Scattering asymptotic conditions in Euclidean relativistic quantum theory

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We discuss the formulation of the scattering asymptotic condition as a strong limit in Euclidean quantum theories satisfying the Osterwalder-Schrader axioms. When used with the invariance principle this provides a constructive method to compute scattering observables directly in the Euclidean formulation of the theory, without an explicit analytic continuation.

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I. INTRODUCTION

The purpose of this paper is to argue that it is possible to calculate scattering observables directly in the Euclidean representation of quantum field theory without analytic continuation. The essential observation, which is a consequence of the Osterwalder Schrader reconstruction theorem [1], is that there is a representation of the physical Hilbert space directly in terms of the Euclidean Green functions without analytic continuation. There is also a representation of the Poincaré Lie algebra on this space. This defines a relativistic quantum theory. Cluster properties of the Schwinger functions suggest that it should be possible to formulate scattering problems directly in this representation.

In (1958) Schwinger [2] argued that as a result of the spectral condition that time-ordered Green functions had analytic continuations to Euclidean space-time variables. The analytically continued functions satisfy Schwinger-Dyson equations and are moments of Euclidean path integrals.

The Osterwalder-Schrader axioms [1][3] define conditions on a collection of Euclidean Green functions (Schwinger functions) that allow the reconstruction of a relativistic quantum theory. The Schwinger functions are formally defined as moments of a Euclidean path integral [4],

$$S_n(\mathsf{x}_1, \cdots, \mathsf{x}_n) = \frac{\int D[\phi] e^{-A[\phi]} \phi(\mathsf{x}_1) \cdots \phi(\mathsf{x}_n)}{\int D[\phi'] e^{-A[\phi']}} \tag{1}$$

where $A[\phi]$ is the classical action functional. The collection of Schwinger functions are Euclidean invariant and satisfy a condition, called reflection positivity. Reflection positivity is used to construct a representation of the Hilbert space of the field theory. On this representation of the Hilbert space there are explicit expressions for the Poincaré generators as differential operators. The remarkable property is that in this representation the Hilbert space inner product is expressed directly in terms of the Euclidean variables, without an explicit analytic continuation.

Given a representation of the Hilbert space and a set of Poincaré generators that satisfy cluster properties it should be possible to calculate quantum mechanical scattering observables. Cluster properties of the Poincaré generators in this representation is a consequence of cluster properties of the Schwinger functions.

In this work we discuss the direct construction of scattering observables in this Euclidean framework. No analytic continuation is used. The input is a set of Schwinger functions. These contain the dynamics and need to be computed as input. Their computation is easier than computing the corresponding Minkowski Green functions. In this work we show that scattering observables can be expressed as a quadratic form with a kernel constructed out of Schwinger functions. This avoids the need for analytic continuation or the need to solve singular integral equations.

We formulate the scattering problem using the two-Hilbert space formulation of time-dependent scattering[5][6]. This separates the internal structure of the asymptotic states from their space-time properties.

Below we list the steps that are needed to construct transition matrix elements. These will be discussed in more detail in the body of this paper. While some of these steps have been discussed in previous work, we include all of them to make this work self contained.

1 An asymptotic Hilbert space is defined as the direct sum of tensor products of irreducible representations of the Poincaré group. The mass and spin of the particles in the target, beam, and measured in the detectors determine the mass and spin labels of the irreducible representations.

2 Mappings from the asymptotic Hilbert space to the Euclidean representation of the physical Hilbert space are constructed. These mappings add the internal structure of the asymptotic particles to the space-time properties, including particle cloud effects. They also control the joint energy-momentum spectrum of the asymptotic states, which leads to strong limits.

3 The scattering asymptotic condition is formulated and sufficient conditions on the connected parts of the Schwinger functions for the existence of the channel Møller wave operators are given.

4 The invariance principle is used to replace H and H_α by $-e^{-\beta H}$ and $-e^{-\beta H_\alpha}$ in the expression for the wave operators, where H is the full Hamiltonian, H_α is the channel α asymptotic Hamiltonian and β is a parameter that defines the energy scale. This allows for a direct Euclidean evaluation of S matrix elements.

5 The computation of S-matrix elements with narrow wave packets in momentum is discussed. Sharp momentum transition matrix elements can be factored out of these S matrix elements provided the transition matrix elements vary slowly on the support of these wave packets.

Several aspects of this program were discussed in previous work [7][8][9]. In [8] a solvable quantum mechanical model was used to demonstrate that sharp-momentum transition matrix elements could be accurately computed for a range of energies (from 50 MeV to 2 GeV) by using the invariance principle to replace H by $-e^{-\beta H}$ in the expression for the Møller wave operators. In these calculations β is a parameter that sets the working energy scale. This computation utilized a uniformly convergent polynomial approximation of e^{inx} for $x \in [0, 1]$. This approximation is not necessary in the Euclidean framework.

In [9] the implementation of this method in the Euclidean framework was discussed. A class of model reflection-positive four-point Schwinger functions was given. The existence of wave operators for these model Schwinger functions was established using a generalization of Cook's method [10]. This assumed that the operators that map the asymptotic Hilbert space to the physical Hilbert space could be constructed.

The primary purpose of this paper is to complete this program by showing how to construct the mappings from the asymptotic Hilbert space to the physical Hilbert space that are needed to establish the existence of wave operators using Cook's method. We also show how the Euclidean representation of the Hilbert space can be used to eliminate the need for the polynomial approximation to $e^{-\beta H}$ that was utilized in [8].

The final result is an expression for scattering observables as a quadrature using the Schwinger functions and narrow wave packets as input.

The mapping from the asymptotic Hilbert space to the Euclidean representation of the Hilbert space is the Euclidean analog of a Haag-Ruelle quasi-local field operator [11][12][13][5][14]. This mapping controls the four-momentum spectrum of the asymptotic states, which is needed to isolate asymptotic states with different mass and the same energy. This can be done in the Euclidean representation because we have explicit representations of the four momentum operators. The result is that the scattering asymptotic condition can be formulated as a strong limit, like it is in non-relativistic quantum mechanics.

In section two the Euclidean representation of the physical Hilbert space given by Osterwalder and Schrader is defined. In section three expressions for the Poincaré generators in this representation of the Hilbert space are given.

The discussion in this paper is limited to spinless particles. Spin does not introduce any additional complications that impact the formulation of the scattering problem. The treatment of particles with spin is discussed in [9].

In section four the results of these sections are illustrated using the Lehmann representation of a two-point Schwinger function. Methods for isolating the discrete part of the Lehmann weight of these functions are central to the formulation of the scattering asymptotic condition. In section five the abstract two-Hilbert space formulation of the scattering asymptotic condition is discussed. This requires "Euclidean Haag-Ruelle" injection operators that map the asymptotic Hilbert space to the physical Hilbert space. The construction of these operators is given in section six. In section seven the construction of sharp momentum transition matrix elements is discussed. It is shown how properties of the Euclidean representation of the Hilbert space can be used to avoid the polynomial approximation used in [8]. Explicit formulas that express S-matrix elements directly in terms of the Euclidean Schwinger functions without analytic continuation are given. The appendix contains the main technical results of this paper. It shows that orthogonal polynomials in the mass squared operator are complete in this representation of the Hilbert space. This is needed to ensure that the injection operators that are used to formulate the asymptotic condition have positive relative-time support, so their range is in the Euclidean representation of the Hilbert space. The material in sections one-five is not new; it is included to make the article as self-contained as possible and limited to material that is essential to the new material. The new material is in sections six, seven and the appendix.

II. HILBERT SPACE

In the Euclidean framework Hilbert space vectors are represented by sequences of Schwartz test functions of the form

$$f(x) := (f_1(\mathsf{x}_{11}), f_2(\mathsf{x}_{21}, \mathsf{x}_{22}), \cdots) \tag{2}$$

where the individual functions have support restrictions

$$f_n(\mathsf{x}_{n1}, \mathsf{x}_{n2}, \cdots, \mathsf{x}_{nn}) = 0 \qquad \text{unless} \qquad 0 < \mathsf{x}^0_{n1} < \mathsf{x}^0_{n2} < \cdots < \mathsf{x}_{nn}. \tag{3}$$

Sequences of functions $\{f_n\}$ satisfying this support condition are called positive relative time functions. This is a linear subspace of the space of sequences of Schwartz test functions in Euclidean variables.

The Euclidean time reflection operator Θ is defined on these sequences. It changes the sign of the Euclidean times in each f_n:

$$\Theta f(x) = f(\theta x) = (f_1(\theta \mathsf{x}_{11}), f_2(\theta \mathsf{x}_{21}, \theta \mathsf{x}_{22}), \cdots) \tag{4}$$

where

$$\theta \mathsf{x} = \theta(x^0_{ij}, \mathbf{x}_{ij}) = (-x^0_{ij}, \mathbf{x}_{ij}). \tag{5}$$

The Hilbert space inner product of f with g is defined by

$$\langle f|g\rangle =$$

$$\sum_{mn} \int d^{4n}x d^{4m}y f^*_n(\mathsf{x}_{n1}, \mathsf{x}_{n2}, \cdots, \mathsf{x}_{nn}) S_{m+n}(\theta \mathsf{x}_{nn}, \cdots, \theta \mathsf{x}_{1n}, \mathsf{y}_{1m}, \cdots, \mathsf{y}_{mm}) g_m(\mathsf{y}_{m1}, \mathsf{y}_{m2}, \cdots, \mathsf{y}_{mm}). \tag{6}$$

It is instructive to write this using the following short-hand notation

$$\langle f|g\rangle = (f, \Pi\Theta S \Pi g) \tag{7}$$

where Π is the projection of the subspace of Euclidean test functions with positive relative time support and Θ is the Euclidean time reversal operator.

Reflection positivity is the condition that this space does not have negative norm states:

$$\langle f|f\rangle = (f, \Pi\Theta S \Pi f) \geq 0. \tag{8}$$

Reflection positivity makes (6) into a Hilbert space inner product. There are states, $|f\rangle$, with zero norm. The physical Hilbert space is obtained by identifying sequences whose difference has zero norm. This space is made complete by identifying Cauchy sequences with vectors in this space. While the Euclidean time supports must be disjoint, the order in (3) does not matter because the Schwinger functions of a local field theory are symmetric [1], so the arguments can be relabeled so (3) is satisfied.

Reflection positivity is not automatic; it is a property of acceptable Schwinger functions. By Euclidean invariance, reflection positivity must hold for any choice of the Euclidean time axis. In addition, without the support restriction, functions that are odd with respect to Euclidean time reflection will have negative norm, so the support condition, or an alternative restriction is necessary for positivity.

In what follows we assume that the collection of Schwinger functions are reflection positive. This condition needs to be verified in models or approximations.

III. THE POINCARÉ LIE ALGEBRA

The Euclidean time reflection breaks the Euclidean invariance of (6). As a result the group of real Euclidean transformations on sequences of Euclidean test functions

$$f_n(\mathsf{x}_{n1}, \mathsf{x}_{n2}, \cdots, \mathsf{x}_{nn}) \to f'_n(\mathsf{x}_{n1}, \mathsf{x}_{n2}, \cdots, \mathsf{x}_{nn}) = f_n(O\mathsf{x}_{n1} - a, O\mathsf{x}_{n2} - a, \cdots, O\mathsf{x}_{nn} - a), \tag{9}$$

where $O \in O(4)$ and a is a Euclidean four vector, becomes a subgroup of the complex Poincaré group with respect to the inner product (6). This is because the covering groups of the Lorentz group and $O(4)$ have the same analytic continuation.

The only technical issue is that these complex transformations do not generally preserve the positive relative time condition, however, Euclidean time translations map positive relative time functions into positive relative time functions when $a^0 \geq 0$. Similarly rotations in Euclidean space-time planes are defined for small angles on a subspace of positive relative-time functions with support on wedge shaped regions $x^0_{kn} > b|\mathbf{x}_{kn}|$ where b is a positive constant. These transformations become contractive Hermetian semigroups and local symmetric semigroups [15] [16] [4] on the

Hilbert space (6) provided the Schwinger functions are sufficiently regular. We assume that the collection of Schwinger functions satisfy these conditions. These transformations are associated with imaginary-time time translations and Lorentz boosts with imaginary rapidity. The important point is that in both cases the generators of contractive Hermetian semigroups and local symmetric semigroups are self-adjoint.

The infinitesimal generators H and $\mathbf{K}$ of time translations and rotationless boosts, along with the generators $\mathbf{J}$ and $\mathbf{P}$ of rotations and translations $\mathbf{P}$ have the following representations as differential operators

$$H f_n(\mathsf{x}_{n1},\mathsf{x}_{n2},\cdots,\mathsf{x}_{nn}) = \sum_{k=1}^{n} \frac{\partial}{\partial \mathsf{x}^0_{nk}} f_n(\mathsf{x}_{n1},\mathsf{x}_{n2},\cdots,\mathsf{x}_{nn})$$

$$\mathbf{P} f_n(\mathsf{x}_{n1},\mathsf{x}_{n2},\cdots,\mathsf{x}_{nn}) = -i\sum_{k=1}^{n} \frac{\partial}{\partial \mathbf{x}_{nk}} f_n(\mathsf{x}_{n1},\mathsf{x}_{n2},\cdots,\mathsf{x}_{nn})$$

$$\mathbf{J} f_n(\mathsf{x}_{n1},\mathsf{x}_{n2},\cdots,\mathsf{x}_{nn}) = -i\sum_{k=1}^{n} \mathbf{x}_{nk}\times \frac{\partial}{\partial \mathbf{x}_{nk}} f_n(\mathsf{x}_{n1},\mathsf{x}_{n2},\cdots,\mathsf{x}_{nn})$$

$$\mathbf{K} f_n(\mathsf{x}_{n1},\mathsf{x}_{n2},\cdots,\mathsf{x}_{nn}) = \sum_{k=1}^{n} (\mathbf{x}_{nk}\frac{\partial}{\partial x^0_{nk}} - x^0_{nk}\frac{\partial}{\partial \mathbf{x}_{nk}}) f_n(\mathsf{x}_{n1},\mathsf{x}_{n2},\cdots,\mathsf{x}_{nn}). \tag{10}$$

Direct computation shows that these operators are Hermetian and satisfy the commutation relations of the Poincaré Lie algebra. It can be shown that the Hamiltonian is bounded from below [3]. Both the boost generators and Hamiltonian do not have the usual factor of i. This is because they are linear in x^0_n or $\frac{\partial}{\partial x^0_m}$ so the usual sign change from complex conjugation is replaced by the Euclidean time reflection.

In the Euclidean framework the Poincaré generators are simple differential operators; the dynamical information is contained in the Schwinger functions. A collection of reflection positive Schwinger functions and the expressions for the Poincaré generators define the relativistic quantum theory.

IV. TWO-POINT FUNCTIONS

The two-point Schwinger function illustrates the relation of the Euclidean representation of a relativistic quantum theory to the conventional treatment of a relativistic particle. The structure of the two-point Schwinger function is also used to formulate the scattering asymptotic condition in the Euclidean framework.

The structure of the two-point Schwinger function is given by its Lehmann representation, which has the form

$$S_2(\mathsf{x}-\mathsf{y}) = \frac{1}{(2\pi)^4}\int \frac{d^4p\rho(m)dm}{p^2+m^2} e^{ip\cdot(x-y)} \tag{11}$$

where $\rho(m)$ is the Lehmann weight. The support of $\rho(m)$ is on the mass spectrum of the states that have non-zero matrix elements with $\phi(x)|0\rangle$, where $\phi(x)$ is the Minkowski Heisenberg field and $|0\rangle$ is the vacuum. The spectral condition requires that the support of $\rho(m)$ is for positive values of m.

Using this general two-point Schwinger function to compute the inner product (6) for vectors represented by a single function with positive-time support gives

$$\langle f|g\rangle = \int f^*(\mathsf{x}) S_2(\theta\mathsf{x}-\mathsf{y}) g(\mathsf{y}) d^4\mathsf{x} d^4\mathsf{y} =$$

$$\frac{1}{(2\pi)^4}\int f^*(\mathsf{x}) \frac{e^{ip^0(-x^0-\mathsf{y}^0)+i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})}\rho(m)}{(p^0)^2+\mathbf{p}^2+m^2} g(\mathsf{y}) dm d^4p d^4\mathsf{x} d^4\mathsf{y} =$$

$$\int \chi^*(\mathbf{p})\frac{\rho(m)}{2\omega_m(\mathbf{p})} dm d\mathbf{p} \psi(\mathbf{p}) \tag{12}$$

where the momentum-space wave functions are

$$\chi(\mathbf{p}) = \int \frac{d\mathbf{x}}{(2\pi)^{3/2}} f(\mathsf{x}^0, \mathbf{x}) e^{-\omega_m(\mathbf{p})\mathsf{x}^0 - i\mathbf{p}\cdot\mathbf{x}} \qquad \phi(\mathbf{p}) = \int \frac{d\mathbf{y}}{(2\pi)^{3/2}} g(\mathsf{x}^0, \mathbf{x}) e^{-\omega_m(\mathbf{p})\mathsf{x}^0 - i\mathbf{p}\cdot\mathbf{y}} \tag{13}$$

and $\omega_m(\mathbf{p}) = \sqrt{m^2 + \mathbf{p}^2}$ is the energy of a particle with mass m and momentum $\mathbf{p}$. Here the support condition on the Euclidean times along with the time reflection on the final state allows the p_0 integral to be evaluated using the residue theorem. There are several important observations about this expression.

1 If $f(\mathsf{x}) = g(\mathsf{x})$ then $\langle f|f\rangle \geq 0$, so the Lehmann representation of two-point Schwinger functions is consistent with reflection positivity.

2 Eq. (11) has the form of an ordinary Lorentz invariant inner product with Lorentz invariant measure $d\mathbf{p}/2\omega_m(\mathbf{p})$. This shows how the physical Hilbert space inner product emerges from this Euclidean quadrature without analytic continuation.

3 The origin of the mass dependence, which provides the dynamical relation between energy and momentum, is in the two-point Schwinger function.

4 The distribution $f(\mathsf{x}) = \delta(\mathsf{x}^0 - c)\tilde{f}(\mathbf{x})$, $(c > 0)$ is square integrable in the inner product (12) when $\tilde{f}(\mathbf{x})$ is a Schwartz test function in three variables. This is due to the non-trivial kernel in the scalar product. This means that the Hilbert space of normalizable vectors includes expressions with delta functions in Euclidean time. This observation has useful computational consequences which will be used in section 8.

5 The mass square Casimir operator is the four-dimensional Euclidean Laplacian:

$$M^2 = \nabla_e^2 := \frac{\partial^2}{\partial \mathsf{x}^{02}} + \boldsymbol{\nabla}_{\mathbf{x}}^2 \tag{14}$$

6 $\rho(m) = \delta(m - m_0)$ is the Lehmann weight for a free particle of mass m_0.

The Lehmann weight $\rho(m)$ has the general structure

$$\rho(m) = \sum_{k=1}^{N} z_k \delta(m - m_k) + \rho_{ac}(m) \tag{15}$$

where

$$0 < m_1 < m_2 < \cdots < m_n < \text{support}(\rho_{ac}). \tag{16}$$

The support of the continuous part of the Lehmann weight, ρ_{ac}, which is associated with multiparticle states, is not bounded from above. The continuous part of the Lehmann weight is polynomially bounded [3]. We assume this condition and that the support of the Lehmann weight also includes discrete masses.

V. SCATTERING ASYMPTOTIC CONDITION

The scattering asymptotic condition is formulated using a two Hilbert space framework [17][18][19][5][6]. Detectors respond to a particle's mass, spin, linear momentum, and spin polarization. The internal structure of the particle is of no consequence to the detector. The two Hilbert space formulation of scattering separates the degrees of freedom that are measured asymptotically from the internal degrees of freedom.

An n_α-particle scattering state in scattering channel α asymptotically looks like a direct product of wave packets in each particle's momentum and magnetic quantum number. These n_α-particle states span a channel subspace defined by

$$\mathcal{H}_\alpha = \otimes_{i=1}^{n_\alpha} \mathcal{H}_{m_i j_i} \tag{17}$$

where $\mathcal{H}_{m_i j_i}$ is a mass $m_i > 0$ spin j_i irreducible representation space of the Poincaré group, associated with mass and spin of each of the n_α asymptotic particles in the scattering channel α. The functions in $\mathcal{H}_{m_i j_i}$ are square integrable functions of the three-momentum and magnetic quantum number of a particle of mass m_i and spin j_i.

A channel α injection operator, J_α,

$$J_\alpha : \mathcal{H}_\alpha \to \mathcal{H} \tag{18}$$

is a mapping from the asymptotic channel α Hilbert space to the full Hilbert space. This operator combines the internal structure degrees of freedom for each asymptotically separated particle in the channel α with their momentum and spin degrees of freedom. The purpose of this paper is to discuss the construction of channel injection operators in the Euclidean representation.

The asymptotic Hilbert space, $\mathcal{H}_\mathcal{A}$, is defined as the direct sum of the channel subspaces over the set of scattering channels $\mathcal{A}$,

$$\mathcal{H}_\mathcal{A} = \oplus_{\alpha\in\mathcal{A}}\mathcal{H}_\alpha \tag{19}$$

and the two-Hilbert space injection operator as the sum of all of the channel injection operators

$$J_\mathcal{A} = \sum_{\alpha\in\mathcal{A}} J_\alpha . \tag{20}$$

While the channel sum includes an infinite number of channels, initial scattering states are vectors in a single channel subspace, $\mathcal{H}_\alpha$, and if the energy of the initial state is bounded, the number of open final channels is finite, so the infinite sum does not lead to convergence problems.

There is a natural unitary representation $U_\mathcal{A}(\Lambda, a)$ of the Poincaré group on the asymptotic Hilbert space. It is the direct sum of tensor products of unitary irreducible representations:

$$U_\mathcal{A}(\Lambda, a) = \oplus_{\alpha\in\mathcal{A}}(\otimes_{i\in\alpha} D^{m_i j_i}(\Lambda, a)) \tag{21}$$

where the $D^{m_i j_i}(\Lambda, a)$ are unitary irreducible representations of the Poincaré group for a particle of mass m_i and spin j_i:

$$D^{m_i j_i}(\Lambda, a) = \langle (m_i, j_i)\mathbf{p}_i, \mu_i | U_i(\Lambda, a) | (m_i, j_i)\mathbf{p}'_i, \mu'_i\rangle \tag{22}$$

where μ_i is the magnetic quantum number; i.e. the projection of a spin (canonical, light front, or helicity) on an axis. These representations are known analytically [20]. In the asymptotic Hilbert space there is no distinction between elementary and composite particles. These distinctions appear in the injection operator.

A scattering state is a solution of the Schrödinger equation that evolves into a state that asymptotically looks like a collection of asymptotically separated free particles. In the two-Hilbert space formalism the asymptotic condition has the form

$$\lim_{t\to\pm\infty} \Vert U(t, I)|\psi_\pm\rangle - J_\mathcal{A} U_\mathcal{A}(t, I)|\psi_\mathcal{A}\rangle\Vert = 0. \tag{23}$$

Using the unitarity of $U(t, I)$ this can be expressed as

$$\lim_{t\to\pm\infty} \Vert |\psi_\pm\rangle - U(-t, I) J_\mathcal{A} U_\mathcal{A}(t, I)|\psi_\mathcal{A}\rangle\Vert = 0 \tag{24}$$

where $|\psi_\mathcal{A}\rangle$ is a normalizable state in the asymptotic Hilbert space $\mathcal{H}_\mathcal{A}$.

The existence of this limit depends on properties of time evolution subgroup of the Poincaré group, $U(t, I)$, and the choice of injection operator, $J_\mathcal{A}$. A sufficient condition for the convergence of this limit is the Cook condition [10], which in the two-Hilbert space framework is

$$\int_a^{\pm\infty} \Vert (H J_\mathcal{A} - J_\mathcal{A} H_\mathcal{A}) U_\mathcal{A}(t, I)|\psi_\mathcal{A}\rangle\Vert dt < \infty \tag{25}$$

where a is finite. For this to hold $(H J_\mathcal{A} - J_\mathcal{A} H_\mathcal{A})$ needs to be short-ranged operator. Since $|\psi_\mathcal{A}\rangle$ is normalizable, this asymptotic condition can be verified one channel at a time, i.e.

$$\int_a^{\pm\infty} \Vert (H J_\alpha - J_\alpha H_\alpha) U_\alpha(t, I)|\psi_\alpha\rangle\Vert dt < \infty, \tag{26}$$

where $|\psi_\alpha\rangle \in \mathcal{H}_\alpha$ is the α-component of $|\psi_\mathcal{A}\rangle$.

The discussion above applies to any representation of a quantum theory. In the Euclidean representation the integrand in the Cook condition can be expressed in terms of the inner product (6):

$$\Vert (H J_\alpha - J_\alpha H_\alpha) U_\alpha(t, I)|\psi_\alpha\rangle\Vert =$$

$$(\psi_\alpha | U_\alpha(-t, I)(J_\alpha^\dagger H - H_\alpha J_\alpha^\dagger)\Theta S (H J_\alpha - J_\alpha H_\alpha) U_\alpha(t, I)|\psi_\alpha)^{1/2}. \tag{27}$$

VI. INJECTION OPERATORS AND ONE-BODY SOLUTIONS

In this section we discuss the construction of injection operators that can be used to formulate the scattering asymptotic condition (23). The basic strategy that we employ is a Euclidean reformulation of Haag-Ruelle [11][12][13][5] scattering.

One difference with quantum theories of a fixed finite number of particles and quantum field theory is that while the field theory Hamiltonian has one-body eigenstates, there are no states corresponding to N *free* particles. The absence of a subspace corresponding to N free particles impacts the formulation of the scattering asymptotic condition. The important observation is that the asymptotic condition defines the initial condition of Schrödinger equation at a time when the particles are asymptotically separated. The condition requires that at this time the solution looks like another state of N asymptotically separated particles. These other states provide labels for different N-particle scattering solutions of the Schrödinger equation. What these other states look like when the particles are not asymptotically separated is irrelevant to the formulation of this initial condition, however the choice of these states impact the labels that distinguish different scattering solutions.

To formulate a suitable asymptotic condition in the field theory case Haag and Ruelle construct localized field operators that create single-particle states out of the vacuum. For scalar particles these quasi-local field operators have the form

$$\phi_h(x) = \frac{1}{(2\pi)^2} \int h(-p^2) e^{ip\cdot(x-y)} \phi(y) d^4p d^4y \tag{28}$$

where $h(-p^2)$ is a smooth function that is 1 when $p^2 = -m^2$ is the mass of the asymptotic particle, and vanishes on the rest of the support of the Lehmann weight of the two-point function. Then they isolate the part of $\phi_h(x)$ that asymptotically behaves like a creation operator for a particle of mass m:

$$a^\dagger(f) = i \int \phi_h(x) \overset{\leftrightarrow}{\partial_t} f(x) d^3x \tag{29}$$

where $f(x)$ is a positive-energy solution of the Klein-Gordon equation for a particle of mass m. When applied to the vacuum this operator creates a time-independent single-particle state out of the vacuum. Products of these operators with different positive-energy Klein Gordon solutions, $f_i(x_i)$ are time dependent, and represent [5]

$$U(-t, I) J_\alpha U_\alpha(t, I) |\psi_\alpha\rangle \tag{30}$$

in (24). Ruelle demonstrated the existence of the strong limits ($t \to \pm\infty$) using locality and the assumed existence of a mass gap. The existence of the strong limits in the Haag-Ruelle case is a result of using the quasi-local fields $\phi_h(x)$, rather than local interpolating fields $\phi(x)$ used in LSZ theory, to formulate the scattering asymptotic condition. Ruelle was able to show that the integrand in (25) asymptotically behaves like $t^{-3(n-1)/2}$, [14] which is sufficient to satisfy the Cook condition.

In order to illustrate the corresponding construction in the Euclidean representation we focus on the Cook condition for elastic two-particle scattering. This can be formulated in terms of the four-point Schwinger function, $S_4(\mathsf{x}_1, \mathsf{x}_2, \mathsf{y}_2, \mathsf{y}_1)$.

In order to support a scattering theory the four-point Schwinger function should have a cluster expansion of the form:

$$S_4(\mathsf{x}_1, \mathsf{x}_2, \mathsf{y}_2, \mathsf{y}_1) = S_2(\mathsf{x}_1, y_1) S_2(\mathsf{x}_2, \mathsf{y}_2) + S_2(\mathsf{x}_1, y_2) S_2(\mathsf{x}_2, \mathsf{y}_1) + S_c(\mathsf{x}_1, \mathsf{x}_2, \mathsf{y}_2, \mathsf{y}_1) \tag{31}$$

where $S_c(\mathsf{x}_1, \mathsf{x}_2, \mathsf{y}_2, \mathsf{y}_1)$ is translationally invariant, but otherwise connected. The two-point functions $S_2(\mathsf{x}_i, \mathsf{y}_j)$ have a standard Lehmann representations of the form (11). In the physically interesting case the Lehmann weight in S_2 has both discrete and continuous parts. For this illustration the Lehmann weight is assumed to have at least two distinct discrete masses, m_1 and m_2.

Equation (23) has two distinct contributions; One from the connected part, $S_c(\mathsf{x}_1, \mathsf{x}_2, \mathsf{y}_2, \mathsf{y}_1)$, of the four-point function and the other from the disconnected parts, $S_2(\mathsf{x}_1, y_1) S_2(\mathsf{x}_2, \mathsf{y}_2)$ and $S_2(\mathsf{x}_1, y_2) S_2(\mathsf{x}_2, \mathsf{y}_1)$

The quantities that appear in (23) are:

$$H_\alpha = \omega_{m_1}(\mathbf{p}_1) + \omega_{m_2}(\mathbf{p}_2) \qquad \omega_{m_1}(\mathbf{p}_1) := \sqrt{m_i^2 + \mathbf{p}_i^2} \tag{32}$$

$$U_\alpha(t, I) = e^{-i(\omega_{m_1}(\mathbf{p}_1) + \omega_{m_2}(\mathbf{p}_2))t} \tag{33}$$

$$H = \frac{\partial}{\partial x_1^0} + \frac{\partial}{\partial x_2^0}. \tag{34}$$

In this example the channel α is assocated with asymptotic two-particle states with particles of mass m_1 and mass m_2.

The matrix elements of the injection operator J_α have the general structure $\langle \mathsf{x}_1, \mathsf{x}_2 | J_\alpha | \mathbf{p}_1, \mathbf{p}_2 \rangle$. Using these expressions in (27) gives

$$\|(HJ_\alpha - J_\alpha H_\alpha) U_\alpha(t,I)|\psi_\alpha\rangle\|^2 =$$

$$\int f_1^*(\mathbf{p}_1) f_2^*(\mathbf{p}_2) e^{i(\omega_{m_1}(\mathbf{p}_1)+\omega_{m_2}(\mathbf{p}_2))t} (\frac{\partial}{\partial x_1^0} + \frac{\partial}{\partial x_2^0} - \omega_{m_1}(\mathbf{p}_1) - \omega_{m_2}(\mathbf{p}_2)) \langle \mathbf{p}_1, \mathbf{p}_2 | J_\alpha^\dagger | \mathsf{x}_1, \mathsf{x}_2 \rangle \times$$

$$(S_2(\theta \mathsf{x}_1, y_1) S_2(\theta \mathsf{x}_2, \mathsf{y}_2) + S_2(\theta \mathsf{x}_1, y_2) S_2(\theta \mathsf{x}_2, \mathsf{y}_1) + S_c(\theta \mathsf{x}_1, \theta \mathsf{x}_2, \mathsf{y}_2, \mathsf{y}_1))$$

$$(\frac{\partial}{\partial y_1^0} + \frac{\partial}{\partial y_2^0} - \omega_{m_1}(\mathbf{p}_1') - \omega_{m_2}(\mathbf{p}_2')) \langle \mathsf{y}_1, \mathsf{y}_2 | J_\alpha | \mathbf{p}_1', \mathbf{p}_2' \rangle$$

$$e^{-i(\omega_{m_1}(\mathbf{p}_1')+\omega_{m_2}(\mathbf{p}_2'))t} f_1(\mathbf{p}_1') f_2(\mathbf{p}_2') d\mathbf{p}_1 d\mathbf{p}_2 d\mathbf{p}_1' d\mathbf{p}_2' d^4x_1 d^4x_2 d^4y_1 d^4y_2 \tag{35}$$

where we represent $\langle \mathbf{p}_1, \mathbf{p}_2 | \psi_\alpha \rangle = f_1(\mathbf{p}_1) f_2(\mathbf{p}_2)$ by a product of narrow wave packets $f_i(\mathbf{p}_i)$ in the three momentum of each asymptotic particle in channel α.

We show that for a suitable choice of injection operator J_α the contribution to (35) from the product of two-point functions vanish. When this is true the entire contribution to (35) comes from the connected part of the four-point function.

To show this we take advantage of the observation that a delta function in Euclidean time multiplied by a Schwartz test function in space variables is a normalizable vector in the Euclidean representation of the Hilbert space.

We consider an injection operator J_α of the form

$$\langle \mathsf{x}_1, \mathsf{x}_2 | J_\alpha | \mathbf{p}_1, \mathbf{p}_2 \rangle = h_1(\nabla_1^2) h_2(\nabla_2^2) \delta(\mathsf{x}_1^0 - \tau_1) \delta(\mathsf{x}_2^0 - \tau_2) \frac{1}{(2\pi)^3} e^{i\mathbf{p}_1\cdot\mathbf{x}_1 + i\mathbf{p}_2\cdot\mathbf{x}_2} \tag{36}$$

where $h_i(m^2)$ is a function that is 1 when m^2 is the square of the asymptotic particles mass and 0 on the rest the support of the Lehmann weight.

A few remarks are in order. First, the Euclidean time support condition can be satisfied by choosing $0 < \tau_1 < \tau_2$. The non-trivial constraint on the $h_i(\nabla_i^2)$ is that these operators map functions with positive Euclidean time support to functions with positive Euclidean time support. The construction of functions with this property will be discussed in the next section. For the purpose of this section we assume that the chosen $h_i(\nabla_i^2)$ preserves the support condition.

With this choice the contribution form one of the disconnected terms in (35) becomes

$$\int f_1^*(\mathbf{p}_1) f_2^*(\mathbf{p}_2) e^{i(\omega_{m_1}(\mathbf{p}_1)+\omega_{m_2}(\mathbf{p}_2))t} (\frac{\partial}{\partial \mathsf{x}_1^0} + \frac{\partial}{\partial \mathsf{x}_2^0} - \omega_{m_1}(\mathbf{p}_1) - \omega_{m_2}(\mathbf{p}_2)) \times$$

$$\langle \mathbf{p}_1, \mathbf{p}_2 | J^\dagger | \mathsf{x}_1, \mathsf{x}_2 \rangle S_2(\theta \mathsf{x}_1, \mathsf{y}_1) S_2(\theta \mathsf{x}_2, \mathsf{y}_2) \times$$

$$(\frac{\partial}{\partial y_1^0} + \frac{\partial}{\partial y_2^0} - \omega_{m_1}(\mathbf{p}_1') - \omega_{m_2}(\mathbf{p}_2')) \langle \mathsf{y}_1, \mathsf{y}_2 | J | \mathbf{p}_1', \mathbf{p}_2' \rangle \times$$

$$e^{-i(\omega_{m_1}(\mathbf{p}_1')+\omega_{m_2}(\mathbf{p}_2'))t} f_1(\mathbf{p}_1') f_2(\mathbf{p}_2') d\mathbf{p}_1 d\mathbf{p}_2 d\mathbf{p}_1' d\mathbf{p}_2' d^4x_1 d^4x_2 d^4y_1 d^4y_2 =$$

$$\int f_1^*(\mathbf{p}_1) f_2^*(\mathbf{p}_2) e^{i(\omega_{m_1}(\mathbf{p}_1)+\omega_{m_2}(\mathbf{p}_2))t} (\frac{\partial}{\partial \mathsf{x}_1^0} + \frac{\partial}{\partial \mathsf{x}_2^0} - \omega_{m_1}(\mathbf{p}_1) - \omega_{m_2}(\mathbf{p}_2)) \times$$

$$h_1(\nabla^2_{\mathsf{x}1})h_2(\nabla^2_{\mathsf{x}2})\delta(\mathsf{x}^0_1-\tau_1)\delta(\mathsf{x}^0_2-\tau_2)\frac{1}{(2\pi)^3}e^{-i\mathbf{p}_1\cdot\mathbf{x}_1-i\mathbf{p}_2\cdot\mathbf{x}_2}\times$$

$$\frac{1}{(2\pi)^6}\frac{d\mathbf{k}_1 d\mathbf{k}_2\rho_1(m)\rho'_2(m')dmdm'}{4\omega_m(\mathbf{k}_1)\omega_{m'}(\mathbf{k}_2)}e^{-\omega_m(\mathbf{k}_1)(\mathsf{x}^0_1+\mathsf{y}^0_1)-\omega_{m'}(\mathbf{k}_2)(\mathsf{x}^0_2-\mathsf{y}^0_2)}e^{i\mathbf{k}_1\cdot(\mathbf{x}_1-\mathbf{y}_1)+i\mathbf{k}_2\cdot(\mathbf{x}_2-\mathbf{y}_2)}\times$$

$$(\frac{\partial}{\partial y^0_1}+\frac{\partial}{\partial y^0_2}-\omega_m(\mathbf{p}'_1)-\omega_{m'}(\mathbf{p}'_2))\times$$

$$h_1(\nabla^2_{\mathsf{y}1})h_2(\nabla^2_{\mathsf{y}2})\delta(\mathsf{y}^0_1-\tau_1)\delta(\mathsf{y}^0_2-\tau_2)\frac{1}{(2\pi)^3}e^{i\mathbf{p}'_1\cdot\mathbf{y}_1+i\mathbf{p}'_2\cdot\mathbf{y}_2}\times$$

$$e^{-i(\omega_{m_1}(\mathbf{p}'_1)+\omega_{m_2}(\mathbf{p}'_2))t}f_1(\mathbf{p}'_1)f_2(\mathbf{p}'_2)d\mathbf{p}_1 d\mathbf{p}_2 d\mathbf{p}'_1 d\mathbf{p}'_2 x^4x_1d^4x^2d^4y_1d^4y_2 =$$

$$\int f^*_1(\mathbf{p}_1)f^*_2(\mathbf{p}_2)(\omega_m(\mathbf{p}_1)-\omega_{m'}(\mathbf{p}_2)-\omega_{m_1}(\mathbf{p}_1)-\omega_{m_2}(\mathbf{p}_2))\times h_1(m^2)h_2(m'^2)\times$$

$$\frac{d\mathbf{p}_1 d\mathbf{p}_2\rho_1(m)\rho'_2(m')dmdm'}{4\omega_m(\mathbf{p}_1)\omega_{m'}(\mathbf{2}_2)}e^{-2\omega_m(\mathbf{p}_1)\tau_1-2\omega_{m'}(\mathbf{p}_2)\tau_2}$$

$$(\omega_m(\mathbf{p}_1)-\omega_{m'}(\mathbf{p}_2)-\omega_{m_1}(\mathbf{p}_1)-\omega_{m_2}(\mathbf{p}_2))h_1(m^2)h_2(m'^2)f_1(\mathbf{p}_1)f_2(\mathbf{p}_2). \qquad (37)$$

with a similar expression for the other disconnected term in (35).

This vanishes identically *because the only contribution to the* m *and* m' *integrals comes from* $m = m_1$ *and* $m' = m_2$ which leads to $(\omega_m(\mathbf{p}_1)-\omega_{m'}(\mathbf{p}_2)-\omega_{m_1}(\mathbf{p}_1)-\omega_{m_2}(\mathbf{p}_2)) = 0$. The disconnected term would not vanish without the $h(m^2)$ functions. Instead there would be non-zero terms that were independent of time (the real time dependence cancels after performing the integrals over the spatial coordinates). This would lead to a violation of the Cook condition. In that case the strong limit would not exist.

The remaining terms are linear in the connected Euclidean four-point Schwinger function. The precise large time behavior depends on the form of the connected four point function. Following arguments in [14], we showed, [9] using a model connected four-point function, the same $t^{-3/2}$ behavior for the integrand in (26) that one gets non-relativistically. This is sufficient to satisfy the Cook condition.

VII. THE h FUNCTION

The key requirement in constructing a suitable injection operator is to construct an operator $h(\nabla^2)$ with the property

$$h(\nabla^2_{\mathsf{x}})f(\mathsf{x}) \qquad (38)$$

has support for positive Euclidean time when $f(\mathsf{x})$ has support for positive Euclidean time. This support condition is clearly preserved when $h(\nabla^2)$ is a polynomial in ∇^2, however it is not obvious that $h(\nabla^2)$ preserves the positive time support condition when $h(m^2)$ has support on the real line. The simplest counter example is the unitary translation operator, which (1) can be formally represented as an infinite series in derivatives and (2) changes (translates) the support of functions.

In the unphysical case of a Lehmann weight with support on N discrete masses an $h(x)$ that selects the j^{th} mass has the form

$$h_j(x) = \prod_{i\neq j}^{N}\frac{x-m^2_i}{m^2_j-m^2_i}. \qquad (39)$$

A more interesting, but still unphysical case, is when the support of the Lehmann weight consists of a single mass (or several masses) and a continuous spectrum with compact support, then the Weierstrass approximation theorem implies $h(x)$ with the required properties can uniformly approximated by a polynomial on the support of the Lehmann weight.

The physically interesting case is when the support of the Lehmann weight has some discrete masses and the support of the continuous part extends from the lowest multiparticle threshold to infinity. The quantity of interest is

$$(f, \Pi\Theta S_2 \Pi h(\nabla^2) f) =$$

$$\int f(\mathbf{p}, x_0) \frac{e^{-\omega_m(\mathbf{p})(x_0+y_0)}}{2\omega_m(\mathbf{p}^2)} \rho(m^2) h(\frac{d^2}{dy_0^2} - p^2) f(\mathbf{p}, y_0) d\mathbf{p} dx_0 dy_0 dm =$$

$$\int f(\mathbf{p}, x_0) \frac{e^{-\omega_m(\mathbf{p})(x_0+y_0)}}{2\omega_m(\mathbf{p}^2)} \rho(m^2) h(m^2) f(\mathbf{p}, y_0) d\mathbf{p} dx_0 dy_0 dm \tag{40}$$

where $\psi(\mathbf{p}, x_0)$ is the partial Fourier transform with respect to the spatial variables of a positive-time support test function.

Our goal is to find states represented by positive time support functions that asymptotically look like single-particle states. The support condition would be trivially satisfied if $h(\nabla^2)$ could be approximated by a finite degree polynomial. The problem is that polynomials on semi-infinite intervals get large, however in this expression the polynomial growth is exponentially suppressed by (see 12):

$$\frac{e^{-\omega_m(\mathbf{p})(x_0+y_0)}}{2\omega_m(\mathbf{p}^2)} \rho(m^2). \tag{41}$$

If the polynomials in m^2 with respect to the weight (41) are complete, then it is possible to approximate the required $h(\nabla^2)$ by a polynomial.

The problem is to show that polynomials in m^2 with respect to the weight (41) are complete. Here it is enough to treat $\mathbf{p}$ as a constant which is sufficient for three-momentum states with compact support. Note that the polynomials must be in m^2 rather than m because the *square* of the mass is the Laplacian in this representation.

Fortunately there is a sufficient condition for the a set of polynomials with respect to a weight on a semi-infinite interval to be complete. The relevant condition, due to Carleman[21], depends on the growth of the Stieltjes moments

$$\gamma_n = \langle \psi | \Theta S_2 (\nabla^2)^n | \psi \rangle = \int_0^\infty \psi(\mathbf{p}, x_0) \frac{e^{-\omega_m(\mathbf{p})(x_0+y_0)}}{2\omega_m(\mathbf{p}^2)} \rho(m^2) m^{2n} \psi(\mathbf{p}, y_0) d\mathbf{p} dx_0 dy_0 dm. \tag{42}$$

The Carleman condition states that if the following sum diverges

$$\sum_{n=1}^\infty \gamma_n^{-\frac{1}{2n}} > \infty \tag{43}$$

then the orthogonal polynomials in m^2 are complete on the half line. For this problem this means that functions $h(m^2)$ with the required properties can be approximated by a polynomial.

In the appendix we show that this condition is satisfied for a polynomially bounded Lehmann weight $\rho(m)$.

There remains the practical question of how to efficiently compute a suitable polynomial approximation to $h(m^2)$.

VIII. COMPUTATIONAL METHODS IN EUCLIDEAN SPACE

The discussion above implies that it is possible to formulate the scattering problem directly in the Euclidean representation without analytic continuation. While this does not mean that the Euclidean representation is ideal from a computational point of view, it is possible to reformulate the the asymptotic condition to take advantage of the Euclidean representation.

The first simplification comes from using the invariance principle. The invariance principle [22][23][5][6] implies that if $g(E)$ is a function of bounded variation with a positive derivative, $g'(E) > 0$, then we can replace H and $H_{\mathcal{A}}$ by $g(H)$ and $g(H_{\mathcal{A}})$ in the asymptotic condition. Using $g(x) = -e^{-\beta x}$, where $\beta > 0$ is a fixed positive constant, the asymptotic condition

$$\lim_{t \to \pm\infty} \| e^{iHt} | \psi_{\mathcal{A}}^{\pm} \rangle - J_{\mathcal{A}} e^{-iH_{\mathcal{A}}t} | \psi_{\mathcal{A}} \rangle \| = 0, \tag{44}$$

is replaced by the equivalent asymptotic condition

$$\lim_{s\to\pm\infty} \|e^{ise^{-\beta H}}|\psi^{\pm}_{\mathcal{A}}\rangle - J_{\mathcal{A}} e^{ise^{-\beta H_{\mathcal{A}}}}|\psi_{\mathcal{A}}\rangle\| = 0. \tag{45}$$

In this expression the real time t is replaced by the dimensionless quantity s. With this substitution $\psi_{\mathcal{A}}$ and $\psi^{+}_{\mathcal{A}}$ remain unchanged.

Using unitarity of $e^{ie^{-\beta H}s}$ the asymptotic condition can be rewritten as

$$\lim_{s\to\pm\infty} \||\psi^{\pm}_{\mathcal{A}}\rangle - e^{-ise^{-\beta H}} J_{\mathcal{A}} e^{ise^{-\beta H_{\mathcal{A}}}}|\psi_{\mathcal{A}}\rangle\| = 0 \tag{46}$$

In what follows, for the purpose of illustration, we discuss the application of these methods to asymptotic conditions involving a single two-particle channel $\alpha \in \mathcal{A}$, with particles of mass 1 and mass 2.

To compute

$$\langle \mathsf{x}_1, \mathsf{x}_2|e^{-ise^{-\beta H}} J_\alpha e^{ise^{-\beta H_\alpha}}|\psi_\alpha\rangle \tag{47}$$

first note that $e^{-\beta H}$ has a spectrum between $[0,1]$ for $H \geq 0$. Since this spectrum is compact it follows that $e^{ise^{-\beta H}}$ can be uniformly approximated by a polynomial in $e^{-\beta H}$:

$$e^{ise^{-\beta H}} = \sum_{n=0}^{\infty} d_n e^{-\beta n H} \tag{48}$$

The uniform convergence means that

$$\|| e^{ise^{-\beta H}} - \sum_{n=0}^{N} d_n e^{-\beta n H}\|| \leq \sup_{x\in[-1,1]}|e^{isx} - \sum_{n=0}^{N} d_n x^n| \tag{49}$$

which can be evaluated by plotting the difference between the functions. While we have shown that this polynomial approximation can be implemented efficiently ([8]) for a wide range of scattering energies, we show that if we use the injection operator in section VI the polynomial can be summed to all orders.

Using the polynomial approximation in the expression for the initial and final scattering wave functions gives

$$\langle \mathsf{x}_1, \mathsf{x}_2|e^{-ise^{-\beta H}} J_\alpha e^{ise^{-\beta H_\alpha}}|\psi_\alpha\rangle \approx$$

$$\sum_{m=0}^{N} d_m \langle \mathbf{x}_1, \tau_1 - m\beta, \mathbf{x}_2, \tau_2 - m\beta | J_\alpha e^{ise^{-\beta H_\alpha}}|\psi_\alpha\rangle =$$

$$\int \sum_{m=0}^{N} d_m h_1(\frac{\partial^2}{\partial\tau_1^2} - \mathbf{p}_1^2) h_2(\frac{\partial^2}{\partial\tau_2^2} - \mathbf{p}_2^2)\delta(\tau_1 - \tau_{10} - m\beta)\delta(\tau_2 - \chi\tau_{10} - m\beta)\frac{d\mathbf{p}_1}{(2\pi)^{3/2}}\frac{d\mathbf{p}_2}{(2\pi)^{3/2}}\times$$

$$e^{i\mathbf{p}_1\cdot\mathbf{x}_1 + i\mathbf{p}_2\cdot\mathbf{x}_2} e^{ise^{-\beta(\omega_{m_1}(\mathbf{p}_1)+\omega_{m_2}(\mathbf{p}_2))}} f_1(\mathbf{p}_1) f_2(\mathbf{p}_2) =$$

$$\int \sum_{m=0}^{N} d_m h_1(-p_1^2) h_2(-p_2^2)\frac{dp_{10}}{2\pi} e^{ip_{10}(\tau_1 - \tau_{10} - m\beta)}\frac{dp_{20}}{2\pi} e^{ip_{20}(\tau_2 - \chi\tau_{10} - m\beta)}\frac{d\mathbf{p}_1}{(2\pi)^{3/2}}\frac{d\mathbf{p}_2}{(2\pi)^{3/2}}\times$$

$$e^{i\mathbf{p}_1\cdot\mathbf{x}_1 + i\mathbf{p}_2\cdot\mathbf{x}_2} e^{ise^{-\beta(\omega_{m_1}(\mathbf{p}_1)+\omega_{m_2}(\mathbf{p}_2))}} f_1(\mathbf{p}_1) f_2(\mathbf{p}_2) \tag{50}$$

where we have taken $\langle \mathbf{p}_1, \mathbf{p}_2|\psi_\alpha\rangle = f_1(\mathbf{p}_1) f_2(\mathbf{p}_2)$ to be a product of localized wave packets in 3-momentum.

We rewrite this in a form that allows a direct expression in terms of the Fourier transform of the Schwinger function:

$$\langle \mathsf{x}_1, \mathsf{x}_2|e^{-ise^{-\beta H}} J_\alpha e^{ise^{-\beta H_\alpha}}|f_\alpha\rangle \approx$$

$$\frac{1}{(2\pi)^5}\int\sum_{m=0}^{\infty}d_m d^4p_1 d^4p_2 e^{ip_1\cdot x_1+ip_2\cdot x_2}e^{ise^{-\beta(\omega_{m_1}(\mathbf{p}_1)+\omega_{m_2}(\mathbf{p}_2))}}h_1(-p_1^2)h_2(-p_2^2)\times$$

$$e^{-ip_{10}(\tau_{10}+m\beta)-ip_{20}(\chi\tau_{10}+m\beta)}f_1(\mathbf{p}_1)f_2(\mathbf{p}_2)=$$

$$\frac{1}{(2\pi)^5}\int d^4p_1 d^4p_2 e^{ip_1\cdot x_1+ip_2 2\cdot x_2}e^{ise^{-\beta(\omega_{mass_1}(\mathbf{p}_1)+\omega_{mass_2}(\mathbf{p}_2))}}h_1(-p_1^2)h_2(-p_2^2)\times$$

$$\sum_m d_m(e^{-i(p_{10}+p_{20})\beta})^m e^{-ip_{10}\tau_{10}-ip_{20}\chi\tau_{10})}f_1(\mathbf{p}_1)f_2(\mathbf{p}_2) \tag{51}$$

In this form the polynomial can be summed explicitly to get

$$\langle \mathsf{x}_1,\mathsf{x}_2|e^{-ise^{-\beta H}}J_\alpha e^{ise^{-\beta H_\alpha}}|f_\alpha^0\rangle=$$

$$\frac{1}{(2\pi)^5}\int e^{ip_1\cdot x_1+ip_2\cdot x_2}e^{ise^{-\beta(\omega_{m_1}(\mathbf{p}_1)+\omega_{m_2}(\mathbf{p}_2))}}h_1(-p_1^2)h_2(-p_2^2)\times$$

$$e^{-ise^{-i\beta(p_{10}+p_{20})}}e^{-ip_{10}\tau_{10}-ip_{20}\chi\tau_{10})}f_1(\mathbf{p}_1)f_2(\mathbf{p}_2) \tag{52}$$

Thus, by using the delta function in τ to define J we can replace the polynomial expansion by an integral. Furthermore, this is now exact.

Using these expressions in the formula for S-matrix elements gives

$$\langle f_{\alpha f}|S|f_{\alpha i}\rangle=\langle f^+_{\alpha_f}|f^-_{\alpha_i}\rangle=$$

$$\int d^4x_1' d^4x_2' d^4x_1 d^4x_2\frac{1}{(2\pi)^{10}}d^4p_1' d^4p_2' d^4p_1 d^4p_2\times$$

$$S_4(\theta x_2',\theta x_1',x_1,x_2)h_{1f}(-p_1'^2)h_{2f}(-p_2'^2)e^{-i(p_1'\cdot x_1'+p_2'\cdot x_2'-p_1\cdot x_1-p_2\cdot x_2)}e^{-ise^{-\beta(\omega_{m_1'}(\mathbf{p}_1')+\omega_{m_2'}(\mathbf{p}_2'))}}e^{2ise^{-i\beta(p_{10}+p_{20})}}\times$$

$$h_{1i}(-p_1^2)h_{2i}(-p_2^2)e^{-ise^{-\beta(\omega_1(\mathbf{p}_1)+\omega_2(\mathbf{p}_2))}}e^{ip_{10}'(\tau_{10})ip_{20}'(\chi\tau_{10})}e^{-ip_{10}(\chi\tau_{20})-ip_{20}(\tau_{10})}\times$$

$$f_1'^*(\mathbf{p}_1')f_2'^*(\mathbf{p}_2')f_1'(\mathbf{p}_1)f_2'(\mathbf{p}_2)=$$

$$\int\frac{d^4p_1' d^4p_2' d^4p_2 d^4p_2}{(2\pi)^2}\tilde{S}_4(-p_2',-p_1',p_1,p_2)e^{-ise^{-i\beta(p_{10}+p_{20})}}e^{ise^{-\beta(\omega_{m_1}(\mathbf{p}_1')+\omega_{m_2}(\mathbf{p}_2'))}}\times$$

$$e^{ise^{-\beta(\omega_1(\mathbf{p}_1)+\omega_2(\mathbf{p}_1))}}e^{-ip_{10}'(\tau_{10})-ip_{20}'(\chi\tau_{10})}e^{-ip_{10}(\tau_{10})-ip_{20}(\chi\tau_{10})}\times$$

$$h_{1f}(-p_1'^2)h_{2f}(-p_2'^2)f_1'(\mathbf{p}_1')f_2'(\mathbf{p}_2')h_{1i}(-p_1^2)h_{2i}(-p_2^2)f_1(\mathbf{p}_1)f_2(\mathbf{p}_2). \tag{53}$$

One set of integrals can be performed due to the translational invariance of the Green function. The result is:

$$\langle f_{\alpha_f}|S|f_{\alpha_i}\rangle=$$

$$\langle f_1' f_2'|S|f_1 f_2\rangle=$$

$$\lim_{s\to\infty} \int f_1'^*(\mathbf{p}_1') f_2'^*(\mathbf{p}_2') h_{1f}(-p_1'^2) h_{2f}(-p_2'^2) d4p_1' d4p_2' d^4p_1 d^4p_2 \times$$

$$e^{-ise^{-\beta(\omega'_{m'_1}(\mathbf{p}'_1)+\omega'_{m'_2}(\mathbf{p}'_2))}} e^{2ise^{-i\beta(p_{10}+p_{20})}} e^{-ise^{-\beta(\omega_{m_1}(\mathbf{p}_1)+\omega_{m_2}(\mathbf{p}_4))}} \times$$

$$e^{-i((p_{10}+p'_{10})\tau_1+(p_{20}+p'_{20})\chi\tau_2)} \tilde{S}_4(-p'_2, -p'_1, p_1, p_2) h_{1i}(-p_1^2) h_{2i}(-p_2^2) f_1(\mathbf{p}_1) f_2(\mathbf{p}_2) \tag{54}$$

This leads to an expression for S matrix elements as a 12 dimensional integral. While the form of this expression is complicated, it involves the computation of a quadratic form involving the Fourier transform of the Euclidean-four point functions without analytic continuation. The initial and final wave packets should be narrow in 3-momentum, which will facilitate the computation of the integrals. It is still necessary to verify that that the connected four point-function satisfies the Cook condition and is consistent with reflection positivity.

The sharp momentum transition matrix elements can be extracted using sharply peaked $f_i(\mathbf{p})$:

$$\langle \mathbf{p}'_1, \mathbf{p}'_2 | T | \mathbf{p}_1, \mathbf{p}_2 \rangle = \frac{i}{2\pi} \frac{\langle f'_1 f'_2 | f_1 f_2 \rangle - \langle f'_1 f'_2 | S | f_1 f_2 \rangle}{\langle f'_1 f'_2 | \delta(E - H_\alpha) | f_1 f'_2 \rangle} \tag{55}$$

The calculation should converge as s gets large. The choice of β is in principle arbitrary, but it should be close to the inverse of the working energy scale. The parameter χ should be of order unity but larger than 1.

IX. SUMMARY

The purpose of this paper was to demonstrate how to formulate the scattering asymptotic condition using strong limits in the formulation of quantum field theory defined by a collection of reflection-positive Schwinger functions. In this formalism the Hilbert space inner product is expressed as a quadratic form with a kernel consisting of the product of a Schwinger functions and a Euclidean time reversal on the final states. Hilbert space vectors have positive Euclidean relative time support.

The main problem addressed in this paper was how to construct the Euclidean analog of Haag-Ruelle quasilocal fields. These are needed to get a scattering theory that can be formulated in terms of strong limits and to treat scattering where the initial and final particles may be composite. In order to achieve the desired result it was necessary to show that certain polynomials are complete with respect to a certain weight function. This was established in section IX. This result was needed to ensure that the application of complicated differential operators that project positive relative time support functions on the desired asymptotic states do not change these support conditions.

To achieve this result we used the fact that there are normalizable vectors in this representation of the Hilbert spare that are proportional to delta functions in the Euclidean time variable. A bi-product of this observation is an expression for scattering matrix elements (54) involving only quadratures involving sharp-momentum wave packets and the Fourier transform of the Schwinger functions. Previous work suggest that this method should be applicable from low to intermediate energies.

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X. APPENDIX

A sufficient condition for the orthogonal polynomials with respect to a measure on $[0, \infty]$ to be complete is that the moments $\{\gamma_n\}$ of the measure satisfy the Carleman condition [24][21]

$$\sum_{n=0}^{\infty} |\gamma_n|^{-\frac{1}{2n}} > \infty \tag{56}$$

The moments of interest have the form

$$\gamma_n := \int_0^\infty \frac{e^{-\sqrt{m^2+\mathbf{p}^2}\tau}}{2\sqrt{m^2+\mathbf{p}^2}}\rho(m)m^{2n}dm \tag{57}$$

where $\tau = \tau_1 + \tau_2 > 0$ and $\rho(m)$ is polynomially bounded. Since $\rho(m)$ is polynomially bounded we replace $\rho(m)$ in (57) by m^k.

$$\gamma_n \to \gamma_n' = \int_0^\infty \frac{e^{-\sqrt{m^2+\mathbf{p}^2}\tau}}{2\sqrt{m^2+\mathbf{p}^2}} m^{2n+k}dm. \tag{58}$$

If we make the substitutions $m = p\sinh(\eta)$ and $p\cosh(\eta) = \sqrt{m^2+\mathbf{p}^2}$ this integral becomes

$$\frac{1}{2}\int_0^\infty \frac{e^{-p\tau\cosh(\eta)}}{\cosh(\eta)}(p\sinh(\eta))^{2n+k}\cosh(\eta)d\eta \tag{59}$$

After the variable change $u = p\tau\cosh(\eta)$ this becomes

$$\frac{1}{2}p^{2n+k}\int_{p\tau}^\infty e^{-u}(\frac{u^2}{p^2\tau^2}-1)^{\frac{2n+k-1}{2}}\frac{du}{p\tau} \leq \frac{1}{2}p^{2n+k}\int_{p\tau}^\infty e^{-u}(\frac{u}{p\tau})^{2n+k-1}\frac{du}{p\tau} \leq$$

$$\frac{1}{2}p^{2n+k}\int_0^\infty e^{-u}(\frac{u}{p\tau})^{2n+k-1}\frac{du}{p\tau} = \frac{1}{2}\tau^{-2n-k}\Gamma(2n+k-2) \tag{60}$$

using the the representation of the Gamma functions [25] (equation 6.1.38):

$$\Gamma(x+1) = \sqrt{2\pi}x^{x+1/2}e^{-x+\theta/12x} \tag{61}$$

where θ is a number between 0 and 1 that depends on x gives

$$\gamma_n \leq \sqrt{\frac{\pi}{2}}\tau^{-2n-k}(2n+k-2)^{2n+k-3/2}e^{-(2n+k-2)+\frac{\theta}{2n+k-2}} \tag{62}$$

which gives

$$\frac{1}{\gamma_n} \geq \sqrt{\frac{2}{pi}}\tau^{2n+k}(2n+k-2)^{-2n-k+3/2}e^{(2n+k-2)-\frac{\theta}{2n+k-2}} \tag{63}$$

and

$$(\frac{1}{\gamma_n})^{\frac{1}{2n}} \geq \sqrt{\frac{2}{\pi}}^{\frac{1}{2n}}\tau^{1+k/2n}(2n+k-2)^{-1-(k-3/2)/2n}e^{(1+(k-2)/2n)-\frac{\theta}{1+(k-2)/2n}} \tag{64}$$

For large n the sum behaves like

$$\sum \frac{1}{n+k/2-1} \tag{65}$$

which diverges. This imples that $h(\nabla^2)$ can be approximated by a polynomial for any well behaved polynomially bounded Lehmann weight.

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