Euclidean formulation of relativistic quantum mechanics

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We discuss preliminary work on a formulation of relativistic quantum mechanics that uses reflection-positive Euclidean Green functions or generating functionals as phenomenological input. This work is motivated by the Euclidean axioms of quantum field theory [1][2]. The key observations are (1) locality is not used to reconstruct the quantum theory and (2) it is possible to construct a fully relativistic quantum theory without performing an explicit analytic continuation.

Hilbert space vectors are represented by wave functionals $A[\phi]$ with inner product

$$A[\phi] = \sum_{j=1}^{n_a} a_j e^{i\phi(f_j)} \qquad \langle A|B\rangle := \sum_{j,k}^{n_a,n_b} a_l^* b_k Z[g_g - \Theta f_j]$$

where a_j are complex constants, f_j are real Schwartz functions on 4 dimensional Euclidean space with positive-time support, Θ is the Euclidean time-reflection operator, and $Z[f]$ is the Euclidean generating functional. Reflection positivity is the condition that $\langle A|A\rangle \geq 0$. For $\beta \geq 0$ and $\mathbf{a} \in \mathbb{R}^3$ we define

$$T(\beta,\mathbf{a})A[\phi] := \sum_{j=1}^{n_a} a_j e^{i\phi(f_{j,\beta,\mathbf{a}})} \qquad f_{j,\beta,\mathbf{a}}(\tau,\mathbf{x}) := f_j(\tau-\beta,\mathbf{x}-\mathbf{a}).$$

The square of the mass operator operating on a wave functional $A[\phi]$ is

$$M^2 A[\phi] = \left(\frac{\partial^2}{\partial\beta^2} + \frac{\partial^2}{\partial\mathbf{a}^2}\right) T(\beta,\mathbf{a})A[\phi]_{|\beta=\mathbf{a}=0}.$$

Solutions of the mass eigenvalue problem with eigenvalue λ can be expanded in terms of an orthonormal set of wave functionals $A_n[\phi]$, $\langle A_n|A_m\rangle = \delta_{mn}$:

$$\Psi_\lambda[\phi] = \sum \alpha_n A_m[\phi].$$

Simultaneous eigenstates of mass, linear momentum, spin, and z component of spin can be constructed from $\Psi_\lambda[\phi]$ using

$$\Psi_{\lambda,j,\mathbf{p},\mu}[\phi] = \int_{SU(2)} dR \int_{\mathbb{R}^3} \frac{d\mathbf{a}}{(2\pi)^{3/2}} e^{-i\mathbf{p}\cdot R\mathbf{a}} U(R) T(0,\mathbf{a}) \Psi_\lambda[\phi] D^{j*}_{\mu j}[R]$$

where $U(R)$ rotates the vector arguments of $f_j(\tau,\mathbf{x})$ in $A[\phi]$. When λ is in the discrete spectrum of M, $\Psi_{\lambda,j,\mathbf{p},\mu}[\phi]$ is a wave functional for a single-particle state that necessarily transforms as a mass λ spin j *irreducible representation.*

Products of suitably normalized single-particle wave functionals define mappings from the product of single-particle irreducible representation spaces of the Poincaré group to the model Hilbert space. Because these wave functionals create only single particle states out of the vacuum, their products are Haag-Ruelle injection operators [3][4] for the two-Hilbert-space formulation [4] of scattering theory. If we define $\Phi[\phi] := \prod_k \Psi_{\lambda_k, j_k, \mathbf{p}_k, \mu_k}[\phi]$, $\otimes g_k = \prod g_k(\mathbf{p}_k, \mu_k)$, and $H_f = \sum_k \sqrt{\lambda_k^2 + \mathbf{p}_k^2}$, then scattering wave operator can be defined by the limit

$$\Omega_{\pm} | \otimes g_k \rangle := \lim_{t \to \pm\infty} e^{iHt} \Phi e^{-iH_f t} | \otimes g_k \rangle.$$

Using the Kato-Birman invariance principle [4] to replace H by $-e^{-\beta H}$ gives

$$\Omega_{\pm} | \otimes g_k \rangle := \lim_{n \to \pm\infty} e^{-ine^{-\beta H}} \Phi e^{ine^{-\beta H_f}} | \otimes g_k \rangle.$$

Since the spectrum of $e^{-\beta H}$ is compact, for large *fixed* n $e^{-ine^{-\beta H}}$ can be uniformly approximated by a polynomial in $e^{-\beta H}$, which is easy to calculate in this framework. These steps provide a means to construct all single-particle states, all scattering states, and compute the action of the Poincaré group on all single-particle states and S-matrix elements, using only the Euclidean generating functional as input. The advantages of this framework are the relative ease with which cluster properties can be satisfied, the close relation to the quantum mechanical interpretation of quantum field theory, and the ability to perform calculations directly in Euclidean space without analytic continuation.

We tested the general method for calculating scattering observables using a solvable quantum mechanical model of the two-nucleon system. These test calculations, which used narrow wave packets, a large finite n and a Chebyshev polynomial expansion of e^{inx}, exhibited convergence to the exact transition matrix elements for a range of relative momenta between about 100 MeV up to 2 GeV. This success warrants further investigation of this framework.

This research was supported by the U.S. Department of Energy.

References

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