Multi-scale methods in quantum field theory
PRD 95,094501 (2017)

W. N. Polyzou, Tracie Michlin and Fatih Bulut

The University of Iowa (WP and TM),Inönü University (FB)

Outline

- Mutli-resolution (wavelet) basis.
- Exact multi-resolution decomposition of quantum fields.
- Decoupling degrees of freedom by scale - construction of coarse-scale effective theories.
- Results, observations and concluding remarks.

Strategy

- **Construct multi-resolution basis.**
- **Exactly decompose fields into discrete degrees of freedom with different resolutions.**
- **Perform resolution and volume truncations on fields**
- **Use truncated fields to construct a truncated Hamiltonian.**
- **Block diagonalize the truncated Hamiltonian by resolution.**
- **Evolve fields with truncated Hamiltonian.**
- **Construct correlation functions.**

Basis functions (Daubechies wavelets)

Generated by the fixed point, $s(x)$, of a renormalization group equation

$$s(x) = \underbrace{D\,(\underbrace{\sum_{l=0}^{2K-1} h_l T^l s(x)}_{\text{block average}})}_{\text{rescale}}.$$

Unitary translations

$$Ts(x) = s(x-1)$$

Unitary scale transformations

$$Ds(x) = \sqrt{2}s(2x).$$

Scale fixing

$$\int dx s(x) = 1$$

Weights h_l (constants) determine properties of basis

Table: **Filter weights for Daubechies K=3 Wavelets**

h_0	$(1+\sqrt{10}+\sqrt{5+2\sqrt{10}})/16\sqrt{2}$
h_1	$(5+\sqrt{10}+3\sqrt{5+2\sqrt{10}})/16\sqrt{2}$
h_2	$(10-2\sqrt{10}+2\sqrt{5+2\sqrt{10}})/16\sqrt{2}$
h_3	$(10-2\sqrt{10}-2\sqrt{5+2\sqrt{10}})/16\sqrt{2}$
h_4	$(5+\sqrt{10}-3\sqrt{5+2\sqrt{10}})/16\sqrt{2}$
h_5	$(1+\sqrt{10}-\sqrt{5+2\sqrt{10}})/16\sqrt{2}$

Determined uniquely (up to reflection) by requiring orthonormal basis functions that can pointwise represent polynomials of degree 2.

Scaling functions, $s_n^k(x)$

Rescale and translate fixed point, $s(x)$

$$s_n^k(x) := D^k T^n s(x) = 2^{k/2} s\left(2^k(x - 2^{-k}n)\right).$$

$\mathcal{S}_k :=$ **resolution** 2^{-k} **subspace,** $\{s_n^k(x)\}$ **basis**

$$\mathcal{S}_k := \{f(x)|f(x) = \sum_{n=-\infty}^{\infty} c_n s_n^k(x), \quad \sum_{n=-\infty}^{\infty} |c_n|^2 < \infty\}.$$

$$\mathcal{S}_k := D^k \mathcal{S}_0$$

$$\mathcal{S}_k \subset \mathcal{S}_{k+n} \qquad n \geq 0$$

$$\mathcal{S}_{k+1} = \mathcal{S}_k \oplus \mathcal{W}_k \qquad \mathcal{W}_k \neq \{\emptyset\}.$$

Multi-resolution decomposition of Hilbert space

$$\mathcal{S}_{k+1} = \mathcal{S}_k \oplus \mathcal{W}_k$$

$$L^2(\mathbb{R}) = \mathcal{S}_k \oplus \mathcal{W}_k \oplus \mathcal{W}_{k+1} \oplus \mathcal{W}_{k+2} \oplus \mathcal{W}_{k+3} \oplus \cdots =$$

$$\cdots \oplus \mathcal{W}_{k-2} \oplus \mathcal{W}_{k-1} \oplus \mathcal{W}_k \oplus \mathcal{W}_{k+1} \oplus \mathcal{W}_{k+2} \oplus \cdots$$

Wavelets (basis for $\mathcal{W}_k$)

$$w(x) := \sum_{l=0}^{2K-1} g_l T^l s(x) \qquad g_l = (-)^l h_{2K-1-l}$$

$$w_n^k(x) := D^k T^n w(x) = 2^{k/2} w\left(2^k(x - 2^{-k} n)\right).$$

Multi-resolution basis

$$\{\xi_m(x)\} = \{s_n^k(x)\}_{n=-\infty}^{\infty} \cup \{w_n^l(x)\}_{n=-\infty}^{\infty}{}_{l=k}^{\infty}$$

- **Complete, orthonormal**
- **Limited smoothness (increases with K)**
- **Compact support** $(s_n^k(x), w_n^k(x)) \subset 2^{-k}[n, (n+2K-1)]$
- **Partition of unity** $1 = 2^{k/2} \sum_m s_m^k(x)$
- $x^n = \sum_m c_m s_m^k(x)$ **(pointwise, $n < K$)**
- $s_n^k(x)$ **resolution** 2^{-k}**,** $w_n^l(x)$ **resolution** 2^{-l+1} **not in** 2^{-l}
- $\int dx x^m w_n^l(x) = 0, \qquad m < K$
- **Basis functions are fractal**

Multi-resolution decomposition of canonical fields

$$\xi_n(x) \in \{s_n^k(x), w_n^l(x)\} \qquad \xi_{\mathbf{n}}(\mathbf{x}) := \xi_{n_1}(x)\xi_{n_2}(y)\xi_{n_3}(z)$$

$$\Phi(\mathbf{x},t) = \sum_{\mathbf{n}} \Phi^k(\mathbf{n},t)\xi_{\mathbf{n}}(\mathbf{x}) \qquad \Phi^k(\mathbf{n},t) = \int d\mathbf{x}\xi_{\mathbf{n}}(\mathbf{x})\Phi(\mathbf{x},t)$$

$$\Pi(\mathbf{x},t) = \sum_{\mathbf{n}} \Pi^k(\mathbf{n},t)\xi_{\mathbf{n}}(\mathbf{x}) \qquad \Pi^k(\mathbf{n},t) = \int d\mathbf{x}\xi_{\mathbf{n}}(\mathbf{x})\Pi(\mathbf{x},t)$$

$$[\Phi(\mathbf{n},t), \Pi(\mathbf{m},t)] = i\delta_{\mathbf{n},\mathbf{m}}$$

$$[\Phi(\mathbf{n},t), \Phi(\mathbf{m},t)] = [\Pi(\mathbf{n},t), \Pi(\mathbf{m},t)] = 0,$$

- **Expansion is exact.**
- **Operator valued distributions replaced by infinite sums of well-defined discrete field operators.**
- **Products of discrete fields are well defined.**
- **Discrete fields satisfy canonical commutation relations.**
- **Decomposes the field into local observables by resolution.**
- **Natural resolution (limit l) and volume (limit n) truncations.**

Resolution and volume truncated fields

$$\Phi_T(\mathbf{x},t) = \sum_{\mathbf{n}\in\mathcal{I}} \Phi^k(\mathbf{n},t)\xi_{\mathbf{n}}(\mathbf{x})$$

$$\Pi_T(\mathbf{x},t) = \sum_{\mathbf{n}\in\mathcal{I}} \Pi^k(\mathbf{n},t)\xi_{\mathbf{n}}(\mathbf{x})$$

- **Restrict index set to finite subset, $\mathcal{I}$ (finite volume, finite resolution).**
- **Truncated fields are still differentiable functions of x**

$$\mathcal{I} = \{k \leq l \leq l_{max}, -n_{l,max} \leq n_l \leq n_{l,max}\}$$

Hamiltonians (exact multi-scale representation)

Mass and kinetic terms

$$\mu^2 \int d\mathbf{x}\Phi^2(\mathbf{x},t) = \mu^2 \sum \Phi(\mathbf{m},t)^2 \qquad \int d\mathbf{x}\Pi^2(\mathbf{x},t) = \sum \Pi(\mathbf{m},t)^2$$

Derivative terms

$$\int d\mathbf{x}\nabla\Phi(\mathbf{x},t)\cdot\nabla\Phi(\mathbf{x},t) = \sum \Phi(\mathbf{m},t) D_{\mathbf{mn}}\Phi(\mathbf{n},t)$$

$$D_{\mathbf{mn}} := \int d\mathbf{x}\nabla\xi_{\mathbf{m}}(\mathbf{x})\cdot\nabla\xi_{\mathbf{n}}(\mathbf{x})$$

Local interactions

$$\int d\mathbf{x}\Phi^N(\mathbf{x},t) = \sum \Gamma_{\mathbf{n}_1\cdots\mathbf{n}_N}\Phi(\mathbf{n}_1,t)\cdots\Phi(\mathbf{n}_N,t)$$

$$\Gamma_{\mathbf{n}_1\cdots\mathbf{n}_N} = \int d\mathbf{x}\xi_{\mathbf{n}_1}(\mathbf{x})\cdots\xi_{\mathbf{n}_N}(\mathbf{x})$$

Terms almost local due to support conditions

All coefficients can be computed analytically using the renormalization group equation, scale fixing condition, and weight coefficients, h_l. For example the non-zero matrix elements

$$D^0_{mn} := \int \frac{ds^0_m(x)}{dx}\frac{ds^0_n(x)}{dx}dx$$

have the following exact rational values

$$D^0_{s;40} = D^0_{s;-40} = -3/560$$

$$D^0_{s;30} = D^0_{s;-30} = -4/35$$

$$D^0_{s;20} = D^0_{s;-20} = 92/105$$

$$D^0_{s;10} = D^0_{s;-10} = -356/105$$

$$D^0_{s;00} = 295/56.$$

Truncated Hamiltonian
replace fields by truncated fields

Dynamics

$$\dot{\Phi}_{\mathbf{n}}(t) = i[H_T, \Phi_{\mathbf{n}}(t)] \qquad \dot{\Pi}_{\mathbf{n}}(t) = i[H_T, \Pi_{\mathbf{n}}(t)]$$

Initial conditions

$$[\Phi(\mathbf{n}, 0), \Pi(\mathbf{m}, 0)] = i\delta_{\mathbf{n},\mathbf{m}}$$

$$[\Phi(\mathbf{n}, 0), \Phi(\mathbf{m}, 0)] = [\Pi(\mathbf{n}, 0), \Pi(\mathbf{m}, 0)] = 0,$$

Hilbert space - the truncated problem well-defined in the free field Fock space

Advantage of wavelet basis

Scaling properties of coefficients (dim $= 1+1$)

$$\mathcal{S}_{k+m} = \mathcal{S}_k \oplus \mathcal{W}_k \oplus \cdots \oplus \mathcal{W}_{k+m}$$

$$\sum D^k_{mn} \Phi^k_m \Phi^k_n = 2^{2k} \sum D^0_{mn} \Phi^k_m \Phi^k_n$$

and

$$\sum \Gamma^k_{n+1 \cdots n_N} \Phi^k_{n_1} \cdots \Phi^k_{n_N} = 2^{3k(\frac{n}{2}-1)} \sum \Gamma^0_{n+1 \cdots n_N} \Phi^k_{n_1} \cdots \Phi^k_{n_N}$$

$$\Downarrow$$

Operator renormalization group equation relates H truncated at different scales (rescale + canonical transformation)

$$H^k(\Phi^k, \Pi^k, m^{2k}, \gamma_N^k) = 2^k H^0(\Phi^0, \Pi^0, 2^{-2k} m^0, 2^{k(N-4)} \gamma_N^0)$$

Canonical transformation

$$\Phi^k = \eta \Phi^0 \qquad \Pi^k = \eta^{-1} \Pi^0 \qquad \eta = 2^{-k/2}$$

can be realized as a unitary transformation

The problem of constructing a coarse-scale effective theory that includes the effects of eliminated fine-scale degrees of freedom is studied first using a free field theory in 1+1 dimension

$$H = \frac{1}{2}\int (\Pi(x,0)\Pi(x,0) + \nabla\Phi(x,0)\cdot\nabla\Phi(x,0) + \mu^2\Phi(x,0)\Phi(x,0))dx,$$

Spatial derivatives give non-trivial scale coupling

$$H_{Ts}^{k+n} = H_{Ts}^{k} + H_{Tw} + H_{Tsw}$$

- **Resolution $1/2^{n+k}$ Hamiltonian = resolution $1/2^k$ Hamiltonian + fine scale corrections + scale coupling terms**
- **Goal - block diagonalize truncated Hamiltonian by resolution**

$$H_T^k := \frac{1}{2}(\sum_n \Pi^k(s,n,0)\Pi^k(s,n,0) + \sum_{mn} \Phi^k(s,m,0) D^k_{s;mn} \Phi^k(s,n,0)$$

$$+\mu^2 \sum_n \Phi^k(s,n,0)\Phi^k(s,n,0)),$$

$$H_w := \frac{1}{2}(\sum_{n,l} \Pi^l(w,n,0)\Pi^l(w,n,0) + \sum_{m,l,n,j} \Phi^l(w,m,0) D^{lj}_{w;mn} \Phi^j(w,n,0)$$

$$+\mu^2 \sum_{l,n} \Phi^l(w,n,0)\Phi^l(w,n,0)),$$

$$H_{sw} := \frac{1}{2} \sum_{m,l,n} \Phi^l(w,m,0) D^{lk}_{sw;mn} \Phi^k(s,n,0).$$

Use similarity renormalization group evolution to decouple scales (eliminate $D^{lk}_{sw;mn}$)

Generate unitarily equivalent Hamiltonians

$$H(\lambda) = U(\lambda)H(0)U^{\dagger}(\lambda)$$

$$\frac{dU(\lambda)}{d\lambda} = \frac{dU(\lambda)}{d\lambda}U^{\dagger}(\lambda)U(\lambda) = K(\lambda)U(\lambda)$$

$$K(\lambda) = \frac{dU(\lambda)}{d\lambda}U^{\dagger}(\lambda) = -K^{\dagger}(\lambda)$$

Use a generator $K(\lambda)$ of the form

$$K(\lambda) = [G(\lambda), H(\lambda)]$$

$$\frac{dH(\lambda)}{d\lambda} = [K(\lambda), H(\lambda)] = [[G(\lambda), H(\lambda)], H(\lambda)] = [H(\lambda), [H(\lambda), G(\lambda)]]$$

Initial conditions

$$H(0) = H_T^{k+n} \qquad G(0) = H_{Tsw}$$

$$G(\lambda) = \textbf{part of } H(\lambda) \quad \textbf{that couples scales}$$

$$H(\lambda) = H_{block}(\lambda) + G(\lambda)$$

Equations separate (free field case)

$$\frac{dH_{block}(\lambda)}{d\lambda} = [G(\lambda), [G(\lambda), H_{block}(\lambda)]]$$

$$\frac{dG(\lambda)}{d\lambda} = -[H_{block}(\lambda), [H_{block}(\lambda), G(\lambda)]]$$

Expand first equation in basis of eigenstates of $H_C := G(\lambda)$, second in eigenstates of $H_B := H_{block}(\lambda)$

$$\frac{dH_{Bmn}(\lambda)}{d\lambda} = (e_{cm}(\lambda) - e_{cn}(\lambda))^2 H_{Bmn}(\lambda)$$

and

$$\frac{dH_{Cmn}(\lambda)}{d\lambda} = -(e_{bm}(\lambda) - e_{bn}(\lambda))^2 H_{Cmn}(\lambda).$$

Integrating

$$H_{Bmn}(\lambda) = e^{\int_0^\lambda (e_{cm}(\lambda') - e_{cn}(\lambda'))^2 d\lambda'} H_{Bmn}(0)$$

$$H_{Cmn}(\lambda) = e^{-\int_0^\lambda (e_{bm}(\lambda') - e_{bn}(\lambda'))^2 d\lambda'} H_{Cmn}(0).$$

- **Formal solution shows coupling terms exponentially suppressed.**

- **Evolves to decoupled effective Hamiltonians involving degrees of freedom on different scales.**

- **Can stall when eigenvalues get close or cross.**

- **The choice of generator is specific to the free field case.**

- **Evolution easily computed - Hamiltonian has discrete canonical fields with constant coefficients.**

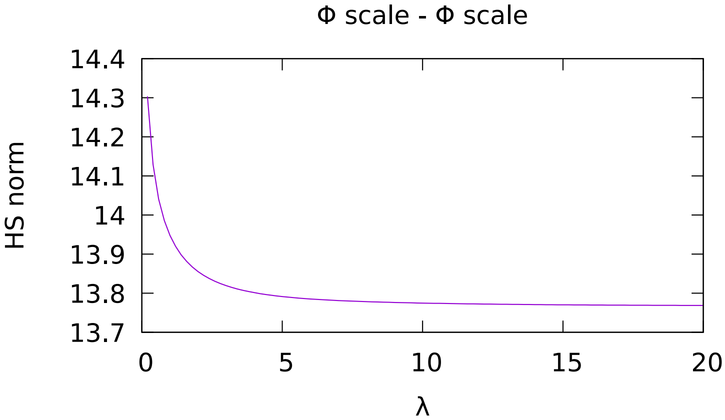
- **Free field case allows a detailed analysis of the evolution of scales.**

Test case: 16 scaling functions, 16 wavelets, BC - vanish at boundary

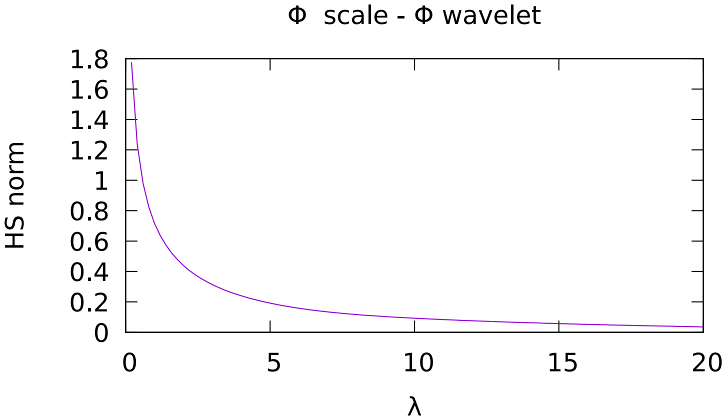
$H(\lambda)$ **has 16 quadratic types obtained by taking products of** $\Phi_s(n,0)$**,** $\Pi_s(m,0)$**,** $\Phi_w(n,0)$**, and** $\Pi_w(m,0)$**.**

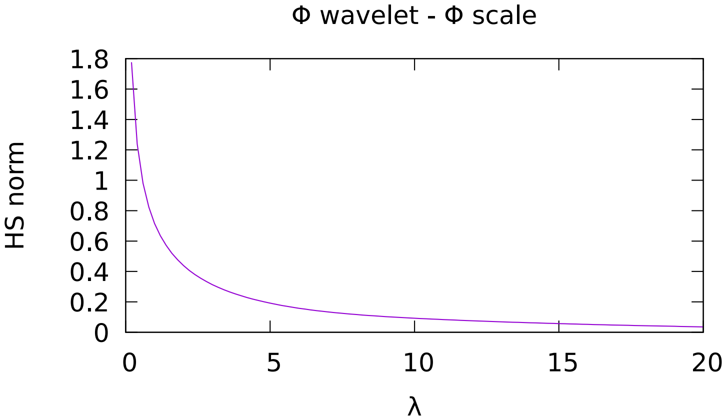
The λ **evolution should drive the coefficients of the** $\Phi_s(n,0)\Phi_w(m,0),\cdots$ **to** 0**.**

Plots show the Hilbert-Schmidt norms of the coefficient matrices of each type of operator as a function of λ**.**

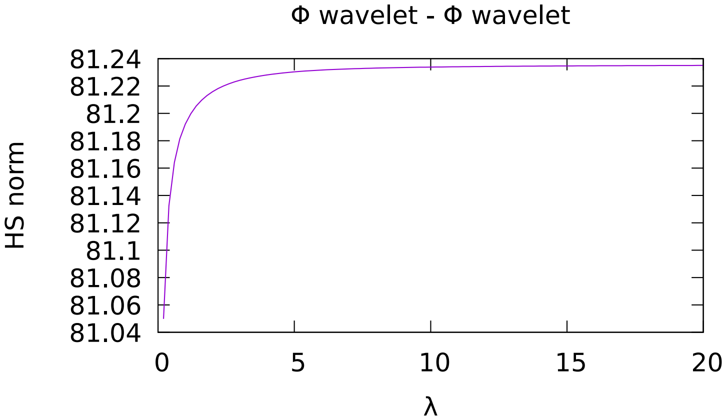


Φ scale - Φ scale
HS norm
14.4
14.3
14.2
14.1
14
13.9
13.8
13.7
0
5
10
15
20
λ


Φ scale - Φ wavelet
HS norm
λ
1.8
1.6
1.4
1.2
1
0.8
0.6
0.4
0.2
0
0
5
10
15
20



Φ wavelet - Φ scale
HS norm
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
0
5
10
15
20
λ


Φ wavelet - Φ wavelet
81.24
81.22
81.2
81.18
81.16
81.14
81.12
81.1
81.08
81.06
81.04
HS norm
0
5
10
15
20
λ

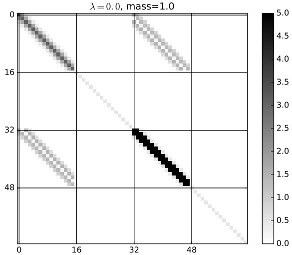


Figure: Full matrix, λ=0, mass=1

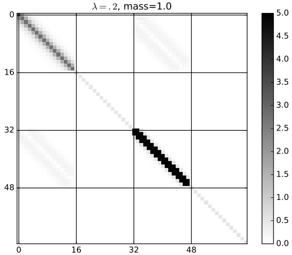


Figure: Full matrix, λ=0.2, mass=1

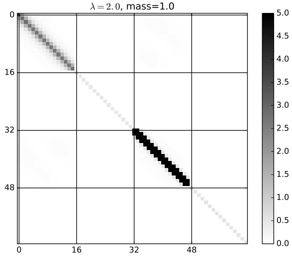


Figure: Full matrix, λ=2.0, mass=1

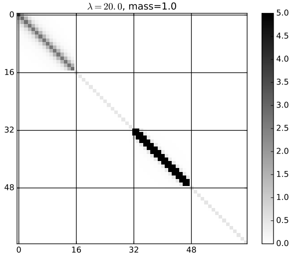


Figure: Full matrix, λ=20.0, mass=1

Observations

- **Coefficients exhibit expected behavior.**
- **The decay of the coefficients of the scale coupling terms is initially fast, but slows significantly.**
- **Truncated free fields are equivalent to coupled oscillators; how does the SRG evolution distribute the normal modes among the blocks?**

To get a more detailed understanding of the effects of the λ flow note that the truncated Hamiltonian has the form

$$H_T = \frac{1}{2}[(\Pi^s, \Pi^w) \begin{pmatrix} I_s & 0 \\ 0 & I_w \end{pmatrix} \begin{pmatrix} \Pi^s \\ \Pi^w \end{pmatrix} +$$

$$(\Phi^s, \Phi^w) \begin{pmatrix} \mu^2 I + D_s & D_{sw} \\ D_{ws} & \mu^2 I + D_w \end{pmatrix} \begin{pmatrix} \Phi^s \\ \Phi^w \end{pmatrix}]$$

$$M := \begin{pmatrix} \mu^2 I + D_s & D_{sw} \\ D_{ws} & \mu^2 I + D_w \end{pmatrix}$$

$$O^t M O = \begin{pmatrix} m^s & 0 \\ 0 & m^w \end{pmatrix}$$

where m^s and m^w are diagonal matrices consisting of eigenvalues of the matrix M.

Transformed discrete fields are related by the canonical transformation

$$\begin{pmatrix} \Phi'^s \\ \Phi'^w \end{pmatrix} := O^t \begin{pmatrix} \Phi^s \\ \Phi^w \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \Pi'^s \\ \Pi'^w \end{pmatrix} := O^t \begin{pmatrix} \Pi^s \\ \Pi^w \end{pmatrix}.$$

$$H' = UHU^\dagger = \frac{1}{2}[(\Pi'^s, \Pi'^w)\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}\begin{pmatrix} \Pi'^s \\ \Pi'^w \end{pmatrix} +$$

$$(\Phi'^s, \Phi'^w)\begin{pmatrix} m^s & 0 \\ 0 & m^w \end{pmatrix}\begin{pmatrix} \Phi'^s \\ \Phi'^w \end{pmatrix}]$$

- **Transformed Hamiltonian is the Hamiltonian for uncoupled harmonic oscillators with frequencies** $\sqrt{m_i}$
- $\Pi\Pi$ **coefficients in flow-evolved Hamiltonian are approximately** $(1/2)(I_s \oplus I_w)$ **so** $\sqrt{}$ **of eigenvalues of** M **correspond to normal mode frequencies.**
- **A general** O **will separate the eigenvalues of** M **into two distinct groups - there is no general relation between normal mode frequencies and scale.**

Table: Normal mode frequencies

$\lambda = 20, \mu = 1$	truncated	exact 1:16	exact 17:32
1.037e+00	1.037e+00	1.041e+00	1.665e+01
1.145e+00	1.146e+00	1.153e+00	1.925e+01
1.326e+00	1.333e+00	1.340e+00	2.208e+01
1.583e+00	1.609e+00	1.604e+00	2.512e+01
1.919e+00	1.995e+00	1.947e+00	2.834e+01
2.341e+00	2.525e+00	2.373e+00	3.167e+01
2.861e+00	3.236e+00	2.890e+00	3.507e+01
3.493e+00	4.161e+00	3.508e+00	3.846e+01
4.263e+00	5.317e+00	4.243e+00	4.178e+01
5.201e+00	6.689e+00	5.112e+00	4.495e+01
6.346e+00	8.232e+00	6.134e+00	4.789e+01
7.722e+00	9.859e+00	7.332e+00	5.053e+01
9.309e+00	1.145e+01	8.729e+00	5.279e+01
1.102e+01	1.289e+01	1.034e+01	5.462e+01
1.274e+01	1.403e+01	1.219e+01	5.597e+01
1.435e+01	1.476e+01	1.429e+01	5.679e+01

The table shows that the SRG flow equation puts the lowest normal modes in coarse scale Hamiltonian and the highest normal mode frequencies in the fine scale Hamiltonian

Conclusions

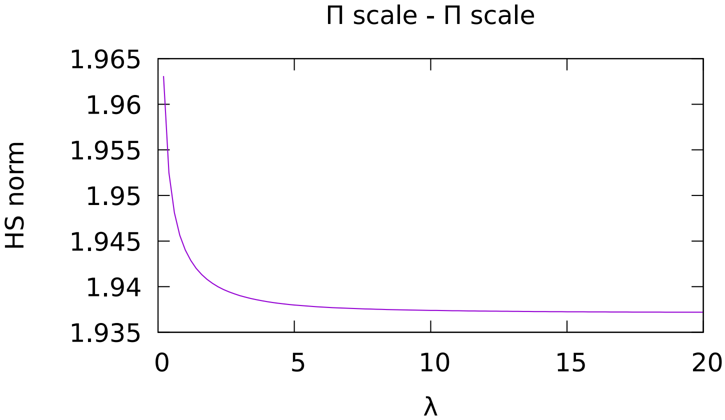
1. **Flow equation methods with a suitable generator can be used to construct an effective field theory with coarse scale degrees of freedom.**
2. **The generator used separates both energy and distance scales.**
3. **Increasing the truncated volume generated new low frequency modes, while increasing the resolution increased the separation between modes. The mass sets a lower bound on the normal mode frequencies.**
4. **The flow equation also exhibited convergence for mass 0.**

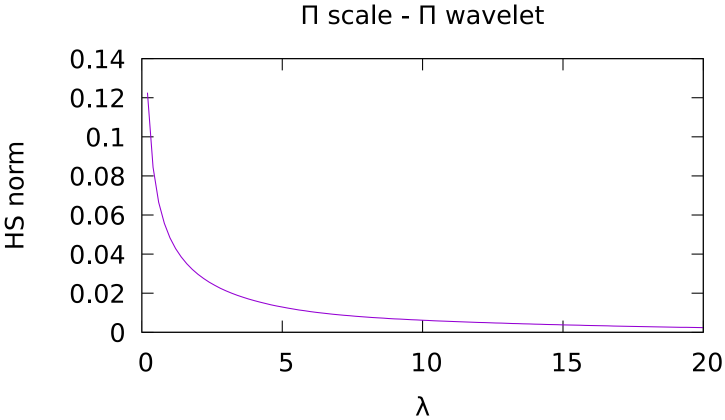
Summary

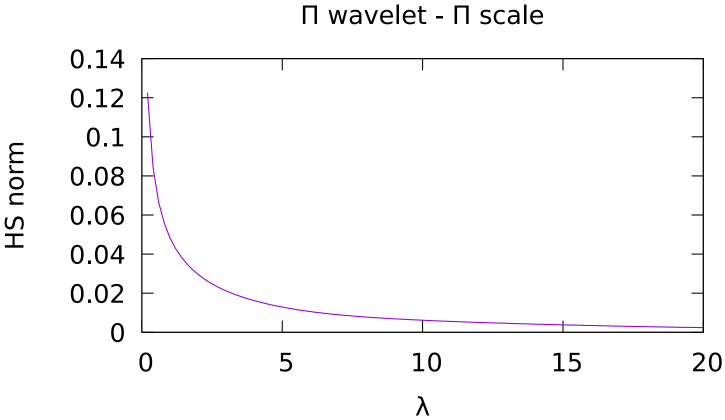
5. For this problem, the flow equation was successfully applied directly to the Hamiltonian, without projecting on a subspace.
6. We found the that flow equation could be integrated using the Euler method, but perturbation theory failed to converge.
7. The final evolved Hamiltonian was approximately local.
8. The spectral properties suggest the in order to approach the continuous spectrum of the exact theory, the volume and resolution truncations need to be removed together.
9. In our test the coefficients of the coupling terms initially fell off quickly, but the rate of fall off slowed down significantly as the flow parameter increased. The method reduced the coupling coefficients by a factors of about 100 for a modest value of the flow parameter.
10. The convergence of the flow equation will slow down as the separation between normal modes decreases.

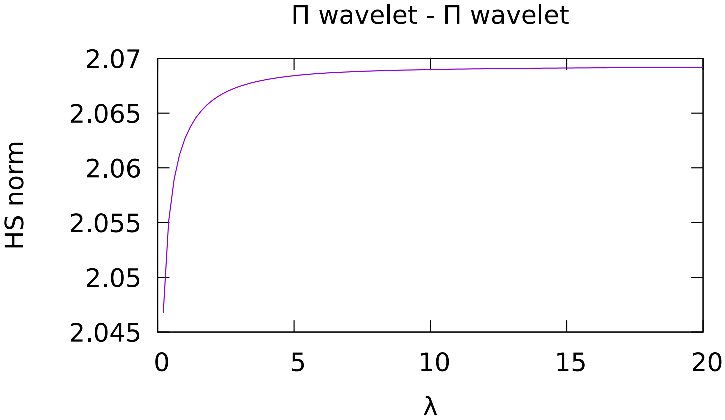
Outlook

1. **It has been shown that the truncated correlation functions converge to the exact free field Wightman functions as the resolution is increased (Singh-Brennan Arxiv 1606:050686).**
2. **Basis functions have only a finite number of derivatives - but the Fourier transform of the test functions are entire.**
3. **Solution of Heisenberg equations of motion + ground state can be used to construct space-real-time correlation functions as functions of continuous variables**
4. **In interacting theories integrating the flow equation generates infinite numbers of operators.**
5. **Fields differentiable functions of x - mathematics discrete.**


Π scale - Π scale
HS norm
λ
1.965
1.96
1.955
1.95
1.945
1.94
1.935
0
5
10
15
20


Π scale - Π wavelet
HS norm
0.14
0.12
0.1
0.08
0.06
0.04
0.02
0
0
5
10
15
20
λ


Π wavelet - Π scale
HS norm
0.14
0.12
0.1
0.08
0.06
0.04
0.02
0
0
5
10
15
20
λ


Π wavelet - Π wavelet
HS norm
2.07
2.065
2.06
2.055
2.05
2.045
0
5
10
15
20
λ

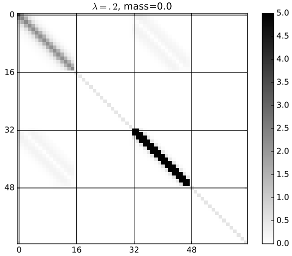


Figure: Full matrix, λ=0.2, mass=0

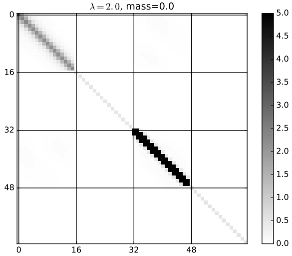


Figure: Full matrix, λ=2.0, mass=0

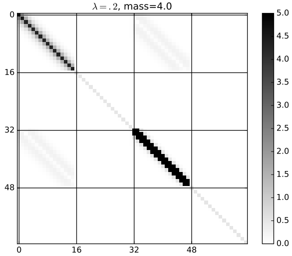


Figure: Full matrix, λ=0.2, mass=4

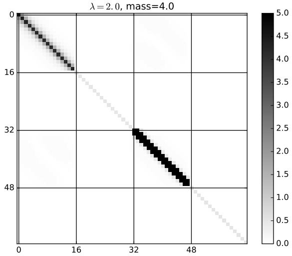


Figure: Full matrix, λ=2.0, mass=4