Cluster properties in Poincaré invariant quantum mechanics *

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Abstract

Using a simple model we provide a quantitative study of the size of the corrections needed to restore cluster properties to the construction of Poincaré invariant dynamical models with kinematic spins, first provided by B. Bakamjian and L. H. Thomas. Our model calculations suggest that these corrections are too small to have a quantitative impact on nuclear physics observables calculated using models with meson and nucleon degrees of freedom.

1 Introduction

We provide a quantitative evaluation of the size of the operators that restore cluster properties in Bakamjian-Thomas formulations of relativistic few-body quantum mechanics.

Relativistic few-body models are an extension of the corresponding non-relativistic models that are exactly Poincaré invariant. By exact Poincaré invariance we mean that quantum probabilities, which are the dimensionless observables of the theory, have the same values in all inertial coordinate systems. Wigner[1] showed that a necessary and sufficient condition for the invariance of quantum probabilities is the existence of a dynamical representation of the Poincaré group on the model Hilbert space.

Dirac[2] showed that at least three of the Poincaré generators must have interactions in order to satisfy the commutation relations in an interacting theory.

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This is because time translation can be expressed in terms of Lorentz boosts and spatial translations.

Beyond Poincaré invariance, the requirement that Poincaré invariance also holds for isolated subsystems requires that the unitary representation of the Poincaré group be well approximated by a tensor product of two representations when evaluated between states representing asymptotically separated subsystems.

Bakamjian and Thomas[3] provided the first non-field theoretic realization of the Poincaré Lie algebra with interactions. Their construction satisfied cluster properties at the two-body level, but not for systems of more than two particles. Coester[4] applied the Bakamjian-Thomas construction to three-body systems and showed that the resulting three-body S-matrix satisfied cluster properties. Unfortunately his result did not extend to the unitary representation of the Poincaré group and did not apply to systems of four or more particles. Sokolov[5][6] provided a complete solution to the problem in terms of certain unitary operators. In this framework the size of the corrections that restore cluster properties are related to how close these unitary operators are to the identity. Sokolov's operators have never been computed in any applications.

The Bakamjian-Thomas construction begins with a tensor product of two irreducible representations of the Poincaré group and decomposes it into a direct integral of irreducible representations using Clebsch-Gordan coefficients for the Poincaré group

$$|(M,j)\mathbf{P},\mu;l,s\rangle = \sum\!\!\!\!\!\!\!\int d\mathbf{p}_1 d\mathbf{p}_2 |(m_1,j_1)\mathbf{p}_1,\mu_1\rangle \otimes |(m_2,j_2)\mathbf{p}_2,\mu_2\rangle \times$$
$$\underbrace{\langle (m_1,j_1)\mathbf{p}_1,\mu_1(m_2,j_2)\mathbf{p}_2,\mu_2|(M,j)\mathbf{P},\mu;l,s\rangle}_{\text{Poincaré group Clebsch-Gordan coeff.}}.$$

Interactions are added to the invariant mass operator of this free-particle irreducible representation that commute with the free spin operator and commute with and are independent of the quantum numbers that label vectors in each irreducible subspace

$$M_I = M_0 + V.$$

In the free-particle irreducible basis these matrix elements have the form

$$\langle (M,j)\mathbf{P},\mu;l,s|V|(M',j')\mathbf{P}',\mu';l',s'\rangle =$$
$$\delta(\mathbf{P}-\mathbf{P}')\delta_{jj'}\delta_{\mu\mu'}\langle M,l,s\|v^j\|M',l',s'\rangle.$$

Simultaneous eigenstates of $M_I, \mathbf{j}^2, j_z, \mathbf{P}$ are constructed by solving the Schrödinger equation for the mass eigenstates

$$(\lambda - M)\phi_{\lambda,j}(M,l,s) = \sum\!\!\!\!\!\!\!\int' dM' \langle M,l,s\|v^j\|M',l',s'\rangle \phi_{\lambda,j}(M,l,s).$$

The eigenstates defined by the wave functions

$$\langle (M,j)\mathbf{P},\mu;l,s|(\lambda,j')\mathbf{P}',\mu'\rangle = \delta(\mathbf{P}-\mathbf{P}')\delta_{jj'}\delta_{\mu\mu'}\phi_{\lambda,j}(M,l,s)$$

are complete and transform irreducibly under the dynamical representation

$$U(\Lambda, a)|(\lambda, j)\mathbf{P}, \mu\rangle$$

$$\sum_{\nu} |(\lambda, j)\mathbf{\Lambda} P, \nu\rangle e^{i\Lambda P\cdot a} \sqrt{\frac{\omega_\lambda(\Lambda P)}{\omega_\lambda(P)}} D^{j}_{\nu\mu}(R_w(\Lambda, P))$$

of the Poincaré group. The original Bakamjian-Thomas construction was for a system of two particles; but a generalization of the construction outlined above works for any number of particles. The key requirement is that the spin in the interacting model is identified with the spin in the non-interacting model.

In the two-body Bakamjian-Thomas representation it is clear that when the two-body interaction is turned off the resulting unitary representation of the Poincaré group becomes the tensor product of two non-interacting irreducible representations, as expected. When the Bakamjian-Thomas construction is applied to systems of three particles, turning off the two-body interactions involving one particle no longer results in a tensor product of a one and two body representation of the Poincaré group.

The Sokolov construction starts with the two-body interactions that appear in the two-body problem and uses them to construct three-body interactions that lead to a dynamical representation of the Poincaré group that clusters to the tensor product of the two-body Bakamjian-Thomas representation and a one-body representation. This construction can be repeated recursively for any number of particles.

To understand the Sokolov construction consider a three-body system where one pair of particles interact. There are two natural constructions of a dynamical representation of the Poincaré group. The first is to take the tensor product of a two-body Bakamjian-Thomas representation with a single-particle irreducible representation. The second is to perform a full three-body Bakamjian Thomas construction where the interaction commutes with the non-interacting three-body spin. Using appropriate choices of two-body interactions, these constructions can be done in a manner that ensures that both representations lead to the same scattering matrix elements and two body-bound state masses. The relevant additions to the two-body invariant mass have the forms

$$\langle \mathbf{P}', j_3', \mu_3', \mathbf{p}_3', m_{12}', j', l', s', \mu' | V_{12}^{TP} | \mathbf{P}, j_3', \mu_3', \mathbf{p}_3, m_{12}, j, l, s, \mu\rangle =$$

$$\delta(\mathbf{P}' - \mathbf{P})\delta_{j_3' j_3}\delta_{\mu_3' \mu_3}\delta(\mathbf{p}_3' - \mathbf{p}_3)\delta_{j' j}\delta_{\mu' \mu}\langle m_{12}', l', s' \| v_{12}^{j} \| m_{12}, l, s\rangle$$

and

$$\langle \mathbf{P}', j_3, \bar{\mu}_3, \mathbf{q}_3', m_{12}', j', l', s', \bar{\mu}' | V_{12}^{BT} | \mathbf{P}, j_3, \bar{\mu}_3, \mathbf{q}_3, m_{12}, j, l, s, \bar{\mu}\rangle =$$

$$\delta(\mathbf{P}' - \mathbf{P})\delta_{j_3' j_3}\delta_{\bar{\mu}_3' \bar{\mu}_3}\delta(\mathbf{q}_3' - \mathbf{q}_3)\delta_{j' j}\delta_{\bar{\mu}' \bar{\mu}}\langle m_{12}', l', s' \| v_{12}^{j} \| m_{12}, l, s\rangle$$

where

$$\mathbf{q}_3 := \mathbf{\Lambda}^{-1}(P/M(k))p_3 \qquad \bar{\mathbf{j}} = R_w(P, p_{12})\mathbf{j} \qquad \bar{\mathbf{j}}_3 = R_w(P, p_3)\mathbf{j}_3.$$

Violations of cluster properties arise because the boost that appears in the definition of $\mathbf{q}_3$ and $\bar{\mathbf{j}}$ depends on m_{12} which does not commute with the potential. The spectator delta functions and spin kronecker delta functions in these two expressions are only equivalent when $m_{12} = m'_{12}$. The S-matrix in both for these representations are equal when the reduced kernels $\langle m'_{12}, l', s' \| v^j_{12} \| m_{12}, l, s\rangle$ are identified. This is because the S-matrix has the form of the reduced kernel

$$\langle m'_{12}, l', s' \| S^j_{12} \| m_{12}, l, s\rangle$$

multiplied by delta functions that become equivalent on shell (when $m_{12} = m'_{12}$). A consequence of this equivalence is that both unitary representations of the Poincaré group are related by an S-matrix preserving unitary transformation [7]

$$A_{12,3} U^{TP}_{12,3}(\Lambda, a) A^\dagger_{12,3} = U^{BT}_{12,3}(\Lambda, a).$$

In the Bakamjian-Thomas representation it is possible to combine the three 2+1 mass operators to get an interacting three-body mass operator that commutes with the non-interacting three-body spin

$$M^{BT} := M^{BT}_{12,3} + M^{BT}_{23,1} + M^{BT}_{31,2} - 2M^{BT}_0 \qquad \mathbf{j}^{BT} := \mathbf{j}_0.$$

Applying the Bakamjian-Thomas construction to this mass operator gives a dynamical unitary representation, $U^{BT}(\Lambda, a)$, of the Poincaré group.

Sokolov defined an S-matrix equivalent representation using a unitary transformation constructed from a symmetrized product of the three $2+1$ unitary operators relating the 2+1 Bakamjian-Thomas representation to the 2+1 tensor product representations:

$$U(\Lambda, a) := A^\dagger U^{BT}(\Lambda, a) A \tag{1}$$

where

$$A = e^{\ln(A_{12,3}) + \ln(A_{23,1}) + \ln(A_{31,2})}.$$

The resulting three-body invariant mass operator can be expressed in terms of these unitary transformations and the mass operators for the $2+1$ tensor product representations:

$$M :=$$
$$A^\dagger \left(A_{12,3} M_{12,3} A^\dagger_{12,3} + A_{23,1} M_{23,1} A^\dagger_{23,1} + A_{31,2} M_{31,2} A^\dagger_{31,2} - 2M^{BT}_0 \right) A.$$

Because the $A_{ij,k} \to I$ when the interaction between particles i and j is turned off, in each asymptotic region this mass operator becomes the mass operator for the associated tensor product representation, which implies that this transformed representation of the Poincaré group satisfies cluster properties. The combined effect of the unitary operators is to generate three-body interactions that restore the Poincaré commutation relations to the cluster expansions of the Poincaré generators.

In the limit that A in (1) becomes the identity the Sokolov representation becomes the Bakamjian-Thomas representation. Thus the size of the difference between these unitary transformations and the identity provides a measure of the size of the violations of cluster properties in the Bakamjian-Thomas representation.

To test the size of the corrections that restore cluster properties we consider a simple four-body model. It consists of a three-particle system where two of the particles interact to form a bound state and an external probe that interacts weakly with the third particle. We assume that there are no interactions between the particles in the bound pair and the third particle or the probe. All particles are treated as spinless particles and the probe is a assumed to interact via a scalar "current". We use a relativistic Malfliet-Tjon type of potential to construct a model with nuclear-physics scales.

We formulate models treating the three-body system as a 2+1 tensor-product representation or a 2+1 Bakamjian-Thomas representation. The current matrix elements in the two cases are related by the unitary transformation $A_{12,3}$

$$\langle 12\otimes 3|j(0)|12\otimes 3'\rangle = \langle (12,3)^{BT}|A_{12,3}j(0)A^{\dagger}_{12,3}|(12,3)'^{BT}\rangle.$$

It follows that the difference between $\langle 12\otimes 3|j(0)|12\otimes 3'\rangle$ and $\langle (12,3)^{BT}|j(0)|(12,3)'^{BT}\rangle$ provides one measure of difference between $A_{12,3}$ and the identity, which is a measure of the size of the operator that restores cluster properties to the Bakamjian-Thomas representation.

In the figures we plot

$$F(\mathbf{p}'_3-\mathbf{p}_3,\mathbf{p}_{12}) := \frac{\int d\mathbf{p}'_{12TP}\langle \mathbf{p}_3,\mathbf{p}_{12},\phi|j(0)|\mathbf{p}'_3,\mathbf{p}'_{12},\phi\rangle_{TP} - \int d\mathbf{p}'_{12BT}\langle \mathbf{p}_3,\mathbf{p}_{12},\phi|j(0)|\mathbf{p}'_3,\mathbf{p}'_{12},\phi\rangle_{BT}}{\int d\mathbf{p}'_{12TP}\langle \mathbf{p}_3,\mathbf{p}_{12},\phi|j(0)|\mathbf{p}'_3,\mathbf{p}'_{12},\phi\rangle_{TP}} \tag{2}$$

In (2) the integral over $\mathbf{p}'_{12}$ removes the dependence on the momentum of the bound pair, $\mathbf{p}_{12}$, in a model that satisfies cluster properties. $F(\mathbf{q},\mathbf{p}_{12})$ must vanish for models satisfying cluster properties, which is illustrated by the flat plane in each of the figures. Any residual dependence on $\mathbf{p}_{12}$ in this expression indicates a violation of cluster properties, which provides a measure of how much the operator $A_{12,3}$ differs from the identity. Figures 1. and 2. show (2) for p_{12} perpendicular and parallel to $q = p'_3 - p_3$ in Dirac's front-form dynamics. These two plots exhibit a small dependence on $\mathbf{p}_{12}$, but the value differs from zero. Figures 3. and 4. show (2) for p_{12} perpendicular and parallel to $q = p'_3 - p_3$ in Dirac's instant-form dynamics. Figures 5. and 6. show (2) for p_{12} perpendicular and parallel to $q = p'_3 - p_3$ in Dirac's point-form dynamics. Both the instant and point-form calculations have more dependence on $\mathbf{p}_{12}$ than the front-form calculation, but the magnitude of the violations of cluster properties are of comparable size in all three cases. While all six plots exhibit clear violations of cluster properties, the size of the violations are a few parts in a thousand which is well within the size of both theoretical and experimental uncertainties in relativistic nuclear physics observables. The violations of cluster properties increase with stronger binding or for wave functions with higher mean momentum, however for scales associated with realistic nuclear-nucleon

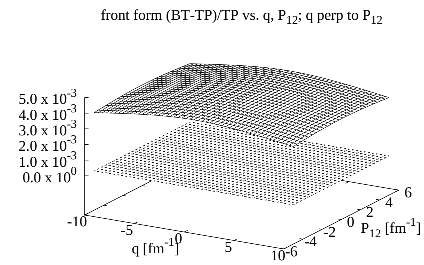

Figure 1: Front form - $p_{12} \perp q$

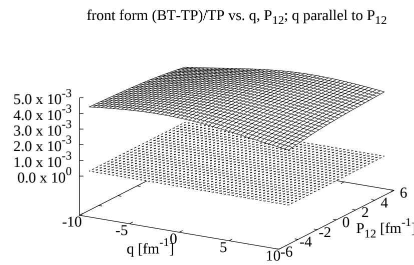

Figure 2: Front form - $p_{12} \| q$

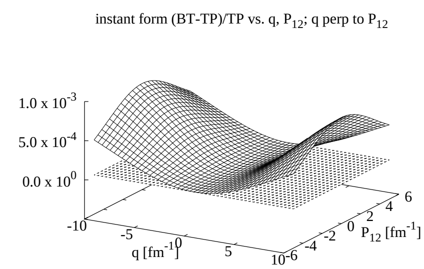

Figure 3: Instant form - $p_{12} \perp q$

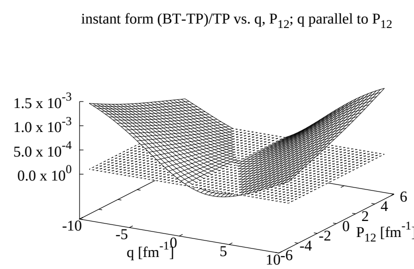

Figure 4: Instant form - $p_{12} \| q$

interactions the violations remain small. This suggests that at current levels of experimental precision there is no real need to compute corrections that restore cluster properties.

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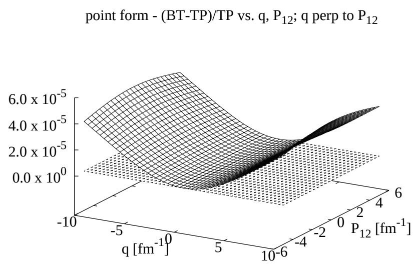

Figure 5: Point form - $p_{12} \perp q$

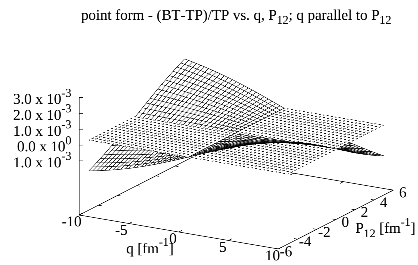

Figure 6: Point form - $p_{12} \| q$

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