# Constraints of cluster separability and covariance on current operators

W. N. Polyzou
*Department of Physics and Astronomy, The University of Iowa, Iowa City, IA 52242*

B. D. Keister
*Physics Division, National Science Foundation, Arlington, VA 22230*
(Dated: December 28, 2011)

Realistic models of hadronic systems should be defined by a dynamical unitary representation of the Poincaré group that is also consistent with cluster properties and a spectral condition. All three of these requirements constrain the structure of the interactions. These conditions can be satisfied in light-front quantum mechanics, maintaining the advantage of having a kinematic subgroup of boosts and translations tangent to a light front. The most straightforward construction of dynamical unitary representations of the Poincaré group due to Bakamjian and Thomas fails to satisfy the cluster condition for more than two particles. Cluster properties can be restored, at significant computational expense, using a recursive method due to Sokolov. In this work we report on an investigation of the size of the corrections needed to restore cluster properties in Bakamjian-Thomas models with a light-front kinematic symmetry. Our results suggest that for models based on nucleon and meson degrees of freedom these corrections are too small to be experimentally observed.

PACS numbers: 21.45+v

## I. INTRODUCTION

Light-front quantum mechanics [1] has the desirable feature that it is easy to construct realistic quantum mechanical models that are exactly Poincaré invariant[2], with a kinematic subgroup that leaves a light front invariant. The most straightforward construction, due to Bakamjian and Thomas [3][4], achieves this by requiring both the generators of the kinematic subgroup and the total light-front spin to be non-dynamical. A dynamical mass operator is defined by adding interactions, that commute with the kinematic generators and the kinematic light-front spin, to the non-interacting invariant mass. The dynamical Poincaré generators are well-defined functions of this mass operator and these kinematical operators.

While the original Bakamjian-Thomas (BT) construction was intended for two-body systems, the construction gives a dynamical representation of the Poincaré group for any number of particles. However, while this construction satisfies exact Poincaré invariance, the Poincaré generators do not satisfy cluster properties for systems of more than two particles.

Sokolov [5] introduced a recursive construction that starts with Bakamjian-Thomas two-body models and builds a many-body unitary representation of the Poincaré group consistent with cluster properties. For the three-particle system the mass Casimir operator operator for this unitary representation in the Sokolov construction is [4]

$$M := A(A^{\dagger}_{(12)(3)}M_{(12)\otimes(3)}A_{(12)(3)} + A^{\dagger}_{(23)(1)}M_{(23)\otimes(1)}A_{(23)(1)} + A^{\dagger}_{(31)(2)}M_{(31)\otimes(2)}A_{(31)(2)} - 2M_0)A^{\dagger} \tag{1.1}$$

where

$$A := \exp(\ln(A_{(12)(3)}) + \ln(A_{(23)(1)}) + \ln(A_{(31)(2)})) \tag{1.2}$$

$$M^2_{(ij)\otimes(k)} := (P^-_{ij} \otimes I_k + I_{ij} \otimes P^-_k)(P^+_{ij} \otimes I_k + I_{ij} \otimes P^+_k) - (\mathbf{P}_{\perp ij} \otimes I_k + I_{ij} \otimes \mathbf{P}_{\perp k})^2 \tag{1.3}$$

$$P^-_{ij} := \frac{M^2_{BTij} + \mathbf{P}^2_{\perp ij}}{P^+_{ij}} \tag{1.4}$$

and $M_{BTij}$ is the two-body Bakamjian-Thomas mass operator. The operators $A_{(ij)(k)}$ are $S$-matrix preserving unitary operators that relate $M_{(ij)\otimes(k)}$ to the three-body $2+1$ Bakamjian-Thomas mass operator $M_{BT(ij)(k)}$, so the mass operator in (1.1) can be expressed in terms of 2+1 Bakamjian-Thomas mass operators as

$$M = AM_{BT}A^{\dagger} = A(M_{BT(12)(3)} + M_{BT(23)(1)} + M_{BT(31)(2)} - 2M_0)A^{\dagger}. \tag{1.5}$$

The existence of the operators $A_{(ij)(k)}$ that preserve the $2+1$ $S$ matrix and the light-front kinematic symmetry are ensured by a theorem of Ekstein [6]. The dynamical Poincaré generators are functions of the mass operator (1.1), kinematic Poincaré generators [7], $\{ \mathbf{E}_{0\perp}, P_0^+, \mathbf{P}_{0\perp}, \hat{\mathbf{z}}\cdot\mathbf{J}_0, \hat{\mathbf{z}}\cdot\mathbf{K}_0\}$ and the kinematic light-front spin $\mathbf{j}_0$,

$$P^- := \frac{M^2+\mathbf{P}_{0\perp}^2}{P_0^+} \tag{1.6}$$

$$\mathbf{J}_\perp := \frac{1}{P_0^+}\left[\frac{(P_0^+ - P^-)}{2}(\hat{\mathbf{z}}\times\mathbf{E}_{0\perp}) - (\hat{\mathbf{z}}\times\mathbf{P}_{0\perp})(\hat{\mathbf{z}}\cdot\mathbf{K}_0) + \mathbf{P}_{0\perp}(\hat{\mathbf{z}}\cdot\mathbf{j}_0) + M\mathbf{j}_{0\perp}\right]. \tag{1.7}$$

These generators cluster into sums of tensor products in the limit that the interactions between particle $i$ and the pair $(jk)$ are turned off:

$$\mathbf{J}_\perp \to \mathbf{J}_{\perp(jk)}\otimes I_i + I_{jk}\otimes\mathbf{J}_{\perp i} = A_{(jk)(i)}\mathbf{J}_{BT\perp(jk)(i)}A^\dagger_{(jk)(i)} \tag{1.8}$$

$$P^- \to P^-_{(jk)}\otimes I_i + I_{jk}\otimes P_i^- = A_{(jk)(i)}P^-_{BT\perp(jk)(i)}A^\dagger_{(jk)(i)}. \tag{1.9}$$

These expression demonstrate that the operators $A_{(jk)(i)}$ are responsible for the failure of cluster properties in the BT generators. The relation (1.5) shows that the mass operator (1.1) gives the same $S$ matrix as the BT mass operator $M_{BT}$; however this is no longer true for systems of four or more particles.

Because of its complexity, there have been no dynamical few-body calculations that utilize the Sokolov construction. On the other hand there have been many calculations based on the Bakamjian-Thomas construction, some involving more than two particles. The size of the unitary transformations $A_{(ij)(k)}$ relative to the identity provides an estimate of the corrections needed to restore cluster properties in Bakamjian-Thomas models.

## II. MODEL AND RESULTS

In this work we use a simple model to investigate how close the unitary transformations, $A_{(ij)(k)}$, are to the identity. The model is a four-body model where an electron scatters off of a proton (particle 3) in the presence of a bound state of a proton and neutron (a deuteron consisting of particles 1 and 2). We assume that the deuteron does not interact with the struck proton or electron. We assume that the interaction of the proton with the electron can be treated in the one-photon-exchange approximation. As a final simplification we assume the nucleons are spinless and replace the four-vector current by a scalar one-body current, $j(x)$.

To study systems with nuclear physics scales we choose the two-body mass Casimir operator to be $M^2_{BT12} := M_0^2 + 4mV$ [8][7] where $V$ is a Malfliet-Tjon [9] nucleon-nucleon interaction, $M_0$ is the non-interacting two-body invariant mass, and $m$ is the nucleon mass. The three-nucleon mass operator (1.1) for this model becomes $M \to M_{(12)(3)} = A_{(12)(3)}M_{BT(12)(3)}A^\dagger_{(12)(3)}$. We calculate

$$F_{TP}(p_3', p_3, p_{12}) := \int \langle d, p_3', p_{12}'|J(0)|d, p_3, p_{12}\rangle dp_{12}'$$

$$= \int {}_{BT}\langle d, p_3', p_{12}'|A^\dagger_{(12)(3)}J(0)A_{(12)(3)}|d, p_3, p_{12}\rangle_{BT} dp_{12}' \tag{2.1}$$

and compare it the corresponding Bakamjian Thomas quantity

$$F_{BT}(p_3', p_3, p_{12}) := \int {}_{BT}\langle d, p_3', p_{12}'|J(0)|d, p_3, p_{12}\rangle_{BT} dp_{12}'. \tag{2.2}$$

The first integral is the form factor for nucleon 3, it is independent of $\mathbf{p}_{12}$ as expected by cluster properties. The second integral treats the $2+1$ three-nucleon system as a Bakamjian-Thomas model. This expression violates cluster properties because it has a non-physical dependence on $\mathbf{p}_{12}$. In addition the result is also sensitive to mass of the deuteron and the momentum dependence of the deuteron wave function.

In figures 1 and 2 we plot the difference $(F_{TP}-F_{BT})/F_{TP}$ as a function of $Q = P_3' - P_3$ and $p_{12}$. We consider frame where the + component of the momentum transfer is zero, which is always possible for spacelike momentum transfers.

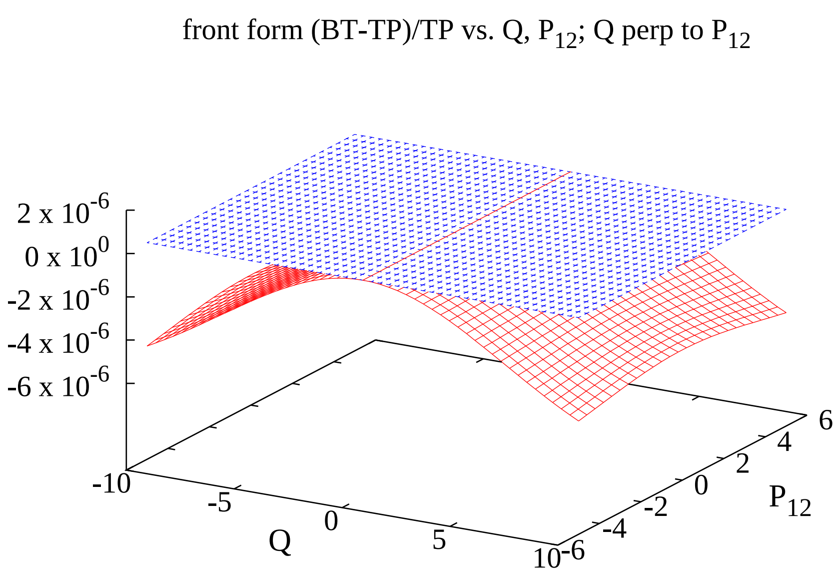


FIG. 1: Front form - $p_{12} \perp Q$

We choose the light front $z + t = 0$, assume that the momentum transfer $Q$ is in the $x$ direction and investigate the dependence on $p_{12}$ in the $x$ (parallel) or $y$ (perp) directions.

The results are shown in figures 1 and 2 and expressed as fractional differences between Bakamjian-Thomas and tensor-product calculations. Deviations from zero therefore exhibit the unphysical dependence upon $Q$ and $p_{12}$ in the Bakamjian-Thomas case. The figures show, for this model, which uses parameters that have scales expected in nuclear physics models with meson-exchange interactions, the size of the corrections needed to restore cluster properties is too small to be measured in laboratory experiments. This investigation suggests that it is reasonable to construct light-front quantum mechanical models of few-nucleon systems using only the Bakamjian-Thomas representation of the Poincaré group, without including the corrections due to the Sokolov operators.

Had we instead constructed our deuteron out of sub-nuclear degrees of freedom involving stronger binding and larger internal momenta, the size of these corrections could be large enough to be observable.


[1] P. A. M. Dirac, Rev. Mod. Phys. **21**,392(1949).
[2] E. P. Wigner, Ann. Math. C **40**, 149 (1939).
[3] B. Bakamjian and L. H. Thomas, Phys. Rev. **92**,1300(1953).
[4] F. Coester and W. N. Polyzou, Phys. Rev. D. **26**,1348(1982).
[5] S. N. Sokolov, Dokl. Akad. Nauk. SSSR **233**,575(1977).
[6] H. Ekstein, Phys. Rev. **117**,1590(1960).
[7] B. D. Keister and W. N. Polyzou, Advances in Nuclear Physics, Volume 20, Ed. J. W. Negele and E. W. Vogt, Plenum Press 1991.
[8] F. Coester, S. C. Pieper and F. J. D Serduke, Phys. Rev. C**11**,1(1975).
[9] R. A. Malfliet and J. A. Tjon, Nucl. Phys. **A127**,161(1969).

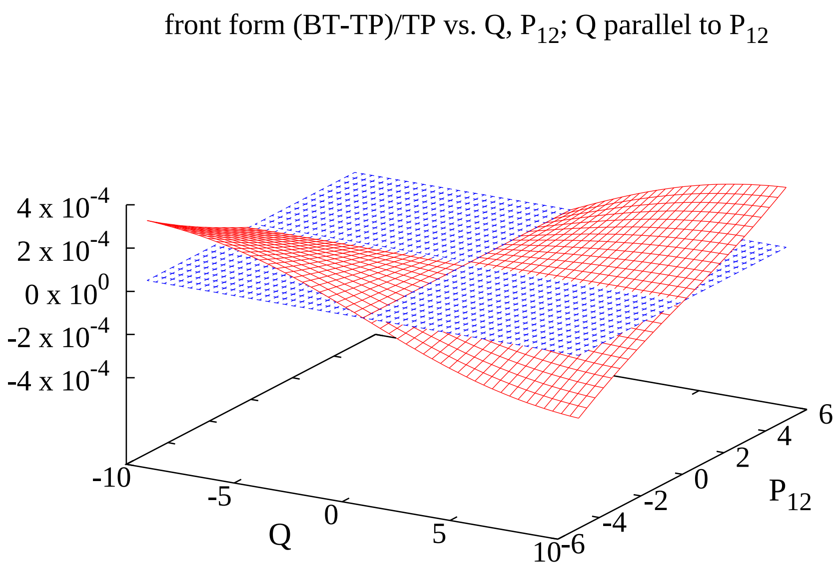


FIG. 2: Front form - $p_{12}||Q$