Scattering and reflection positivity in relativistic Euclidean quantum mechanics

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Abstract

In this paper I discuss a formulation of relativistic few-particle scattering theory where the dynamical input is a collection of reflection-positive Euclidean covariant Green functions. This formulation of relativistic quantum mechanics has the advantage that all calculations can be performed using Euclidean variables. The construction of the Hilbert space inner product requires that the Green functions are reflection positive, which is a non-trivial constraint on the dynamics. In this paper I discuss the structure of reflection positive Green functions. I show that scattering wave operators exist when the dynamics is given by a Green function with the structure discussed in this work. Techniques for computing scattering observables in this Euclidean framework are discussed.

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I. INTRODUCTION

In this paper I discuss a formulation of relativistic few-body quantum mechanics where the dynamics is introduced through a collection of Euclidean "Green" functions. The motivation for this approach is similar to the motivation for using Euclidean methods in quantum field theory. The complicated singularities of the Minkowski Green functions are replaced by well-behaved Euclidean Green functions. In addition, while the input can in principle be directly calculated from a field theory, for the purpose of making realistic models it can also be treated phenomenologically. Cluster properties, which are difficult to realize in direct interaction formulations of relativistic quantum mechanics, are relatively simple to realize in the Euclidean formulation. The surprising features of this approach are (1) all relevant calculations can be performed entirely in Euclidean space, (2) there is numerical evidence that GeV-scale scattering calculations can be performed without analytic continuation (3) and finite Poincaré transformations can be performed without leaving Euclidean space.

In the next section I discuss the general structure of this Euclidean formulation of relativistic quantum mechanics. In section three I discuss the construction of the model Hilbert space and the construction of a representation of the Poincaré Lie algebra on this space. This defines what I mean by a relativistic quantum theory. In section four I illustrate how the usual Minkowski description of a single particle is realized in the Euclidean framework. In section five I discuss the important constraint of reflection positivity, which is a required property of acceptable Euclidean Green functions. I illustrate some of the difficulties with establishing reflection positivity and generalize a known one-dimensional result to demonstrate the existence of a large class of reflection positive four-point Green functions. In section six I discuss the formulation of the scattering problem when the dynamics appears in the kernel of a Hilbert space inner product. I show that Møller wave operators exist for the class of reflection positive Green function introduced in the previous section. In section seven I discuss methods for the numerical computation of scattering observables in this formalism. Section eight provides a summary of the results and a discussion of open problems. The appendix generalizes the discussion in section four to massive particles with any spin.

II. INPUT

The input to a Euclidean relativistic quantum theory is a collection of Euclidean invariant distributions, which I refer to as quasi-Schwinger functions

$$\{S_{m:n}(\mathsf{x}_m, \cdots, \mathsf{x}_1; \mathsf{y}_1, \cdots, \mathsf{y}_n)\}. \tag{2.1}$$

Here san-serif variables $\mathsf{x}_i = (\mathsf{x}_i^0, \mathbf{x}_i)$ and $\mathsf{y}_i := (\mathsf{y}_i^0, \mathbf{y}_i)$ are Euclidean space-time variables. In principle the collection of quasi-Schwinger functions could be infinite, however finite collections are appropriate for making few-body models. The y and x variables are associated with initial and final degrees of freedom respectively. For purpose of illustration I consider the spinless case. The treatment of particles with spin is discussed in the appendix.

The quasi-Schwinger functions are assumed to have the following properties:

E.1 Euclidean invariance:

$$S_{m:n}(\mathsf{x}_m, \cdots, \mathsf{x}_1; \mathsf{y}_1, \cdots, \mathsf{y}_n) =$$
$$S_{m:n}(O\mathsf{x}_m + \mathsf{a}, \cdots, O\mathsf{x}_1 + \mathsf{a}; O\mathsf{y}_1 + \mathsf{a}, \cdots, O\mathsf{y}_n + \mathsf{a}) \tag{2.2}$$

where a is a constant Euclidean 4-vector and O is a four-dimensional orthogonal transformation.

E.2 Reflection positivity:

for any $\{f_n(\mathsf{y}_1, \cdots, \mathsf{y}_n)\}$ with support for $0 < \mathsf{y}_1^0 < \mathsf{y}_2^0 < \cdots < \mathsf{y}_{n-1}^0 < \mathsf{y}_n^0$:

$$(f, f)_M =$$
$$\sum_{k,n} \int f_k^*(\mathsf{x}_k, \cdots, \mathsf{x}_1) S_{k:n}(\theta\mathsf{x}_k, \cdots, \theta\mathsf{x}_1; \mathsf{y}_1, \cdots, \mathsf{y}_n) f_n(\mathsf{y}_1, \cdots, \mathsf{y}_n) d^{4k}\mathsf{x} d^{4n}\mathsf{y} \geq 0 \tag{2.3}$$

where $\theta\mathsf{y} := \theta(\mathsf{y}^0, \mathbf{y}) := (-\mathsf{y}^0, \mathbf{y})$ is the Euclidean time reflection operator.

In addition the quasi-Schwinger functions must satisfy

$$S_{k:n}(\mathsf{x}_k \cdots \mathsf{x}_1; \mathsf{y}_1 \cdots \mathsf{y}_n) = S^*_{n:k}(\mathsf{y}_n \cdots \mathsf{y}_1; \mathsf{x}_1 \cdots \mathsf{x}_k) \tag{2.4}$$

so $(f, g)_M = (g, f)^*_M$.

E.3 Cluster properties:

$$S_{k:n}(\mathsf{x}_k, \cdots, \mathsf{x}_1; \mathsf{y}_1, \cdots, \mathsf{y}_n) = \sum \prod S^c_{k_i:n_i}(\mathsf{x}_{ki}, \cdots, \mathsf{x}_{1i}; \mathsf{y}_{1i}, \cdots, \mathsf{y}_{ni}) \qquad (2.5)$$

where the $S^c_{k_i:n_i}$ satisfy [**E.1-E.2**], $\sum k_i = k$ and $\sum n_i = n$. The sum runs over all partitions of the initial and final variables into clusters containing at least one initial and one final coordinate in each cluster. For a given partition the products are over clusters in the partition. The $S^c_{k_i:n_i}$ vanish as distributions whenever any of the coordinates are asymptotically separated.

For quasi-Schwinger functions describing identical particles the initial and final variables are separately symmetric or anti-symmetric under interchange of the Euclidean space time coordinates. In quantum field theory locality identifies the n-point Schwinger functions with different numbers of initial and final degrees of freedom [1]. In this work the locality requirement is relaxed which allows for the possibility of having different n-point functions for different combinations of initial and final particles.

In this formalism the connected n-point quasi-Schwinger functions replace the n-body interactions in conventional potential models. The problem is to construct a set of model quasi-Schwinger functions that lead to predictions consistent with a large class of experiments. The virtue of this type of phenomenology is that if the Schwinger functions of the field theory could be calculated, the model input could be replaced by the exact Schwinger functions.

III. THE QUANTUM THEORY

In any quantum theory the fundamental observables are probability amplitudes. These are represented by Hilbert space inner products of unit normalized vectors. The first step in the construction of a relativistic quantum model is to specify the structure of the model Hilbert space. This is a specification of the model degrees of freedom, however in the Euclidean framework the relation of the representation of the Hilbert space to the particle content of the theory is more abstract.

Given the representation of the Hilbert space, the invariance of the probability amplitudes under change of inertial coordinate system requires a unitary representation of the Poincaré group [2]. An equivalent requirement is a set of ten self-adjoint operators that satisfy the commutation relations of the Poincaré group. These operators generate unitary one-parameter groups corresponding to time translation, space translation, rotations and rotationless Lorentz transformations. Finite Poincaré transformations can be expressed as products of these one-parameter unitary groups.

In this section I define the model Hilbert space of Euclidean relativistic quantum mechanics and construct a representation of the Poincaré Lie algebra on this space.

I assume that a collection of quasi-Schwinger functions satisfying **E.1-E.3** is given. The quasi-Schwinger functions can be used to construct two quadratic forms. The first one is a pure Euclidean form where the $S_{k:n}$ serve as kernels of the quadratic form:

$$(f,g)_E = \sum_{m,n} \int f^*_m(\mathsf{x}_1,\cdots,\mathsf{x}_m) S_{m:n}(\mathsf{x}_m,\cdots,\mathsf{x}_1;\mathsf{y}_1,\cdots,\mathsf{y}_n) g_n(\mathsf{y}_1,\cdots,\mathsf{y}_n) d^{4m}\mathsf{x} d^{4n}\mathsf{y}. \quad (3.1)$$

Here vectors are represented by collections of test functions $\{g_n(\mathsf{y}_n,\cdots,\mathsf{y}_1)\}$ of Euclidean space-time variables.

Following Osterwalder and Schrader [1] I use the Euclidean time-reflection operator to define a second quadratic form

$$(f,g)_M = \sum_{m,n} \int f^*_m(\mathsf{x}_1,\cdots,\mathsf{x}_m) S_{m:n}(\theta\mathsf{x}_m,\cdots,\theta\mathsf{x}_1;\mathsf{y}_1,\cdots,\mathsf{y}_n) g_n(\mathsf{y}_1,\cdots,\mathsf{y}_n) d^{4m}\mathsf{x} d^{4n}\mathsf{y}, \quad (3.2)$$

where $\theta\mathsf{x} = \theta(\tau,\mathbf{x}) = (-\tau,\mathbf{x})$ is the Euclidean time reflection operator.

In both of these forms the integrals are over Euclidean space-time variables. The two quadratic forms differ by the presence of the Euclidean-time reflections, θ, in all of the final-state variables. The second quadratic form can be expressed abstractly in terms of the first form

$$(f,g)_M = (\Theta f,g)_E \quad (3.3)$$

where Θ represents the operator that reflects all of the final Euclidean times. I distinguish these forms by the subscripts E for Euclidean and M for Minkowski. The Minkowski designation of the second form is not obvious at this point.

To understand the motivation for introducing the second form note that the Euclidean invariance property, **E.1**, ensures that the representation $U_E(O,a)$ of the four-dimensional Euclidean group, defined by

$$((U_E(O,a)f_m)(\mathsf{y}_1,\cdots,\mathsf{y}_m) := f_m(O\mathsf{y}_1+a,\cdots,O\mathsf{y}_m+a), \quad (3.4)$$

satisfies the unitarity condition

$$(U_E(O,a)f, U_E(O,a)g)_E = (f,g)_E \tag{3.5}$$

in the sense that it leaves the first quadratic form invariant.

Each of the one-parameter subgroups of $U_E(O,a)$ can be formally expressed as the exponential of i multiplied by a Hermitian infinitesimal generator in the $(,)_E$ quadratic form. The Euclidean time-reversal operator Θ changes the sign of the infinitesimal generators of $U_E(O,a)$ that are linear in Euclidean time or derivatives with respect to Euclidean time. There are four generators with one of these properties. They are the generator of Euclidean time translation and the three generators of rotations in Euclidean space-time planes, $(\mathsf{x}^0,x,0,0),(\mathsf{x}^0,0,y,0),(\mathsf{x}^0,0,0,z)$. The additional sign change makes these four generators anti-Hermitian in the $(,)_M$ quadratic form. These can be made Hermitian with respect to the $(,)_M$ quadratic form by multiplying each one by $-i$. With this change these four generators become Hermitian with respect to the quadratic form $(,)_M$. In addition, a direct calculation shows that when combined with generators of ordinary rotations and space translations, the resulting set of 10 operators satisfy the Poincaré commutation relations.

The reason for this connection is because the covering groups of both the complex orthogonal groups and complex Lorentz groups are identical. Because of this relationship the group of real orthogonal transformations can be considered to be a subgroup of the complex Lorentz group.

To understand this relation note that the determinants of the matrices

$$X := \begin{pmatrix} t+z & x-iy \\ x+iy & t-z \end{pmatrix} \qquad \text{and} \qquad \mathsf{X} := \begin{pmatrix} i\tau+z & x-iy \\ x+iy & i\tau-z \end{pmatrix} \tag{3.6}$$

are (up to sign) the Minkowski and Euclidean line elements respectively. The determinant of both X and X are preserved by the same linear $SL(2,\mathbb{C})\times SL(2,\mathbb{C})$ transformation

$$X' = AXB^T \qquad \mathsf{X}' = A\mathsf{X}B^T \tag{3.7}$$

where A and B are $SL(2,\mathbb{C})$ matrices. While these transformations preserve the Euclidean and Minkowski displacements, they do not generally preserve the reality of the corresponding coordinates. Thus, the resulting transformations can be considered representations of either complex orthogonal or complex Lorentz transformations.

The Hermiticity of the modified generators on $(,)_M$ implies that they can be exponentiated to construct a unitary representation of the Poincaré group with respect to the form $(,)_M$. The only problem is that when the test functions, $\{f_m(\mathsf{x}_m, \cdots, \mathsf{x}_1)\}$, are unrestricted, this quadratic form has negative norm vectors. This is easily seen by choosing $f(\cdots)$ to be odd with respect to Euclidean time reflection. An indefinite inner product is not suitable for a quantum-mechanical interpretation.

Osterwalder and Schrader [1] solve this problem by restricting the test functions to have support for positive relative Euclidean times $0 < \mathsf{x}_1^0 < \mathsf{x}_2^0 < \cdots < \mathsf{x}_{m-1}^0 < \mathsf{x}_m^0$ and limiting the types of acceptable Schwinger functions to what they refer to as reflection-positive Schwinger functions.

Following Osterwalder and Schrader we call a collection of quasi-Schwinger functions $\{S_{k:n}\}$ satisfying

$$(f, f)_M \geq 0 \tag{3.8}$$

for functions with positive time and positive relative time support *reflection positive*. Reflection positivity is a property of the collection of quasi-Schwinger functions. The quadratic form $(,)_M$ can be completed to a Hilbert space inner product if the quasi Schwinger functions are reflection positive. In this case vectors are equivalence classes of collections of positive relative time test functions, where $f \sim g$ if and only if $(f - g, f - g)_M = 0$. This space is completed by including Cauchy sequences of these equivalence classes. The resulting Hilbert space is denoted by $\mathcal{H}_M$. For identical particles the symmetry of the quasi-Schwinger functions means that test functions with non-overlapping Euclidean time support can always be reordered to have positive relative time support.

One problem with this realization of the Hilbert space is that the Euclidean time translations and rotations in Euclidean space-time planes do not generally preserve the positive Euclidean time-support constraint. Nevertheless, for suitable quasi-Schwinger functions, semigroup methods can be used to establish that the generators, extracted from infinitesimal versions of these transformations, are self-adjoint operators on $\mathcal{H}_M$. Formally the one-parameter group of Euclidean time translation on $\mathcal{H}_E$ becomes a contractive Hermitian semigroup [3] on $\mathcal{H}_M$ and the one-parameter groups of rotations in Euclidean space-time planes in $\mathcal{H}_E$ become local symmetric semigroups [4][5][6] of Lorentz transformations with imaginary rapidity on $\mathcal{H}_M$. Both of these semigroups have self-adjoint generators on $\mathcal{H}_M$ for quasi-Schwinger functions where the continuity properties of the semigroups can be established. In what follows I assume that the model quasi-Schwinger functions satisfy these conditions.

Explicit expressions for the Poincaré generators are constructed by computing the generators of the one-parameter subgroups of $U_E(O,a)$ on suitable domains and multiplying by the appropriate factors of 1 for the generators that commute with Θ and $-i$ by the generators that anti-commute with Θ. The parameters of the Euclidean group are angles of rotation and displacements of spacetime coordinates. The resulting Poincaré generators have the following representations:

$$\langle \mathsf{x}|H|\mathbf{f}\rangle := -\frac{d}{da}\langle \mathsf{x}-(a,0,0,0)|\mathbf{f}\rangle|_{a=0}e =$$
$$\{0, \frac{\partial}{\partial \mathsf{x}^0_{11}} f_1(\mathsf{x}_{11}), \left(\frac{\partial}{\partial \mathsf{x}^0_{21}} + \frac{\partial}{\partial \mathsf{x}^0_{22}}\right) f_2(\mathsf{x}_{21},\mathsf{x}_{22}), \cdots\} \tag{3.9}$$

$$\langle \mathsf{x}|\mathbf{P}|\mathbf{f}\rangle = i\boldsymbol{\nabla}_{\mathbf{a}}\langle \mathsf{x}-(0,\mathbf{a})|\mathbf{f}\rangle|_{|\mathbf{a}=0} =$$
$$\{0, -i\frac{\partial}{\partial \boldsymbol{\mathsf{x}}_{11}} f_1(\mathsf{x}_{11}), -i\left(\frac{\partial}{\partial \boldsymbol{\mathsf{x}}_{21}} + \frac{\partial}{\partial \boldsymbol{\mathsf{x}}_{22}}\right) f_2(\mathsf{x}_{21},\mathsf{x}_{22}), \cdots\} \tag{3.10}$$

$$\langle \mathsf{x}|\mathbf{J}|\mathbf{f}\rangle := \{0, -i\boldsymbol{\mathsf{x}}_{11}\times\frac{\partial}{\partial \boldsymbol{\mathsf{x}}_{11}} f_1(\mathsf{x}_{11}),$$
$$-i\left(\vec{\mathsf{x}}_{21}\times\frac{\partial}{\partial \boldsymbol{\mathsf{x}}_{21}} + \boldsymbol{\mathsf{x}}_{22}\times\frac{\partial}{\partial \boldsymbol{\mathsf{x}}_{22}}\right) f_2(\mathsf{x}_{21},\mathsf{x}_{22}), \cdots\} \tag{3.11}$$

$$\langle \mathsf{x}|\mathbf{K}|\mathbf{f}\rangle := \{0, \left(\boldsymbol{\mathsf{x}}_{11}\frac{\partial}{\partial \mathsf{x}^0_{11}} - \mathsf{x}^0_{11}\frac{\partial}{\partial \boldsymbol{\mathsf{x}}_{11}}\right) f_1(\mathsf{x}_{11}),$$
$$\left(\boldsymbol{\mathsf{x}}_{21}\frac{\partial}{\partial \mathsf{x}^0_{21}} - \mathsf{x}^0_{21}\frac{\partial}{\partial \boldsymbol{\mathsf{x}}_{21}} + \boldsymbol{\mathsf{x}}_{22}\frac{\partial}{\partial \mathsf{x}^0_{22}} - \mathsf{x}^0_{22}\frac{\partial}{\partial \boldsymbol{\mathsf{x}}_{22}}\right) f_2(\mathsf{x}_{21},\mathsf{x}_{22}), \cdots\}. \tag{3.12}$$

Here H is the Hamiltonian, $\mathbf{P}$ is the linear momentum operator, $\mathbf{J}$ is the angular momentum operator and $\mathbf{K}$ is the rotationless boost generator. These expressions can be used to get the following representation for the square of the mass operator

$$M^2 = H^2 - P^2 \tag{3.13}$$

$$\langle \mathsf{x}|M^2|\mathbf{f}\rangle := \{0, -\nabla^2_{11} f_1(\mathsf{x}_{11}), -(\nabla_{21}+\nabla_{22})^2 f_2(\mathsf{x}_{21},\mathsf{x}_{22}), \cdots\} \tag{3.14}$$

where ∇^2 is the 4-dimensional Euclidean Laplacian.

The easiest dynamical quantities to calculate in this representation are matrix elements of $e^{-\beta H}$, which are the result of translating all of the initial Euclidean time variables to the right by the same positive constant β:

$$\langle \mathsf{x}|e^{-\beta H}|\mathbf{f}\rangle := \{f_0, f_1(\mathsf{x}_{21}-\beta\hat{\boldsymbol{\tau}}), f_2(\mathsf{x}_{21}-\beta\hat{\boldsymbol{\tau}}, \mathsf{x}_{22}-\beta\hat{\boldsymbol{\tau}}), \cdots\}. \tag{3.15}$$

The representation of the Poincaré Lie algebra (3.9-3.12) on the model Hilbert space $\mathcal{H}_M$ defines the Euclidean representation of a relativistic quantum theory.

In addition to giving a Euclidean representation of the Hilbert space scalar product, reflection positivity and repeated applications of the Schwartz inequality [7] give the inequalities

$$\|e^{-\beta H}|\psi\rangle\|_M \leq \|e^{-2^n\beta H}|\psi\rangle\|_M^{1/2^n} \||\psi\rangle\|_M^{1-1/2^n} \leq \||\psi\rangle\|_E^{1/2^n} \||\psi\rangle\|_M^{1-1/2^n} \tag{3.16}$$

for and n and $\beta > 0$. Taking the limit as $n \to \infty$

$$\|e^{-\beta H}|\psi\rangle\|_M \leq \||\psi\rangle\|_M \tag{3.17}$$

shows that $e^{-\beta H}$ is a contractive semigroup on $\mathcal{H}_M$, which means that H has a spectrum bounded from below.

For most purposes $e^{-\beta H}$ can be used to replace the Hamiltonian, where β is treated as an adjustable scale setting parameter. $e^{-\beta H}$ has the same eigenstates as H, the eigenvalues are simply related, and $-e^{-\beta H}$ can be used in place of the Hamiltonian to formulate a scattering theory. The operator $e^{-\beta H}$ has the additional advantage that it is bounded.

IV. REFLECTION POSITIVITY AND PARTICLES

The Minkowski designation of the scalar product $(,)_M$, where everything is Euclidean, is surprising at first glance since no analytic continuations have been performed.

The best way to understand this is to consider the example of the two-point Schwinger function for a particle of mass m. If this is the only Schwinger function in the model then the "Minkowski" inner product is [7]

$$(f,f)_M = \int f^*(\mathsf{x}) S_{1:1}(\theta\mathsf{x};\mathsf{y}) f(\mathsf{y}) d^4\mathsf{x} d^4\mathsf{y}$$

$$= \frac{1}{(2\pi)^4}\int f^*(\mathsf{x}) \frac{e^{i\mathsf{p}\cdot(\theta\mathsf{x}-\mathsf{y})}}{\mathsf{p}^2+m^2} f(\mathsf{y}) d^4\mathsf{x} d^4\mathsf{y} d^4\mathsf{p} \tag{4.1}$$

$$= \frac{1}{(2\pi)^4} \int d^4\mathsf{x} d^4\mathsf{y} d^4\mathsf{p} f(\mathsf{x}) \frac{e^{-i\mathsf{p}_0\cdot(\mathsf{x}_0+\mathsf{y}_0)+i\mathsf{p}\cdot(\mathsf{x}-\mathsf{y})}}{(\mathsf{p}^0+i\omega_m(\mathsf{p}))(\mathsf{p}^0-i\omega_m(\mathsf{p}))} f(\mathsf{y})$$

$$= \int \frac{d^3\mathsf{p}}{2\omega_m(\mathsf{p})} |\psi(\mathsf{p})|^2 \geq 0 \tag{4.2}$$

where $\omega_m(\mathsf{p}) = \sqrt{m^2+\mathsf{p}^2}$ is the energy of a particle of mass m and momentum p. The wave function $\psi(\mathbf{p})$, expressed as a square integrable function of momentum, is related to the Euclidean positive-time support function $f(\mathsf{x})$ by

$$\psi(\mathbf{p}) := \frac{1}{(2\pi)^{3/2}} \int_0^\infty d\mathsf{x}_0 \int d\mathbf{x} f(\mathsf{x}) e^{-\omega_m(\mathbf{p})\mathsf{y}_0 - i\mathbf{p}\cdot\mathbf{x}}. \tag{4.3}$$

In this example the calculation of the norm involves only Euclidean quantities, while the final result has the familiar Minkowski form involving the Lorentz invariant measure for a free particle of mass m. This calculation also demonstrates that the Schwinger function for a scalar field of mass m is reflection positive.

Another simple calculation shows that the representation (3.9) of the Hamiltonian has the correct interpretation:

$$(f, Hf)_M = \int f^*(x) S_{1:1}(\theta x; y) \frac{\partial}{\partial \mathsf{y}^0} f(y) d^4x d^4y \tag{4.4}$$

$$= \int \frac{d^3\mathsf{p}}{2\omega_m(\mathbf{p})} \omega_m(\mathbf{p}) |\Psi(\mathbf{p}, m)|^2 \tag{4.5}$$

where integration by parts was used to show that the Hamiltonian has the interpretation of the energy of a free particle of mass m. The spectral condition is clearly satisfied.

Direct calculations also shows that for this Schwinger function, any f is an eigenstate of M^2 with eigenvalue m^2,

$$(g, M^2 f)_M = \int g^*(x) S_{1:1}(\theta x; y) \nabla^2_{\mathsf{y}} f(y) d^4x d^4y \tag{4.6}$$

$$= m^2 (g, M^2 f)_M. \tag{4.7}$$

Reflection positive two-point Schwinger functions for particles with any $m > 0$ and spin are given in the appendix.

This one-particle example illustrates how Lorentz invariant inner products emerge from calculations involving *purely Euclidean quantities*. This example also shows how reflection positivity gives both the Minkowski scalar product as well as the spectral condition on the Hamiltonian.

V. REFLECTION POSITIVITY

One difficulty with constructing reflection positive multipoint functions is that there are non-trivial functions associated with zero norm vectors. This can be seen in the integral, (4.3), which can be expressed as a partial Fourier transform over the spatial variables and Laplace transform over the Euclidean time variable

$$\psi(\mathbf{p}) := \int_0^\infty d\mathsf{y}_0 \int d^3\mathsf{y}\tilde{f}(\mathsf{y}^0, \mathbf{p}) e^{-\omega_m(\mathbf{p})\mathsf{y}_0}. \tag{5.1}$$

If $\tilde{f}(\mathsf{y}^0, \mathbf{p})$ is orthogonal to $e^{-\omega_m(\mathbf{p})\mathsf{y}_0}$ as a function of y_0 for each value of $\mathbf{p}$, then this vanishes, leading to a non-trivial positive Euclidean-time-support function, $\tilde{f}(\mathsf{y}^0, \mathbf{p})$, corresponding to a 0 norm vector.

If a free quasi-Schwinger functions is perturbed by adding a small connected perturbation that is only required to be Euclidean invariant, then a function representing a zero norm vector with respect to the product of the free quasi-Schwinger functions might have a non-zero contribution due to the perturbation. This contribution can always be made negative using the freedom to adjust the sign of the perturbation. The means that reflection positivity is not stable with respect to small Euclidean invariant perturbations. The practical consequence of this observation is that the solution of the Euclidean Bethe-Salpeter equation,

$$S_{2:2} = S_{1:1}S_{1:1} + S_{1:1}S_{1:1}KS_{2:2}, \tag{5.2}$$

with a model Euclidean invariant kernel, is not automatically reflection positivity, even if the kernel is small. I am not aware of any general results about what kind of restrictions are needed on Euclidean invariant Bethe-Salpeter kernels K for $S_{2:2}$ to be reflection positive.

An alternative strategy is to explore the direct construction of connected reflection positive four-point functions. The resulting four-point quasi-Schwinger functions could be used directly or used to deduce sufficient properties of Bethe-Salpeter kernels by expressing them in terms of known reflection-positive quasi-Schwinger functions:

$$K = S_{1:1}^{-1}S_{1:1}^{-1} - S_{2:2}^{-1}. \tag{5.3}$$

In this expression the inverses are interpreted as inverses of operators on the "Euclidean" Hilbert space.

Since cluster properties imply that the four-point function can be expressed as a sum of products of two-point functions and a connected four-point function,

$$S_{2:2} = \sum S_{1:1} S_{1:1} + S^{c}_{2:2}, \tag{5.4}$$

in order to show $S_{2:2}$ is reflection positive it is sufficient to show that the connected four-point function, $S^{c}_{2:2}$, is reflection positive.

In one dimension there is a result due to Widder [8][9][10] from classical analysis that gives the general structure of reflection positive two-point functions. Widder's theorem points out that any kernel $k(s)$ satisfying the reflection positivity condition

$$\int f(\theta s) k(s-t) f(t) ds dt = \int f(s) k(-s-t) f(t) ds dt \geq 0 \tag{5.5}$$

can be expressed in the exponential form

$$k(-\tau' - \tau) = \int e^{-\lambda(\tau'+\tau)} \rho(\lambda) d\lambda \tag{5.6}$$

for some positive density $\rho(\lambda)$. Since in this example, $\tau', \tau > 0$, we can write the kernel as

$$k(-\tau' - \tau) = \int_0^\infty \frac{\lambda}{\pi} \rho(\lambda) d\lambda \int_{-\infty}^{\infty} ds \frac{e^{-is(\tau'+\tau)}}{s^2 + \lambda^2}. \tag{5.7}$$

This has the form of a one-dimensional version of the Källén-Lehmann representation of a two-point Schwinger function.

The Widder result suggests that connected four-point quasi-Schwinger functions with the structure

$$\int d^4\mathsf{p}_1 d^4\mathsf{p}_2 d^4\mathsf{p}_3 dm_a dm_c dm_b e^{-i\mathsf{p}_1\cdot(\mathsf{y}'-\mathsf{x}')} e^{-i\mathsf{p}_2\cdot(\mathsf{x}'-\mathsf{x})} e^{-i\mathsf{p}_3\cdot(\mathsf{x}-\mathsf{y})} \times$$
$$\frac{v(m_a, \mathsf{p}_1, m_c, \mathsf{p}_2, m_b, \mathsf{p}_3)}{(\mathsf{p}_1^2 + m_a^2)(\mathsf{p}_2^2 + m_c^2)(\mathsf{p}_3^2 + m_b^2)}. \tag{5.8}$$

would be reflection positive for suitable Euclidean invariant kernels, $v(m_a, \mathsf{p}_1, m_c, \mathsf{p}_2, m_b, \mathsf{p}_3)$. This structure does not provide a general representation for a Euclidean invariant reflection positive four point functions, as one gets in Widder's theorem. On the other hand Widder's theorem suggests that reflection positivity and Euclidean covariance strongly constrain the class of reflection positive four-point functions.

The "Minkowski" scalar product over Euclidean spacetime variables for quasi-Schwinger functions of the form (5.8) is

$$\int d^4\mathsf{x} d^4\mathsf{y} d^4\mathsf{x}' d^4\mathsf{y} d^4\mathsf{p}_1 d^4\mathsf{p}_2 d^4\mathsf{p}_3 dm_a dm_c dm_b f_e^*(-\mathsf{y}^{0\prime}, \mathbf{y}') g_e^*(-\mathsf{x}^{0\prime}, \mathbf{x}') e^{-i\mathsf{p}_1\cdot(\mathsf{y}'-\mathsf{x}')} \times$$

$$e^{-i\mathsf{p}_2\cdot(\mathsf{x}'-\mathsf{x})}e^{-i\mathsf{p}_3\cdot(\mathsf{x}-\mathsf{y})}\frac{v(m_a,\mathsf{p}_1,m_c,\mathsf{p}_2,m_b,\mathsf{p}_3)}{(\mathsf{p}_1^2+m_a^2)(\mathsf{p}_2^2+m_2^2)(\mathsf{p}_3^2+m_b^2)}f_e(\mathsf{y}^0,\mathbf{y})g_e(\mathsf{x}^0\mathbf{x}). \tag{5.9}$$

I assume that the functions satisfy the support condition, $f_e(\mathsf{y}^0,\mathbf{y})$ and $g_e(\mathsf{x}^0,\mathbf{x})$ can be non-zero only for $0<\mathsf{x}^0<\mathsf{y}^0$.

The most straightforward assumption is to choose $v(m_a,\mathsf{p}_1,m_2,\mathsf{p}_2,m_b,\mathsf{p}_3)$ analytic in the upper-half p_i^0 planes. In this case the p_i^0 integrals can be performed as in the 1 dimensional case, where the convergence in the upper-half plane is ensured by the Euclidean time-support constraints provided the kernel is polynomially bounded, and the Minkowski boundary value is a tempered distribution. The integral over the vector variables leads to Fourier transforms of the vector variables and results in the expression

$$(2\pi)^9\int d\mathsf{x}^0 d\mathsf{y}^0 d\mathsf{x}^{0\prime} d\mathsf{y}^{0\prime} d\mathsf{p}_1 d\mathsf{p}_2 d\mathsf{p}_3 dm_a dm_c dm_b \tilde{f}_e^*(\mathsf{y}^{0\prime},-\mathbf{p}_1)\tilde{g}_e^*(\mathsf{x}^{0\prime},\mathbf{p}_1-\mathbf{p}_2)\times$$
$$e^{-\omega_{m_a}(\mathbf{p}_1)(\mathsf{y}^{0\prime}-\mathsf{x}^{0\prime})}e^{-\omega_{m_c}(\mathbf{p}_2)(\mathsf{x}^{0\prime}+\mathsf{x}^0)}e^{-\omega_{m_b}(\mathbf{p}_3)(\mathsf{y}^0-\mathsf{x}^0)}\times$$
$$\frac{v(m_1,(i\omega_{m_a}(\mathbf{p}_1),\mathbf{p}_1),m_c,(i\omega_{m_c}(\mathbf{p}_2),\mathbf{p}_2),(i\omega_{m_b}(\mathbf{p}_3),\mathbf{p}_3),m_b)}{2\omega_{m_a}(\mathbf{p}_1)2\omega_{m_c}(\mathbf{p}_2)2\omega_{m_b}(\mathbf{p}_3)}\tilde{f}_e(\mathsf{y}^0,\mathbf{p}_3-\mathbf{p}_2)\tilde{g}_e(\mathsf{x}^0,-\mathbf{p}_3). \tag{5.10}$$

The kernel, which was initially a Euclidean invariant function of Euclidean scalar product becomes a Lorentz invariant function of Lorentz invariant inner products when all of the $\mathsf{p}^0\to i\omega_{m_i}(\mathbf{p}_i)$. This connected quasi-Schwinger function will be reflection positive if

$$v(m_a,(i\omega_{m_a}(\mathbf{p}_1),\mathbf{p}_1),m_c,(i\omega_{m_c}(\mathbf{p}_2),\mathbf{p}_2),(i\omega_{m_b}(\mathbf{p}_3),\mathbf{p}_3),m_b) \tag{5.11}$$

is a positive symmetric matrix in $\mathbf{p}_1,m_a$ and $\mathbf{p}_3,m_b$ for all values of $\mathbf{p}_2$ and m_c. This is not a difficult condition to realize.

This simple construction demonstrates the existence of a large class of reflection positive connected four-point quasi-Schwinger functions. Exchange symmetry puts additional constraints on the quasi-Schwinger functions. The elements of this construction provide a framework for investigating the reflection positivity of a larger class of quasi-Schwinger functions. In what follows this class of reflection-positive quasi-Schwinger functions will be used to test the existence of scattering wave operators in this Euclidean formulation of relativistic quantum mechanics.

VI. SCATTERING THEORY

In theories like quantum field theory, where the dynamics is encoded in the kernel of the quantum mechanical scalar product, there is no natural asymptotic dynamics on the physical Hilbert space to formulate scattering asymptotic conditions.

One way to formulate scattering in this situation [11][12] is to use a method that is sometimes used in non-relativistic scattering theory, which involves introducing a secondary Hilbert space of scattering asymptotes [13] [14], where the asymptotic states are treated as free particles on a second Hilbert space and wave functions defining the internal structure of the asymptotic separated particles are treated as mappings into the dynamical Hilbert space. In the field theory case [15][16] this approach has the advantage that scattering observables can be computed using strong limits, as in non-relativistic quantum mechanics. The price for this is that it is necessary to solve the subsystem "bound state" problems to formulate the scattering asymptotic condition. The benefit is that it is possible to do scattering calculations using purely Euclidean methods.

While the two-Hilbert space formulation of scattering can be used more generally, I consider the special case of a model with a fixed number of particles. In this case the dynamics is given by a single 2n-point quasi Schwinger function

$$S_{n:n}(\mathsf{x}_1, \cdots, \mathsf{x}_n : \mathsf{y}_1, \cdots, \mathsf{y}_n). \tag{6.1}$$

The cluster condition (**E.3**) means that for a given partition of the coordinates into asymptotically separated clusters of a partition a, the quasi-Schwinger functions can be expressed as a product of quasi-Schwinger functions for each cluster plus a remainder that vanishes when the clusters are asymptotically separated:

$$S_{n:n} = \prod_{i=1}^{n_a} S_{n_{a_i}:n_{a_i}} + S^a_{n:n}. \tag{6.2}$$

In this expression n_a denotes the number of non-empty clusters in the partition a, n_{a_i} are the number of elements in the i^{th} cluster of a, where $\sum_{i=1}^{n_a} n_{a_i} = n$. $S^a_{n:n}$ is the remainder that vanishes as a distribution when the coordinates in the different clusters are asymptotically separated.

There are scattering channels α associated with the partition a if the quasi-Schwinger functions, $S_{n_{a_i}:n_{a_i}}$, for each cluster of a has point spectrum eigenstates of the square of the cluster mass operator (3.13).

These asymptotic one-particle eigenstates are represented by functions of the Euclidean variables,

$$\langle \mathsf{x}_1, \cdots, \mathsf{x}_{n_{a_i}} | m_i \rangle . \tag{6.3}$$

associated with the i^{th} cluster of a satisfying the mass eigenvalue problem

$$\langle \mathsf{x}_1, \cdots, \mathsf{x}_{n_{a_i}} | M^2 | m_i \rangle = m_i^2 \langle \mathsf{x}_1, \cdots, \mathsf{x}_{n_{a_i}} | m_i \rangle \tag{6.4}$$

where M^2 is the square of the mass operator defined by (3.13).

These mass eigenstates can be decomposed into simultaneous eigenstates of this mass, spin, linear momentum, and 3-component of the canonical spin using the translation and rotation subgroups of the Poincaré group:

$$\langle \mathsf{x}_1, \cdots, \mathsf{x}_{n_{a_i}} | m_i, \mathbf{p}_i \rangle . = \int \frac{d\mathsf{a}}{(2\pi)^{3/2}} e^{-i\mathbf{p}\cdot\mathbf{a}} \langle \mathsf{x}_1 + \mathsf{a}, \cdots, \mathsf{x}_{n_{a_i}} + \mathsf{a}) | m_i \rangle \tag{6.5}$$

and

$$\langle \mathsf{x}_1, \cdots, \mathsf{x}_{n_{a_i}} | (m_i, j_i), \mathbf{p}_i, \mu_i \rangle =$$

$$\int_{SU(2)} dR \sum_{\nu=-j_i}^{j_i} \langle R^{-1}\mathsf{x}_1, \cdots, R^{-1}\mathsf{x}_{n_{a_i}} | m_i, R^{-1}\mathbf{p}_i \rangle D^{j_i *}_{\mu_i,\nu}(R). \tag{6.6}$$

where the integral in (6.6) is over the $SU(2)$ Haar measure. Note that if the original mass eigenstates have compact positive Euclidean time support, this condition is unchanged by spatial translations or rotations. The resulting irreducible states $|(m_i, j_i), \mathbf{p}_i, \mu_i\rangle$ can be renormalized as desired.

The $SU(2)$ representation property follows by direct computation

$$\langle \mathsf{x}_1, \cdots, \mathsf{x}_{n_{a_i}} | U(R) | (m_i, j_i), \mathbf{p}_i, \mu_i \rangle =$$

$$\int_{SU(2)} dR' \sum_{\nu=-j_i}^{j_i} \langle R'^{-1}R^{-1}\mathsf{x}_1, \cdots, R'^{-1}R^{-1}\mathsf{x}_{n_{a_i}} | m_i, R'^{-1}\mathbf{p}_i \rangle D^{j_i *}_{\mu_i,\nu}(R') =$$

$$\int_{SU(2)} dR'' \sum_{\nu,\mu_i'=-j_i}^{j_i} \langle R''^{-1}\mathsf{x}_1, \cdots, R''^{-1}\mathsf{x}_{n_{a_i}} | m_i, R''^{-1}R\mathbf{p}_i \rangle D^{j_i *}_{\mu_i',\nu}(R'') D^{j_i}_{\mu_i',\mu_i}(R) =$$

$$\sum_{\mu_i'=-j_i}^{j_i} \langle \mathsf{x}_1, \cdots, \mathsf{x}_{n_{a_i}} | (m_i, j_i), R\mathbf{p}_i, \mu_i' \rangle D^{j_i}_{\mu',\mu_i}(R). \tag{6.7}$$

By construction, the states

$$\langle \mathsf{x}_1, \cdots, \mathsf{x}_{n_{a_i}} | (m_i, j_i), \mathbf{p}_i, \mu_i \rangle \tag{6.8}$$

transform irreducibly with respect to the Poincaré group:

$$\langle \mathsf{x}_1, \cdots, \mathsf{x}_{n_{a_i}} | U_{\alpha_i}(\Lambda, a) | (m_i, j_i), \mathbf{p}_i, \mu_i \rangle =$$

$$\sum_{\nu_i} \langle \mathsf{x}_1, \cdots, \mathsf{x}_{n_{a_i}} | (m_i, j_i), \mathbf{\Lambda} p_i, \nu_i \rangle D^{j_i}_{\nu_i \mu_i}(R_w(\Lambda, p_i)) \sqrt{\frac{\omega_{m_i}(\mathbf{\Lambda} p_i)}{\omega_{m_i}(\mathbf{p}_i)}} e^{i\Lambda p_i \cdot a} \tag{6.9}$$

where the factor $\sqrt{\frac{\omega_{m_i}(\mathbf{\Lambda} p_i)}{\omega_{m_i}(\mathbf{p}_i)}}$ arises because I have chosen to normalize these states with a $\delta(\mathbf{p} - \mathbf{p}')$ normalization. This demonstrates how to perform finite Poincaré transformations on Poincaré irreducible eigenstates associated with masses in the point spectrum of the mass operator.

These one-particle states can be used to construct a mapping from the mass m_i spin j_i irreducible subspace of the Poincaré group to the Hilbert space associated with cluster quasi-Schwinger function $S_{n_{a_i}:n_{a_i}}$ for the variables in the i^{th} cluster of a. I define the mapping Φ_{α_i} from the irreducible representation space, $\mathcal{H}_{m_i j_i}$, of square integrable functions of $\psi_i(\mathbf{p}_i, \mu_i)$ to functions of the Euclidean space-time variables $(\mathsf{x}_1, \cdots, \mathsf{x}_{n_{a_i}})$ by

$$\langle \mathsf{x}_1, \cdots \mathsf{x}_{n_{a_i}} | \Phi_{\alpha_i} | \psi_i \rangle := \int d\mathbf{p}_i \sum_{\mu_i = -j_i}^{j_i} \langle \mathsf{x}_1, \cdots, \mathsf{x}_{n_{a_i}} | (m_i, j_i), \mathbf{p}_i, \mu_i \rangle \psi_i(\mathbf{p}_i, \mu_i). \tag{6.10}$$

Taking tensor products defines the channel α injection operator Φ_α:

$$\Phi_\alpha := \otimes_{i=1}^{n_a} \Phi_{\alpha_i} \tag{6.11}$$

from the tensor product of the n_a irreducible representation spaces of the Poincaré group associated with the scattering channel α:

$$\mathcal{H}_\alpha = \otimes_{i=1}^{n_a} \mathcal{H}_{m_{\alpha_i}, j_{\alpha_i}} \tag{6.12}$$

to the Hilbert space $\mathcal{H}_M$. In (6.12) α denotes the scattering channel labeled by the subsystem bound states $\alpha_1 \cdots \alpha_{n_i}$ corresponding to each cluster of the partition a. In general a given partition a can be associated with any number of channels. The multi-particle interpretation of these states is only realized in asymptotic regions where the residual part, $S^a_{n:n}$, of $S_{n:n}$ in (6.2) does not contribute to the inner product.

In order to be in the Hilbert space $\mathcal{H}_M$ the function needs to have positive relative time support. This is true for the individual cluster mass eigenstates. The supports of each of these one-particle solutions can be shifted to the right using Euclidean time translations:

$$\langle \mathsf{x}_1, \cdots, \mathsf{x}_{n_{a_i}} | e^{-\beta H} | (m_i, j_i), \mathbf{p}_i, \mu_i \rangle = \langle \mathsf{x}_1, \cdots, \mathsf{x}_{n_{a_i}} | (m_i, j_i), \mathbf{p}_i, \mu_i \rangle e^{-\beta \sqrt{\mathbf{p}_i^2 + m_i^2}}$$

$$\langle \mathsf{x}_1 - \beta\hat{\boldsymbol{\tau}}, \cdots, \mathsf{x}_{n_{a_i}} - \beta\hat{\boldsymbol{\tau}} | (m_i, j_i), \mathbf{p}_i, \mu_i \rangle. \tag{6.13}$$

Since the Hamiltonian commutes with the linear momentum, all components of the spin and the mass, the resulting irreducible eigenstates are unchanged. Comparing these expressions leads to the identification

$$\langle \mathsf{x}_1, \cdots, \mathsf{x}_{n_{a_i}} | (m_i, j_i), \mathbf{p}_i, \mu_i \rangle =$$
$$\langle \mathsf{x}_1 - \beta\hat{\boldsymbol{\tau}}, \cdots, \mathsf{x}_{n_{a_i}} - \beta\hat{\boldsymbol{\tau}} | (m_i, j_i), \mathbf{p}_i, \mu_i \rangle e^{\sqrt{\mathbf{p_i}^2 + m_i^2}\beta}. \tag{6.14}$$

In this case the equality means that both functions belong to the same equivalence class. The support of the second function representing this class is shifted to the right in the Euclidean time variables by a factor β relative to the function on the left. Using different Euclidean time translations on each cluster, the support of each cluster bound state can be translated so they have positive relative time support with respect to some ordering of the Euclidean times.

This shows that Euclidean time translations behave something like gauge transformations when they act on irreducible eigenstates in the sense that they transform functions representing irreducible basis vectors to different functions in the same equivalence class.

The asymptotic Hilbert space $\mathcal{H}_{\mathcal{A}}$ is defined to be the direct sum of all of the channel subspaces (including one-cluster channels if applicable)

$$\mathcal{H}_{\mathcal{A}} = \oplus_\alpha \mathcal{H}_\alpha. \tag{6.15}$$

The injection operator

$$\Phi : \mathcal{H}_{\mathcal{A}} \to \mathcal{H}_M \tag{6.16}$$

is a mapping from the asymptotic Hilbert space to the "Minkowski" Hilbert space defined by the sum of the channel injection operators

$$\Phi := \sum_\alpha \Phi_\alpha. \tag{6.17}$$

There is a natural unitary representation of the Poincaré group on $\mathcal{H}_{\mathcal{A}}$ defined by

$$U_{\mathcal{A}}(\Lambda, a) = \oplus_{\alpha \in \mathcal{A}} \left(\otimes_{i=1}^{n_a} U_{\alpha_i}(\Lambda, a) \right) \tag{6.18}$$

where the $U_{\alpha_i}(\Lambda, a)$ are given by (6.9).

Scattering wave operators are mappings from the asymptotic Hilbert space $\mathcal{H}_{\mathcal{A}}$ to the "Minkowski" Hilbert space defined by the strong limits

$$\Omega_{\pm} := \lim_{t\to\pm\infty} e^{iHt}\Phi e^{-iH_{\mathcal{A}}t} \tag{6.19}$$

where $H_{\mathcal{A}}$ is the Hamiltonian associated with (6.18). The existence of these limits depends on properties of the quasi-Schwinger functions.

Conventional methods [17] can be used to derive sufficient conditions for this limit to exist. The first step is to write the limit as the integral of a derivative

$$\lim_{t\to\pm\infty} e^{iHt}\Phi e^{-iH_{\mathcal{A}}t} = \Phi + \int_0^{\pm\infty} \frac{d}{dt}\left(e^{iHt}\Phi e^{-iH_{\mathcal{A}}t}dt\right) =$$

$$\Phi + i\int_0^{\pm\infty} e^{iHt}(H\Phi - \Phi H_{\mathcal{A}})e^{-iH_{\mathcal{A}}t}dt. \tag{6.20}$$

A sufficient condition for the existence of the wave operator is the convergence of the integral (6.20). A sufficient condition for the convergence of this integral is the Cook condition [17], which exploits the unitarity of the time evolution operator

$$\int_a^{\infty} \|(H\Phi - \Phi H_{\mathcal{A}})e^{\mp iH_{\mathcal{A}}t}|\psi\rangle\|dt < \infty \tag{6.21}$$

and provides a bound on the norm of the integral in (6.20).

The quantities appearing in (6.21) depend on the quasi-Schwinger functions. It is sufficient to consider this integral one channel at a time. Then the integrand becomes

$$\|(H\Phi_\alpha - H_\alpha\Phi_\alpha)e^{\mp iH_\alpha t}|\psi\rangle\| =$$

$$(\psi_\alpha, e^{\pm iH_\alpha t}(\Phi^\dagger H - \Phi^\dagger H_\alpha)(\Theta\prod S_{n_{a_i}:n_{a_i}} + \Theta S^a_{n:n})(H\Phi_\alpha - \Phi_\alpha H_\alpha)\psi_\alpha)^{1/2} \tag{6.22}$$

where I have used (6.2) to express the quasi-Schwinger function as the sum of a term that involves a product of the subsystem quasi-Schwinger functions associated with a partition a into asymptotically separated subsystems plus a remainder term $S^a_{n:n}$ that vanishes asymptotically.

I consider the example of the four-point quasi-Schwinger function

$$\int_0^{\infty} \left(\psi, e^{\mp iH_0 t}(\Phi^\dagger H - H_0\Phi^\dagger)e^{\pm iH_0 t}\Theta S_{2:2}(H\Phi - \Phi H_0)e^{\mp iH_0 t}\psi\right)^{1/2} dt < \infty. \tag{6.23}$$

In (6.23) $H\Phi - \Phi H_0$ replaces the non-relativistic potential. $U_0(t)$ is the time evolution operator for the asymptotically free eigenstates. H_0 is the sum of the energies of the particles in each asymptotically cluster

$$H_0 = \sum \sqrt{m^2_{a_i} + \mathbf{p}_{a_i}}. \tag{6.24}$$

I will argue that for sufficiently well-behaved four-point quasi-Schwinger functions the integrand in (6.23) falls off like $t^{-3/2}$, which is sufficient for the Cook condition (6.21) to be satisfied.

To see this first note that cluster properties imply that the quasi-Schwinger function is the sum of a product of two-point Schwinger functions and a connected term. I consider a connected term that has the structure discussed in (5.8)

$$\int d^4\mathsf{q}_1 d^4\mathsf{q}_2 d^4\mathsf{q}_3 dm_a dm_c dm_b \frac{e^{i\mathsf{q}_1\cdot(\mathsf{x}_1-\mathsf{x}_2)+i\mathsf{q}_2\cdot(\mathsf{x}_2-\mathsf{x}_3)+i\mathsf{q}_3\cdot(\mathsf{x}_3-\mathsf{x}_4)}}{(\mathsf{q}_1^2+m_a^2)(\mathsf{q}_2^2+m_c^2)(\mathsf{q}_3^2+m_b^2)} v(\mathsf{q}_1, m_a, \mathsf{q}_2, m_c, \mathsf{q}_3, m_b) \tag{6.25}$$

where $m_c > m_a, m_b$ and the spectrum of m_c should be continuous for scattering. I have already established that this connected quasi Schwinger function is reflection positive for suitable $v(\mathsf{q}_1, m_a, \mathsf{q}_2, m_c, \mathsf{q}_3, m_b)$.

First I consider the structure of the asymptotic states. If the two-point functions have only a single mass eigenstate, as in (4.1), then the one-body problem becomes trivial. If the two-point function has several discrete mass eigenvalues then

$$\Pi_j = \prod_{i\neq j} \frac{\nabla^2 - m_i^2}{m_j^2 - m_i^2}. \tag{6.26}$$

projects the initial or final state on the asymptotic j-th mass subspace. If the two-point Lehmann weight also has continuum eigenstates then it is necessary to project on the appropriate one-particle subspace. In what follows I consider that simplest case where mass operator associated with the two-point function has a single discrete eigenvalue. This means that the two point functions have the from (4.1).

In this case it is enough to consider asymptotic states $\Phi|e^{\mp iH_0 t}|\psi\rangle$ of the form

$$\langle \mathsf{x}_1, \mathsf{x}_2|\Phi|e^{\mp iH_0 t}\psi\rangle = \prod_{i=1}^{2} f_i(\mathsf{x}_i^0) \int \frac{d\mathbf{p}_i}{(2\pi)^{3/2}} \psi_i(\mathbf{p}_i) e^{i\mathbf{p}_i\cdot\mathbf{x}_i \mp i\omega_{m_i}(\mathbf{p}_i)t} \tag{6.27}$$

where the $f_i(\mathsf{x}_i^0)$ are sharply peaked with compact support and integrate to 1.

With this choice of wave packets

$$\langle \mathsf{x}_1, \mathsf{x}_2|(H\Phi - \Phi H_0)e^{\mp iH_0 t}|\psi\rangle =$$

$$(2\pi)^3 \left(\frac{\partial}{\partial \mathsf{x}_1^0} + \frac{\partial}{\partial \mathsf{x}_2^0} - \omega_{m_1}(\mathbf{p}_1) - \omega_{m_2}(\mathbf{p}_2)\right) f_1(\mathsf{x}_1^0) f_2(\mathsf{x}_2^0)\times$$

$$\int \frac{d\mathbf{p}_1 d\mathbf{p}_2}{(2\pi)^3} \psi_1(\mathbf{p}_1)\psi_2(\mathbf{p}_2) e^{i\mathbf{p}_1\cdot\mathbf{x}_1 \mp i\omega_{m_1}(\mathbf{p}_1)t + i\mathbf{p}_2\cdot\mathbf{x}_2 \mp i\omega_{m_2}(\mathbf{p}_2)t}. \tag{6.28}$$

The integrand in the Cook condition uses (6.25) and (6.28) in

$$(\psi, e^{\mp iH_0 t}(\Phi^\dagger H - H_0 \Phi^\dagger)\Theta(S_{1:1}S_{1:1} + S^c_{2:2})(H\Phi - \Phi H_0)e^{\pm iH_0 t}\psi)_E. \tag{6.29}$$

In this expression the partial derivatives in (6.28) with respect to the Euclidean times can be integrated by parts. When $(-)$ the Euclidean time derivatives act on the product of the two-point functions $S_{1:1}S_{1:1}$ they generate a energy factors (4.5) that exactly cancel the asymptotic energy factors in (6.28), making (6.29) vanish.

This means that only the connected part of the four-point quasi-Schwinger functions contributes to the integral (6.29). Thus, the connected part of the four-point function plays an analogous role to the interaction in the non-relativistic case.

What remains has the form

$$(2\pi)^6 \int d^4\mathsf{q}_1 d^4\mathsf{q}_2 d^4\mathsf{q}_3 dm_a dm_c dm_b d\mathsf{x}^0_1 d\mathsf{x}^0_2 d\mathsf{x}^0_3 d\mathsf{x}^0_4 \, (\omega_{m_c}(\mathbf{q}_2) - \omega_{m_1}(\mathbf{q}_1) - \omega_{m_2}(\mathbf{q}_2 - \mathbf{q}_1)) \times$$

$$f_1(\mathsf{x}^0_1) f_2(\mathsf{x}^0_2) \psi^*_1(\mathbf{q}_1)\psi^*_2(\mathbf{q}_2 - \mathbf{q}_1) e^{\pm i\omega_{m_1}(\mathbf{q}_1)t \pm i\omega_{m_2}(\mathbf{q}_2 - \mathbf{q}_1)t} \times$$

$$\frac{e^{-\omega_{m_a}(\mathbf{q}_1)(\mathsf{x}^0_1 - \mathsf{x}^0_2) - \omega_{m_c}(\mathbf{q}_2)\cdot(\mathsf{x}^0_2 + \mathsf{x}^0_3) - \omega_{m_b}(\mathbf{q}_3)(\mathsf{x}^0_3 - \mathsf{x}^0_4)}}{(\mathsf{q}^2_1 + m^2_a)(\mathsf{q}^2_2 + m^2_c)(\mathsf{q}^2_3 + m^2_b)} \times$$

$$e^{\pm i\omega_{m_1}(\mathbf{q}_1)t \mp i\omega_{m_2}(\mathbf{q}_2 - \mathbf{q}_1)t} v(\mathsf{q}_1, m_a, \mathsf{q}_2, m_c, \mathsf{q}_3, m_b) \times$$

$$(\omega_{m_c}(\mathbf{q}_2) - \omega_{m_3}(\mathbf{q}_3) - \omega_{m_4}(\mathbf{q}_2 - \mathbf{q}_3)) \, f_1(\mathsf{x}^0_3) f_2(\mathsf{x}^0_4) \psi_1(\mathbf{q}_3)\psi_2(\mathbf{q}_2 - \mathbf{q}_3) \tag{6.30}$$

The large-time behavior of this integral is relevant for the Cook condition. To estimate the large time behavior write

$$-\omega_{m_a}(\mathbf{q}_1)(\mathsf{x}^0_1 - \mathsf{x}^0_2) \pm i\omega_{m_1}(\mathbf{q}_1)t =$$

$$-(\omega_{m_a}(\mathbf{q}_1) - \omega_{m_1}(\mathbf{q}_1))(\mathsf{x}^0_1 - \mathsf{x}^0_2)$$

$$-\omega_{m_1}(\mathbf{q}_1)(\mp it + (\mathsf{x}^0_1 - \mathsf{x}^0_2)) \tag{6.31}$$

and

$$-\omega_{m_b}(\mathbf{q}_3)(\mathsf{x}^0_3 - \mathsf{x}^0_2) \pm i\omega_{m_3}(\mathbf{q}_3)t =$$

$$-(\omega_{m_b}(\mathbf{q}_3) - \omega_{m_3}(\mathbf{q}_3))(\mathsf{x}^0_3 - \mathsf{x}^0_2)$$

$$-\omega_{m_3}(\mathbf{q}_3)(\mp it + (\mathsf{x}^0_1 - \mathsf{x}^0_2)). \tag{6.32}$$

We assume that the $m_b \geq m_3$ and $m_a \geq m_1$, which is the easiest case to consider.

To put (6.29) in a manageable form I make some simplifying assumptions. First I assume that the $f_i(\mathsf{x}^0)$ are sharply peaked to factor the integrand out of the integral. The resulting approximation leads to in a constant multiplied by the integrand evaluated at points at x_i^0 in the support of $f_i(\mathsf{x}^0)$. Similarly I change to total and relative momentum variables and use the translational invariance to eliminate the center of momentum degrees of freedom. Specifically I assume that the total 3-momentum support of the wave functions is near zero. One total momentum integral is eliminated by the momentum conserving delta function. The second is approximated by setting the total momentum equal to zero and multiplying by the volume of the total momentum support. What remains, up to a multiplicative constant, has the form

$$\Big(\psi, U_0(\pm t)(H\Phi - \Phi H_0)U_0^\dagger(t)\psi, \theta S^c_{2:2}(H\Phi - \Phi H_0)U_0(\mp t)\psi\Big) \rightarrow$$

$$\int (\omega_{m_c}(\mathbf{0}) - \omega_{m_3}(\mathbf{k}') - \omega_{m_4}(\mathbf{k}')) \times$$

$$\psi_3^*(-\mathbf{k}')\psi_4^*(\mathbf{k}')e^{i\omega_{m_4}(\frac{1}{2}\mathbf{k}')t + i\omega_{m_3}(\frac{1}{2}\mathbf{k}')t}$$

$$e^{-\omega_{m_a}(\mathbf{k}')(\mathsf{x}_4^0 - \mathsf{x}_3^0) - \omega_{m_c}(\mathbf{0})(\mathsf{x}_3^0 + \mathsf{x}_1^0) - \omega_{m_b}(\mathbf{k})(\mathsf{x}_1^0 - \mathsf{x}_2^0)}$$

$$\frac{d\mathbf{k}d\mathbf{k}'}{\omega_{m_a}(\mathbf{k}')\omega_{m_c}(\mathbf{0})\omega_{m_b}(\mathbf{k})}$$

$$\left(\omega_{m_c}(\mathbf{0}) - \omega_{m_1}(\frac{1}{2}\mathbf{k}) - \omega_{m_2}(\frac{1}{2}\mathbf{k})\right) \times$$

$$\psi_1(\mathbf{k})\psi_2(-\mathbf{k})e^{\mp i\omega_{m_1}(\mathbf{k})t \mp i\omega_{m_2}(\mathbf{k})t}. \tag{6.33}$$

where $\mathbf{k}$ is the momentum of one of the particles in the zero momentum frame.

The time dependence in this expression comes from the $\mathbf{k}$ and $\mathbf{k}'$ integrals. If we use (6.31-6.32) in these expression, assuming that $m_a \geq m_1$ and $m_b \geq m_2$ the integral 6.33 has the general form

$$\int_0^\infty \frac{k^2 dk}{\omega_{m_1}(k^2)} h(k^2) e^{-\omega_{m_1}(k^2)(\mathsf{x}_1^0 - \mathsf{x}_2^0 \pm 2it)} \tag{6.34}$$

where $h(k^2)$ is a well-behaved function of $\mathbf{k}^2$. This can be put in a form where it is possible to see the asymptotic behavior by making the substitution $k = m_1 \sinh(\eta)$ and $\omega_{m_1}(k^2) = m\cosh(\eta)$ which leads to an integral of the form

$$m_2^3 \int_0^\infty \sinh^2(\eta) e^{-m_1\cosh(\eta)(\mathsf{x}_1^0 - \mathsf{x}_2^0 - 2it)} h(\sinh^2(\eta)) d\eta. \tag{6.35}$$

If I ignore the $h(\sinh^2(\eta))$ this integral can be done exactly. The time dependence is

$$I \to \frac{1}{m_1(\mathsf{x}_1^0 - \mathsf{x}_2^0 \mp 2it)} K_1(m_1(\mathsf{x}_1^0 - \mathsf{x}_2^0 \mp 2it)) \tag{6.36}$$

where for large arguments

$$K_n(z) \to \sqrt{\frac{\pi}{2z}} e^{-z}(1 + \frac{n^2}{2z} + \cdots). \tag{6.37}$$

Using (6.37) in (6.36) shows that this integral falls off like $\frac{1}{t^{3/2}}$ for large time. Each extra power of k in $h(\sinh^2(\eta))$ introduces another factor of $t^{-1/2}$. The same factor is generated by both the k and k' integrals. This shows that the Cook condition (6.21) is satisfied for the reflection quasi-Schwinger functions of the form (5.8). The large-time behavior is asymptotically identical to what one finds in non-relativistic scattering theory, however in this case it is controlled in part by the relative Euclidean time support condition.

The relativistic invariance of the S matrix can be established using similar methods. The required condition in terms of the wave operators are the intertwining conditions

$$U(\Lambda, a)\Omega_{\pm\alpha} = \Omega_{\pm\alpha} U_\alpha(\Lambda, a) \tag{6.38}$$

for both asymptotic conditions. For the space translations and rotations this condition is a consequence of the translational and rotational invariance of the injection operators, Φ_α. For time translations this follows from the existence of the wave operators. For the boosts a sufficient condition is

$$\lim_{t\to\pm\infty} \|(\mathbf{K}\Phi_\alpha - \Phi_\alpha \mathbf{K}_\alpha) e^{\mp i H_\alpha t}|\psi\rangle\| = 0 \tag{6.39}$$

As in the scattering case, the non-zero contributions to this expression before taking the limit come from the part $S^a_{n:n}$ (see 6.22) of the quasi-Schwinger function that vanishes as the clusters of a are separated. In the two-body example above, this depends on properties of the connected four-point function. When (6.39) holds finite Lorentz transformations on the scattering eigenstates can be realized by transforming the asymptotic states using (6.9)

VII. COMPUTATIONAL ISSUES

This paper gives a representation of a model Hilbert space whose vectors are equivalence classes of functions of *Euclidean spacetime variables* with certain support conditions, and a representation of the Poincaré Lie algebra on this space. Given a Hilbert space representation and a set of Poincaré generators in this representation it is in principle possible to calculate any dynamical quantity.

The new features in this Euclidean formulation are that the dynamics enters in the kernel of the Hilbert space inner product and the Hilbert space vectors are represented by equivalence classes of functions. Computational strategies must be designed with these properties in mind.

What is easy to calculate in this formalism are matrix elements of $e^{-\beta H}$ which only involve a translation of the all of the Euclidean time variables in the wave functions by a common constant and a quadrature.

The brute force way to treat the large degeneracy would be use start with a dense set of vectors, use the Gram Schmidt method to generate an orthonormal basis and compute matrix elements of $e^{-\beta H}$ in that basis. Working in an orthonormal basis eliminates the large degeneracy, and having matrix elements of ${}^{-\beta H}$ in a complete orthonormal basis is in principle sufficient to compute any dynamical quantity.

The most interesting observables are scattering observables. While these could in principle be calculated using the method discussed in the previous paragraph, it is not the most efficient way to proceed. An alternative is to utilize the time-dependent representation of the S matrix elements, which in the two-Hilbert space representation has the form

$$\langle \psi_{\alpha f} | S | \psi_{\gamma i} \rangle = \lim_{t \to \infty} \langle \psi_{\alpha f} | e^{iH_\alpha t} \Phi_\alpha^\dagger e^{-2iHt} \Phi_\gamma e^{iH_\gamma t} | \psi_{\gamma_i} \rangle. \tag{7.1}$$

In normal formulations of scattering theory [18] this is the starting point. The derivation of the usual time-independent formula assumes that the wave packets are sufficiently narrow and the kernel of the transition operator is sufficiently smooth that the kernel of the transition operator can be factored out of the integral, resulting in a representation of the cross section that is independent of the choice of wave packet.

This can be turned around by using sufficiently narrow wave packets centered about $\mathbf{p}_1, \cdots, \mathbf{p}_{n_\alpha}$ and $\mathbf{p}_1, \cdots, \mathbf{p}_{n_\gamma}$ and keeping them for the entire calculation. In both cases the result leads to (approximate) sharp-momentum values the transition matrix

$$\langle \mathbf{p}'_1, \cdots, \mathbf{p}'_{n_\alpha} | t(E_\gamma + i0) | \mathbf{p}_1, \cdots, \mathbf{p}_{n_\gamma} \rangle \approx \frac{i}{2\pi} \frac{\langle \psi_{\alpha f} | (S - I) | \psi_{\gamma i} \rangle}{\langle \psi_{\alpha f} | \delta(E_\alpha - E_\gamma) | \psi_{\gamma i} \rangle}. \tag{7.2}$$

All of the quantities in this expression can be computed from (7.1).

While (7.2) may be difficult to compute directly, one expects that the Kato-Birman invariance principle [19][20] can be used to make the replacement $H \to -e^{-\beta H}$ in (7.1) giving the following equivalent expression for the S-matrix elements

$$\langle \psi_{\alpha f} | S | \psi_{\gamma i} \rangle = \lim_{n \to \infty} \langle \psi_{\alpha f} | e^{-ine^{-\beta H_\alpha}} \Phi_\alpha^\dagger e^{2ine^{-\beta H}} \Phi_\gamma e^{-ine^{-\beta H_\gamma}} | \psi_{\gamma_i} \rangle . \tag{7.3}$$

This is an identity rather than an approximation. β is a parameter that can be used to set an energy scale. The result is independent of β, but the wrong choice of β would make the limit difficult to evaluate.

While (7.3) looks more complicated than (7.1), what is relevant is that because of the spectral condition (3.16), $e^{-\beta H}$, has a spectrum in the unit interval. Since e^{2inx} can be uniformly approximated by a polynomial on the compact interval $[0, 1]$, it follows that if

$$|P(x) - e^{2inx}| < \epsilon \qquad \forall\, x \in [0, 1] \tag{7.4}$$

then

$$\| P(e^{-\beta H}) - e^{2ine^{-\beta H}} \| < \epsilon \tag{7.5}$$

where the same ϵ appears in both expressions. This is important because it means that it is sufficient to test the convergence of the approximation using (7.4) rather than the more complex (7.5).

The result of this sequence of approximations is that sharp transition matrix elements can formally be expressed in terms of matrix elements of polynomials in $e^{-\beta H}$, which in the Euclidean framework can be calculated using (3.15).

The calculation outlined above involves four approximations that must be performed in a specific order.

1. Solve for Φ_α. This requires solving the one-body problem for subsystems. This can be trivial or difficult depending on the spectral properties of the quasi-Schwinger functions. The accuracy of the subsequent approximations will be sensitive to how well the one-body solution is isolated from the rest of the subsystem mass eigenstates.

2. The next approximation is to choose sufficiently narrow wave packets. These are needed to extract sharp momentum transition operators. The error on the sharp momentum transition matrix elements decreases as the width of the wave packet decreases.

3. The large n limit in (7.3). This replaces the large time limit in the expressions for the wave operators. The value of n needed for convergence will increase as the wave packets become narrower. The parameter β can be used to minimize the values of n need for convergence. The convergence is strong for well-behave connected quasi-Schwinger functions.

4. The degree of the polynomial needed to approximate e^{2inx} increases as n increases. This can be tested directly using (7.4).

The virtue is that each of the above approximations converges. Since each approximation effects the subsequent approximations it is clear that these approximations must be performed in the order prescribed above. The last three approximations were tested in the context of a solvable model, where it was possible to independently investigate the accuracy of each of the last three approximations [21]. What was found, in the context of a simple model, was that in order to get three-significant figure accuracy on sharp-momentum two-body transition matrix elements, the wave packet width needed to be about 1/10 of the beam momentum. This was observed for incident momenta between .05 to 2 GeV.

Ten significant figure accuracy could be realized in the approximation of S-matrix elements with n. These results were realized using wave packets that were sufficiently narrow to get three significant figure accuracy in the sharp-momentum transition operators. Values of n needed for ten significant figure accuracy for these wave packets were in the $100 - 400$ range for incident momenta between .05 - 2.0 GeV, when β was chosen close to the inverse of the working energy scale.

Ten significant figure accuracy could be realized in the approximation of $e^{2ine^{-\beta H}}$ by Chebyshev polynomials in $e^{-\beta H}$. In the model calculations of [21] the degree of the polynomials was between 300-700. The coefficients of the Chebyshev polynomial expansion of e^{2inx} were determined using Gauss-Chebychev quadrature, which made it possible to handle polynomials of large degree.

The demonstrated convergence of the last three of the four approximations listed above in the model calculations of [21] strongly suggest that few-GeV scale scattering calculations are possible. The largest source of error is associated with the width of the wave packets. The three significant figure accuracy for the sharp momentum transition matrix elements achieved in these model calculations is sufficient for few-GeV scattering calculations. In the two-body case, where the two-point functions are given by *free* Euclidean Green functions, the one-body problem becomes trivial. What remains are the three approximations that were successfully tested in the model calculations. This suggests that the approximation methods discussed above should be sufficient to solve a two-body model with a four-point function of the form (5.4), where $S_{1:1}$ are free field Schwinger function and the connected four-point function has the form (5.8). The existence of the limits that define the wave operators suggest the possibility that time-independent equations based on (7.3) might also be possible.

A number of open problems remain. The first one is to develop a strategy for constructing quasi-Schwinger functions that define realistic models. The difficulty with the Bethe-Salpeter approach, in addition to the stability discussed in section V, is that because $S_{1:1}$ does not map the positive-time subspace to the positive-time subspace, the reflection positivity applied to the kernel involves the full matrix of the kernel, rather than the restriction the positive Euclidean time subspace. The construction of realistic reflection positive quasi-Schwinger functions is most important remaining open problem.

A second open problem is how to obtain accurate solutions to the one-body problem when the subsystem quasi-Schwinger functions also contains continuum mass eigenstates. The Lehmann weights for realistic two-point Schwinger functions have both discrete and continuous mass eigenstates. The challenge is how to construct a projection operator on the one-body subspace of $\mathcal{H}_M$. These are needed in order to realize the strong convergence of the scattering theory.

The third open problem is how to perform quadratures with sufficient accuracy to successfully implement the above strategy. This problem is not special to this formalism.

Finally the ultimate goal is to apply these methods to QCD, where the connection with the spectral condition (3.16) suggests that reflection positivity may only be satisfied for initial and final states that are color singlets. The methods discussed above can in principle be applied in this case, since the asymptotic one-body states would be color singlets, and the S matrix should map singlets to singlets.

VIII. SUMMARY

In this paper I exhibited a class connected four-point quasi-Schwinger functions satisfying reflection positivity. Their structure was motivated by a theorem of Widder which applies only to the one-dimensional case. Widder's result suggests that reflection positivity is a fairly restrictive condition. The general structure of reflection positive four-point functions is still an open problem. More importantly, the construction of quasi-Schwinger functions for realistic models remains an open problem. In this paper I used time-dependent methods to demonstrate the existence of scattering wave operators for models based on the reflection positive quasi Schwinger functions of the form (5.8). The basic observation is that the Cook condition that is normally used as a sufficient condition for the existence of non-relativistic wave operators can be applied in this formulation of Euclidean relativistic quantum mechanics. The $t^{-3/2}$ asymptotic behavior that ensures the existence of the wave operator for short-ranged potentials in the non-relativistic case is realized in the relativistic case for sufficiently well behaved connected quasi-Schwinger functions.

Time-dependent methods for computing transition matrix elements using purely Euclidean methods were suggested. Models calculations suggest that these methods should in principle be applicable to few-GeV scale scattering calculations.

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IX. APPENDIX - SPIN

Quasi-Schwinger functions that lead to any positive-mass irreducible representation space of the Poincaré group are constructed in this appendix.

A basis for vectors in a positive-mass irreducible representation space of the Poincaré group are simultaneous eigenstates of the mass, spin, linear momentum, and z-component of some kind of spin (canonical, Jacob-Wick helicity, light-front, $\cdots$). These states have the following transformation property

$$U(\Lambda, a)|(m,j)\mathbf{p}\mu\rangle =$$

$$\sum_{\nu} e^{-i\Lambda p\cdot a}|\mathbf{\Lambda} p,\nu\rangle D^j_{\mu\nu}[B^{-1}(\Lambda p/m),a)\Lambda B(p/m)]\sqrt{\frac{\omega_m(\Lambda p)}{\omega_m(p)}} \tag{9.1}$$

where $B^{-1}(\Lambda p/m), a)\Lambda B(p/m)$ is a Wigner rotation. The choice of Lorentz boost, $B(p/m)^{\mu}{}_{\nu}$, in the Wigner rotation determines the type of spin [22]. For any kind of spin the Wigner D functions, which are also finite dimensional representations of $SL(2,\mathbb{C})$ and it can be factored into products. Multiplication of both sides of (7.1) by

$$D^{j}_{\mu\nu}[B^{-1}(p/m)] \tag{9.2}$$

leads to

$$U(\Lambda,a)|(m,j)\mathbf{p}\mu\rangle D^{j}_{\mu\sigma}[B^{-1}(p/m)]\sqrt{\omega_m(p)} =$$

$$e^{-i\Lambda p\cdot a}|\mathbf{\Lambda}p,\nu\rangle D^{j}_{\mu\sigma'}[B^{-1}(\Lambda p/m)]\sqrt{\omega_m(\Lambda p)}D^{j}_{\sigma'\sigma}[\Lambda]. \tag{9.3}$$

The vectors

$$|p,j,\sigma\rangle := |(m,j)\mathbf{p}\mu\rangle D^{j}_{\mu\sigma}[B^{-1}(p/m)]\sqrt{\omega_m(p)} \tag{9.4}$$

transform in a Lorentz covariant manner

$$U(\Lambda,0)|p,j,\sigma\rangle = e^{-i\Lambda p\cdot a}|\Lambda p,j,\sigma\rangle D^{j}_{\sigma'\sigma}[\Lambda]. \tag{9.5}$$

The transformation $U(\Lambda,0)$ is unitary with respect to the inner product

$$\psi(p,j,\sigma) = \langle p,j,\sigma|\psi\rangle, \tag{9.6}$$

$$\langle\psi|\phi\rangle = \int \psi^*(p,j,\mu)D^{j}_{\mu\sigma}[B(p/m)B^{\dagger}(p/m)]m\frac{d\mathbf{p}}{\omega_m(p,j,\sigma)}\phi(p) \tag{9.7}$$

where $p_0 = \omega_m(\mathbf{p})$ is the energy. The kernel simply removes the momentum-dependent $SL(2,C)$ Wigner functions from the covariant representation. Because the $SL(2,\mathbb{C})$ matrices cancel in computing matrix elements - the result is the same independent of whether the right or left-handed representations of $SL(2,\mathbb{C})$ are used.

Note that in $SL(2,\mathbb{C})$ a general boost has a polar decomposition

$$B(p) = P(p)R(p) \tag{9.8}$$

where $P(p)$ is the positive Hermitian operator,

$$P(p) = e^{\boldsymbol{\rho}\cdot\boldsymbol{\sigma}/2}, \tag{9.9}$$

$\boldsymbol{\rho}$ is the rapidity vector and $R(p)$ is an $SU(2)$ matrix (generalized Melosh rotation) that determines the type of spin. It follows that

$$B(p/m)B^{\dagger}(p/m) = P(p)R(p)R^{\dagger}(p)P(p) = P^2(p) = e^{\boldsymbol{\rho}\cdot\boldsymbol{\sigma}} = \sigma\cdot p. \tag{9.10}$$

In this expression the Melosh rotations cancel, so the result is independent of the choice of spins. Thus this scalar product can be expressed as

$$\int \psi^*(p,j,\alpha) D^j_{\alpha\beta}[p\cdot\sigma] m \frac{d\mathbf{p}}{\omega_m(p)} \psi(p,j,\beta) = \tag{9.11}$$

$$\int \psi^*(p,j,\alpha) D^j_{\alpha\beta}[p\cdot\sigma] 2m d^4p \delta(p^2+m^2)\psi(p,j,\beta). \tag{9.12}$$

This is essentially identical to the form found in [23] (see eq. 1.57). The important observation is that $\sigma\cdot p$ is a positive Hermetian matrix for timelike p. The same holds for $D^j_{\mu\sigma}[p\cdot\sigma]$, $D^j_{\mu\sigma}[p\cdot\sigma^*]$, and $D^j_{\mu\sigma}[p\cdot\sigma^{-1}]$.

The following quasi-Schwinger function is a Euclidean covariant rather than Euclidean invariant distribution

$$\int \frac{2mD^j_{\alpha\beta}[p_e\cdot\sigma_e]}{p_e^2+m^2} d^4p_e e^{ip_e\cdot(\mathsf{x}-\mathsf{y})} \tag{9.13}$$

that leads exactly to the representation (9.7) of a mass m spin j irreducible representation.

These considerations show that the following Euclidean scalar product is reflection positive

$$\int g^*_\alpha(\Theta\mathsf{x}) \frac{2mD^j_{\alpha\beta}[p_e\cdot\sigma_e]}{p_e^2+m^2} d^4p_e e^{ip_e\cdot(\mathsf{x}-\mathsf{y})} g_\beta(\mathsf{y}) d^4x d^4y d^4p_e =$$

$$\psi^*(p,j,\alpha) D^j_{\alpha\beta}[p\cdot\sigma] 2md^4p\delta(p^2+m^2)\psi(p,j,\beta) \tag{9.14}$$

with

$$\psi(p,j,\beta) = \int g(\mathsf{x}^0,\mathsf{x}) e^{-\omega_m(\mathbf{p})\mathsf{x}^0} e^{i\mathbf{p}\cdot\mathsf{x}}. \tag{9.15}$$

This shows how to construct reflection-positive quasi-Schwinger functions for any irreducible representation of the Poincaré group. While I did not choose to double the representation, doubled representations can be realized by replacing $D^j_{\alpha\beta}[p\cdot\sigma]$ by a direct sum of a right and left handed representation , $D^j_{\alpha'\beta'}[p\sigma_2\cdot\sigma*\sigma_2]$, which is also positive for positive energy timelike p.

The kernel of the resulting Lorentz covariant measure is modified by the presence of $D^j_{\mu\mu'}[p\cdot\sigma]$ where $p=(\omega_m(\mathbf{p}),\mathbf{p})$:

$$\frac{d\mathbf{p}}{2\omega_m(\mathbf{p})} D^j_{\mu\mu'}[p\cdot\sigma] D^j_{\nu\nu'}[p\cdot\sigma_\pi]]$$

where $\boldsymbol{\sigma}_\pi = \sigma_2 \boldsymbol{\sigma}^* \sigma_2$. The transformation properties of multipoint quasi-Schwinger function are the same as products of these two-point quasi Schwinger functions.

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