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Euclidean relativistic quantum mechanics - scattering asymptotic conditions

Abstract We discuss the formulation of the scattering asymptotic condition in a relativistic quantum theory formulated in terms of reflection positive Euclidean green functions.

Keywords Scattering · Euclidean field theory · Relativistic quantum theory

1 Introduction

Schwinger [1] showed that the spectral condition implies that the Green functions of a local quantum field theory have an analytic continuation to Euclidean times. This observation has motivated Euclidean formulations of quantum field theory that have been used successfully in numerical lattice discretizations of the theory. One difficulty with the Euclidean formulation is that the imaginary time leads to problems in formulating scattering problems [2] that involve asymptotic time limits. Osterwalder and Schrader [3] proved a reconstruction theorem that showed how to construct a relativistic quantum field theory from a set of Euclidean invariant Green functions satisfying a condition called reflection positivity. The proof of the reconstruction theorem has three important attributes. First, it is constructive, leading to an explicit representation of the physical Hilbert space and a set of self-adjoint operators on that space satisfying the Poincaré Lie algebra. Second, the locality property is logically independent of the other properties that are needed to construct a relativistic quantum theory. This implies that it should be possible to make approximations or truncations of a local theory that still retain all of the remaining properties of a relativistically invariant quantum theory. Third, the construction of the relativistic quantum theory does not require explicit analytic continuation of the Euclidean times. This paper is based on [4][5] where we argue that given a Hilbert space representation and a set of Poincaré generators on that space that satisfy cluster properties, it should be possible to formulate scattering directly in the Euclidean representation of the Hilbert space.

2 Reconstruction of quantum mechanics

A dense set of vectors in the Hilbert space consists of sets of normalizable functions of Euclidean space-time variables,

$$\mathbf{f} = (f_0, f_1(x_{e11}), f_2(x_{e21}, x_{e22}), \cdots), \tag{1}$$

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with support for positive relative Euclidean times, $0 < x^0_{en1} < x^0_{en2} < x^0_{en3} \cdots$.

The Hilbert-space inner product of two such vectors is expressed in terms of a quadratic form involving the Euclidean Green functions, $G^e_n(x_{e1}, \cdots, x_{en})$, by

$$\langle \mathbf{f}|\mathbf{g}\rangle = \tag{2}$$

$$\sum_{mn} \int f^*_m(x_{em1}, \cdots, x_{emm}) G^e_{m+n}(T_e x_{emm}, \cdots, T_e x_{em1}, y_{en1}, \cdots, y_{enn}) \times \tag{3}$$

$$g_n(y_{en1}, \cdots, y_{enn}) d^{4m}x d^{4n}y \tag{4}$$

where T_e Euclidean time reflection. The collection of Euclidean Green functions are called reflection positive if $\langle \mathbf{f}|\mathbf{f}\rangle \geq 0$ for all $\mathbf{f}$ satisfying the support condition. This represents the physical Hilbert space inner product, even though all of the integrals in (3) are over Euclidean variables and there is no analytic continuation. This inner product has zero norm vectors, so Hilbert space vectors are actually equivalence class of functions $\mathbf{f}$ subject to the equivalence relation $\mathbf{f} \sim \mathbf{g}$ if and only if $\langle \mathbf{f}-\mathbf{g}|\mathbf{f}-\mathbf{g}\rangle = 0$.

The infinitesimal generators of the Poincaré group are represented by the following self-adjoint operators on this space

$$H f_n(x_{en1}, x_{en2}, \cdots, x_{enn}) = \sum_{k=1}^{n} \frac{\partial}{\partial x^0_{ekn}} f_n(x_{en1}, x_{en2}, \cdots, x_{enn}) \tag{5}$$

$$\mathbf{P} f_n(x_{en1}, x_{en1}, \cdots, x_{1nn}) = -i \sum_{k=1}^{n} \frac{\partial}{\partial \mathbf{x}_{enk}} f_n(x_{en1}, x_{en2}, \cdots, x_{enn}) \tag{6}$$

$$\mathbf{J} f_n(x_{en1}, x_{en2}, \cdots, x_{ne}) = -i \sum_{k=1}^{n} \mathbf{x}_{enk} \times \frac{\partial}{\partial \mathbf{x}_{enk}} f_n(x_{en1}, x_{en2}, \cdots, x_{enn}) \tag{7}$$

$$\mathbf{K} f_n(x_{en1}, x_{en2}, \cdots, x_{ne}) = \sum_{k=1}^{n} (\mathbf{x}_{enk} \times \frac{\partial}{\partial x^0_{enk}} - x^0_{enk} \frac{\partial}{\partial \mathbf{x}_{enk}}) f_n(x_{en1}, x_{en2}, \cdots, x_{enn}). \tag{8}$$

It is straightforward to show that these operators are Hermetian with respect to the inner product (3) and satisfy the Poincaré commutation relations. They are also self-adjoint by virtue of being generators of unitary one-parameter groups, contractive semigroups, or local symmetric semigroups [6][7][8] on this Hilbert space. The Hamiltonian H also satisfies a spectral condition as a consequence of reflection positivity [9].

If the Euclidean Green functions satisfy cluster properties,

$$\lim_{\mathbf{a}\to\infty} (G^e_{m+n}(X_{em} + \mathbf{a}, Y_{en}, \cdots) - G^e_m(X_{em}) G^e_n(Y_{em})) \to 0 \tag{9}$$

$$X_{em} = (x_{em1}, \cdots, x_{emm}) \tag{10}$$

then the Poincaré generators defined by (5-8) also satisfy cluster properties.

3 Formulation of scattering

Scattering matrix elements, $S_{fi} = \langle \mathbf{f}_+|\mathbf{f}_-\rangle$, are defined as the inner product of initial and final states at a given time that asymptotically look like systems of non-interacting particles long before and long after a collision. The relation between these states and the corresponding asymptotic states is given by the scattering asymptotic condition, which replaces the initial condition on the initial or final states by the scattering asymptotic conditions

$$\lim_{t\to\pm\infty} \| e^{-iHt}|\mathbf{f}_\pm\rangle - \Phi e^{-iH_f t}|\chi\rangle \| = 0. \tag{11}$$

The quantity $|\chi\rangle$ on the right describes free-particle wave packets. For a two-particle final state it has the form

$$\Phi e^{-iH_{f12}t}|\chi_{12}\rangle := \sum_{\mu_1\mu_2} \int \underbrace{|\phi_{b1}, \mathbf{p}_1, \mu_1, \phi_{b2}, \mathbf{p}_2, \mu_2\rangle}_{\Phi} \times \tag{12}$$

$$\underbrace{e^{-i(E_1(\mathbf{p}_1)+E_2(\mathbf{p}_2))t}}_{e^{-iH_{f12}t}} d\mathbf{p}_1 d\mathbf{p}_2 \underbrace{\chi_1(\mathbf{p}_1,\mu_1)\chi_2(\mathbf{p}_1,\mu_1)}_{wave\,packets} \tag{13}$$

where

$$M_{bi}|\phi_{bi}\rangle = m_{b_i}|\phi_{bi}\rangle \qquad E_i(\mathbf{p}_i) = \sqrt{m_{b_i}^2 + \mathbf{p}_i^2} \tag{14}$$

and m_{b_i} is in the point-spectrum of the subsystem mass operator. It follows that the operator Φ is a mapping from a two-particle asymptotic Hilbert space of square integrable functions of $\mathbf{p}_1, \mu_1, \mathbf{p}_2.\mu_2$, to the physical Hilbert space with inner product (3).

A sufficient condition for the existence of these limits is the Cook condition, which in this notation has the form

$$\int_a^\infty \|(H\Phi - \Phi H_f)e^{\mp iH_f t}|\chi\rangle\| dt < \infty \tag{15}$$

where $(H\Phi - \Phi H_f)$ replaces the short-range potential in the original formulation of Cook's method[10].

This condition can be formulated in the Euclidean representation. Two conditions are needed to establish convergence in the Euclidean case. First, the Green functions need to satisfy cluster properties. Second, the operator Φ must eliminate the contribution from the disconnected parts of the Green function in (11).

To illustrate how this works consider the case of $2-2$ scattering. Cluster properties lead to a four-point green function that can be expressed as the sum of products of two-point functions and a connected fourpoint function:

$$G_4 = G_{2e}G_{2e} + G_{4ec}. \tag{16}$$

In general there will be additional disconnected terms - the analysis below can be easily extended to treat these cases.

In the Euclidean representation the square of integrand of (11) for the case of 2-2 scattering has the form

$$\begin{gathered}
\|(H\Phi - \Phi H_f)e^{-iH_f t}|\chi_{0\pm}(0)\rangle\|^2 = \\
(\chi_f, e^{iH_f t}(\Phi^\dagger H - H_f\Phi^\dagger), T_e(G_{2e}G_{2e} + G_{4ec})(H\Phi - \Phi H_f)e^{-iH_i t}\chi_i) = \\
\int \chi_1^*(\mathbf{p}_1)\chi_2^*(\mathbf{p}_2)e^{i(E_{m_1}(\mathbf{p}_1)+E_{m_2}(\mathbf{p}_2))t} d\mathbf{p}_1 d\mathbf{p}_2 \times \\
(\frac{\partial}{\partial x_1^0} + \frac{\partial}{\partial x_2^0} - E_{m_1}(\mathbf{p}_1) - E_{m_2}(\mathbf{p}_2))\langle \mathbf{p}_1, \mathbf{p}_2|\Phi^\dagger|x_1, x_2\rangle \times \\
d^4x_1 d^4x_2 (G_{2e}(T_e x_1, y_1)G_{2e}(T_e x_2, y_2) + G_{4ec}(T_e x_1, T_e x_2, y_2, y_1)) d^4y_1 d^4y_2 \times \\
(\frac{\partial}{\partial y_1^0} + \frac{\partial}{\partial y_2^0} - E_{m_1}(\mathbf{p}_1') - E_{m_2}(\mathbf{p}_2'))\langle y_1, y_2|\Phi|\mathbf{p}_1', \mathbf{p}_2'\rangle \times \\
e^{-i(E_{m_1}(\mathbf{p}_1')+E_{m_2}(\mathbf{p}_2'))t}\chi_1(\mathbf{p}_1')\chi_2(\mathbf{p}_2') d\mathbf{p}_1' d\mathbf{p}_2'
\end{gathered} \tag{17}$$

where $E_{m_i}(\mathbf{p}) = \sqrt{m_i^2 + \mathbf{p}^2}$ are the energies of the outgoing particles, and G_{4ec} is the connected part of the Euclidean 4-point Green function.

We expect that the contribution to this expression from the connected Green functions will fall off like $1/t^3$ for large time, as it does in the Minkowski case (see [11] p. 132). For the disconnected terms it is necessary to study the contribution from two-point Green functions. These can be expressed in terms of the Kallën-Lehmann representation of the Euclidean two-point function:

$$\begin{gathered}
\langle\xi|\psi\rangle = \int \xi^*(x_e)G_{2e}(T_e x_e - y_e)\psi(y_e) d^4x_e d^4y_e = \\
\int \frac{d^4p_e d^4x_e d^4y_e}{(2\pi)^4}\xi^*(x_e)\frac{e^{ip_e^0(-x_e^0-y_e^0)+i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})}\rho(m)}{(p_e^0)^2 + \mathbf{p}^2 + m^2}\psi(y_e) dm = \\
\int \underbrace{\xi^*(x_e)\frac{d^4x_e}{(2\pi)^{3/2}}e^{-E_m(\mathbf{p})x_e^0 + i\mathbf{p}\cdot\mathbf{x}}}_{\tilde{\xi}^*(\mathbf{p},m)} \frac{\rho(m)dm d\mathbf{p}}{2E_m(\mathbf{p})} \underbrace{e^{-E_m(\mathbf{p})y_e^0 - i\mathbf{p}\cdot\mathbf{y}}\frac{d^4y_e}{(2\pi)^{3/2}}\psi(y_e)}_{\tilde{\psi}(\mathbf{p},m)}
\end{gathered} \tag{18}$$

The Euclidean time support condition of the wave functions $\psi(y_e)$ and $\xi^*(x_e)$ allows the p_0 integral to be computed by contour integration.

The subsystem mass operator can be seen to be the Euclidean 4-Laplacian by integration by parts in (18):

$$\int \frac{\rho(m)}{2E_m(\mathbf{p})} e^{-E_m(\mathbf{p})y_e^0 - i\mathbf{p}\cdot\mathbf{y}} \frac{d^4y_e}{(2\pi)^{3/2}} \nabla_{ye}^2 \psi(y_e) = \tag{19}$$

$$\int \frac{\rho(m)}{2E_m(\mathbf{p})} e^{-E_m(\mathbf{p})y_e^0 - i\mathbf{p}\cdot\mathbf{y}} \frac{d^4y_e}{(2\pi)^{3/2}} m^2 \psi(y_e) \tag{20}$$

The problem is to find subsystem point mass eigenstates $\psi(y_e)$ and $\xi(x_e)$ with the mass of the asymptotic particle that are also consistent with the Euclidean time-support condition.

The spectrum of the subsystem mass operator is identical to the support of the Lehmann weight, $\rho(m)$. The Lehmann weight is assumed to have the form

$$\rho(m) = \sum z_i \delta(m - m_i) + \rho_{ac}(m) \tag{21}$$

where the support of the continuous part, $\rho_{ac}(m)$ of $\rho(m)$ is a half line that starts at the sum of the masses of the lightest possible intermediate particles.

In order to select the desired asymptotic particle it is necessary to choose wave functions $\psi(x_e)$ that only get contributions from the Lehmann weight corresponding to mass of the asymptotic particle. This corresponds to one of the delta functions in (21).

One way to do this, motivated by the method used by Haag and Ruelle [12][13][11], is to multiply the wave function by a function $h(m^2)$ that is 1 when m^2 is the square of the mass of the asymptotic particle and 0 on the rest of the Lehmann weight, $\rho(m^2)$. Since the square of the cluster mass operator is the Euclidean Laplacian, a wave function of the form

$$\psi(x_e) = h(\nabla^2) g(x_e) \tag{22}$$

has the desired properties provided it satisfies the support condition. Here $g(x_e)$ is another function satisfying the support condition. The simplest choice of $g(x_e)$ satisfying the support condition is a function of the form

$$g(x_e) = \delta(x_e^0 - \tau)\tilde{g}(\mathbf{x}). \tag{23}$$

It is interesting that in spite of presence of the delta function that enforces the support condition, this is a normalizable vector with respect to the scalar product (3).

The question is whether the support conditions on $h(m^2)$ and $h(\nabla^2)g(x_e)$ are compatible. The concern is that on one hand an infinite series of derivatives, like a translation operator, can change support conditions. On the other hand $h(m^2)$ cannot even be be analytic if the Lehmann weight has any continuous spectrum because $h(m^2)$ it must vanish on the continuous support of the Lehmann weight.

In the unphysical situation, where the support of the Lehmann weight consists of a finite number of discrete points, a finite-degree polynomial $h(\nabla^2)$ can be constructed that selects the desired mass and automatically preserves the support condition. Also, in the case where the support of the continuous portion of the Lehmann weight is compact, the Weierstrass approximation theorem implies that $h(\nabla^2)$ can be uniformly approximated by a polynomial. While both cases are unphysical, both preserve the support condition of the wave function.

This leads to the question of whether it is possible to make a polynomial approximation to $h(m^2)$ that converges to any continuous function of the half line. It turns out that there is a sufficient condition called the Carleman condition that provides a suitable sufficient (but not necessary) condition.

The concern is that applying $h(\nabla^2)$ to $g(x_e)$ does not automatically preserve the positive Euclidean-time support condition of $g(x_e)$.

The Carleman condition [14][15]gives a sufficient condition

$$\sum_{n=1}^{\infty} \gamma_n^{-\frac{1}{2n}} > \infty \tag{24}$$

on an infinite collection of moments γ_n on the half line

$$\gamma_n = \int_0^\infty w(x)dx x^n \tag{25}$$

to uniquely solve for the measure, $w(x)dx$.

When this condition is satisfied, multiplication by x has a unique-self adjoint extension on the Hilbert space with weight $w(x)$ and the orthogonal polynmials with respect to the weight $w(x)$ are complete.

For the Euclidean two-point function the relevant moments are

$$\gamma_n := (\psi, T_e G_{2e} (\nabla^2)^n \psi) = \int_0^\infty w(m) m^{2n}. \tag{26}$$

For the scattering application we actually need the completeness of polynomials in m^2 rather than m which is why the n^{th} moment involves m^{2n}.

For the two-point example above the relevant moments are

$$\gamma_n := \int_0^\infty \frac{e^{-\sqrt{m^2+\mathbf{p}^2}\tau}}{2\sqrt{m^2+\mathbf{p}^2}} \rho(m) m^{2n} dm \tag{27}$$

In order to estimate these moments we assume (see theorem 6.2.4 in [9]) that the continuous part of the Lehmann weight is polynomially bounded. In this case it is sufficient to replace the moments by

$$\gamma_n \to \gamma'_n = \int_0^\infty \frac{e^{-\sqrt{m^2+\mathbf{p}^2}\tau}}{2\sqrt{m^2+\mathbf{p}^2}} m^{2n+k} dm. \tag{28}$$

where k is a constant (the polynomial bound). With some algebra it is possible to prove the inequality

$$\sum_{n=1}^\infty \gamma_n'^{-\frac{1}{2n}} > \sum_{n=0}^\infty \frac{c}{2n+k-2} > \infty \tag{29}$$

which is sufficient to show that $h(\nabla^2)$ can be approximated by a polynomial, thus ensuring that it is possible to construct point-spectrum mass eigenstates that satisfy the Euclidean time support condition.

Returning to equation (17) it is now possible to see how the condition above eliminates the disconnected terms in this equation. The first step is to replace the wave packets $\chi(x_e)$ by $h_n(\nabla^2)g(x_e)$, where h_n is a polynomial approximation to $h(m^2)$. Integrating the Laplacians by parts means that they can be replaced by $h_n(m^2)$, where m^2 is integrated over the support of the Lehmann weight. The completeness of the polynomials in m^2 means that we can make the contribution from the continuous part of the Lehmann weight as small as desired. Integrating the Euclidean time derivatives by parts in the two-point functions leads to replacements of the form $\frac{\partial}{\partial x^0} \to \omega_m(\mathbf{p})$. When this is integrated over m the only term that is picked up is the $m = m_i$ selected by $h(m^2)$. This term cancels the energy factors that come form ΦH_f. What remains is just the contribution from the connected four-point green function, which is expected to vanish at t gets large.

This shows that it is possible to compute S-matrix elements in the Euclidean representation. As a practical matter e^{iHt} is not easy to compute in the Euclidean representation. There are a number of approaches that can be used given a representation for H. One that we have tested makes use of the fact that given a set of reflection positive Green functions, matrix elements of polynomials in $(e^{-\beta H})$ are easy to calculate using the inner product (3).

To take advantage of this note that the invariance principle [16][17] implies

$$\lim_{t\to\pm\infty} e^{iHt}\Phi e^{-iH_f t}|\chi\rangle = \lim_{n\to\pm\infty} e^{-ine^{-\beta H}}\Phi e^{ine^{-\beta H_f}}|\chi\rangle \tag{30}$$

Because the spectrum of $e^{-\beta H}$ is compact, for any fixed n, $e^{-ine^{-\beta H}}$ can be uniformly approximated by a polynomial in $e^{-\beta H}$. Matrix elements of $e^{-\beta m H}$ simply translate the argument of the Euclidean wave functions by $-\beta m$. In principle β is any positive number; in practice it sets the energy scale of the

problem. We have demonstrated in a solvable model that these methods can be applied to calculate sharp momentum cross sections over a wide range of energies [18]. There are many other possibilities given that the reconstruction theorem provides all of the elements necessary to formulate the scattering problem.

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