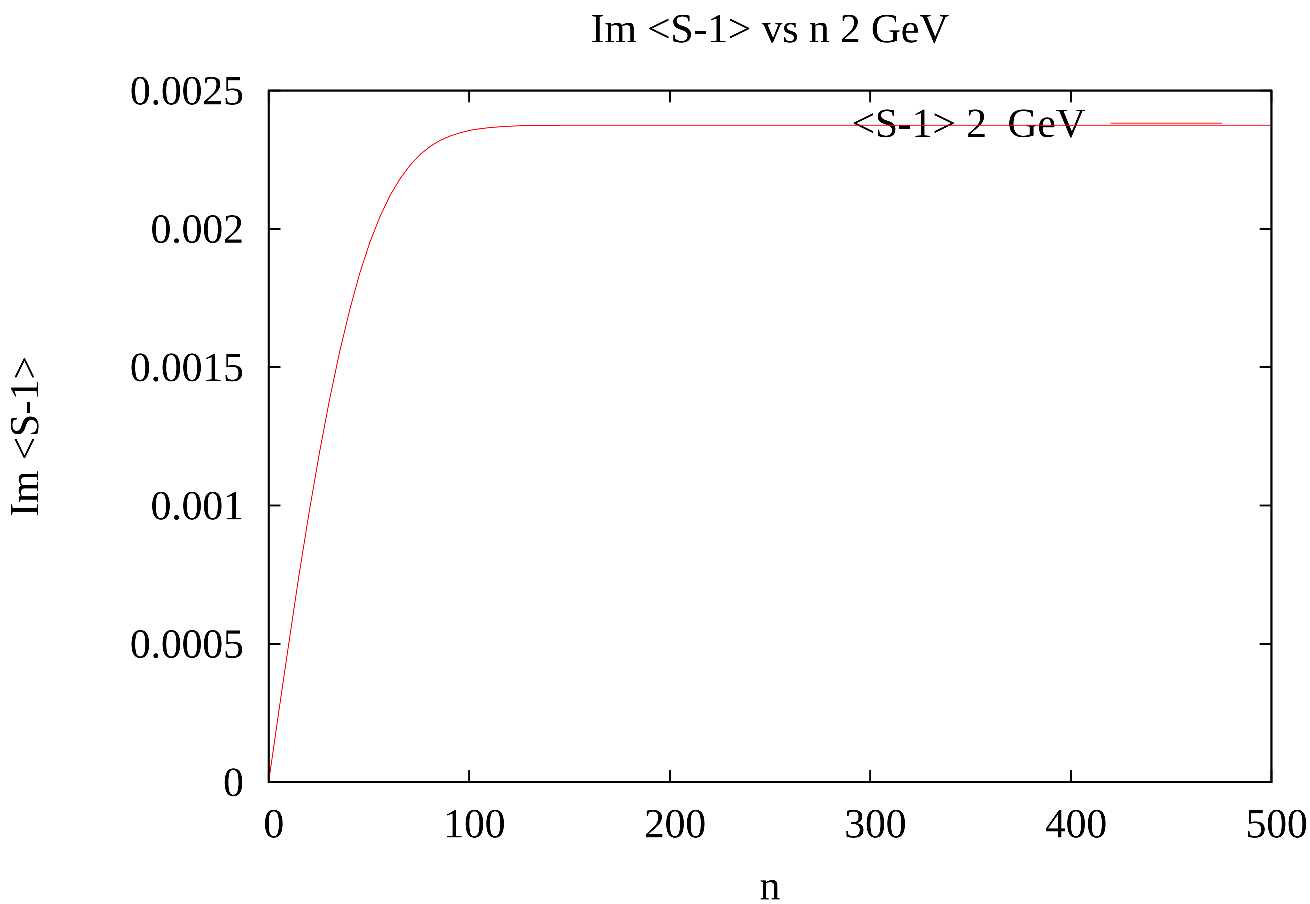

FIG. 10: Im($\langle\phi|(S-1)|\phi\rangle$) vs n 2[GeV] $\alpha = 250[\mathrm{GeV}^{-2}]$

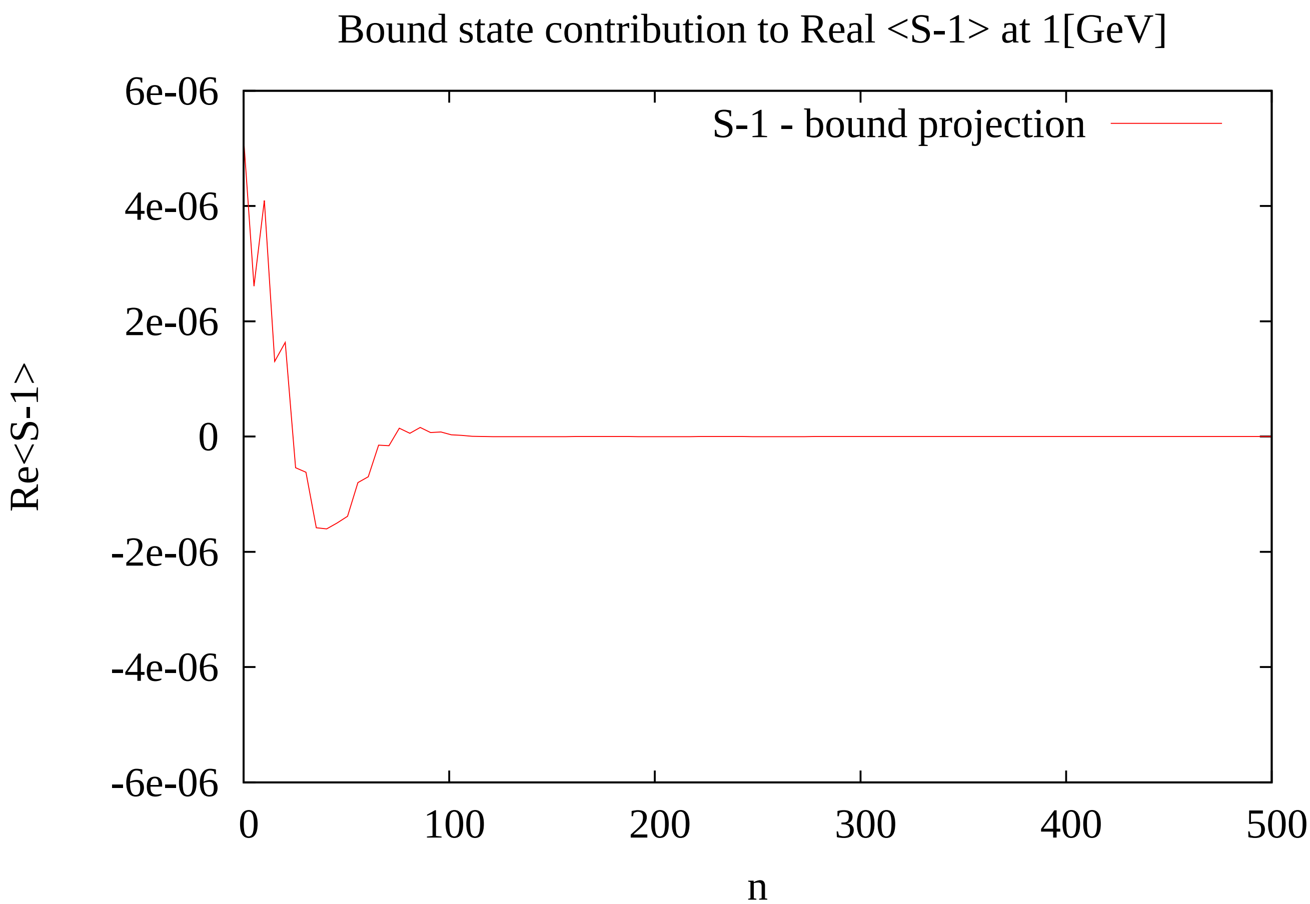

FIG. 11: Bound state contribution to Re $(\langle\phi|(S-1)|\phi\rangle)$ vs n 1[GeV] $\alpha = 600[\mathrm{GeV}^{-2}]$

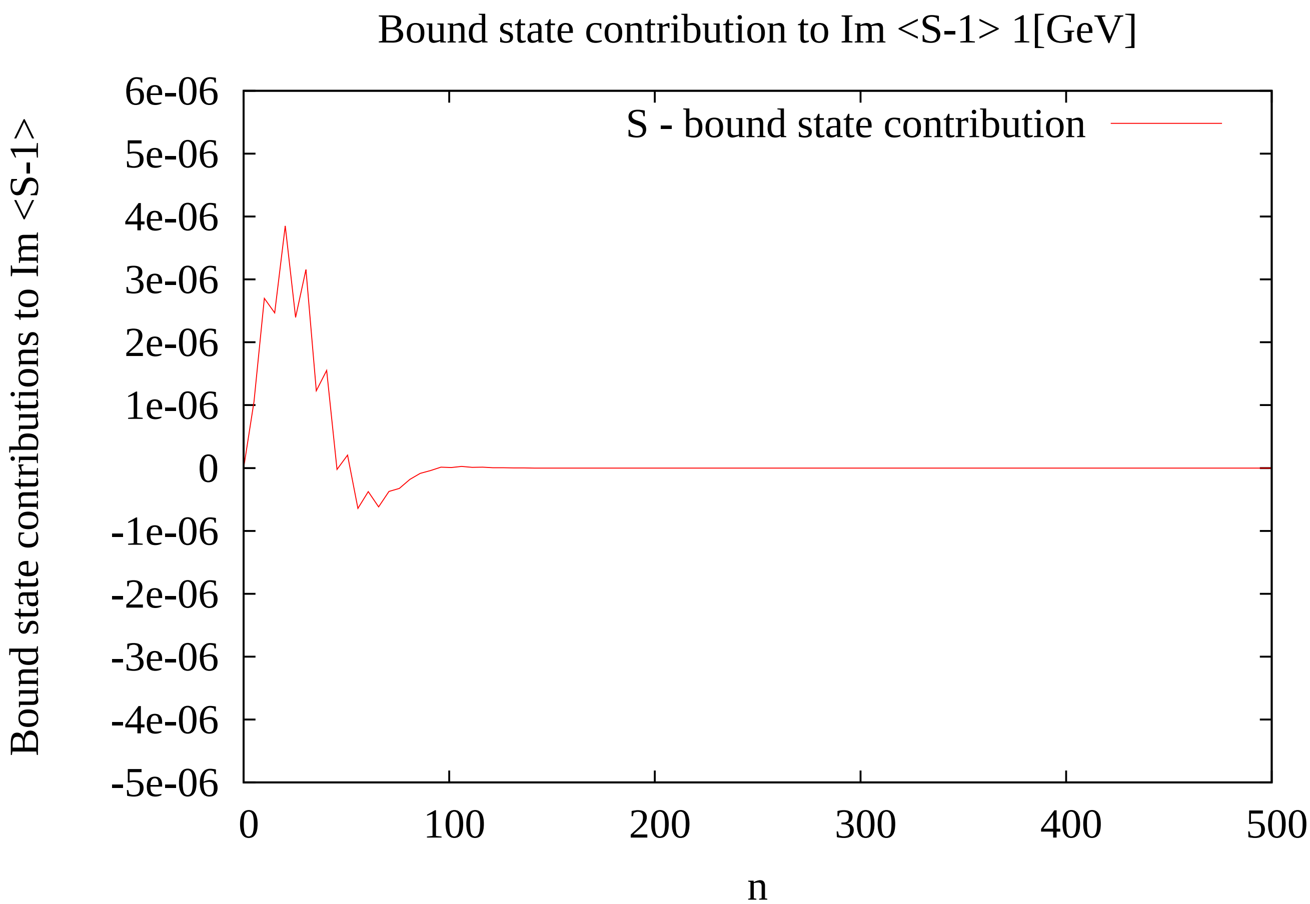

FIG. 12: Bound state contribution to Im $(\langle\phi|(S-1)|\phi\rangle)$ vs n 1[GeV] $\alpha = 600[\mathrm{GeV}^{-2}]$