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The Relativistic Three-Body Bound State in Three-Dimensions

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Abstract. Studying of the relativistic three-body bound state in a three-dimensional (3D) approach is a necessary first step in a process to eventually perform scattering calculations at GeV energies, where partial-wave expansions are not useful. To this aim we recently studied relativistic effects in the binding energy and for the first time, obtained the relativistic 3B wave function [1]. The relativistic Faddeev integral equations for the bound state are formulated in terms of momentum vectors, and relativistic invariance is incorporated within the framework of Poincaré invariant quantum mechanics.

1 Introduction and Formalism

While most three-nucleon calculations utilize partial-wave (PW) representations, at laboratory energies above about $\simeq 300$ MeV those become inefficient. For applications in the GeV regime one needs to consider vector formulations to implement the dynamics [2–4]. Convergence in the GeV regime has been demonstrated [2] in a relativistic one channel model using this approach. In this work we review the formulation of the 3B bound state problem in vector variables, which demonstrates that the relativistic problem can be treated using a straightforward generalization of methods that have been successfully used in the non-relativistic three-body problem.

In a Faddeev formulation, the relativistic bound state of three particles with mass m interacting with pairwise forces, is described by

$$|\psi\rangle = (M_t - M_0)^{-1}\, T(M_t)\, P\, |\psi\rangle, \tag{1}$$

where $M_t = E_t + 3m$ is the $3B$ mass eigenvalue, M_0 is the non-interacting $3B$ mass operator, $T(z) := V + V(z - M)^{-1}V$ is the boosted two-body (2B) $t-$matrix embedded in the 3B Hilbert space, and $P = P_{ij}P + P_{ik}P$ is the permutation operator for three identical particles. Projecting Eq. (1) on relativistic momentum states $|\mathbf{p}, \mathbf{k}\rangle$ leads to the integral equation,

$$\langle \mathbf{p}, \mathbf{k}|\psi\rangle = \int d\mathbf{p}'\, d\mathbf{k}'\, d\mathbf{p}''\, d\mathbf{k}'' \frac{\delta(\mathbf{k} - \mathbf{k}')}{M_t - M_0(p,k)}\, T\Big(\mathbf{p}, \mathbf{p}'; M_t - \omega_m(k)\Big) \langle \mathbf{p}', \mathbf{k}'|P|\mathbf{p}'', \mathbf{k}''\rangle\, \langle \mathbf{p}'', \mathbf{k}''|\psi\rangle. \tag{2}$$

The relativistic basis states contain two Jacobi momentum vectors $\mathbf{p}$ and $\mathbf{k}$. The relativistic Jacobi momentum $\mathbf{k}$ is constructed by boosting the single particle momentum to the 3B rest frame, whereas

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the momentum $\mathbf{p}$ is obtained by boosting the momentum $\mathbf{k}$ to the 2B rest frame. In the kernel of relativistic Faddeev integral equation one needs the fully off-shell boosted 2B $t-$matrix $T(z)$ which can be calculated from the non-relativistic $t-$matrix t_{nr} by a two-step process, which guarantees that the nonrelativistic and relativistic 2B observables and CM wave functions remain unchanged. Those steps are

1. The relativistic right-half-shell 2B $t-$matrix is related to the non-relativistic right-half-shell by a multiplicative function F from [5],

$$T\left(\mathbf{p},\mathbf{p}';\sqrt{m_0^2(\mathbf{p}')+k^2}+i0^+\right)=F(p,p',k)\,t_{nr}\left(\mathbf{p},\mathbf{p}';\frac{p'^2}{m}+i0^+\right). \tag{3}$$

2. The relativistic fully off-shell t-matrix is obtained from the relativistic right-half-shell $t-$matrix by solving a first resolvent type equation,

$$T(z_j)=T(z)+T(z_j)\Big(g_0(z_j)-g_0(z)\Big)T(z). \tag{4}$$

   Here $g_0(z)$ is the relativistic two-body propagator, with z being the right-half-shell energy and z_j the fully-off-shell energy. This procedure was proposed by Keister and Polyzou [6] and implemented for the first time by Lin et al. in relativistic Faddeev equations for 3B scattering [2].

By following this strategy one can directly obtain the fully off-shell boosted $t-$matrix from nonrelativistic one.
The permutation operator P can be evaluated in relativistic basis states as

$$\begin{aligned}\langle\mathbf{p}',\mathbf{k}'|P|\mathbf{p}'',\mathbf{k}''\rangle &= {}_1\langle\mathbf{p}',\mathbf{k}'|\mathbf{p}'',\mathbf{k}''\rangle_2+{}_1\langle\mathbf{p}',\mathbf{k}'|\mathbf{p}'',\mathbf{k}''\rangle_3\\ &= N(\mathbf{k}',\mathbf{k}'')\Big\{\delta^3\Big(\mathbf{p}'-\mathbf{k}''-\frac{1}{2}\mathbf{k}'C(\mathbf{k}',\mathbf{k}'')\Big)\delta^3\Big(\mathbf{p}''+\mathbf{k}'+\frac{1}{2}\mathbf{k}''C(\mathbf{k}'',\mathbf{k}')\Big)\\ &+ \delta^3\Big(\mathbf{p}'+\mathbf{k}''+\frac{1}{2}\mathbf{k}'C(\mathbf{k}',\mathbf{k}'')\Big)\delta^3\Big(\mathbf{p}''-\mathbf{k}'-\frac{1}{2}\mathbf{k}''C(\mathbf{k}'',\mathbf{k}')\Big)\Big\},\end{aligned} \tag{5}$$

where

$$N(\mathbf{k}',\mathbf{k}'')=\mathcal{N}^{-1}(-\mathbf{k}'-\mathbf{k}'',\mathbf{k}'')\,\mathcal{N}^{-1}(-\mathbf{k}'-\mathbf{k}'',\mathbf{k}'). \tag{6}$$

Here $\mathcal{N}$ is the square root of the Jacobian of the basis change,

$$\mathcal{N}(\mathbf{k}_j,\mathbf{k}_k)=\left|\frac{\partial(\mathbf{k}_j,\mathbf{k}_k)}{\partial(\mathbf{p}_{jk},\mathbf{k}_{jk})}\right|^{1/2}=\left[\frac{\omega_m(p_{jk})+\omega_m(p_{jk})}{\omega_m(k_j)+\omega_m(k_k)}\,\frac{\omega_m(k_j)\omega_m(k_k)}{\omega_m(p_{jk})\omega_m(p_{jk})}\right]^{1/2}\neq 1. \tag{7}$$

The function C is given by

$$C(\mathbf{k},\mathbf{k}')=1+\frac{\omega_m(k')-\omega_m(|\mathbf{k}+\mathbf{k}'|)}{\omega_m(k')+\omega_m(|\mathbf{k}+\mathbf{k}'|)+\sqrt{\Big(\omega_m(k')+\omega_m(|\mathbf{k}+\mathbf{k}'|)\Big)^2-k^2}}, \tag{8}$$

where $\omega_m(k)=\sqrt{m^2+k^2}$. In the large mass limit, i.e. $m\gg|\mathbf{p}|,|\mathbf{k}|$, the permutation coefficient C and the Jacobian function N approach the nonrelativistic limit and are equal to one.

After evaluating the permutation operator and the matrix elements of the fully off-shell two-body $t-$matrix, the relativistic Faddeev integral equation reads,

$$\psi(\mathbf{p}\,,\mathbf{k}) = \frac{1}{M_t - M_0(p,k)} \int d\mathbf{k}'\, N(\mathbf{k},\mathbf{k}')\, T^{sym}\Big(\mathbf{p},\tilde{\boldsymbol{\pi}}; M_t - \omega_m(k)\Big)\psi(\boldsymbol{\pi},\mathbf{k}'), \tag{9}$$

where $T^{sym}(\mathbf{p},\mathbf{p}';\epsilon) = T(\mathbf{p},\mathbf{p}';\epsilon) + T(-\mathbf{p},\mathbf{p}';\epsilon)$ is symmetrized boosted two-body t-matrix and the shifted momentum arguments are given by

$$\tilde{\boldsymbol{\pi}} = \mathbf{k}' + \frac{1}{2}C(\mathbf{k},\mathbf{k}')\,\mathbf{k}, \quad \boldsymbol{\pi} = \mathbf{k} + \frac{1}{2}C(\mathbf{k}',\mathbf{k})\,\mathbf{k}'. \tag{10}$$

The relativistic and non-relativistic interaction models are defined so that they have the same two-body center-of-momentum scattering cross sections and wave functions. In the three-body space they are fixed by S-matrix cluster properties and Poincaré or Galilean invariance, respectively. Differences between the relativistic and nonrelativistic Faddeev equations arise from:

1. the Jacobian function N, representing from the change of the 3B basis states,
2. the coefficient C, which appears in the shifted arguments of the momenta,
3. the relation between the relativistic and non-relativistic t-matrices defined by function F,
4. the relations between the relativistic and non-relativistic free Green's functions.

2 Results and Discussion

For our numerical analysis we used two models of a spin-independent Malfliet-Tjon type potential, MT-V [3] and MT-Vc [4]. The model MT-Vc contains an additional cutoff function of dipole type. The relativistic Faddeev integral equation is solved using a Lanczos technique [7]. The calculated relativistic and nonrelativistic 3B binding energies for MT-V potential are $E_r = -7.4825$ (MeV) and $E_{nr} = -7.7382$ (MeV), which indicates a reduction of about $(E_r - E_{nr})/E_{nr} = 3.3\,\%$ in the 3B binding energy. This reduction is consistent with the partial wave calculation of Ref. [8], truncated to s-wave, with about 2.70 % reduction in the 3B binding energy. The similar behavior can also be seen for MT-Vc potential, where the relativistic effects leads to about 3.2 % reduction in 3B binding energy. The obtained nonrelativistic and relativistic 3B binding energies for MT-Vc potential are $E_{nr} = -7.7177$ (MeV) and $E_r = -7.4732$ (MeV), respectively.

While our calculations indicate that the overall relativistic effect is small, this results from large cancellations in the four relativistic corrections mentioned above. In Fig. 1, we show the contribution of each correction to the 3B relativistic binding energy of MT-V potential, where E_{appr} is the 3B binding energy when one of the ingredients is replaced by the corresponding non-relativistic quantity. By setting the Jacobian function N to one in our relativistic calculations, the 3B binding energy has a small increase of about 0.8 %, while setting the permutation coefficient C to one leads to a small reduction of about 0.6 %. Replacing the relativistic right-half-shell $t-$matrix with the non-relativistic one, i.e. setting $F = 1$ in Eq. (3), leads to about 2.4 % increase in the 3B binding energy. Finally, replacing the relativistic free propagator with the nonrelativistic one, the 3B binding energy is reduced by about 1.84 %. By setting both Jacobian functions, N and C, to one in our relativistic calculations, the net contribution of the dynamic ingredients to the 3B binding energy is a small increase of about 0.15 %. Similarly, the net contribution of the kinematic ingredients obtained by setting $F = 1$ and replacing the relativistic free propagator with the nonrelativistic one, leads to an increase in the 3B binding energy of about 0.22 %.

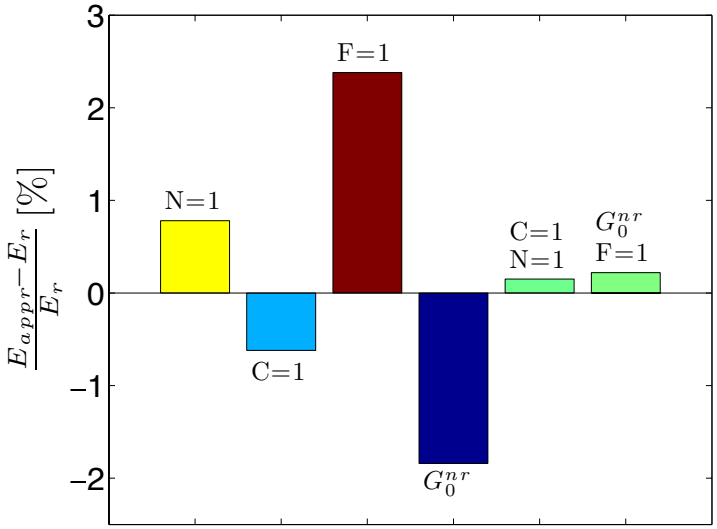

Figure 1. Contributions of different relativistic corrections to the 3B binding energy of MT-V potential. (1) $N = 1$: the Jacobian of the permutation operator (2) $C = 1$ relativistic Jacobi momenta $\rightarrow$ NR Jacobi momenta, (3) $F = 1$ the relativistic half-shell t-matrix $\rightarrow$ NR half-shell t-matrix (4) G_0: free resolvent $G_0 \rightarrow G_0^{NR}$.

The extension of the formulation of the relativistic Faddeev integral equations based on realistic nucleon-nucleon (NN) interactions is in progress. The input here is an operator representation of spin-dependent NN interaction in a helicity representation, which depends on the total spin and the relative momentum in the NN subsystems. Since the 3D approach automatically considers all partial waves the number of equations is fixed. The successful implementation in $3N$ bound state will pave the way to $3N$ scattering in the few-GeV energy range.

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