Scattering and reflection positivity in relativistic Euclidean quantum mechanics

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Abstract

In this paper I exhibit a class of reflection positive Euclidean invariant four-point functions that can be used formulate a Poincaré invariant quantum theory. I demonstrate the existence of scattering wave operators, which can be calculated without analytic continuation in this representation.

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I. INTRODUCTION

In this paper I discuss the existence of scattering wave operators in a representation of relativistic quantum mechanics where the dynamics is introduced through a collection of reflection positive Euclidean "Green functions".

This work is motivated by the observation that the physical Hilbert space inner product in the Osterwalder-Schrader reconstruction theorem of Euclidean field theory [1] can be expressed as a sesquilinear form involving only Euclidean Green functions and Euclidean test functions. Furthermore, the properties of the Euclidean Green functions that are needed to reconstruct a relativistic quantum theory are subset of the properties needed to construct a local field theory. There are explicit representations of all ten Poincaré generators as self-adjoint operators in this Euclidean representation of the Hilbert space. The important observation is that this representation of the relativistic quantum theory does *not* require analytic continuation.

The general formalism was presented in a previous paper [2]. In that paper we introduced methods to compute sharp-momentum transition operators using matrix elements of $e^{-\beta H}$ in normalizable states. These matrix elements can be computed using only quadratures in the Euclidean representation. The computational methods that we introduced were tested using a solvable model and convergence to the exact transition matrix elements was demonstrated for energies between 50 MeV and 2 GeV.

The dynamical input is a set of multi-point Euclidean Green functions that are defined as moments of Euclidean path integrals or solutions of Schwinger-Dyson equations. Actual computations use finite collections of approximate or model Green functions. The Euclidean Green functions that define an acceptable quantum theory satisfy a condition called reflection positivity, which is equivalent to the requirement that vectors in the Euclidean representation of the Hilbert space have non-negative norm.

Reflection positivity in continuum theories is a restrictive condition. This is expected since it ensures the existence of an analytic continuation of spacetime variables to Minkowski space. It is difficult to check because the sum of a reflection positive Green function and a small Euclidean invariant perturbation is not necessarily reflection positive [3]. In addition, products of reflection positive operators are not necessarily reflection positive. These observations imply that the solution of the Bethe-Salpeter equation with reflection positive input is not automatically reflection positive. A natural question that arises is are there real fundamental problems with finding reflection positive Green functions, or do the issues discussed above just make it difficult to demonstrate? Having a class of non-trivial examples would make it clear that there are no fundamental problems in constructing reflection positive Green functions.

This paper addresses two questions which were not addressed in [2]. The first question, discussed in the previous paragraph, is "Are there any non-trivial reflection-positive multipoint Green functions?" The second question, assuming that first question is answered in the affirmative, is "Do these reflection positive Green functions lead to a non-trivial scattering theory?" This is the Euclidean version of determining what class of potentials lead to a non-trivial scattering in non-relativistic quantum mechanics.

This paper does not provide a general solution to this problem, but it exhibits a representative class of reflection positive Euclidean invariant four-point functions that are motivated by a one-dimensional structure theorem. This paper also shows that scattering wave operators exist for this class of four-point functions. This paper also includes a short discussion of issues related to applying these methods to QCD as well as a discussion of how to extend the framework presented in [2] to treat states of any spin.

The advantages of this approach are (1) all relevant calculations can be performed entirely in Euclidean space, (2) there is numerical evidence that GeV-scale scattering calculations can be performed without analytic continuation (3) and finite Poincaré transformations can be performed without leaving Euclidean space.

In the next section I introduce notation and summarize needed results from [2]. In section three I exhibit a class of reflection positive four-point Green functions. In section four I discuss the formulation of the scattering problem when the dynamics is in the kernel of the scalar product and demonstrate the existence of scattering wave operators based on the Green functions introduced in section three. In section five I briefly review computational methods that can be applied to the examples of sections three and four. Section six provides a brief summary of the key results of this paper and a discussion of remaining open problems. The appendix discusses the structure of two-point Euclidean Green functions for arbitrary spin.

II. DEFINITIONS AND ASSUMPTIONS

This section summarizes the essential background material from [2].

A relativistic Euclidean quantum theory is defined by a collection of Euclidean invariant distributions, $S_{m:n}(\mathsf{x}_m, \cdots, \mathsf{x}_1; \mathsf{y}_1, \cdots, \mathsf{y}_n)$, where the x_i and y_j are final and initial Euclidean space-time variables. The collection can be finite or infinite.

A dense set of Hilbert space vectors are represented by test functions

$$\{f_m(\mathsf{x}_1, \cdots, \mathsf{x}_m)\} \tag{2.1}$$

with support for positive relative Euclidean times

$$0 < \mathsf{x}_1^0 < \mathsf{x}_2^0 < \cdots < \mathsf{x}_m^0. \tag{2.2}$$

The physical Hilbert space inner product is

$$\langle f|g\rangle = \sum_{m,n} \int f_m^*(\mathsf{x}_1, \cdots, \mathsf{x}_m) S_{m:n}(\theta\mathsf{x}_m, \cdots, \theta\mathsf{x}_1; \mathsf{y}_1, \cdots, \mathsf{y}_n) g_n(\mathsf{y}_1, \cdots, \mathsf{y}_n) d^{4m}\mathsf{x} d^{4n}\mathsf{y}, \tag{2.3}$$

where $\theta\mathsf{x} = \theta(\mathsf{x}^0, \mathbf{x}) = (-\mathsf{x}^0, \mathbf{x})$ is the Euclidean time reflection operator. The Hilbert space generated by the functions (2.1) with inner product (2.3) is denoted by $\mathcal{H}$.

The collection of Green functions in (2.3) is reflection positive if

$$\langle f|f\rangle \geq 0. \tag{2.4}$$

This inner product has zero norm-vectors, so the actual Hilbert space vectors are equivalence classes of functions of the form (2.1-2.2), under the equivalence relation that the norm of the difference between equivalent functions is zero.

In quantum field theory microscopic locality requires that the collection of Green functions is infinite and there is no distinction between the initial and final variables; in relativistic quantum mechanical models both of these conditions can be relaxed. Relaxing these conditions leads to violations of microscopic locality, however all of the other observable requirements (axioms) of a relativistic quantum theory remain satisfied [1]. Computable models involve finite numbers of degrees of freedom, so the full set of requirements of the field theory will not be realized, but it is still desirable to retain the relativistic invariance of the quantum theory.

In addition, the collection of Green functions should be Hermitian,

$$S_{k:n}(\mathsf{x}_k \cdots \mathsf{x}_1; \mathsf{y}_1 \cdots \mathsf{y}_n) = S^*_{n:k}(\mathsf{y}_n \cdots \mathsf{y}_1; \mathsf{x}_1 \cdots \mathsf{x}_k), \tag{2.5}$$

so $\langle f|g\rangle = \langle g|f\rangle^*$ and satisfy cluster properties,

$$S_{k:n}(\mathsf{x}_k, \cdots, \mathsf{x}_1; \mathsf{y}_1, \cdots, \mathsf{y}_n) \to \prod S_{k_i:n_i}(\mathsf{x}_{ki}, \cdots, \mathsf{x}_{1i}; \mathsf{y}_{1i}, \cdots, \mathsf{y}_{ni}), \tag{2.6}$$

as distributions, which means that they become products of fewer-point Green functions when different clusters of initial and final variables are asymptotically separated.

The Poincaré generators on $\mathcal{H}$ have simple expressions. The Hamiltonian and square of the invariant mass operator are

$$\langle \mathsf{x}|H|\mathbf{f}\rangle := -\frac{d}{da}\langle \mathsf{x} - (a,0,0,0)|\mathbf{f}\rangle_{|a=0} =$$

$$\{0, \frac{\partial}{\partial \mathsf{x}^0_{11}} f_1(\mathsf{x}_{11}), \left(\frac{\partial}{\partial \mathsf{x}^0_{21}} + \frac{\partial}{\partial \mathsf{x}^0_{22}}\right) f_2(\mathsf{x}_{21}, \mathsf{x}_{22}), \cdots\} \tag{2.7}$$

and

$$M^2 = H^2 - P^2 \tag{2.8}$$

$$\langle \mathsf{x}|M^2|\mathbf{f}\rangle := \{0, \, -\nabla^2_{11} f_1(\mathsf{x}_{11}), -(\nabla_{21} + \nabla_{22})^2 f_2(\mathsf{x}_{21}, \mathsf{x}_{22}), \cdots\} \tag{2.9}$$

where ∇^2 is the 4-dimensional Euclidean Laplacian. The generators of spatial translations and rotations follow in the usual way from the translational and rotational invariance of the Green functions. Matrix elements of the form $\langle f|e^{-\beta H}|g\rangle$ are important in applications and can be computed by quadrature by replacing

$$g_n(\mathsf{y}_1, \cdots, \mathsf{y}_n) \to g_n(\mathsf{y}_1 - (\beta, \mathbf{0}), \cdots, \mathsf{y}_n - (\beta, \mathbf{0})) \tag{2.10}$$

in (2.3).

III. REFLECTION POSITIVITY

One difficulty with constructing reflection positive multipoint functions is that there are non-trivial functions associated with zero norm vectors.

If a free Euclidean Green function is perturbed by adding a small connected perturbation that is only required to be Euclidean invariant, then a function representing a zero norm vector with respect to the product of the free Euclidean Green functions might have a non-zero contribution due to the perturbation. This contribution can always be made negative using the freedom to adjust the sign of the perturbation. The means that reflection positivity is not stable with respect to small Euclidean invariant perturbations. The practical consequence of this observation is that the solution of the Euclidean Bethe-Salpeter equation,

$$S_{2:2} = S_{1:1}S_{1:1} + S_{1:1}S_{1:1}KS_{2:2}, \tag{3.1}$$

with a model Euclidean invariant kernel, K, is not automatically reflection positive, even if the kernel is small. I am not aware of any general results about what kind of restrictions are needed on Euclidean invariant Bethe-Salpeter kernels for $S_{2:2}$ to be reflection positive. In addition, reflection positivity ensures the existence of an analytic continuation to Minkowski space in space-time variables, which suggests that it is a restrictive condition.

The absence of any general methods for constructing reflection positive Green functions is a problem if one wants to make phenomenological models of Euclidean Green functions to use in this framework. Fortunately, a general structure theorem exits for the two-point function in the one-dimensional case. In what follows this one-dimensional result will be used to motivate the construction of a class of reflection positive four-point functions.

Since cluster properties imply that four-point functions can be expressed as a sum of products of reflection-positive two-point functions and a connected four-point function,

$$S_{2:2} = \sum S_{1:1}S_{1:1} + S^c_{2:2}, \tag{3.2}$$

in order to show $S_{2:2}$ is reflection positive it is sufficient to show that the connected four-point function, $S^c_{2:2}$, is reflection positive.

In one dimension there is a result due to Widder [4][5][6] from classical analysis that gives the general structure of reflection positive two-point functions. Widder's theorem points out that any kernel $k(s)$ satisfying the reflection positivity condition

$$\int f(\theta s)k(s-t)f(t)dsdt = \int f(s)k(-s-t)f(t)dsdt \geq 0 \tag{3.3}$$

can be expressed in the exponential form

$$k(-\tau'-\tau) = \int e^{-\lambda(\tau'+\tau)}\rho(\lambda)d\lambda \tag{3.4}$$

for some positive density $\rho(\lambda)$. Since in this example, $\tau', \tau > 0$, we can write the kernel as

$$k(-\tau'-\tau) = \int_0^\infty \frac{\lambda}{\pi}\rho(\lambda)d\lambda \int_{-\infty}^\infty ds \frac{e^{-is(\tau'+\tau)}}{s^2+\lambda^2}. \tag{3.5}$$

This has the form of a one-dimensional version of the Källén-Lehmann representation of a two-point Euclidean Green function.

The Widder result suggests that connected four-point Euclidean Green functions with the structure

$$S^c_{2:2}(\mathsf{x}_2, \mathsf{x}_2; \mathsf{y}_1, \mathsf{y}_2) =$$
$$\int d^4\mathsf{p}_1 d^4\mathsf{p}_2 d^4\mathsf{p}_3 dm_a dm_c dm_b e^{-i\mathsf{p}_1\cdot(\mathsf{x}_2-\mathsf{x}_1)} e^{-i\mathsf{p}_2\cdot(\mathsf{x}_1-\mathsf{y}_1)} e^{-i\mathsf{p}_3\cdot(\mathsf{y}_1-\mathsf{y}_2)} \times$$
$$\frac{v(m_a, \mathsf{p}_1, m_c, \mathsf{p}_2, m_b, \mathsf{p}_3)}{(\mathsf{p}_1^2 + m_a^2)(\mathsf{p}_2^2 + m_c^2)(\mathsf{p}_3^2 + m_b^2)} \tag{3.6}$$

would be reflection positive for suitable Euclidean invariant kernels, $v(m_a, \mathsf{p}_1, m_c, \mathsf{p}_2, m_b, \mathsf{p}_3)$. The relevant observation is that this reduces to the one-dimensional case if there is enough symmetry between the initial and final variables. This structure does not provide a general representation for a Euclidean invariant reflection positive four point functions, as one gets in Widder's theorem. On the other hand, Widder's theorem suggests that reflection positivity and Euclidean covariance strongly constrain the class of reflection positive four-point functions.

The contribution to the Hilbert space norm from connected Green functions of the form (3.6) is

$$\||fg\rangle\|^2_c =$$
$$\int d^4\mathsf{x}_1 d^4\mathsf{x}_2 d^4\mathsf{y}_1 d^4\mathsf{y}_2 d^4\mathsf{p}_1 d^4\mathsf{p}_2 d^4\mathsf{p}_3 dm_a dm_c dm_b f_e^*(-\mathsf{x}_2^0, \mathbf{x}_2) g_e^*(-\mathsf{x}_1^0, \mathbf{x}_1) e^{-i\mathsf{p}_1\cdot(\mathsf{x}_2-\mathsf{x}_1)} \times$$
$$e^{-i\mathsf{p}_2\cdot(\mathsf{x}_1-\mathsf{y}_1)} e^{-i\mathsf{p}_3\cdot(\mathsf{y}_1-\mathsf{y}_2)} \frac{v(m_a, \mathsf{p}_1, m_c, \mathsf{p}_2, m_b, \mathsf{p}_3)}{(\mathsf{p}_1^2 + m_a^2)(\mathsf{p}_2^2 + m_2^2)(\mathsf{p}_3^2 + m_b^2)} f_e(\mathsf{y}_2^0, \mathbf{y}_2) g_e(\mathsf{y}_1^0, \mathbf{y}_1) \tag{3.7}$$

where the functions satisfy the support condition, $g_e(\mathsf{y}_1^0, \mathbf{y}_1)$ and $f_e(\mathsf{y}_2^0, \mathbf{y}_2)$ can be non-zero only for $0 < \mathsf{y}_1^0 < \mathsf{y}_2^0$.

The most straightforward assumption is to choose $v(m_a, \mathsf{p}_1, m_2, \mathsf{p}_2, m_b, \mathsf{p}_3)$ to be symmetric with respect to $\mathbf{p}_1, m_a \leftrightarrow \mathbf{p}_3, m_b$ and analytic in the upper-half p_i^0 planes. In this case the p_i^0 integrals can be performed as in the one-dimensional case, where the convergence in the upper-half plane is ensured by the Euclidean time-support constraints provided the kernel is polynomially bounded, and the Minkowski boundary value is a tempered distribution. The integral over the vector variables leads to Fourier transforms of the vector variables and results in the expression

$$(2\pi)^9 \int d\mathsf{x}^0 d\mathsf{y}^0 d\mathsf{x}^{0\prime} d\mathsf{y}^{0\prime} d\mathsf{p}_1 d\mathsf{p}_2 d\mathsf{p}_3 dm_a dm_c dm_b \tilde{f}_e^*(\mathsf{x}_2^{0\prime}, -\mathbf{p}_1) \tilde{g}_e^*(\mathsf{x}_1^{0\prime}, \mathbf{p}_1 - \mathbf{p}_2) \times$$

$$e^{-\omega_{m_a}(\mathbf{p}_1)(\mathsf{x}_2^0-\mathsf{x}_1^0)}e^{-\omega_{m_c}(\mathbf{p}_2)(\mathsf{x}_1^0+\mathsf{y}_1^0)}e^{-\omega_{m_b}(\mathbf{p}_3)(\mathsf{y}_2^0-\mathsf{y}_1^0)}\times$$

$$\frac{v(m_1,(i\omega_{m_a}(\mathbf{p}_1),\mathbf{p}_1),m_c,(i\omega_{m_c}(\mathbf{p}_2),\mathbf{p}_2),(i\omega_{m_b}(\mathbf{p}_3),\mathbf{p}_3),m_b)}{2\omega_{m_a}(\mathbf{p}_1)2\omega_{m_c}(\mathbf{p}_2)2\omega_{m_b}(\mathbf{p}_3)}\tilde{f}_e(\mathsf{y}_2^0,\mathbf{p}_3-\mathbf{p}_2)\tilde{g}_e(\mathsf{y}_1^0,-\mathbf{p}_3). \quad (3.8)$$

The kernel, which was initially a Euclidean invariant function of Euclidean scalar products becomes a Lorentz invariant function of Lorentz invariant inner products when the residues are evaluated at the poles $\mathsf{p}^0 \to i\omega_{m_i}(\mathbf{p}_i)$. This connected Euclidean Green function will be reflection positive if

$$v(m_a,(i\omega_{m_a}(\mathbf{p}_1),\mathbf{p}_1),m_c,(i\omega_{m_c}(\mathbf{p}_2),\mathbf{p}_2),(i\omega_{m_b}(\mathbf{p}_3),\mathbf{p}_3),m_b) \quad (3.9)$$

is a positive symmetric matrix in $\mathbf{p}_1, m_a$ and $\mathbf{p}_3, m_b$ for all values of $\mathbf{p}_2$ and m_c. This is not a difficult condition to realize.

This construction demonstrates the existence of a large class of reflection positive connected four-point Euclidean Green functions. Exchange symmetry puts additional constraints on the Euclidean Green function functions. This construction provides a class of non-trivial reflection positive Green functions that can be used to investigate properties of non-trivial models.

The second question that can now be addressed is whether the relativistic quantum models defined by these Green functions have non-trivial scattering operators.

IV. SCATTERING THEORY

In theories like quantum field theory, where the dynamics is encoded in the kernel of the quantum mechanical scalar product, there is no natural asymptotic dynamics on the physical Hilbert space to formulate scattering asymptotic conditions.

Instead, in order to formulate the scattering theory, cluster properties of the Green functions are utilized to formulate the asymptotic conditions. Cluster properties imply that in each asymptotic region the Green functions break up into products of subsystem Green functions. Since the Green functions define the structure of the Hilbert space, in each asymptotic region the Hilbert space looks like the tensor product of Hilbert spaces associated the dynamics of asymptotically separated subsystems. Scattering asymptotic conditions can be formulated by finding eigenfunctions associated with the point spectrum of each of the subsystem mass operators using (2.9). These can be further decomposed into simultaneous eigenstates of mass, linear momentum, spin and spin projection using the space translations and rotations (see (4.4) equations and (4.6) of [2]). These vectors are a basis for an irreducible representation space of the Poincaré group. Products of these subsystem bound-state eigenfunctions define a mapping from a Hilbert space of scattering asymptotes, which is the tensor product of Poincaré irreducible representation spaces, to $\mathcal{H}$.

In order to illustrate the main concepts I consider the special case of two-particle scattering. Also for simplicity I assume that 1 and 2 correspond to different types of scalar particles. In this case the dynamics is given by a 4-point Euclidean Green function

$$S_{2:2}(\mathsf{x}_2, \mathsf{x}_1 : \mathsf{y}_1, \mathsf{y}_2). \tag{4.1}$$

The cluster condition means that the four-point function has the form

$$S_{2:2}(\mathsf{x}_2, \mathsf{x}_1 : \mathsf{y}_1, \mathsf{y}_2) =$$

$$S_{1:1}(\mathsf{x}_1 : \mathsf{y}_1) S_{1:1}(\mathsf{x}_2 : \mathsf{y}_2) + S^c_{2:2}(\mathsf{x}_2, \mathsf{x}_1 : \mathsf{y}_1, \mathsf{y}_2) \tag{4.2}$$

where $S^c_{2:2}(\mathsf{x}_2, \mathsf{x}_1 : \mathsf{y}_1, \mathsf{y}_2)$ is a connected four-point function and $S_{1:1}(\mathsf{x}_i : \mathsf{y}_i)$ are two-point functions.

I also assume that $S_{1:1}(\mathsf{x}_i : \mathsf{y}_i)$ is the two-point Euclidean Green function for a free scalar particle of mass m_i. In this case every vector in the Hilbert space associated with $S_{1:1}(\mathsf{x}_i : \mathsf{y}_i)$ is a mass eigenstate with eigenvalue m_i so there is no need to solve the mass eigenvalue problem discussed above. These asymptotic one-particle eigenstates are represented by functions of the Euclidean variables,

$$\langle \mathsf{x}_i | m_i \rangle = \psi_i(\boldsymbol{\mathsf{x}}_i) h_i(\mathsf{x}^0_i) = h_i(\mathsf{x}^0_i) \int \frac{d\mathbf{p}}{(2\pi)^{3/2}} e^{i\boldsymbol{\mathsf{p}}\cdot\boldsymbol{\mathsf{x}}_i} \tilde{\psi}_i(\mathbf{p}) \tag{4.3}$$

where $h_i(\mathsf{x}^0_i)$ is a smooth function that has compact support for positive Euclidean time. The freedom to choose the functions $h_i(\mathsf{x}^0_i)$ is related to the fact the vectors are represented by equivalence classes of functions. Specifically, in the asymptotic region, the different h_i's are all associated with the same one-body Poincaré irreducible basis states.

I define a mapping, Φ from the tensor product of the space of square integrable functions of the $\boldsymbol{\mathsf{p}}_i$ to the Hilbert space $\mathcal{H}$ by

$$\langle \mathsf{x}_1, \mathsf{x}_2 | \Phi | \psi_1 \otimes \psi_2 \rangle = h_1(\mathsf{x}^0_1) h_2(\mathsf{x}^0_2) \int \frac{d\boldsymbol{\mathsf{p}}_1}{(2\pi)^{3/2}} \frac{d\boldsymbol{\mathsf{p}}_2}{(2\pi)^{3/2}} e^{i\boldsymbol{\mathsf{p}}_1\cdot\boldsymbol{\mathsf{x}}_1} \tilde{\psi}_1(\boldsymbol{\mathsf{p}}_1) e^{i\boldsymbol{\mathsf{p}}_2\cdot\boldsymbol{\mathsf{x}}_2} \tilde{\psi}_2(\boldsymbol{\mathsf{p}}_2) \tag{4.4}$$

where the h_i can be chosen so their supports do not intersect. Since the particles are spin zero particles, this defines a mapping from the tensor product of two irreducible representation spaces of the Poincaré group (mass m_1, spin 0 and mass m_2, spin 0), $\mathcal{H}_0$, to the Hilbert space, $\mathcal{H}$. The functions $h_1(\mathsf{x}_1^0)h_2(\mathsf{x}_2^0)$ are considered as part of the mapping.

Scattering wave operators are mappings from the asymptotic Hilbert space $\mathcal{H}_0 := \mathcal{H}_{m_1} \otimes \mathcal{H}_{m_2}$ to $\mathcal{H}$ defined by the strong limits

$$\Omega_\pm := \lim_{t\to\pm\infty} e^{iHt}\Phi e^{-iH_0 t} \tag{4.5}$$

where $H_0 = \sqrt{m_1^2 + \mathbf{p}_1^2} + \sqrt{m_2^2 + \mathbf{p}_2^2}$ is the Hamiltonian associated with the asymptotically free particles. The existence of these limits depends on properties of the connected four-point Euclidean Green function. This will be shown below.

Conventional methods can be used to derive sufficient conditions for this limit to exist. The first step is to write the limit as the integral of a derivative

$$\lim_{t\to\pm\infty} e^{iHt}\Phi e^{-iH_0 t} = \Phi + \int_0^{\pm\infty} \frac{d}{dt}\left(e^{iHt}\Phi e^{-iH_0 t} dt\right) =$$

$$\Phi + i\int_0^{\pm\infty} e^{iHt}(H\Phi - \Phi H_0)e^{-iH_0 t}dt. \tag{4.6}$$

A sufficient condition for the existence of the wave operator is the strong convergence of the integral (4.6). A sufficient condition for the convergence of this integral is the Cook condition [7], which exploits the unitarity of the time evolution operator

$$\int_a^\infty \|(H\Phi - \Phi H_0)e^{\mp iH_0 t}|\psi\rangle\| dt < \infty \tag{4.7}$$

and provides a bound on the norm of the integral in (4.6).

The quantities appearing in (4.7) depend on the Euclidean Green function functions. In this case the integrand has the form

$$\|(H\Phi - H_0\Phi)e^{\mp iH_0 t}|\psi\rangle\| =$$

$$(\psi, e^{\pm iH_0 t}(\Phi^\dagger H - \Phi^\dagger H_0)(\theta S_{1:1}S_{1:1} + \theta S^c_{2:2})(H\Phi - \Phi H_0)\psi)^{1/2} \tag{4.8}$$

where $(,)$ represents the Euclidean inner product and I have used (4.2) to express the four-point Green function as the sum of a product of two-point functions with a connected four-point function that vanishes asymptotically.

The second goal of this paper will be realized by showing that condition (4.7) holds for the four-point Green functions (3.6), which is sufficient for the existence of the strong limits (4.5). To do this it is necessary to show

$$\int_0^\infty \left(\psi, e^{\mp iH_0 t}(\Phi^\dagger H - H_0 \Phi^\dagger) e^{\pm iH_0 t} \theta S_{2:2}(H\Phi - \Phi H_0) e^{\mp iH_0 t}\psi\right)^{1/2} dt < \infty. \tag{4.9}$$

In (4.9) $H\Phi - \Phi H_0$ replaces the non-relativistic potential. $U_0(t)$ is the time-evolution operator for the asymptotically free eigenstates.

I will argue that the integrand in (4.9) falls off like $t^{-3/2}$, which is sufficient for the Cook condition (4.7) to be satisfied.

To see this first note that cluster properties imply that the Euclidean Green function is the sum of a product of two-point Euclidean Green functions and a connected term. I consider a connected term that has the structure discussed in (3.6)

$$S(\mathsf{x}_1, \mathsf{x}_2; \mathsf{x}_3, \mathsf{x}_4) =$$

$$\int d^4\mathsf{q}_1 d^4\mathsf{q}_2 d^4\mathsf{q}_3 dm_a dm_c dm_b \frac{e^{i\mathsf{q}_1\cdot(\mathsf{x}_1-\mathsf{x}_2)+i\mathsf{q}_2\cdot(\mathsf{x}_2-\mathsf{x}_3)+i\mathsf{q}_3\cdot(\mathsf{x}_3-\mathsf{x}_4)}}{(\mathsf{q}_1^2+m_a^2)(\mathsf{q}_2^2+m_c^2)(\mathsf{q}_3^2+m_b^2)} v(\mathsf{q}_1, m_a, \mathsf{q}_2, m_c, \mathsf{q}_3, m_b) \tag{4.10}$$

where $m_c > m_a, m_b$ and the spectrum of m_c should be continuous for scattering. I have already established that this connected Euclidean Green function is reflection positive for suitable $v(\mathsf{q}_1, m_a, \mathsf{q}_2, m_c, \mathsf{q}_3, m_b)$.

In this case the asymptotic states $\Phi|e^{\mp iH_0 t}|\psi\rangle$ have the form

$$\langle \mathsf{x}_1, \mathsf{x}_2|\Phi|e^{\mp iH_0 t}\psi\rangle = \prod_{i=1}^{2} h_i(\mathsf{x}_i^0) \int \frac{d\mathbf{p}_i}{(2\pi)^{3/2}} \psi_i(\mathbf{p}_i) e^{i\mathbf{p}_i\cdot\mathbf{x}_i \mp i\omega_{m_i}(\mathbf{p}_i)t} \tag{4.11}$$

where I choose the functions $h_i(\mathsf{x}_i^0)$ to be sharply peaked with compact support and integrate to 1.

With this choice of wave packets

$$\langle \mathsf{x}_1, \mathsf{x}_2|(H\Phi - \Phi H_0)e^{\mp iH_0 t}|\psi\rangle =$$

$$(2\pi)^3 \left(\frac{\partial}{\partial \mathsf{x}_1^0} + \frac{\partial}{\partial \mathsf{x}_2^0} - \omega_{m_1}(\mathbf{p}_1) - \omega_{m_2}(\mathbf{p}_2)\right) h_1(\mathsf{x}_1^0) h_2(\mathsf{x}_2^0) \times$$

$$\int \frac{d\mathbf{p}_1 d\mathbf{p}_2}{(2\pi)^3} \psi_1(\mathbf{p}_1)\psi_2(\mathbf{p}_2) e^{i\mathbf{p}_1\cdot\mathbf{x}_1 \mp i\omega_{m_1}(\mathbf{p}_1)t + i\mathbf{p}_2\cdot\mathbf{x}_2 \mp i\omega_{m_2}(\mathbf{p}_2)t}. \tag{4.12}$$

The integrand in the Cook condition uses (4.10) and (4.12) in

$$(\psi, e^{\mp iH_0 t}(\Phi^\dagger H - H_0\Phi^\dagger)\Theta(S_{1:1}S_{1:1} + S_{2:2}^c)(H\Phi - \Phi H_0)e^{\pm iH_0 t}\psi). \tag{4.13}$$

In this expression the partial derivatives in (4.12) with respect to the Euclidean times can be integrated by parts. When the Euclidean time derivatives act on the product of the two-point functions $S_{1:1}S_{1:1}$ they generate energy factors (see equations (6.4) and (6.5) of [2]) that exactly cancel the asymptotic energy factors in (4.12), making the $S_{1:1}S_{1:1}$ terms in (4.13) vanish.

This means that only the connected part of the four-point Euclidean Green function contributes to the integral (4.13). Thus, the connected part of the four-point function plays an analogous role to the interaction in the non-relativistic case.

What remains has the form

$$(2\pi)^6 \int d^4\mathsf{q}_1 d^4\mathsf{q}_2 d^4\mathsf{q}_3 dm_a dm_c dm_b d\mathsf{x}_1^0 d\mathsf{x}_2^0 d\mathsf{x}_3^0 d\mathsf{x}_4^0 \left(\omega_{m_c}(\mathbf{q}_2) - \omega_{m_1}(\mathbf{q}_1) - \omega_{m_2}(\mathbf{q}_2 - \mathbf{q}_1)\right) \times$$
$$h_1(\mathsf{x}_1^0) h_2(\mathsf{x}_2^0) \psi_1^*(\mathbf{q}_1) \psi_2^*(\mathbf{q}_2 - \mathbf{q}_1) e^{\pm i\omega_{m_1}(\mathbf{q}_1)t \pm i\omega_{m_2}(\mathbf{q}_2 - \mathbf{q}_1)t} \times$$
$$\frac{e^{-\omega_{m_a}(\mathbf{q}_1)(\mathsf{x}_1^0 - \mathsf{x}_2^0) - \omega_{m_c}(\mathbf{q}_2)\cdot(\mathsf{x}_2^0 + \mathsf{x}_3^0) - \omega_{m_b}(\mathbf{q}_3)(\mathsf{x}_3^0 - \mathsf{x}_4^0)}}{(\mathsf{q}_1^2 + m_a^2)(\mathsf{q}_2^2 + m_c^2)(\mathsf{q}_3^2 + m_b^2)} \times$$
$$e^{\pm i\omega_{m_1}(\mathbf{q}_3)t \mp i\omega_{m_2}(\mathbf{q}_2 - \mathbf{q}_3)t} v(\mathsf{q}_1, m_a, \mathsf{q}_2, m_c, \mathsf{q}_3, m_b) \times$$
$$\left(\omega_{m_c}(\mathbf{q}_2) - \omega_{m_1}(\mathbf{q}_3) - \omega_{m_2}(\mathbf{q}_2 - \mathbf{q}_3)\right) h_1(\mathsf{x}_3^0) h_2(\mathsf{x}_4^0) \psi_1(\mathbf{q}_3) \psi_2(\mathbf{q}_2 - \mathbf{q}_3). \qquad (4.14)$$

The large-time behavior of this integral is relevant for the Cook condition. To estimate the large time behavior write

$$-\omega_{m_a}(\mathbf{q}_1)(\mathsf{x}_1^0 - \mathsf{x}_2^0) \pm i\omega_{m_1}(\mathbf{q}_1)t =$$
$$-(\omega_{m_a}(\mathbf{q}_1) - \omega_{m_1}(\mathbf{q}_1))(\mathsf{x}_1^0 - \mathsf{x}_2^0)$$
$$-\omega_{m_1}(\mathbf{q}_1)(\mp it + (\mathsf{x}_1^0 - \mathsf{x}_2^0)) \qquad (4.15)$$

and

$$-\omega_{m_b}(\mathbf{q}_3)(\mathsf{x}_3^0 - \mathsf{x}_2^0) \pm i\omega_{m_1}(\mathbf{q}_3)t =$$
$$-(\omega_{m_b}(\mathbf{q}_3) - \omega_{m_1}(\mathbf{q}_3))(\mathsf{x}_3^0 - \mathsf{x}_2^0)$$
$$-\omega_{m_3}(\mathbf{q}_3)(\mp it + (\mathsf{x}_1^0 - \mathsf{x}_2^0)). \qquad (4.16)$$

We assume that the $m_a, m_b \geq m_1$, which is the easiest case to consider. These assumptions ensure that the integrals are all convergent.

To put (4.13) in a manageable form I make some simplifying assumptions. First I assume that the $h_i(\mathsf{x}^0)$ are sharply peaked to factor the integrand out of the integral. The resulting approximation leads to a constant multiplied by the integrand evaluated at points at x_i^0 in the support of $h_i(\mathsf{x}^0)$. Similarly, I change to total and relative momentum variables and use the translational invariance to eliminate the center of momentum degrees of freedom. I assume that the total 3-momentum support of the wave functions is near zero. One total momentum integral is eliminated by the momentum conserving delta function. The other total momentum integral is approximated by setting the total momentum to zero and multiplying by the volume of the support of the total momentum. What remains, up to a multiplicative constant, has the form

$$\Big(\psi, U_0(\pm t)(H\Phi - \Phi H_0)U_0^\dagger(t)\psi, \theta S_{2:2}^c(H\Phi - \Phi H_0)U_0(\mp t)\psi\Big) \to$$

$$\int (\omega_{m_c}(\mathbf{0}) - \omega_{m_1}(\mathbf{k}') - \omega_{m_2}(\mathbf{k}')) \times$$

$$\psi_3^*(-\mathbf{k}')\psi_4^*(\mathbf{k}')e^{\pm i\omega_{m_2}(\mathbf{k}')t \pm i\omega_{m_1}(\mathbf{k}')t}$$

$$e^{-\omega_{m_a}(\mathbf{k}')(\mathsf{x}_4^0 - \mathsf{x}_3^0) - \omega_{m_c}(\mathbf{0})(\mathsf{x}_3^0 + \mathsf{x}_1^0) - \omega_{m_b}(\mathbf{k})(\mathsf{x}_1^0 - \mathsf{x}_2^0)}$$

$$\frac{d\mathbf{k}d\mathbf{k}'}{\omega_{m_a}(\mathbf{k}')\omega_{m_c}(\mathbf{0})\omega_{m_b}(\mathbf{k})}$$

$$(\omega_{m_c}(\mathbf{0}) - \omega_{m_1}(\mathbf{k}) - \omega_{m_2}(\mathbf{k})) \times$$

$$\psi_1(\mathbf{k})\psi_2(-\mathbf{k})e^{\mp i\omega_{m_1}(\mathbf{k})t \mp i\omega_{m_2}(\mathbf{k})t}. \tag{4.17}$$

where $\mathbf{k}$ is the momentum of one of the particles in the zero momentum frame.

The time dependence in this expression comes from the $\mathbf{k}$ and $\mathbf{k}'$ integrals. If I use (4.15-4.16) in this expression, assuming that $m_a \geq m_1$ and $m_b \geq m_2$ the integral (4.17) has the general form

$$\int \frac{d\mathbf{k}}{\omega_{m_1}(\mathbf{k})} g(\mathbf{k}) e^{-\omega_{m_1}(\mathbf{k}^2)(\mathsf{x}_1^0 - \mathsf{x}_2^0 \pm 2it)} \tag{4.18}$$

where $g(\mathbf{k})$ is a well-behaved function of $\mathbf{k}$.

It follows from the lemma on page 157 of [8] that, for the case that the wave functions $\psi_i(\mathbf{k})$ are smooth with compact support, integrals of this form fall off like $t^{-3/2}$ for large time.

This shows that the Cook condition (4.7) is satisfied for the reflection positive Euclidean Green functions of the form (3.6). The asymptotic large-time behavior is identical to the behavior found in non-relativistic scattering theory.

The relativistic invariance of the S matrix can be established using similar methods. The required condition in terms of the wave operators are the intertwining conditions

$$U(\Lambda, a)\Omega_{\pm} = \Omega_{\pm} U_0(\Lambda, a) \tag{4.19}$$

for both asymptotic conditions. For the space translations and rotations this condition is a consequence of the translational and rotational invariance of the injection operators, Φ. For time translations this follows from the existence of the wave operators. For the boosts a sufficient condition is

$$\lim_{t\to\pm\infty} \|(\mathbf{K}\Phi - \Phi\mathbf{K}_0)e^{\mp iH_0 t}|\psi\rangle\| = 0 \tag{4.20}$$

where expressions for the boost generators are given in [2]. As in the scattering case, the non-zero contributions to this expression before taking the limit come from $S^c_{2:2}$ (see 4.8). In the two-body example above, this depends on properties of the connected four-point function. When (4.20) holds finite Lorentz transformations on the scattering eigenstates can be realized by transforming the asymptotic states using (4.19).

V. COMPUTATIONAL ISSUES

In this paper we have demonstrated the existence of a class of reflection positive Euclidean Green functions and shown that this class of Euclidean Green functions leads to non-trivial scattering operators. The scattering operators were constructed using conventional time-dependent methods, where cluster properties of the Green functions were used to formulate the scattering asymptotic conditions.

The result is that having established the existence of wave operators and knowing how to compute matrix elements of $e^{-\beta H}$ in a basis of normalizable states (see 2.10), there is enough information to compute transition matrix elements.

The strategy adopted in [2] to perform this computation utilized three controlled approximations. The first is to use narrow wave packets to extract sharp momentum transition matrix elements from S-matrix elements

$$\langle \mathbf{p}'_1, \cdots, \mathbf{p}'_{n_\alpha}|t(E_\gamma + i0)|\mathbf{p}_1, \cdots, \mathbf{p}_{n_\gamma}\rangle \approx \frac{i}{2\pi}\frac{\langle\psi_{\alpha f}|(S-I)|\psi_{\gamma i}\rangle}{\langle\psi_{\alpha f}|\delta(E_\alpha - E_\gamma)|\psi_{\gamma i}\rangle}. \tag{5.1}$$

The convergence of these approximations is determined by the smoothness of the transition matrix elements.

The S matrix elements needed as input were expressed, using the invariance principle [9][10], as matrix elements of $e^{i2ne^{-\beta H}}$ in normalizable eigenstates

$$\langle \psi_{\alpha f} | S | \psi_{\gamma i} \rangle = \lim_{n \to \infty} \langle \psi_{\alpha f} | e^{-ine^{-\beta H_\alpha}} \Phi_\alpha^\dagger e^{2ine^{-\beta H}} \Phi_\gamma e^{-ine^{-\beta H_\gamma}} | \psi_{\gamma_i} \rangle. \tag{5.2}$$

The convergence with n depends on the width of the wave packets. Ten significant figure accuracy was achieved in the test model of ref ([2]).

The third approximation that we used was to uniformly approximate $e^{2ine^{-\beta H}}$ by a polynomial in $e^{-\beta H}$. This was possible because the spectrum of $e^{-\beta H}$ is bounded. Because of the uniform convergence the error is identical to the error in approximating

$$|e^{2inx} - P(x)| < \epsilon \qquad x \in [0, 1] \tag{5.3}$$

by a polynomial. Ten significant figure accuracy was again realized using Chebyshev polynomials with Gauss-Chebyshev quadratures. This method requires that the approximations be performed in the specified order.

In more realistic models an additional approximation is needed, which is the solution of point spectrum mass eigenstates of the subsystem Green functions. These appear in the multi-channel generalization of the mapping Φ and are needed to get the strong convergence needed to satisfy the Cook condition (4.7). They can also appear in the two-point function if it has a non-trivial Lehmann weight.

In general, given an explicit Hilbert space representation and knowing that the scattering theory exists, there are many other techniques that could be used to calculate scattering observables without analytic continuation. The method discussed above provides one method that has been tested, but it may not be the most efficient method available.

VI. QCD

Ultimately one would like to use Euclidean methods to compute GeV scale scattering observables in QCD. Lattice, path integral, and Schwinger-Dyson formulations of QCD all yield Euclidean Green functions.

In QCD, because of confinement, the Euclidean Green functions of the theory are not expected to be reflection positive. However reflection positivity should hold for color singlet initial and final states. In addition the scattering asymptotic states should also be reflection positive. The Euclidean methods discussed in this paper are still be applicable if these two conditions hold.

VII. SUMMARY

In this paper the existence of non-trivial reflection positive Euclidean Green functions was demonstrated by exhibiting an explicit class of reflection positive connected four-point Euclidean Green functions. The structure of this class of Green functions was motivated by a theorem of Widder that exhibited the structure of a general one-dimensional reflection-positive two-point function. The general structure of reflection positive four-point functions is still an open problem. More importantly, the structure of Euclidean Green functions for realistic models remains an open problem.

In this paper time-dependent scattering methods were used to demonstrate the existence of scattering wave operators for models based on the reflection positive Euclidean Green functions of the form (3.6). The basic observation is that the Cook condition that is normally used as a sufficient condition for the existence of non-relativistic wave operators can be applied in this formulation of Euclidean relativistic quantum mechanics. The $t^{-3/2}$ asymptotic behavior of the integrand in (4.7) that ensures the existence of the wave operator for short-ranged potentials in the non-relativistic case is realized in the relativistic case for sufficiently well-behaved connected Euclidean Green function functions.

The results of this paper imply that the Euclidean methods tested in [2], when applied to models defined by the class of reflection positive Green function in section three, should converge to transition matrix elements for a range of energies up to the few GeV scale.

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VIII. APPENDIX - SPIN

Two point Euclidean Green functions that lead to any positive-mass irreducible representation space of the Poincaré group are constructed in this appendix.

A basis for vectors in a positive-mass irreducible representation space of the Poincaré group consists of simultaneous eigenstates of the mass, spin, linear momentum, and z-

component of some kind of spin (canonical, Jacob-Wick helicity, light-front, $\cdots$). These states have the following transformation property

$$U(\Lambda,a)|(m,j)\mathbf{p},\mu\rangle =$$

$$\sum_{\nu} e^{-i\Lambda p\cdot a}|\mathbf{\Lambda} p,\nu\rangle D^j_{\nu\mu}[B^{-1}(\Lambda p/m)\Lambda B(p/m)]\sqrt{\frac{\omega_m(\Lambda p)}{\omega_m(p)}} \tag{8.1}$$

where $B^{-1}(\Lambda p/m)\Lambda B(p/m)$ is a Wigner rotation. The choice of Lorentz boost, $B(p/m)^{\mu}{}_{\nu}$, in the Wigner rotation determines the type of spin [11]. For any kind of spin the Wigner D functions, which are also finite dimensional representations of $SL(2,\mathbb{C})$, can be factored into products. Multiplication of both sides of (8.1) by

$$D^j_{\mu\nu}[B^{-1}(p/m)] \tag{8.2}$$

leads to

$$\sum_{\mu} U(\Lambda,a)|(m,j)\mathbf{p},\mu\rangle D^j_{\mu\sigma}[B^{-1}(p/m)]\sqrt{\omega_m(p)} =$$

$$\sum_{\nu\sigma'} e^{-i\Lambda p\cdot a}|\mathbf{\Lambda} p,\nu\rangle D^j_{\nu\sigma'}[B^{-1}(\Lambda p/m)]\sqrt{\omega_m(\Lambda p)}D^j_{\sigma'\sigma}[\Lambda]. \tag{8.3}$$

The vectors

$$|p,j,\sigma\rangle := \sum_{\mu} |(m,j)\mathbf{p},\mu\rangle D^j_{\mu\sigma}[B^{-1}(p/m)]\sqrt{\omega_m(p)} \tag{8.4}$$

transform in a Lorentz covariant manner

$$U(\Lambda,0)|p,j,\sigma\rangle = \sum_{\sigma'} e^{-i\Lambda p\cdot a}|\Lambda p,j,\sigma'\rangle D^j_{\sigma'\sigma}[\Lambda]. \tag{8.5}$$

The transformation $U(\Lambda,0)$ is unitary with respect to the inner product

$$\psi(p,j,\sigma) = \langle p,j,\sigma|\psi\rangle, \tag{8.6}$$

$$\langle\psi|\phi\rangle = \sum_{\mu\sigma}\int \psi^*(p,j,\mu)D^j_{\mu\sigma}[B(p/m)B^{\dagger}(p/m)]\frac{md\mathbf{p}}{\omega_m(p)}\phi(p,j,\sigma) \tag{8.7}$$

where $p_0 = \omega_m(\mathbf{p})$ is the energy. The kernel simply removes the momentum-dependent $SL(2,C)$ Wigner functions from the covariant representation. Because the $SL(2,\mathbb{C})$ matrices cancel in computing matrix elements - the result is the same independent of whether the right or left-handed representations of $SL(2,\mathbb{C})$ are used.

Note that in $SL(2,\mathbb{C})$ a general boost has a polar decomposition

$$B(p) = P(p)R(p) \tag{8.8}$$

where $P(p)$ is the positive Hermitian operator,

$$P(p) = e^{\boldsymbol{\rho}\cdot\boldsymbol{\sigma}/2}, \tag{8.9}$$

$\boldsymbol{\rho}$ is the rapidity vector and $R(p)$ is an $SU(2)$ matrix (generalized Melosh rotation) that determines the type of spin. It follows that

$$B(p/m)B^{\dagger}(p/m) = P(p)R(p)R^{\dagger}(p)P(p) = P^2(p) = e^{\boldsymbol{\rho}\cdot\boldsymbol{\sigma}} = \sigma \cdot p. \tag{8.10}$$

In this expression the Melosh rotations cancel, so the result is independent of the choice of spins. Thus this scalar product can be expressed as

$$\sum_{\alpha\beta} \int \psi^*(p,j,\alpha) D^j_{\alpha\beta}[p\cdot\sigma] m \frac{d\mathbf{p}}{\omega_m(p)} \phi(p,j,\beta) = \tag{8.11}$$

$$\sum_{\alpha\beta} \int \psi^*(p,j,\alpha) D^j_{\alpha\beta}[p\cdot\sigma] 2m d^4p \delta(p^2+m^2) \phi(p,j,\beta). \tag{8.12}$$

This is essentially identical to the form found in [12] (see eq. 1.57). The important observation is that $\sigma \cdot p$ is a positive Hermetian matrix for timelike p. The same holds for $D^j_{\mu\sigma}[p\cdot\sigma^t]$, $D^j_{\mu\sigma}[p\cdot\sigma^*]$, and $D^j_{\mu\sigma}[p\cdot\sigma^{-1}]$.

The following Green function is a Euclidean covariant rather than Euclidean invariant distribution

$$\int \frac{2mD^j_{\alpha\beta}[p_e\cdot\sigma_e]}{p_e^2+m^2} d^4p_e e^{ip_e\cdot(\mathsf{x}-\mathsf{y})} \tag{8.13}$$

that leads exactly to the representation (8.7) of a mass m spin j irreducible representation.

These considerations show that the following Euclidean scalar product is reflection positive

$$\sum_{\alpha\beta} \int g^*_\alpha(\theta\mathsf{x}) \frac{2mD^j_{\alpha\beta}[p_e\cdot\sigma_e]}{p_e^2+m^2} d^4p_e e^{ip_e\cdot(\mathsf{x}-\mathsf{y})} g_\beta(\mathsf{y}) d^4x d^4y d^4p_e =$$

$$\sum_{\alpha\beta} \psi^*(p,j,\alpha) D^j_{\alpha\beta}[p\cdot\sigma] 2m d^4p \delta(p^2+m^2) \psi(p,j,\beta) \tag{8.14}$$

with

$$\psi(p,j,\beta) = \int g_\beta(\mathsf{x}^0,\mathsf{x}) e^{-\omega_m(\mathbf{p})\mathsf{x}^0} e^{i\mathbf{p}\cdot\mathsf{x}}. \tag{8.15}$$

This shows how to construct reflection-positive two-point Euclidean Green functions for any irreducible representation of the Poincaré group. While I did not choose to double the representation, doubled representations can be realized by replacing $D^j_{\alpha\beta}[p\cdot\sigma]$ by a direct sum of a right and left handed representation , $D^j_{\alpha'\beta'}[p \cdot \sigma_2 \sigma^* \sigma_2]$, which is also positive for positive energy timelike p.

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