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Multi-scale methods in quantum field theory

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Abstract Daubechies wavelets are used to make an exact multi-scale decomposition of quantum fields. For reactions that involve a finite energy that take place in a finite volume, the number of relevant quantum mechanical degrees of freedom is finite. The wavelet decomposition has natural resolution and volume truncations that can be used to isolate the relevant degrees of freedom. The application of flow equation methods to construct effective theories that decouple coarse and fine scale degrees of freedom is examined.

1 Introduction

Daubechies wavelets [1][2] are used to decompose quantum fields into localized degrees of freedom on all distance scales. For reactions that involve a finite energy that take place in a finite volume, the number of relevant quantum mechanical degrees of freedom is finite. While a truncation to these degrees of freedom leads to a mathematically well-defined framework, a realistic treatment of the dynamics may still require a prohibitively large number of degrees of freedom for computation.

The Daubechies wavelet decomposition has natural resolution and volume truncations that can be used to identify the relevant degrees of freedom. It is

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desirable to construct an effective theory that eliminates degrees of freedom that are important for a realistic treatment of the dynamics, but not directly related to the scales of experimental interest. The application of flow equation methods to construct effective theories that decouple degrees of freedom with different resolutions is examined.

2 Daubechies basis

The basis functions are constructed from the fixed point, $s(x)$, of the renormalization group equation

$$s(x) = \underbrace{D\,\underbrace{(\sum_{l=0}^{2K-1} h_l T^l s(x))}_{\text{block average}}}_{\text{rescale}} \tag{1}$$

where T and D are unitary translation and dilatation operators

$$(Tf)(x) = f(x-1) \qquad (Df)(x) = \sqrt{2} f(2x).$$

The weights h_l are real numbers determined by the conditions that the functions $s_n(x) := s(x-n)$ are orthonormal and locally finite linear combinations of these functions can pointwise represent polynomials of degree $m < K$. In this work $K = 3$ is chosen. For this choice $s(x)$ has one continuous derivative and has compact support on $[0, 5]$. The condition $\int s(x)dx = 1$ is imposed on the fixed point $s(x)$ to fix an initial scale. Functions of finer resolution are constructed by applying powers of D to the $s_n(x)$:

$$s_n^k(x) := D^k T^n s(x) = 2^{k/2} s\left(2^k (x - 2^{-k} n)\right).$$

These functions have support on intervals of width $(2K-1) \times 2^{-k}$, They are orthonormal and span a resolution 2^{-k} linear subspace of $L^2(\mathbb{R})$ defined by:

$$\mathcal{S}_k := \{f(x) | f(x) = \sum_{n=-\infty}^{\infty} c_n s_n^k(x), \quad \sum_{n=-\infty}^{\infty} |c_n|^2 < \infty\}.$$

Equation (1) implies that the subspaces for different resolutions are nested

$$\mathcal{S}_k \subset \mathcal{S}_{k+n} \qquad n \geq 0$$

and successive subspaces have non-trivial orthogonal complements

$$\mathcal{S}_{k+1} = \mathcal{S}_k \oplus \mathcal{W}_k \qquad \mathcal{W}_k \neq \{\emptyset\}.$$

Iterating this identity gives the exact decomposition of $L^2(\mathbb{R})$ into mutually orthogonal subspaces of different resolutions

$$L^2(\mathbb{R}) = \mathcal{S}_k \oplus \mathcal{W}_k \oplus \mathcal{W}_{k+1} \oplus \mathcal{W}_{k+2} \oplus \mathcal{W}_{k+3} \oplus \cdots . \tag{2}$$

An orthonormal basis for $\mathcal{W}_k$ is given by functions $\{w_n^k(x)\}_n$, that are constructed from the fixed point $s(x)$ by

$$w_n^k(x) := D^k T^n w(x) = 2^{k/2} w\left(2^k(x - 2^{-k}n)\right)$$

where

$$w(x) := \sum_{l=0}^{2K-1} g_l T^l s(x) \qquad \text{and} \qquad g_l = (-)^l h_{2K-1-l}.$$

It follows from (2) that for any fixed starting resolution, 2^{-k}, the functions

$$\xi_{\mathbf{n}}(\mathbf{x}) := \xi_{n_1}(x)\xi_{n_2}(y)\xi_{n_3}(z) \qquad \xi_n(x) \in \{s_n^k(x), w_n^l(x)\}_{n=-\infty;l\geq k}^{\infty} \tag{3}$$

are an orthonormal multi-resolution basis of $L^2(\mathbb{R}^3)$ of functions that have compact support and one continuous derivative.

3 Multiresolution decomposition of quantum fields

The basis (3) can be used to decomposition the quantum fields by resolution [3][4]:

$$\Phi(\mathbf{x}, t) = \sum_{\mathbf{n}} \Phi^k(\mathbf{n}, t)\xi_{\mathbf{n}}(\mathbf{x}) \qquad \Phi^k(\mathbf{n}, t) = \int d\mathbf{x}\xi_{\mathbf{n}}(\mathbf{x})\Phi(\mathbf{x}, t). \tag{4}$$

While the basis functions are not Schwartz functions, direct computation shows that the resulting discrete fields are well-defined operators in the free-field case.

Given a pair of canonical fields, $\Phi(\mathbf{x}, t)$ and $\Pi(\mathbf{x}, t)$, that satisfy canonical equal-time commutation relations, the corresponding discrete fields will satisfy discrete equal-time canonical commutation relations:

$$[\Phi(\mathbf{n}, t), \Pi(\mathbf{m}, t)] = i\delta_{\mathbf{n},\mathbf{m}}$$

with all other commutators vanishing. These expansions can be used in field-theoretic Hamiltonians. In this discrete representation the integral over the Hamiltonian density is replaced by an infinite sum. Terms in a typical Hamiltonian are replaced by the infinite sums

$$\mu^2 \int d\mathbf{x}\Phi^2(\mathbf{x}, 0) \to \mu^2 \sum_{\mathbf{m}} \Phi^2(\mathbf{m}, 0) \qquad \int d\mathbf{x}\Pi^2(\mathbf{x}, 0) \to \sum_{\mathbf{m}} \Pi^2(\mathbf{m}, 0)$$

$$\int d\mathbf{x}\boldsymbol{\nabla}\Phi(\mathbf{x}, 0) \cdot \boldsymbol{\nabla}\Phi(\mathbf{x}, 0) \to \sum_{\mathbf{mn}} D_{\mathbf{mn}}\Phi(\mathbf{m}, 0)\Phi(\mathbf{n}, 0)$$

$$\lambda \int d\mathbf{x}\Phi^n(\mathbf{x}, 0) \to \lambda \sum_{\mathbf{m}_1 \cdots \mathbf{m}_n} \Gamma_{\mathbf{m}_1 \cdots \mathbf{m}_n}\Phi(\mathbf{m}_1, 0) \cdots \Phi(\mathbf{n}_n, 0)$$

where

$$D_{\mathbf{mn}} := \int d\mathbf{x}\boldsymbol{\nabla}\xi_{\mathbf{m}}(\mathbf{x}) \cdot \boldsymbol{\nabla}\xi_{\mathbf{n}}(\mathbf{x}) \qquad \text{and} \qquad \Gamma_{\mathbf{n}_1 \cdots \mathbf{n}_N} = \int d\mathbf{x}\xi_{\mathbf{n}_1}(\mathbf{x}) \cdots \xi_{\mathbf{n}_N}(\mathbf{x}) \tag{5}$$

are constant coefficients. They can be computed exactly using the renormalization group equation and properties of the basis [5][3] .

4 Truncated quantum fields

There are natural volume and/or resolution truncations of quantum fields represented by a Daubechies expansion. Truncations are defined by retaining the terms in the expansion (4) whose basis functions have support in a given volume and is larger than a minimal support volume. The resulting truncated fields are still differentiable functions of x; they are expressed as a finite sum

$$\Phi_T(\mathbf{x},t) = \sum_{\mathbf{n}\in\mathcal{I}} \Phi^k(\mathbf{n},t)\xi_{\mathbf{n}}(\mathbf{x}). \tag{6}$$

A truncated Hamiltonian is defined by replacing the fields in the exact Hamiltonian by the truncated fields. This truncation limits the volume and finest resolution of the theory. The resulting truncated Hamiltonian has a finite number of degrees of freedom. The scaling properties of the integrals [3],

$$D^k_{mn} = 2^{2k}\sum D^0_{mn} \qquad \Gamma^k_{n_1\cdots n_N} = 2^{3k(\frac{n}{2}-1)}\Gamma^0_{n_1\cdots n_N}$$

which follow from (5), lead to an exact renormalization group equation for infinite volume truncated Hamiltonians with different resolutions

$$H^k(\Phi^k,\Pi^k,m^{2k},\gamma^k_N) = 2^k H^0(\Phi^0,\Pi^0,2^{-2k}m^0,2^{k(N-4)}\gamma^0_N)$$

where the fields in these two Hamiltonians ($1+1$ dimension) are related by the canonical transformation

$$\Phi^k = \eta\Phi^0 \qquad \Pi^k = \eta^{-1}\Pi^0 \qquad \eta = 2^{-k/2}.$$

The vacuum of the truncated theory can be constructed by decomposing the canonical fields into creation and annihilation parts

$$a_{\mathbf{n}} := \frac{1}{\sqrt{2}}(\alpha\Phi_{\mathbf{n}} + i\frac{1}{\alpha}\Pi_{\mathbf{n}})$$

where α is any constant; and then solving the coupled cluster equations [6]

$$e^{-S}H_T e^S|0\rangle_0 = 0 \qquad a_{\mathbf{n}}|0\rangle_0 = 0 \qquad S = \sum S^m_{\mathbf{n}_1\cdots\mathbf{n}_n} a^\dagger_{\mathbf{n}_1}\cdots a^\dagger_{\mathbf{n}_m}$$

for the coefficients $S^m_{n_1\cdot n_n}$. The vacuum of the truncated theory is $|0\rangle = Ne^S|0\rangle_0$ where N is a normalization constant.

The truncated fields are solutions of the Heisenberg equations

$$\dot{\Phi}_{\mathbf{n}}(t) = i[H_T,\Phi_{\mathbf{n}}(t)] \qquad \dot{\Pi}_{\mathbf{n}}(t) = i[H_T,\Pi_{\mathbf{n}}(t)]$$

with initial conditions

$$[\Phi(\mathbf{n},0),\Pi(\mathbf{m},0)] = i\delta_{\mathbf{n},\mathbf{m}} \qquad [\Phi(\mathbf{n},0),\Phi(\mathbf{m},0)] = [\Pi(\mathbf{n},0),\Pi(\mathbf{m},0)] = 0.$$

The solutions have the form (4). Correlation functions are defined as vacuum expectation values of products of the truncated fields. They are differentiable functions of the space-time coordinates.

5 Effective theories

A benefit of decomposing fields into degrees of freedom with different resolutions is that it becomes possible to formulate the problem of constructing an effective theory with physical-scale degrees of freedom by eliminating dynamically important small-scale degrees of freedom. This can be achieved by a block diagonalization of the Hamiltonian by resolution. While defining the exact Hamiltonian requires renormalization, a truncated Hamiltonian that has all of the degrees of freedom relevant to a given energy and volume is well-defined and should accurately describe the system of interest.

The problem of decoupling scales is examined for the case of a free field theory where the scale coupling appears in the constant matrices $D_{\mathbf{mn}}$. The advantage of the free field case is that a block diagonalization by scale can be performed at the operator level and nature of the degrees of freedom in each block can be examined. This test uses a truncated Hamiltonian for a free scalar field in 1+1 dimensions. The fields are truncated to include 16 scaling basis functions, $s_n(x)$, and 16 wavelet basis functions, $w_n(x)$. These functions represent degrees of freedom on scales that differ by factor of 2.

The truncated Hamiltonian has general form

$$H_T = \frac{1}{2}[(\Pi^s, \Pi^w) \begin{pmatrix} I_s & 0 \\ 0 & I_w \end{pmatrix} \begin{pmatrix} \Pi^s \\ \Pi^w \end{pmatrix} +$$

$$(\Phi^s, \Phi^w) \begin{pmatrix} \mu^2 I_s + D_s & D_{sw} \\ D_{ws} & \mu^2 I_w + D_w \end{pmatrix} \begin{pmatrix} \Phi^s \\ \Phi^w \end{pmatrix}]$$

where each block represents a 16×16 matrix and the non-diagonal elements D_{sw} and D_{ws} are the integrals of products of derivatives (5) of the basis functions. The upper block represents the coarse-scale degrees of freedom while the lower block represents the fine-scale degrees of freedom. The goal is to find a unitarily equivalent Hamiltonian, $H(\lambda)$ where the block coupling terms are absent or small.

This is tested [4] using flow-equation methods due to Wegner [7] [8]

$$H(\lambda) = U(\lambda)H(0)U^\dagger(\lambda) \qquad \frac{dU(\lambda)}{d\lambda} = \frac{dU(\lambda)}{d\lambda}U^\dagger(\lambda)U(\lambda) = K(\lambda)U(\lambda).$$

The generator $K(\lambda)$ is chosen to be

$$K(\lambda) = \frac{dU(\lambda)}{d\lambda}U^\dagger(\lambda) := [G(\lambda), H(\lambda)]$$

where $G(\lambda)$ is the block diagonal part of $H(\lambda)$. This leads to a differential equation directly for $H(\lambda)$ that becomes a set of coupled equations for the block diagonal, $G(\lambda)$ and coupling parts $H_c(\lambda)$ of $H(\lambda)$:

$$\frac{dG(\lambda)}{d\lambda} = [H_c(\lambda), [H_c(\lambda), G(\lambda)]]$$

$$\frac{dH_c(\lambda)}{d\lambda} = -[G(\lambda), [G(\lambda), H_c(\lambda)]].$$

These equations can be solved in the bases of eigenstates of $G(\lambda)$ and $H_c(\lambda)$ respectively

$$G_{mn}(\lambda) = e^{\int_0^\lambda (e_{cm}(\lambda') - e_{cn}(\lambda'))^2 d\lambda'} G_{mn}(0) \tag{7}$$

$$H_{cmn}(\lambda) = e^{-\int_0^\lambda (e_{bm}(\lambda') - e_{bn}(\lambda'))^2 d\lambda'} H_{cmn}(0). \tag{8}$$

The solutions indicate that the scale-coupling parts of the matrix are exponentially suppressed as the flow parameter, λ, is increased.

To test this the Hilbert-Schmidt norms of the coupling matrices as functions of the flow parameter λ are computed. Figures 1 and 2 compare the evolution of the Hilbert Schmidt norms for a representative set of scale coupling coefficients (figure 1) to fixed-scale coefficients (figure 2). Figure 1 shows that the scale coupling terms are driven to 0, but the rate of decrease falls off as the flow parameter is increased, while figure 2 shows that Hilbert-Schmidt norm of the fixed scale coefficients converges to a finite size.

Figures 3 and 4 give a more detailed picture of the evolution of each coefficient, initially (figure 3) and when $\lambda = 20$ (figure 4). The first quadrant shows the 16×16 matrix of coefficients for the Φ-Φ scaling function fields, the next diagonal quadrant shows the coefficients for the Π-Π scaling function fields. The third diagonal quadrant shows the coefficients for the Φ-Φ wavelet function fields and the fourth diagonal quadrant shows the coefficients for the Φ-Φ scaling function fields. The off-diagonal terms in figure 3 are the coefficients of the scale coupling terms. Figure 4 shows that they are driven to 0 for $\lambda = 20$.

One advantage of the free-field is that the truncated Hamiltonian is the Hamiltonian for 32 coupled harmonic oscillators. The block diagonalization will put oscillators with 16 normal modes into the coarse-scale block and the other 16 into the fine-scale block. At the level of the approximation (there is still a small coupling at $\lambda = 20$) the flow equation method used here to separate scales put the 16 lowest normal modes in the coarse scale block and the 16 highest normal modes in the fine scale block. This is both the expected and desired behavior.

While this example demonstrates how flow equation methods can be used to separate scales, a number of issues remain. This investigation showed that increasing the resolution adds higher energy normal modes, but does not reduce the separation between normal mode frequencies that one would expect in a continuum limit. In order to reduce this separation the truncated volume must be increased. This means the continuum limit requires that the resolution and volume limits be taken together. Another concern is that as the normal mode frequencies get closer, the separation of the eigenvalues in (7-8) will get smaller, resulting much slower convergence of the flow equation. A third concern is that for interacting theories each iteration of the flow equation will generate new many-body interactions. In order to control the growth in the number of generated interactions, the scaling properties of each generated interaction need to be investigated so irrelevant ones can be eliminated. Also, the operator form of the flow equation that worked for the free field, may not be possible with interactions; however it can be made to work using projection operators.

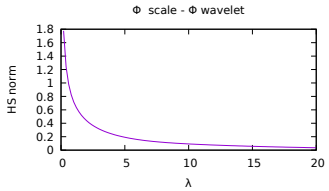

Fig. 1 ϕ scale-ϕ wavelet

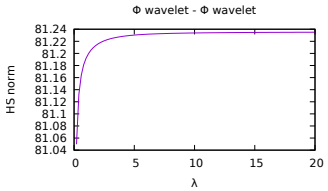

Fig. 2 ϕ wavelet-ϕ wavelet

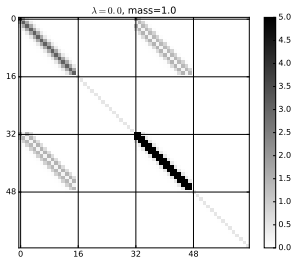

Fig. 3 Full matrix, λ=0, mass=1

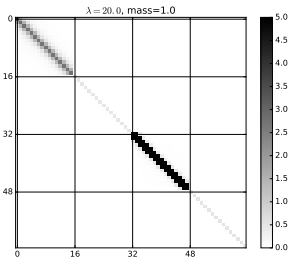

Fig. 4 Full matrix, λ=20, mass=1

The extension to 3-dimensions is straightforward. Convergence of the flow equation method was also established for 0 mass.

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