Spin and dynamics in relativistic quantum theories

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Abstract

The role of relativity and dynamics in defining the spin and orbital angular momentum content of hadronic systems is discussed.

1 Introduction

There is a great deal of interest in the distribution of spin and orbital angular momentum in hadronic systems. In general the underlying dynamics of partons in hadrons is relativistic. In the relativistic case the coefficients that relate the parton spins to hadronic spins are momentum-dependent and are necessarily influenced by the momentum-dependence of the hadronic wave function. This momentum dependence appears in both relativistic quantum mechanics and relativistic quantum field theory. In addition, in QCD, because constituent partons are confined, their "mass" becomes an additional dynamical variable. The purpose to this paper is to show how this dynamical dependence enters the relation between the hadronic and partonic spins and to show the equivalence of the treatment spin in Poincaré and Lorentz covariant theories.

The treatment of spin in relativistic systems is different than it is in non-relativistic systems. In a relativistic system the spin of a parton is identified with the angular momentum of the parton in its rest frame while the spin of the hadron is defined as the angular momentum of the hadron in its rest frame. Transforming a parton from its rest frame to the hadrons rest frame, where the spins can be coupled, involves boosts which generate dynamical rotations.

These rotations transform the parton spins and also impact the relative orbital angular momentum before they can be coupled. The spin of the constituents, the internal orbital angular momentum and the spin of the system are related by Clebsch-Gordan coefficients of the Poincaré group[1][2][3] . The Poincaré group Clebsch-Gordan coefficients are labeled by eigenvalues of mass and spin Casimir operators, which are dynamical operators.

An additional complication is that boosts to the rest frame are not unique; a boost to the rest frame followed by a momentum-dependent rotation is a different boost to the rest frame. There are as many different kinds of boosts as there are momentum-dependent rotations. Each boost defines a different spin observable. For example, there are distinct boosts that are used to define the helicity, light front spin or canonical spin. These are three specific choices, that are distinguished by useful properties, out of an infinite number of possibilities. In many-body systems there is another relevant spin, which we call the constituent spin[3], which is distinguished by the property that spins and orbital angular momenta can be combined using ordinary SU(2) coupling methods to get the hadronic spin.

While all of the spins satisfy $SU(2)$ commutation relations, the different spins observables are related by the momentum dependent (Melosh) rotations[4] that relate different boosts. Because different spin observables differ by momentum-dependent rotations, partial derivatives with respect to momentum that hold one spin observable constant will not commute with a different spin observable. This means not only are there an infinite number of possible spin observables, but each one is associated with a different quantity that can be identified with an orbital angular momentum. As a result the spin and orbital angular momentum content of a hadron is dynamical and representation dependent. In what follows we discuss some of the relevant issues.

The dependence of the spin on the choice of boost is seen in the relations between the spin and angular momentum

$$j_a^l := \frac{1}{2}\epsilon^{lmn} B_a^{-1}(p)^m{}_\mu B_a^{-1}(p)^n{}_\nu J^{\mu\nu} \tag{1}$$

where $B_a^{-1}(p)^m{}_\mu$ is a boost that maps p to its rest frame. Here a is an index used to distinguish different types of Lorentz boosts.

This definition can be equivalently expressed in terms of the polarization vectors, $e_a^m{}_\mu(p) := B_a^{-1}(p)^m{}_\mu$, $m = 1, 2, 3$, that are three orthonormal space-like vectors that are orthogonal to the four momentum:

$$j_a^l = \frac{1}{2}\epsilon^{lmn} e_a^m{}_\mu(p) e_a^n{}_\nu(p) J^{\mu\nu}. \tag{2}$$

Spins constructed using different boosts (labeled by a and b) are related by momentum-dependent Melosh rotations

$$j_a^l = R_{ab}(p)^l{}_m j_b^m \qquad \text{where} \qquad R_{ab}(p)^l{}_m = B_a^{-1}(p)^l{}_\mu B_b(p)^\mu_m. \tag{3}$$

Because the different types of spin observables differ by momentum-dependent rotations, "Position operators" [5][6][2]that involve partial derivatives with respect to momentum need to specify which kind of spins is being held constant during the differentiation,

$$[\boldsymbol{\nabla}_{P_{|_{j_a}}}, \mathbf{j}_a] = 0 \Rightarrow [\boldsymbol{\nabla}_{P_{|_{j_a}}}, \mathbf{j}_b] \neq 0. \tag{4}$$

These partial derivatives can be written in terms of the Poincaré generators with $\mathbf{V} = \mathbf{P}/M$ by [6][2]

$$X_a^k = i\boldsymbol{\nabla}_{\mathbf{P}_{|_{\mathbf{j}_a}}} = -\frac{1}{2}\{H^{-1}, K^k\} + iH^{-1}C_{1a}^{kl}(\mathbf{V})j_a^l \tag{5}$$

where H is the Hamiltonian, $M = \sqrt{H^2 - \mathbf{P}^2}$ is the invariant mass operator and $\mathbf{V}$ is the four velocity. In terms of these operators the spins and angular momentum are related by

$$J^j = (\mathbf{X}_a \times \mathbf{P})^j + C_{2a}^{jk}(\mathbf{V})j_a^k \tag{6}$$

where the operators $C_{1a}^{jk}(\mathbf{V})$ and $C_{2a}^{jk}(\mathbf{V})$ are the following functions of the Poincaré generators:

$$C_{1a}^{jk}(\mathbf{V}) = \frac{1}{2}\mathbf{Tr}[B_a(\mathbf{V})^{-1}\sigma_j B_a(\mathbf{V})\sigma_k] - V^0\mathbf{Tr}[B_a(\mathbf{V})^{-1}\frac{\partial}{\partial V_l}B_a(\mathbf{V})\sigma_m] \tag{7}$$

$$C_{2a}^{jk}(\mathbf{V}) = \frac{1}{2}\mathbf{Tr}[B_a(\mathbf{V})^{-1}\sigma_j B_a(\mathbf{V})\sigma_k] + i\epsilon_{jlm}\mathbf{Tr}[B_a(\mathbf{V})^{-1}\frac{\partial}{\partial V_l}B_a(\mathbf{V})\sigma_m] \tag{8}$$

and $B_a(\mathbf{V})$ is the $SL(2,\mathbb{C})$ representation of the a-boost. The quantity $\mathbf{X}_a \times \mathbf{P}$ is the associated orbital angular momentum [6][2]

Three components of the four momentum and the projection of any of these spin observables on a given axis are labels for vectors in irreducible subspaces. Products of two such irreducible representations can be expressed as direct integrals of composite irreducible representations using the Clebsch-Gordan coefficients for the Poincaré group. Like any set of Clebsch-Gordan coefficients, the actual coefficients depend on the choice of irreducible basis. The Poincaré group Clebsch-Gordan coefficients for a basis labeled by the a-type spin are

$$\begin{aligned}
&{}_a\langle (M_1, j_1)\mathbf{P}_1, \mu_1 (M_2, j_2)\mathbf{P}_2, \mu_2 | k, j(M_1, j_1, M_2, j_2)\mathbf{P}, \mu, l, s_{12}\rangle_a = \\
&\sum_{\mu_1', \mu_2', \mu_1'', \mu_2'', \mu_s, m} \delta(\mathbf{P} - \mathbf{P}_1 - \mathbf{P}_2)\frac{\delta(k - k(\mathbf{P}_1, \mathbf{P}_2))}{k^2} \times \\
&\sqrt{\frac{\omega_{M_1}(\mathbf{k})\omega_{M_2}(\mathbf{k})}{\omega_{M_1}(\mathbf{P}_1)\omega_{M_2}(\mathbf{P}_2)}}\sqrt{\frac{\omega_{M_1}(\mathbf{P}_1) + \omega_{M_2}(\mathbf{P}_1)}{\omega_{M_1}(\mathbf{k}) + \omega_{M_2}(\mathbf{k})}} \times \\
&D^{j_1}_{\mu_1 \mu_1'}[R_{wa}(B_a(V), k_1)] D^{j_1}_{\mu_1' \mu_1''}[R_{ac}(k_1)] \times \\
&D^{j_2}_{\mu_2 \mu_2'}[R_{wa}(B_a(V), k_2)] D^{j_2}_{\mu_2' \mu_2''}[R_{ac}(k_2)] \times \\
&Y^l_m(\hat{\mathbf{k}}(\mathbf{P}_1, \mathbf{P}_2))\langle j_1, \mu_1'', j_2, \mu_2'' | s_{12}, \mu_s\rangle \langle l, m, s_{12}, \mu_s | j, \mu\rangle
\end{aligned} \tag{9}$$

These involve two types of spin rotations. There are Wigner rotations $R_{wa}(B_a(V), k_i)$ that arise from the a-boosts that relate the system and parton rest frames and generalized Melosh rotations, $R_{ac}(k_2)$, that transform the resulting spins to the canonical spin representation where all of the spins and orbital angular momenta Wigner rotate together so they can be added using ordinary $SU(2)$ spin addition.

The spins obtained by applying these two rotations to the hadronic spins are the constituent spins mentioned earlier. These are the spins associated with the magnetic quantum numbers μ_i'' in (9). It is apparent from this equation that when these spins are combined with the orbital angular momentum using spherical harmonics and SU(2) Clebsch-Gordan coefficients the result is the total spin.

The Poincaré group Clebsch-Gordan coefficients (9) simplify in special bases. If the spins are defined using the standard rotationless boosts there are no Melosh rotations, if the rotationless boost is replaced by a light-front boost there are no Wigner rotations, and if the rotationless boost is replaced by a helicity boost the Wigner rotations become multiplication by a phase.

One result of the momentum-dependence of the rotations is that the momentum-dependence of the hadronic wave function affects the expectation values of both the spins and orbital angular momentum. Since the momentum dependence of the hadronic wave function is determined by the dynamics, the dynamics enters in the spin coupling when the Poincaré Clebsch-Gordan coefficients are integrated against the hadronic wave functions.

When one couples two interacting subsystems, one has to ask whether the masses in the Poincaré Clebsch-Gordan coefficients are the physical masses of the subsystems or the invariant masses of their constituents. For example, the mass of a meson or the invariant mass of a quark antiquark pair? So far we have treated them as invariant masses of the constituents. Cluster properties suggest that one should really use the physical mass operators of the subsystems. Fortunately there is a unitary transformation that removes the interaction dependence from the hadronic spin[7][8]. In this representation the spins can be coupled by sequential coupling using the Clebsch-Gordan coefficients of the Poincaré group as if the particles were not interacting. This unitary transformation changes the Hamiltonian, generating many-body interactions. It also changes the representation of the wave function in a way that preserves probabilities, expectation values, as well as scattering observables. As a result of this all of the dynamical spin effects can be absorbed by changing the representation of the wave function. In QCD the quark masses themselves are also not constant.

A second ambiguity with spin has to do with whether the dynamics is formulated using Poincaré covariant or Lorentz covariant bases. Field theories are normally formulated using Lorentz covaraint bases while relativistic quantum mechanics is typically formulated using Poincaré covariant bases. These are simply related; the dynamics enters both representations, but in different but equivalent ways. To understand this note that the unitary representation of the Poincaré group on positive-mass positive-energy irreducible basis states has the form

$$U(\Lambda,0)|(M,j)\mathbf{P},\mu\rangle_a = \sum |(M,j)\mathbf{\Lambda}P,\nu\rangle_a D^j_{\nu\mu}(R_{wa}(\Lambda,P)). \tag{10}$$

The Wigner rotation can be decomposed into the composition of a boost followed by a Lorentz transformation followed by an inverse boost with the transformed four momentum

$$R_{wa}(\Lambda,P) = B_a^{-1}(\Lambda P)\Lambda B_a(P). \tag{11}$$

The group representation property can be used to split the Wigner function apart. The finite dimensional representations of $SU(2)$ are related to finite dimensional representation of $SL(2,\mathbb{C})$ by analytic continuation[9][3], so we can still use the group representation property. Absorbing the Wigner functions of the boosts into the states gives the Lorentz spinor representation of the states:

$$|(m,j)\mathbf{P},b\rangle := \sum_\mu |(m,j)\mathbf{P},\mu\rangle_a D^j_{\mu b}[B_a^{-1}(P/M)]. \tag{12}$$

Here the boosts are represented by $2\times 2\, SL(2,\mathbb{C})$ transformations. These spinor basis states (12) have the following Lorentz covariant transformation property

$$U(\Lambda,0)|(m,j)\mathbf{P},b\rangle = \sum_{b'} |(m,j)\mathbf{\Lambda}P,b'\rangle D^j_{b'b}[\Lambda]. \tag{13}$$

The price paid for using the covariant representation is that the Hilbert space inner product becomes dynamical

$$\langle\psi|\phi\rangle =$$

$$\int \langle\psi|(m,j)\mathbf{P},b\rangle d^4P\theta(P^0)\delta(P^2+M^2)D^j_{bb'}[P^\mu\sigma_\mu/M]\langle(m,j)\mathbf{P},b'|\phi\rangle \tag{14}$$

where we have used the hermiticity of the $SL(2,\mathbb{C})$ representation of the rotationless boost which gives

$$B_a(V)B_a^\dagger(V) = B_c(V)R_{ca}(V)R_{ca}^\dagger B_c^\dagger(V) = B_c(V)B_c^\dagger(V) = B_c^2(V) = P^\mu\sigma_\mu/M \tag{15}$$

independent of the type (a) of boost. The zero component of σ^μ in (15) is the identity and the other three components are the Pauli matrices. In (14) the dynamics is appears in the mass-shell condition, which makes the Wigner function into a positive matrix. The inner product (14) is identical to the original Poincaré covariant inner product.

Unlike representations of $SU(2)$, the representations of $SL(2,C)$ are not equivalent to the complex conjugate representations. This means that we could alternatively replace (12) by

$$|(m,j)\mathbf{P},\dot{b}\rangle := \sum_\mu |(m,j)\mathbf{P},\mu\rangle_a D^j_{\mu\dot{b}}[B_a^\dagger(P/M)]. \tag{16}$$

and (13) by

$$U(\Lambda,0)|(m,j)\mathbf{P},\dot{b}\rangle = \sum_{b'} |(m,j)\mathbf{\Lambda}P,\dot{b}'\rangle D^j_{\dot{b}'\dot{b}}[((\Lambda)^\dagger)^-1)]. \tag{17}$$

This gives a representation of the scalar product that has the same form as (14) with the replacement

$$D^j_{bb'}[P^\mu\sigma_\mu/M] \to D^j_{\dot b\dot b'}[P^\mu\sigma_2\sigma^*_\mu\sigma_2/M]. \tag{18}$$

While the inner products in all three representations are identical, the Lorentz covariant and its complex conjugate representations are related by space reflection. Space reflection changes the kernel of the Hilbert-space scalar product in the covariant representations. Space reflection can be represented as an operator on states by replacing the representations (12) and (16) by a direct sum of both representations. In the direct sum representation the wave function becomes a $2\times(2j+1)$ component spinor

$$\psi(P,b) \to \begin{pmatrix} \xi(P,b) \\ \chi(P,\dot b) \end{pmatrix} \tag{19}$$

and the kernel of the inner product becomes

$$d^4P\theta(P^0)\delta(P^2+M^2)\begin{pmatrix} D^j_{bb'}[P^\mu\sigma_\mu/M] & 0 \\ 0 & D^j_{\dot b\dot b'}[P^\mu\sigma_2\sigma^*_\mu\sigma_2/M] \end{pmatrix}. \tag{20}$$

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One desirable feature of the Lorentz covariant representation is that the basis-dependent features are hidden in the wave functions. To see this note that for rotations the upper and lower components have identical transformations laws

$$U(R,0)|(M,j)\mathbf{P},b\rangle = \sum_{\dot b'}|(M,j)R\mathbf{P},b'\rangle D^j_{b'b}[R]. \tag{21}$$

and

$$U(R,0)|(M,j)\mathbf{P},\dot b\rangle = \sum_{\dot b'}|(M,j)R\mathbf{P},\dot b'\rangle D^j_{\dot b'\dot b}[R]. \tag{22}$$

which is the standard rotational transformation law that leads to the standard relation

$$\mathbf{J} = \mathbf{X}\times\mathbf{P}+\mathbf{j} \tag{23}$$

in the covariant representation.

In this representation the relation between the spin, angular momentum, and orbital angular momentum looks very much like the corresponding non-relativistic quantities. The price paid for this simplification is that the Hilbert space inner product has a non-trivial kernel. This kernel contains all of the dynamical effects discussed in the context of Poincaré irreducible spins. The covariant spin is related to the Poincaré irreducible spin of a particle by a boost. For spin 1/2 particles the usual u and v spinors are direct sum representations of a Lorentz boost. The choice (helicity, canonical, light front spin) appears in the representation of these spinors. The Poincaré irreducible labels are the parameters that normally label asymptotic states in the S-matrix.

In the end there are many different kinds of spin observables. In order to measure the spin we need to know how the various spin operators couple to the electroweak current operators. This will be different for each type of spin observable.

The conclusion is that for relativistic systems the coupling of spins and orbital angular momenta involves momentum and mass-dependent Poincaré group Clebsch-Gordan coefficients. In QCD even the parton masses that appear in these coefficients become variables. The result is that the dynamics cannot be ignored in attempts to identify the different terms the contribute to the hadronic spin.

In addition, there are many different spin and orbital angular momentum observables. How these different quantities contribute to the hadronic spin is representation dependent. As a consequence, is important to know how these operators are precisely defined and how they are related to experimental observables.

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References

[1] P. Moussa and R. Stora (1965), in Lectures in Theoretical Physics, Vol VIIA Lorentz Group, Ed. W. E. Brittin and A. O. Barut, The University of Colorado Press, Boulder, CO pp 66-69.

[2] B. D. Keister and W. N. Polyzou (1991), Relativistic Hamiltonian Dynamics, Advances in Nuclear Physics, Volume 20, p. 225, Ed J. Negele and Erich Vogt, Plenum Press, New York-London .

[3] W. N. Polyzou, W. Glöckle and H. Witała (2013), Spin in relativistic quantum theory, Few-Body Systems, 54, p. 1667.

[4] H. J. Melosh Phys (1974), Quarks: Currents and constituents, Rev. D **9**, p. 1095.

[5] T. D. Newton and E. P. Wigner (1949), Localized States for Elementary Systems, Rev. Mod. Phys. 21, p. 400.

[6] W. N. Polyzou (1989), Relativistic Two-Body Models, Annals of Physics, N.Y. 193, p. 367.

[7] F. Coester and W. N. Polyzou (1982), Relativistic Quantum Mechanics of Particles with Direct Interactions Phys. Rev. D26, 1348.

[8] W. N. Polyzou (2010), Examining the equivalence of Bakamjian-Thomas mass operators in different forms Phys. Rev. C 82,064001 .

[9] Weinberg, S, (1964), Feynman Rules for any spin, Phys. Rev. **133**, p. B1318-1332.