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Mini review of Poincaré invariant quantum theory

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Abstract We review the construction and applications of exactly Poincaré invariant quantum mechanical models of few-degree of freedom systems. We discuss the construction of dynamical representations of the Poincaré group on few-particle Hilbert spaces, the relation to quantum field theory, the formulation of cluster properties, and practical considerations related to the construction of realistic interactions and the solution of the dynamical equations. Selected applications illustrate the utility of this approach.

Keywords relativistic quantum mechanics · relativistic scattering theory

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1 Introduction

While there is strong evidence that QCD is the theory of the strong interactions, direct calculations of scattering observables in QCD with mathematically controlled errors are difficult at some important energy scales. These difficulties are particularly significant at the few-GeV scale, where perturbative methods are not applicable. This is an interesting energy scale because it is the scale where sensitivity to sub-nuclear degrees of freedom is expected to begin. Mathematical models that are motivated by QCD may provide useful insight into the dynamics at these energy scales.

Poincaré invariant quantum mechanics is one of a number of approaches that can be used to model systems of strongly interacting particles at the few GeV energy scale. At the simplest level it is quantum mechanics with an underlying Poincaré symmetry. While Poincaré invariant quantum mechanics can be treated as a phenomenology that is independent of QCD, it can also be related to QCD. Poincaré invariant quantum mechanics has proved to be useful in applications, but there are no textbook treatments of the subject.

Historically, Poincaré invariant quantum mechanics was first articulated by Wigner [1], who pointed out that a necessary and sufficient condition for a quantum theory to be relativistically invariant is the existence of a unitary ray representation of the Poincaré group on the quantum mechanical Hilbert space. Wigner's work did not have a significant impact on applications of quantum field theory, but it directly motivated attempts to provide an axiomatic [2–4] foundation for quantum field theory. These axioms provide a Hilbert space formulation of quantum field theory that can be directly related to Poincaré invariant quantum mechanics.

Dirac [5] studied the problem of constructing the Poincaré Lie algebra for systems of interacting particles. He observed that the presence of interactions in the Hamiltonian implied that at most a sub-algebra of the Poincaré Lie algebra could be free of interactions. He identified the three largest sub-algebras, and classified dynamical models according to which sub-algebra remained free of interactions. Dirac used the terms instant, point, and front-forms of dynamics to label the different kinematic sub-algebras. Bakamjian and Thomas [6] provided the first construction of the full Poincaré Lie algebra for a system of two interacting particles in Dirac's instant-form of the dynamics. Coester [7] generalized Bakamjian and Thomas' construction to systems of three interacting particles. His construction also led to a S matrix that satisfied space-like cluster properties. Sokolov [8] provided a complete construction of the Poincaré Lie Algebra for a system of N interacting particles in Dirac's point-form of the dynamics that was consistent with a stronger form of spacelike cluster properties, where the Poincaré generators satisfy cluster properties. This stronger form of cluster properties provides a simple relation between the few and many-body systems that is difficult to realize in theories satisfying only S-matrix cluster properties. Coester and Polyzou [9] provided the complete solution for systems of N-particles in all three of Dirac's forms of the dynamics satisfying the strong form of cluster properties. A more general construction based on only group representations, that has Dirac's form of dynamics as special cases, was given in [10, 11]. The subject was reviewed by Keister and Polyzou in [12].

There have been many applications of Poincaré invariant quantum mechanics in all three of Dirac's forms of dynamics. The earliest applications involved the study of electromagnetic probes on mesons, nucleons, and nuclei. Some of the relevant papers are [13–27]. The first three-nucleon bound state calculation using this framework was performed by Glöckle, Coester and Lee [28]. Calculations of the triton binding energy with realistic interactions have been performed recently [29]. Applications to nuclear reactions appear in [30–35] which include reactions with particle production [36].

This mini-review is limited to theories that are formulated by constructing exact unitary representations of the Poincaré group on few-particle Hilbert spaces. There are many other approaches to relativistic quantum mechanics that have been successfully applied at the few GeV scale. Each one emphasizes different desirable features of the full field theory, however when the number of degrees of freedom is limited, it is impossible to satisfy all of the axioms of the underlying field theory. Our preference for using Poincaré invariant quantum mechanics is based on three observations: (1) many computational methods successfully used in non-relativistic quantum mechanics can be directly applied in Poincaré invariant quantum mechanics, (2) the theories involve a finite number of degrees of freedom, allowing exact numerical calculations of model predictions, (3) the theories share most of the axiomatic properties of quantum field theory and there is a direct relation to the Hilbert space formulation of field theory. The fundamental property of the quantum field theory that is given up in order to have a theory of a finite number of degrees of freedom is microscopic locality. The justification for this choice is that microscopic locality is not an experimentally testable property since probing a system at arbitrarily short distance scales requires arbitrarily large energy transfers. In addition, Poincaré invariant quantum mechanics does not have a large enough algebra of observables to localize particles in arbitrarily small spacetime regions. One manifestation of this is the absence of a reasonable position operator [37] in relativistic quantum theories of a finite number of degrees of freedom.

In the next section we discuss the construction of representations of single-particle Hilbert spaces. In section 3 we discuss irreducible representations of the Poincaré group that act on the single-particle Hilbert spaces. In section 4 we construct a dynamical representation of the Poincaré group by adding interactions to the mass Casimir operator of a non-interacting irreducible representation constructed from tensor products of single particle representations. The strong and weak form of cluster properties are discussed in section 5. The formulation of the three-body problem is discussed in section 6. The relation to quantum field theory is discussed in section 7. Selected few-nucleon applications are discussed in section 8.

2 Particles, Hilbert spaces and irreducible representations

Experiments measure observables that describe the state of free particles by considering how the particles interact with classical electromagnetic fields. A complete experiment measures the linear momentum and spin state of each initial and final particle. There is a natural connection with these single-particle observables and irreducible representations of the Poincaré group. The Poincaré group has ten infinitesimal generators. These Hermitian operators include the Hamiltonian which generates time translations, the linear momentum operators which generate space translations, the angular momentum operators which generate rotations, and the rotationless boost generators which generate transformations that change the momentum of the particle. From these ten infinitesimal generators it is possible to construct two Casimir invariants, four independent commuting Hermitian observables and four conjugate operators. The Casimir invariants fix the mass and spin of the particle. Eigenvalues of the commuting observables label the states of the particle, and the conjugate operators determine the spectrum of the commuting observables and thus the allowed states of the particle.

For a standard description of a particle, the commuting observables can be taken to be the three components of the linear momentum, and a component of a spin operator. The spectrum of the momentum is $\mathbb{R}^3$, while the spectrum of a component of the spin vector takes on discrete values in integer steps from $-j$ to j. In this case the Hilbert space is

$$\mathcal{H}_{mj} = L^2(\mathbb{R}^3) \otimes \mathbb{C}^{2j+1}. \tag{1}$$

Single-particle states are represented by wave functions, $\psi(\mathbf{p}, \mu) = \langle (m,j)\mathbf{p}, \mu|\psi\rangle$. The relations of the operators $m, j^2, \mathbf{p}$ and $\mathbf{j}\cdot\hat{\mathbf{z}}$ to the Poincaré Lie Algebra determines a unitary representation, $U_1(\Lambda, a)$, of Poincaré group on $\mathcal{H}_{mj}$:

$$\langle (m,j)\mathbf{p}, \mu|U_1(\Lambda,a)|\psi\rangle = \int \sum_{\mu'=-j}^{j} \mathcal{D}^{m,j}_{\mathbf{p},\mu;\mathbf{p}',\mu'}[\Lambda, a] d\mathbf{p}' \psi(\mathbf{p}',\mu') = \psi'(\mathbf{p},\mu) \tag{2}$$

where the Poincaré group Wigner function is

$$\mathcal{D}^{m,j}_{\mathbf{p},\mu;\mathbf{p}',\mu'}[\Lambda, a] := \langle (m,j)\mathbf{p}, \mu|U_1(\Lambda,a)|(m,j)\mathbf{p}',\mu'\rangle =$$

$$\delta(\mathbf{p} - \boldsymbol{\Lambda} p')\sqrt{\frac{\omega_m(\mathbf{p})}{\omega_m(\mathbf{p}')}} e^{ip\cdot a} D^j_{\mu\mu'}[R_{wc}(\Lambda, p)] \tag{3}$$

and $R_{wc}(\Lambda, p)$ is a Wigner rotation.

Because a sequence of Lorentz boosts that start and end at the rest frame generally define a rotation, in order to obtain an unambiguous definition of a spin vector for all values of the particle's momentum, it is necessary to define a standard way to measure a spin observable. The above representation implicitly defines the spin projection by its value in the particle's rest frame after the particle is transformed to the rest frame with a rotationless Lorentz transformation. This is one of an infinite number of possible choices of spin observables. This choice is consistent with the "canonical" spin that appears in standard Dirac u and v spinors. Different spin observables are related by momentum-dependent rotations that lead to different couplings to the electromagnetic field. This ensures that measurable physical quantities are independent of the observables used to label single particle states.

These single-particle representations are irreducible, and all positive-mass positive-energy irreducible representations of the Poincaré group can be put in this general form. These irreducible representations will be important in formulating dynamical models. In general, any unitary representation of the Poincaré group can be decomposed into a direct sum or direct integral (for continuous mass eigenvalues) of irreducible representations. We will build the dynamical unitary representation of the Poincaré group out of the non-interacting irreducible representations.

3 Poincaré group Wigner functions and kinematic subgroups

In the previous section we represented single-particle wave functions in the basis of generalized eigenstates $|(m,j),\mathbf{p},\mu\rangle$.

The state of the particle could be also alternatively determined by measuring the particles' four velocity, $v^\mu = (\sqrt{1-\mathbf{v}\cdot\mathbf{v}}, \mathbf{v})$, and spin projection:

$$|(m,j),\mathbf{v},\mu\rangle = |(m,j),\mathbf{p}(\mathbf{v},m),\mu\rangle m^{3/2}, \tag{4}$$

or the light-front components of the four momentum $p^+ = \sqrt{m^2+\mathbf{p}^2} + \mathbf{p}\cdot\hat{\mathbf{z}}$, $\mathbf{p}_\perp = (\mathbf{p}\cdot\hat{\mathbf{x}}, \mathbf{p}\cdot\hat{\mathbf{y}})$, and light-front spin projection:

$$|(m,j),p^+,\mathbf{p}_\perp,\mu\rangle = \sum_{\mu'=-j}^{j} |(m,j),\mathbf{p}(p^+,\mathbf{p}_\perp,m),\mu'\rangle \sqrt{\frac{\omega_m(\mathbf{p})}{p^+}} D^j_{\mu'\mu}[B_c^{-1}(p)B_f(p)] \tag{5}$$

where $B_c^{-1}(p)B_f(p)$ is a Melosh rotation [38], defined by a light-front-preserving boost followed by the inverse of a rotationless boost. The different basis choices are related to the basis $|(m,j),\mathbf{p},\mu\rangle$ by the unitary transformations in (4) and (5). The light-front preserving boosts have the desirable property that they form a group, which means that there are no Winger rotations for any sequence of light-front preserving boosts that start and end in the rest frame; the price paid for this is that the Wigner rotation of a pure rotation is not equal to the rotation.

The Poincaré group Wigner functions depend on the choice of basis. The Wigner functions

$$\mathcal{D}^{m,j}_{\mathbf{v},\mu;\mathbf{v}',\mu'}[\Lambda,a] := \langle (m,j)\mathbf{v},\mu|U_1(\Lambda,a)|(m,j)\mathbf{v}',\mu'\rangle \tag{6}$$

$$\mathcal{D}^{m,j}_{p^+,\mathbf{p}_\perp,\mu;p'^+,\mathbf{p}'_\perp,\mu'}[\Lambda,a] := \langle (m,j)p^+,\mathbf{p}_\perp,\mu|U_1(\Lambda,a)|(m,j)p'^+,\mathbf{p}'_\perp,\mu'\rangle \tag{7}$$

are related to the Wigner function (3) by the unitary transformations (4) and (5).

While the concept of a kinematic subgroup does not make sense for a single particle, the kinematic subgroup for an instant-form dynamics is the subgroup of the Poincaré group that leaves the Wigner function (3) independent of mass; the kinematic subgroup for a point-form dynamics is the subgroup of the Poincaré group that leaves the Wigner function (6) independent of mass; the kinematic subgroup for a front-form dynamics is the subgroup of the Poincaré group that leaves the Wigner function (7) independent of mass. Different mass-independent subgroups appear in different irreducible bases because the unitary transformations relating the irreducible bases (4) and (5) to the basis $|(m,j)\mathbf{p},\mu\rangle$ depend on the particles' mass. These mass-independent subgroups become kinematic subgroups in dynamical models because the mass acquires an interaction while the other operators used to construct dynamical irreducible bases remain interaction free. More generally, it is possible to define perfectly good single-particle bases where the identity is the only subgroup where the corresponding Wigner function is independent of mass.

4 Two-body models - Clebsch-Gordan coefficients

The two-body Hilbert space is a tensor product of two single-particle Hilbert spaces, $\mathcal{H} = \mathcal{H}_{m_1 j_1} \otimes \mathcal{H}_{m_2 j_2}$. The non-interacting representation of the Poincaré group on $\mathcal{H}$ is the tensor product of two single-particle (irreducible) representations of the Poincaré group, $U_0(\Lambda,a) := U_1(\Lambda,a)\otimes U_2(\Lambda,a)$. While the single-particle representations of the Poincaré group are irreducible, their tensor product is reducible. Formally the tensor product representation can be expressed as a direct integral of irreducible representations,

$$U_0(\Lambda, a) = \sum\!\!\!\!\!\!\int_{jls}^{\oplus} dm U_{0,m,j,l,s}(\Lambda, a), \tag{8}$$

where $U_{0,m,j,l,s}(\Lambda, a)$ are mass m spin j irreducible representations of the Poincaré group. The quantum numbers l and s are invariant degeneracy parameters that distinguish multiple copies of the irreducible representations with the same m and j. They have the same quantum numbers as the spin and orbital angular momentum. The mass m is the two-particle invariant mass that has a continuous spectrum starting from $m_1 + m_2$. The Poincaré group Clebsch-Gordan coefficients relate the tensor product representation to the direct integral of irreducible representations and satisfy

$$\sum \int \mathcal{D}^{m,j}_{\mathbf{p},\mu;\mathbf{p}',\mu'}[\Lambda, a] d\mathbf{p}' \langle (m, j, l, s), \mathbf{p}', \mu' | (m_1, j_1), \mathbf{p}_1, \mu_1; (m_2, j_2), \mathbf{p}_2, \mu_2 \rangle =$$

$$\sum \int \langle (m, j, l, s), \mathbf{p}, \mu | (m_1, j_1), \mathbf{p}'_1, \mu'_1; (m_2, j_2), \mathbf{p}'_2, \mu'_2 \rangle d\mathbf{p}'_1 \mathbf{p}'_2 \times$$

$$\mathcal{D}^{m_1,j_1}_{\mathbf{p}'_1,\mu'_1;\mathbf{p}_1,\mu_1}[\Lambda, a] \mathcal{D}^{m_2,j_2}_{\mathbf{p}'_2,\mu'_2;\mathbf{p}_2,\mu_2}[\Lambda, a]. \tag{9}$$

The Clebsch-Gordan coefficients, $\langle (m, j, d), \mathbf{p}, \mu | (m_1, j_1), \mathbf{p}_1, \mu_1; (m_2, j_2), \mathbf{p}_2, \mu_2 \rangle$, $d := \{l, s\}$, are basis-dependent and are known in all three of the representations ([39, 7, 40, 12]).

The two-body irreducible basis states look similar to relative and center of mass variables in non-relativistic quantum mechanics; but they differ in the structure of the Poincaré group Clebsch-Gordan coefficients, which contain momentum-dependent spin rotation functions and non-trivial kinematic factors that ensure unitarity.

The basis states $\{|(m, j, l, s), \mathbf{p}, \mu\rangle\}$ transform irreducibly under $U_0(\Lambda, a)$:

$$U_0(\Lambda, a)|(m, j, d), \mathbf{p}, \mu\rangle = \sum{}' \int d\mathbf{p}' |(m, j, d), \mathbf{p}', \mu'\rangle \mathcal{D}^{m,j}_{\mathbf{p}',\mu';\mathbf{p},\mu}[\Lambda, a]. \tag{10}$$

While (10) is *not* the dynamical representation of the Poincaré group, by working in this non-interacting irreducible basis it is possible to construct dynamical representations by adding an interaction v, which in this basis has a kernel of the form

$$\langle (m', j', d'), \mathbf{p}', \mu' | v | (m, j, d), \mathbf{p}, \mu \rangle = \delta_{j'j} \delta_{\mu'\mu} \delta(\mathbf{p}' - \mathbf{p}) \langle m', d' \| v^j \| m, d \rangle, \tag{11}$$

to the non-interacting two-body mass operator. This interaction has the same form as a typical Galilean invariant non-relativistic interaction if we replace $m = \sqrt{m_1^2 + k^2} + \sqrt{m_2^2 + k^2}$ by k and $d = \{l, s\}$ by (l, s).

We define the dynamical mass operator $M := \sqrt{m_1^2 + k^2} + \sqrt{m_2^2 + k^2} + v$. Simultaneous eigenstates of M, $\mathbf{p}$, $\mathbf{j}^2$ and $\mathbf{j} \cdot \hat{\mathbf{z}}$ can be constructed by diagonalizing M in the irreducible non-interacting basis. These eigenfunctions have the form

$$\langle (k', j', l', s'), \mathbf{p}', \mu' | (\lambda, j), \mathbf{p}, \mu \rangle = \delta_{j'j} \delta_{\mu'\mu} \delta(\mathbf{p}' - \mathbf{p}) \phi_{\lambda,j}(\mathbf{k}^2, l, s) \tag{12}$$

where the wave function, $\phi_{\lambda,j}(k^2, l, s)$, is the solution of the mass eigenvalue problem with eigenvalue λ:

$$(\lambda - \sqrt{m_1^2 + k^2} - \sqrt{m_2^2 + k^2}) \phi_{\lambda,j}(k, l, s) =$$

$$\int_0^\infty k'^2 dk' \sum_{s'} \sum_{l'=|j-s|}^{j+s} \langle k,l,s|V^j|k',l',s'\rangle \phi_{\lambda,j}(k',l',s'). \tag{13}$$

The dynamical unitary representation of the Poincaré group is defined on this complete set of states, $|(\lambda,j),\mathbf{p},\mu\rangle$ by

$$U(\Lambda,a)|(\lambda,j),\mathbf{p},\mu\rangle = \sum_{\mu'=-j}^{j} \int d\mathbf{p}' |(\lambda,j),\mathbf{p}',\mu'\rangle \mathcal{D}^{\lambda,j}_{\mathbf{p}',\mu';\mathbf{p},\mu}[\Lambda,a]. \tag{14}$$

The relevant dynamical feature is that the Poincaré group Wigner function now depends on the eigenvalue λ of the dynamical mass operator, which requires solving (13).

Because of the choice of basis, the Poincaré group Wigner function $\mathcal{D}^{\lambda,j}_{\mathbf{p}',\mu';\mathbf{p},\mu}[\Lambda,a]$ has the same structure as the Wigner function (3) and thus has the property that when (Λ,a) is in the three-dimensional Euclidean subgroup, it is independent of the mass eigenvalue λ, which means that for this dynamical model the kinematic subgroup is dictated by the *choice of representation used to define the irreducible basis*.

Even though the dynamics has a non-trivial interaction dependence, it is only necessary to solve (13), which is analogous to solving the center of mass Schrödinger equation in the non-relativistic case.

This construction can be repeated using different irreducible bases, such as (4) or (5), where the Wigner functions have different mass-independent symmetry groups. For these bases if we choose to use the interactions

$$\langle (k',j',l',s'),\mathbf{v}',\mu'|v_{point}|(k,j,l,s),v,\mu\rangle = \delta_{j'j}\delta_{\mu'\mu}\delta(\mathbf{v}'-\mathbf{v})\langle k',l',s'\|v^j\|k,l,s\rangle \tag{15}$$

$$\langle (k',j',l',s'),p'^+,\mathbf{p}'_\perp,\mu'|v_{front}|(k,j,l,s),p^+,\mathbf{p}_\perp,\mu\rangle =$$

$$\delta_{j'j}\delta_{\mu'\mu}\delta(\mathbf{p}'_\perp-\mathbf{p}_\perp)\delta(p'^+-p^+)\langle k',l',s'\|v^j\|k,l,s\rangle, \tag{16}$$

where the reduced kernels $\langle k',l',s'\|v^j\|k,l,s\rangle$ are the same in (11), (15), and (16) in the bases (2), (4), and (5), respectively, and construct dynamical eigenstates of the form

$$|(\lambda,j),\mathbf{v},\mu\rangle, \qquad |(\lambda,j),p^+,\mathbf{p}_\perp,\mu\rangle\ , \tag{17}$$

then equation (13) still determines the binding energy and scattering phase shifts. It follows that the resulting two-body models have the same bound-state and scattering observables, however each of the resulting unitary representations of the Poincaré group has a different kinematic subgroup. The dynamical irreducible eigenstates transform like $|(\lambda,j),\mathbf{p},\mu\rangle$ with the Wigner function (14) replaced by (6) or (7) where m is replaced by λ. This makes these unitary transformations dynamical.

The mass operators and interactions, v, v_{point} and v_{front} are distinct operators, but the three representations are related by unitary transformation that leave the binding energies and scattering observables unchanged. The dynamical calculations are identical in all three cases and are given by solving (13). This shows that dynamical models with different kinematic subgroups are equivalent and cannot be distinguished on the basis of any experimental observations.

5 Cluster properties - Ekstein's theorem

An important feature of non-relativistic quantum mechanics is that the same interactions appear in the few and many-body problems. Specifically, the Hamiltonian becomes a sum of subsystem Hamiltonians when the short-ranged interactions between particles in different subsystems are turned off. In the relativistic case the corresponding requirement is that the unitary time-translation group breaks up into a tensor product of subsystem groups when the system is asymptotically separated into independent subsystems. We call this the strong form of cluster properties.

The observable requirement is that the S-matrix clusters. We call this the weak form of cluster properties because it follows from the strong form of cluster properties, however because different Hamiltonians can have the same S-matrix, the weak form of cluster properties does not imply that the same interactions appear in the few and many-body Hamiltonians. Because of this, in order to maintain a simple relation between the few and many-body problem, we require that Poincaré invariant quantum theories satisfy the strong form of cluster properties.

A theorem of Ekstein [41] provides necessary and sufficient conditions for two short-ranged interactions to give the same S matrix. The requirement is that the Hamiltonians are related by a unitary transformation A satisfying the asymptotic condition

$$\lim_{t\to\pm\infty} \|(I-A)U_0(t)|\psi\rangle\| = 0 \tag{18}$$

where $U_0(t)$ is the non-interacting time translation operator. We refer to unitary transformations with this property as scattering equivalences. It is important that this condition be satisfied for both time limits; to appreciate the relevance of this condition consider two Hamiltonians with different repulsive potentials. Because these Hamiltonians have the same spectrum and multiplicities they are related by a unitary transformation, however the derived S-matrices may have different phase shifts. The phase shifts differ if and only if two time limits do not agree.

Scattering equivalences that preserve weak cluster properties but not strong cluster properties exist and are the key to restoring the strong form of cluster properties in Poincaré invariant quantum theory. The strategy is illustrated in the formulation of the three-body problem in the next section.

6 Three-body problem

The strong form of cluster properties implies that given a set of dynamical two-body generators, the three-body generators necessarily can be expressed as sums of one, two and three-body operators

$$H = H_1 + H_2 + H_3 + H_{12} + H_{23} + H_{31} + H_{123} \tag{19}$$

$$\mathbf{P} = \mathbf{P}_1 + \mathbf{P}_2 + \mathbf{P}_3 + \mathbf{P}_{12} + \mathbf{P}_{23} + \mathbf{P}_{31} + \mathbf{P}_{123} \tag{20}$$

$$\mathbf{J} = \mathbf{J}_1 + \mathbf{J}_2 + \mathbf{J}_3 + \mathbf{J}_{12} + \mathbf{J}_{23} + \mathbf{J}_{31} + \mathbf{J}_{123} \tag{21}$$

$$\mathbf{K} = \mathbf{K}_1 + \mathbf{K}_2 + \mathbf{K}_3 + \mathbf{K}_{12} + \mathbf{K}_{23} + \mathbf{K}_{31} + \mathbf{K}_{123}. \tag{22}$$

The one and two-body operators in (19-22) are the same operators that appear in the two-body problems, while the three-body operators, $H_{123}, \mathbf{P}_{123}, \mathbf{J}_{123}$ and $\mathbf{K}_{123}$ are the only new ingredients in the three-particle generators.

It is easy to show that if the generators have this form it is impossible to satisfy the Poincaré commutation relations if all of the three-body operators vanish. However, although the commutation relations put non-linear constraints on these operators, it will become clear that the solutions are not unique.

To avoid solving the non-linear problem of satisfying the commutation relations, it is more productive to start by first satisfying the commutation relations at the expense of strong cluster properties. This can be done by applying the method of section 4 directly to the three-body problem. This involves adding suitable interactions to the non-interacting invariant three-body mass operator.

To begin the construction we consider a three-body system where only one pair of particles interact. The relevant basis is a non-interacting three-body irreducible representation of the Poincaré group. It is constructed by successive pairwise coupling using the Poincaré group Clebsch-Gordan coefficients. If we assume that particles one and two are the interacting pair then preferred order of coupling would be $(12) \to ((12)(3))$:

$$|\mathbf{p}_1, \mu_1\rangle \otimes |\mathbf{p}_2, \mu_2\rangle \to |(k_{12}, l_{12}, s_{12}, j_{12})\mathbf{p}_{12}, \mu_{12}\rangle \tag{23}$$

$$|(k_{12}, l_{12}, s_{12}, j_{12})\mathbf{p}_{12}, \mu_{12}\rangle \otimes |\mathbf{p}_3, \mu_3\rangle \to$$
$$|(q, L_{(12)(3)}, S_{(12)(3)} J_{(12)(3)}, k_{12}, l_{12}, s_{12}, j_{12})\mathbf{p}, \mu\rangle. \tag{24}$$

We introduce the following shorthand notation for the basis states in these equations. We write (23) as $|1 \otimes 2\rangle \to |(12)\rangle$ and (24) as $|(12) \otimes 3\rangle \to |(12)(3)\rangle$. Using this notation we define two different embeddings of the two-body interaction in the three-body Hilbert space using the two representation in (24):

$$\langle (12)' \otimes 3' | v_{12\otimes} | (12) \otimes 3\rangle =$$
$$\langle k'_{12}, l'_{12}, s'_{12} \| v^j \| k_{12}, l_{12}, s_{12}, \rangle \delta(\mathbf{p}'_{12} - \mathbf{p}_{12}) \delta(\mathbf{p}'_3 - \mathbf{p}_3) \delta_{j'_{12} j_{12}} \delta_{j'_3 j_3} \tag{25}$$

and

$$\langle (12)'(3)' | v_{12} | (12)(3)\rangle =$$
$$\langle k'_{12}, l'_{12}, s'_{12} \| v^j \| k_{12}, l_{12}, s_{12}, \rangle \delta(\mathbf{p}' - \mathbf{p}) \times$$
$$\frac{\delta(q' - q)}{q^2} \delta_{j'_{(12)(3)} j_{(12)(3)}} \delta_{j'_{12} j_{12}} \delta_{L'_{(12)(3)} L_{(12)(3)}} \delta_{S'_{(12)(3)} S_{(12)(3)}} \delta_{\mu' \mu} \tag{26}$$

where the reduced kernel, $\langle k'_{12}, l'_{12}, s'_{12} \| v^j \| k_{12}, l_{12}, s_{12}, \rangle$, is identical in (25) and (26). These expressions define different interactions, $(v_{12\otimes} \neq v_{12})$.

We use these two interactions to define two different $2 + 1$-body mass operators $M_{(12)\otimes(3)}$ and $M_{(12)(3)}$ defined by

$$M_{(12)\otimes(3)} := \sqrt{\left(\sqrt{\left(\sqrt{m_1^2 + k_{12}^2} + \sqrt{m_2^2 + k_{12}^2} + v_{12\otimes}\right)^2 + \mathbf{p}_{12}^2}\right) + \sqrt{m^2 + \mathbf{p}_3^2})^2 - \mathbf{p}^2)} \tag{27}$$

$$M_{(12)(3)} := \sqrt{\left(\sqrt{m_1^2 + k_{12}^2} + \sqrt{m_2^2 + k_{12}^2} + v_{12}\right)^2 + q^2} + \sqrt{m_3^2 + q^2}. \tag{28}$$

Because of the invariance principle [42–44] the S-matrix can be computed by replacing the Hamiltonian by the mass operator (this is equivalent to evaluating the S-matrix in the three-body rest frame) in the standard time-dependent representation of the scattering operator.

$M_{(12)\otimes(3)}$ is the mass operator of the tensor product of a two-body representation involving particles one and two and a spectator representation of the Poincaré group associated with particle three, $U_{12}(\Lambda,a)\otimes U_3(\Lambda,a)$. By construction it is consistent with the strong form of cluster properties. The mass operator $M_{(12)(3)}$ commutes with the three-body spin and commutes with and is independent of the total three-body momentum and z-component of the three-body canonical spin. Simultaneous eigenstates of $M_{(12)(3)},\mathbf{p},j^2,j_z$ are complete and transform irreducible with respect to the Poincaré group. This defines a dynamical unitary representation of the Poincaré group, $U_{(12)(3)}(\Lambda,a)$, on the three-body Hilbert space following the construction of section 4.

The scattering operators associated with both of these operators are related by

$$\langle(12)\otimes(3)|S_{(12)\otimes(3)}|(12)\otimes(3)\rangle =$$

$$\langle k'_{12},l'_{12},s'_{12}\|S^j\|k_{12},l_{12},s_{12},\rangle\delta(\mathbf{p}'_{12}-\mathbf{p}_{12})\delta(\mathbf{p}'_3-\mathbf{p}_3)\delta_{j'_{12}j_{12}}\delta_{j'_3j_3} \tag{29}$$

and

$$\langle(12)(3)|S_{(12)(3)}|(12)(3)\rangle = \langle k'_{12},l'_{12},s'_{12}\|S^j\|k_{12},l_{12},s_{12},\rangle\delta(\mathbf{p}'-\mathbf{p})\times$$

$$\frac{\delta(q'-q)}{q^2}\delta_{j'_{(12)(3)}j_{(12)(3)}}\delta_{j'_{12}j_{12}}\delta_{L'_{(12)(3)}L_{(12)(3)}}\delta_{S'_{(12)(3)}S_{(12)(3)}}\delta_{\mu'\mu} \tag{30}$$

where the reduced two-body kernels $\langle k'_{12},l'_{12},s'_{12}\|S^j\|k_{12},l_{12},s_{12},\rangle$ are identical. Because the delta functions become equivalent when they are evaluated on shell, the S matrices in both representations are identical. Ekstein's theorem implies the scattering equivalence $A_{(12)(3)}U_{12}(\Lambda,a)\otimes U_3(\Lambda,a)A^\dagger_{(12)(3)} = U_{(12)(3)}(\Lambda,a)$.

To construct a dynamical representation of the Poincaré group with all three particles interacting we first construct the mass operator

$$M = M_{(12)(3)} + M_{(23)(1)} + M_{(31)} - 2M_0. \tag{31}$$

Because each term in (31) commutes with $\mathbf{p},j^2,j_z$, and is independent of $\mathbf{p}$ and j_z it follows that simultaneous eigenstates of $M,\mathbf{p},j^2,j_z$ are complete and transform irreducibly, thus defining a dynamical unitary representation, $U(\Lambda,a)$, of the Poincaré group on the three-nucleon Hilbert space.

Because each of the $2+1$ mass operators in (31) is scattering equivalent to $2+1$ mass operators associated with a tensor product representation, it follows that M can be expressed as

$$M = A_{(12)(3)}M_{(12)\otimes(3)}A^\dagger_{(12)(3)} + A_{(23)(1)}M_{(23)\otimes(1)}A^\dagger_{(23)(1)} +$$

$$A_{(31)(2)}M_{(31)\otimes(2)}A^\dagger_{(31)(2)} - 2M_0. \tag{32}$$

From this representation it follows that when interaction between the i^{th} particle and the other two particles are turned off that

$$U(\Lambda,a) \to A_{(jk)(i)}U_{(jk)}(\Lambda,a)\otimes U_i(\Lambda,a)A^\dagger_{(jk)(i)} \tag{33}$$

which formally violates the strong form of cluster properties.

The strong form of cluster properties can be restored by transforming $U(\Lambda,a)$ with the product $A^\dagger = A^\dagger_{(12)(3)}A^\dagger_{(31)(2)}A^\dagger_{(23)(1)}$. Because products of scattering equivalences are scattering equivalences, this does not change the three-body S matrix. This transformation also restores strong-cluster properties, because $A^\dagger \to A^\dagger_{(jk)(i)}$ when the interactions between particle i and the other two particles are turned off, canceling off the extra unitary transformations in (33). The undesirable feature of A is that the individual $A_{(jk)(i)}$'s do not commute, so it introduces an exchange asymmetry that does not affect the S-matrix. The exchange symmetry can be manifestly restored by replacing the product of the $A_{(jk)(i)}$'s by a symmetrized product, such as:

$$A := e^{\ln(A_{(12)(3)})+\ln(A_{(23)(1)})+\ln(A_{(31)(2)})} \tag{34}$$

$$U_\otimes(\Lambda, a) = A^\dagger U(\Lambda, a) A. \tag{35}$$

Equation (35) defines a unitary representation, $U_\otimes(\Lambda, a)$, of the Poincaré group that satisfies the strong form of cluster properties because

$$A \to A_{(jk)(i)} \tag{36}$$

when the interactions between particle i and the other two particles are turned off. Thus

$$\begin{aligned} U_\otimes(\Lambda, a) \to A^\dagger_{(jk)(i)} U_{(jk)(i)}(\Lambda, a) A_{(jk)(i)} = \\ A^\dagger_{(jk)(i)} A_{(jk)(i)} U_{(jk)}(\Lambda, a) \otimes U_i(\Lambda, a) A^\dagger_{(jk)(i)} A_{(jk)(i)} = \\ U_{(jk)}(\Lambda, a) \otimes U_i(\Lambda, a) \end{aligned} \tag{37}$$

This property ensures that the infinitesimal generators have the additive form (19-22) and (35) generates the required three-body interactions.

Because there are many other ways to construct symmetric products of non-commuting operators and because it is possible to add a three-body interaction to M that commutes with and is independent of the total momentum and spin, it is clear the three-body parts of the generators that are required to restore the commutation relations are not unique. It is also important to note that it is not possible to use the freedom to add three-body interactions to eliminate the three-body interactions required to restore the commutation relations; in this representation the generated three-body interactions do not commute with the non-interacting spin.

This construction can be extended to formulate dynamical models satisfying the strong form of cluster properties for any fixed number of particles, isobar models in any of Dirac's form of dynamics. It is even possible to treat production beyond isobar types of models.

Models with different kinematic subgroups can be constructed by starting with different irreducible bases (4), (5). As long as the reduced kernels of the interactions are identical, all of the Bakamjian-Thomas three-body mass operators, M, will give identical bound-state and scattering observables. They are related by scattering equivalences constructed by applying the unitary transformations

$$|(\lambda, j), \mathbf{v}, \mu, \cdots\rangle = |(\lambda, j), \mathbf{p}(\mathbf{v}, \lambda), \mu, \cdots\rangle \lambda^{3/2}, \tag{38}$$

or

$$|(\lambda, j), p^+, \mathbf{p}_\perp, \mu, \cdots\rangle = |(\lambda, j), \mathbf{p}(p^+, \mathbf{p}_\perp, \lambda), \mu', \cdots\rangle \sqrt{\frac{\omega_\lambda(\mathbf{p})}{p^+}} D^j_{\mu'\mu}[B_c^{-1}(p) B_f(p)] \tag{39}$$

on each invariant subspace of the associated mass operator. Each of these representation is in turn scattering equivalent to a representation that satisfies strong cluster properties and has the same kinematic subgroup.

Because A is a scattering equivalence, it is only necessary to solve the Faddeev equations for the mass operator M in a non-interacting irreducible basis. Furthermore, since all bound state and scattering observables can be computed using only the internal mass operator, with the delta functions in $\mathbf{p}$ and μ removed, this equation is the same in all of Dirac's forms of dynamics when expressed in terms of the kinematic mass and kinematically invariant degeneracy quantum numbers. The operators A and the choice of kinematic subgroup are only needed if the three-body system is embedded in the four-body Hilbert space or if the eigenstates are used to construct electroweak current matrix elements.

7 Connection with quantum field theory

Poincaré invariant quantum mechanics as formulated by Bakamjian and Thomas resembles non-relativistic quantum mechanics more than quantum field theory. The Hilbert spaces have the same structure as non-relativistic Hilbert spaces, the theory is not manifestly covariant, spin 1/2 particles are treated using two-component spinors. In spite of these apparent differences there is a direct connection to quantum field theory which we outline below.

To develop the connection we assume the existence of an underlying quantum field theory with a Poincaré invariant vacuum and a collection of Heisenberg fields, $\boldsymbol{\phi}_i(x)$, where the bold face indicates a multi-component field. The index i distinguishes different types of fields.

In quantum field theory Hilbert-space vectors are constructed by applying functions of smeared fields,

$$\phi_i(f) = \int d^4x \mathbf{f}(x) \cdot \boldsymbol{\phi}_i(x) \tag{40}$$

to the physical vacuum $|0\rangle$.

Polynomials in the smeared fields applied to the physical vacuum generate a dense set of vectors. The field theoretic unitary representation of the Poincaré group $U^\dagger(\Lambda, a)$ acts covariantly on the smeared fields:

$$U^\dagger(\Lambda, a)\boldsymbol{\phi}_i(\mathbf{f})U(\Lambda, a) = \int d^4x \mathbf{f}(\Lambda x + a) S(\Lambda) \boldsymbol{\phi}_i(x) \tag{41}$$

where $S(\Lambda)$ is the finite dimensional representation of the Lorentz group appropriate to the field. The covariance of the fields implies Poincaré transformation properties of test functions that leave the scalar product invariant.

If the field theory has one-particle states, then there are functions, A, of smeared fields with the property that $A|0\rangle$ is a one particle state. One-particle eigenstates that transform irreducibly with respect to the Poincaré group can be constructed by projecting $A|0\rangle$ on states of sharp linear momentum and canonical spin. This can be done using the unitary representation (41) of the Poincaré group

$$|(m,j)\mathbf{p},\mu\rangle = A(\mathbf{p},\mu)|0\rangle :=$$

$$\sum_{\nu=-j}^{j} \int dR dp^0 d^4x e^{ip\cdot x} U(R,x) A|0\rangle D^{j*}_{\mu\nu}(R) \delta(p^2+m^2)\theta(P^0) \tag{42}$$

where R is a rotation, dR is the $SU(2)$ Haar measure, $U(R,x)$ is the unitary representation of the Poincaré group restricted to rotations and spacetime translations, and $D^{j*}_{\mu\mu}(R)$ is a $SU(2)$ Wigner function.

The normalization of these states can be chosen so

$$\langle (m',j')\mathbf{p}',\mu' | (m,j)\mathbf{p},\mu\rangle = \delta(\mathbf{p}'-\mathbf{p})\delta_{m'm}\delta_{j'j}\delta_{\mu'\mu}. \tag{43}$$

It follows from the definitions and the group representation properties that these states transform as mass m spin j irreducible representations of the Poincaré group:

$$U(\Lambda,a)|(m,j)\mathbf{p},\mu\rangle = \sum_{\mu'=-j}^{j} |(m,j)\boldsymbol{\Lambda} p,\mu'\rangle e^{i\Lambda p\cdot a} D^{j*}_{\mu'\mu}[B^{-1}(\Lambda(p))\Lambda B(p)]\sqrt{\frac{\omega_m(\boldsymbol{\Lambda} p)}{\omega_m(\mathbf{p})}} =$$

$$\sum_{\mu'=-j}^{j}\int d\mathbf{p}' |(m,j)\mathbf{p}',\mu'\rangle \mathcal{D}'^{mj}_{\mu',\mathbf{p}';\mathbf{p},\mu}[\Lambda,a] \tag{44}$$

To construct scattering states define $C(\mathbf{p},\mu) := (\sqrt{m^2+\mathbf{p}^2}A(\mathbf{p},\mu)) - [H, A(\mathbf{p},\mu)]_-)$. Scattering states are then given by the Haag-Ruelle method: [45][46]

$$|(\mathbf{p}_1,\mu_1,\cdots,\mathbf{p}_N,\mu_N)^{\pm}\rangle = \lim_{t\to\pm\infty} U(-t)\prod_j [C_j(\mathbf{p}_j,\mu_j)e^{-it\omega_{m_j}(\mathbf{p}_j)}]|0\rangle \tag{45}$$

where the limits are strong limits after smearing over suitable momentum wave packets.

The operators $\prod_j [C_j(\mathbf{p}_j,\mu_j)|0\rangle$ can be considered as mappings from an N particle channel Hilbert space, $\mathcal{H}_\alpha$, to the Hilbert space of the field theory. Vectors in the N-particle channel Hilbert space are square integrable functions in the variables $\mathbf{p}_1,\mu_1,\cdots,\mathbf{p}_N,\mu_N$. We denote these operators by $\Omega_{\alpha\pm}$ where α indicates the channel.

The direct sum of all of the channel Hilbert spaces, including the one-particle channels, defines an asymptotic Hilbert space. We define $\Omega_\pm$ that maps the asymptotic Hilbert space to the physical Hilbert space by

$$\Omega_\pm \begin{pmatrix} |\mathbf{f}_{\alpha_1}\rangle \\ |\mathbf{f}_{\alpha_2}\rangle \\ \vdots \end{pmatrix} = \sum_\alpha \Omega_{\alpha_i\pm}|\mathbf{f}_{\alpha_i}\rangle. \tag{46}$$

By construction these wave operators satisfy the intertwining relations [46]

$$U(\Lambda,a)\Omega_\pm = \Omega_\pm \oplus_\alpha U_\alpha(\Lambda,a). \tag{47}$$

The Poincaré invariant S operator of the field theory is given by

$$S = \Omega_+^\dagger \Omega_- \tag{48}$$

where each $U_\alpha(\Lambda,a)$ is a tensor product of single particle irreducible representations of the Poincaré group on the channel subspace $\mathcal{H}_\alpha$

Poincaré invariant quantum mechanics formulated in the previous sections has the same basic structure. The primary difference is that the asymptotic Hilbert space for the field theory has an infinite number of channels and describes physics at all energy scales, while the Poincaré invariant quantum mechanical wave operators involve only a subset of these channels that are experimentally relevant only up to a given energy scale.

If Π is a Poincaré invariant projection operator on the asymptotic subspaces corresponding channels of a Poincaré invariant quantum model that also limits the maximum invariant mass of the asymptotic states, then the following operator

$$W = \Omega_{f+} \Pi \Omega^{\dagger}_{qm+} = \Omega_{f-} \Pi \Omega^{\dagger}_{qm-} \tag{49}$$

maps an invariant subspace of the quantum mechanical Hilbert space to an invariant subspace of the field theory Hilbert space in a manner that satisfies

$$\Pi S_{qm} \Pi = \Pi S_f \Pi \tag{50}$$

$$W U_p(\Lambda, a) = U_f(\Lambda, a) W \ . \tag{51}$$

These mappings define the relevant relation between the Poincaré invariant quantum theory and the underlying field theory.

Thus, for asymptotic scattering states in the range of Π the Poincaré invariant quantum mechanical theory can be designed to give identical results to the field theory. Obviously the two theories differ on asymptotic states that are not in the range of Π.

Even though the Poincaré invariant quantum theory does not satisfy microscopic locality, we see that it can give the same S matrix elements as the full field theory at a given energy scale.

8 Few nucleon applications

In this section we discuss an illustrative set of applications to few nucleon problems. A realistic nucleon-nucleon interaction is needed for these applications. The invariant mass operator for two free nucleons can be expressed in terms of a relative momentum as

$$m_{012} =: \sqrt{k^2 + m_1^2} + \sqrt{k^2 + m_2^2}. \tag{52}$$

It is always possible to express the two-body interaction as an addition to k^2:

$$M_{12} = m_{12} + v_{12} := \sqrt{k^2 + 2\mu v_{nn} + m_1^2} + \sqrt{k^2 + 2\mu v_{nn} + m_2^2} \tag{53}$$

where following [47] v_{nn} is a realistic nucleon-nucleon interaction [48,49] and μ is the reduced mass

$$\mu = \frac{m_1 m_2}{m_1 + m_2}. \tag{54}$$

In this representation the dynamical two-body mass operator becomes a function of the non-relativistic center of mass Hamiltonian:

$$M_{12} = \sqrt{2\mu h_{nr} + m_1^2} + \sqrt{2\mu h_{nr} + m_2^2} \tag{55}$$

where

$$h_{nr} = \frac{\mathbf{k}^2}{2\mu} + v_{nr}. \tag{56}$$

It is a consequence of the Kato-Birman [42–44] invariance principle that the relativistic wave operators for (53) and non-relativistic wave operators for (56) are identical

$$\Omega_{nr\pm} := \lim_{t\to\pm\infty} e^{iH_{nr}t} e^{-iH_{nr0}t} = \lim_{t\to\pm\infty} e^{ih_{nr}t} e^{-ih_0 t} = \lim_{t\to\pm\infty} e^{iMt} e^{-iM_0 t} =$$

$$\lim_{t\to\pm\infty} e^{iM^2 t} e^{-iM_0^2 t} = \lim_{t\to\pm\infty} e^{iH_r^2 t} e^{-H_{r0}^2 t} = \lim_{t\to\pm\infty} e^{iH_r t} e^{-iH_{r0} t} = \Omega_{r\pm} \tag{57}$$

where $M = M_{12}$ in (57). The identity (57) ensures that both scattering operators are identical as functions of $\mathbf{k}^2$:

$$S_{nr} = \Omega^{\dagger}_{nr+}\Omega_{nr-} = \Omega^{\dagger}_{r+}\Omega_{r-} = S_r \tag{58}$$

Here the relativistic and non-relativistic S are related because the interactions are fit to the same two-body data correctly transformed to the center of momentum frame. The non-relativistic Hamiltonian (56) is NOT the non-relativistic limit of (53).

This construction, which first appeared in [47], shows that existing realistic interactions can be directly used in the formulation of a Poincaré invariant two-body problem. Equation (55) implies that the wave functions of (56) and (53) are identical functions of $\mathbf{k}^2, l, s$.

If we replace the interaction in (11) by the interaction v_{12} in (53) and use this in the three-body calculation discussed in section 6 then the three-body S-matrix can be expressed in terms of three-body mass operators:

$$\bar{S}_{ac} = \delta_{ac} - 2\pi i \delta(M_a - M_c) T^{ac}(z_c) \tag{59}$$

which are functions of the transition operators

$$T^{ac}(z) = T^{ac}(z) = V^c + V^a R(z) V^c \tag{60}$$

where $a, b, c \in \{(12)(3), (23)(1), (31)(2)\}$,

$$M_{(ij)(k)} = \sqrt{(\sqrt{m_i^2 + k^2 + 2\mu v_{nn}} + \sqrt{m_j^2 + k^2 + 2\mu v_{nn}})^2 + q^2} + \sqrt{m_k^2 + q^2}. \tag{61}$$

$$V_a = M_a - M_0 \qquad V^c = \sum_{a\neq b} V_a \qquad R(z) = (z - M)^{-1} \tag{62}$$

$$R_c(z) = (z - M_0 - V_c)^{-1} \tag{63}$$

$$R(z) = R_c(z) + R_c(z) V^c R(z) \tag{64}$$

and v_{nn} is the nucleon-nucleon interaction that appears in (53) embedded in the three-nucleon Hilbert space with the delta functions in (26). The Faddeev equations can be derived using standard methods

$$T^{ab}(z) = V^b + \sum_{c\neq a} V_c R_c(z) T^{cb}(z). \tag{65}$$

While it does not make any sense to study the non-relativistic limit of interactions that are constructed by fitting to two-body bound and scattering data, we can compare the relativistic and non-relativistic three-body calculations that use the same two-body interaction, v_{nn}, as input. In the Poincaré invariant quantum mechanics case the Faddeev equations have the form

$$\langle a|T^{ab}(z)|b\rangle = \langle a|V^b|b\rangle + \sum_{c\neq a} \int \langle a|c\rangle\langle c|V_c R_c(z)|c\rangle\langle c|T^{cb}(z)|b\rangle \tag{66}$$

where $\langle a|c\rangle$ are Poincaré group Racah coefficients, which are the unitary transformation that relate three-body Poincaré irreducible bases constructed using pairwise coupling in different orders. These coefficients, which have the form

$$\langle a|c\rangle = \delta(\mathbf{p}-\mathbf{p}')\delta_{\mu\mu'}\delta(m-m')\delta_{jj'}\mathcal{R}^{mj}(d_a,d_c), \tag{67}$$

with d_a and d_b distinct sets of invariant degeneracy parameter, replace the non-relativistic permutation operators.

The construction of the kernel is facilitated by the fact that the two-body eigenfunctions of (55) and (56) are identical. The kernel of the relativistic Faddeev equation can be directly related to the non-relativistic two-body t using the following relations:

$$\langle c'|V_c R_c(z_c)|c\rangle = \langle c'|T_c(z_c)(z_c-M_0)^{-1}|c\rangle =$$

$$\langle c'|V_c|c^-\rangle(z_c-M_0)^{-1} = \langle c'|M_c-M_0|c^-\rangle(z_c-M_0)^{-1} =$$

$$\frac{2\mu}{\omega_1\omega_2+\omega_1'\omega_2'}\frac{(\omega_1+\omega_2)^2+(\omega_1'+\omega_2')^2}{\sqrt{(\omega_1+\omega_2)^2+q^2}+\sqrt{(\omega_1'+\omega_2')^2+q^2}}\langle c'|t_{nr}(k_c)|c\rangle(z_c-M_0)^{-1} \tag{68}$$

where

$$\omega_i = \sqrt{k^2+m_i^2}\ , \tag{69}$$

which holds for the half shell kernel; this can be used as input to construct the fully off-shell kernel using the first resolvent identity [50]

$$T_c(z') = T_c(z_c) + T_c(z')\frac{(z'-z_c)}{(z'-M_0)(z_c-M_0)}T_c(z_c) \qquad z'\neq z_c \tag{70}$$

Alternatively, this kernel has also been computed using an iterative procedure based on a non-linear integral equation[51].

The differences with the non-relativistic three-body calculations are the different off-shell dependence dictated by (68), the differences in the Poincaré group Racah coefficients (67) and the non-relativistic permutation operators. These differences show up for the first time in the three-body system, since our two-body interactions are designed to reproduce the same experimental two-body cross sections as the non-relativistic calculations.

Solving these equations leads to three types of predictions: binding energies,

$$M|\Psi\rangle = \lambda|\Psi\rangle \tag{71}$$

$$|\Psi\rangle = E(V)|\Psi_{bt}\rangle \qquad M_{bt}|\Psi_{bt}\rangle = \lambda|\Psi_{bt}\rangle, \tag{72}$$

scattering probabilities ($N=3$ only),

$$|S_{fi}|^2 = |\langle\Psi_f^+|\Psi_i^-\rangle|^2 = |\langle\Psi_{btf}^+|\Psi_{bti}^-\rangle|^2\ , \tag{73}$$

electromagnetic and weak current matrix elements

$$\langle\Psi_f|I^\nu(0)|\Psi_i\rangle = \langle\Psi_f|AI^\nu(0)A^\dagger|\Psi_i\rangle\ , \tag{74}$$

where $I^\mu(0)$ is a current that is conserved, covariant, and clusters in the representation (35) of the three-body dynamics.

In what follows we discuss three applications of Poincaré invariant quantum mechanics that illustrate its ability to model a variety reactions where relativity may be important.

8.1 Relativistic spin rotations in low energy A_y

A calculation by Miller and Schewnk [52] suggested that Wigner rotations might have an observable effect on the polarization observable A_y for low-energy p-d scattering. Comparison of three-body calculations based on Poincaré invariant quantum mechanics [34] and non-relativistic quantum mechanics using the same realistic CD-Bonn interaction [49] as input indeed show a surprising sensitivity of A_y to Wigner rotations. These calculations, which are shown in Fig. 1, compare the non-relativistic result (dotted curve), the relativistic result without Wigner rotations (dashed curve) and the full relativistic calculation (solid curve) to data [53, 54]. While the relativistic effects move the calculations away from the data, this calculation illustrates that the relativistic effects cannot be ignored in these calculations, even at these low energies.

8.2 Relativistic Effects in Exclusive pd Breakup

The value of Poincaré invariant quantum mechanics is that it provides a consistent framework to study strong interactions in the few GeV energy scale. At this scale it is more efficient to perform calculations using direct integration [55, 56] rather than with partial wave expansions. The feasibility of using Poincaré invariant quantum mechanics to treat nucleon deuteron-scattering at these energy scales was established by solving the Faddeev equation of section 6 using Malfliet-Tjon [57] interactions to model the nucleon-nucleon potential. The two-body interactions were included using the method discussed above. Convergence of the solutions of the Faddeev equations was established up to 2GeV [31–33]. In three-body reactions there are many observables that can be used to test the sensitivity of relativistic effects. One interesting observable is the cross section when the outgoing protons in a breakup reaction are measured at symmetric angles relative to the beam direction. These cross sections were computed [32] in non-relativistic and Poincaré invariant three-body models using the same Malfliet-Tjon two-body interactions as input.

Fig. 3 shows cross sections for different choices of angles symmetric about the beam direction. The solid curve is the relativistic calculation while the long dashed curve is the non-relativistic one. The other two curves compare the exact calculation to the first terms in the multiple scattering series both for the relativistic and non-relativistic cases. As the angle is increased the relativistic and non-relativistic curves, exhibit different behavior. For this kinematic configuration the multiple scattering series converges quickly, although this result depends on what is measured. Fig. 2 shows similar plots for non-symmetric angles. Again the first order multiple scattering calculations work reasonably well and there is a definite difference between the relativistic and non-relativistic predictions. In both cases the data [58], has the same qualitative behavior as the relativistic calculations.

8.3 Exchange currents in electron-deuteron scattering

The last application involves electron scattering off of nuclear targets at values of momentum transfer Q^2 appropriate to J-lab experiments. In Poincaré invariant quantum mechanics electron scattering observables in the one-photon-exchange approximation can be expressed in terms of matrix elements of a conserved covariant current $I^\mu(x)$ which should have a cluster expansion

$$I^\mu(x) = \sum I_i^\mu(x) + \sum I_{ij}^\mu(x) + \sum I_{ijk}^\mu(x) + \cdots . \tag{75}$$

Both Poincaré covariance, current conservation, and cluster properties put dynamical constraints on the current operator.

The deuteron is the simplest electromagnetic target that is sensitive to the two-body part of the current. While a general method for constructing $I^\mu(x)$ based on dynamical considerations is not known, the constraints can be satisfied by using the Wigner-Eckart theorem for the Poincaré group, which amounts to computing a maximal set of linearly independent current matrix elements and using covariance and current conservation to generate the remaining matrix elements. Different model two-body currents can be tested in this framework. For elastic scattering off of a deuteron there are three independent observables which can be taken as, $A(Q^2)$, $B(Q^2)$, and $T_{20}(Q^2, 70^o)$. The input to a calculation is a deuteron wave function, a dynamical representation of the Poincaré group, nucleon form factors [59–63], and a model exchange current [64]. The calculations illustrated in Figs. 4-6 use a model of the deuteron with a light front kinematic symmetry. The dynamical representation of the Poincaré group is constructed from the Argonne V18 interaction[48], the nucleon form factors are from ref. [61] (labeled BBBA) and the exchange current is the long-range part of a "pair current" derived from the one-pion-exchange part of the V18 interaction. Figs. 4, 5, and 6 show comparisons of $A(Q^2)$, $B(Q^2)$, and $T_{20}(Q^2, 70^o)$ to experimental data with and without the exchange current. Two different implementations of the Poincaré group Wigner-Eckart theorem are responsible for the small difference in the curves labeled II and III.

These three calculations illustrate both the power and flexibility of Poincaré invariant quantum mechanics as a tool to study systems of strongly interacting particles at scales up to a few GeV. Data shown for A are from [65–75], for B are from [76, 77, 71, 78, 72, 65], and for T_{20} from [79–84].

These calculations demonstrate that Poincaré invariant quantum mechanics is a useful framework for making realistic models of system of strongly interacting particles at the few-GeV energy scale. Some of these effects extend to surprisingly low energies.

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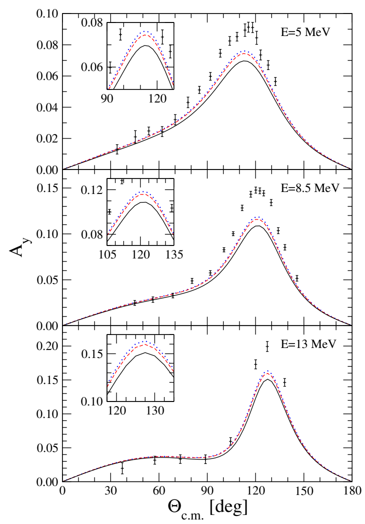


Fig. 1 Comparison of relativistic and non-relativistic calculations of the observable A_y at low energies.

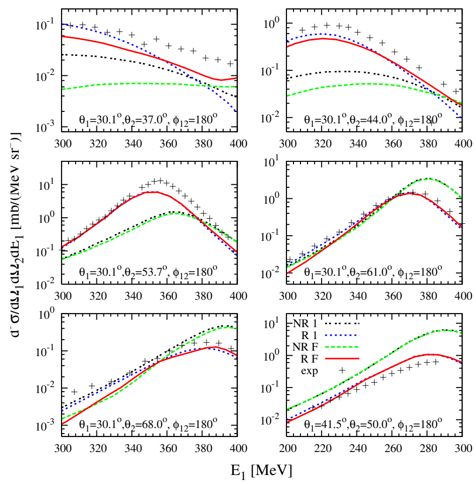


Fig. 2 Comparison of relativistic and non-relativistic calculations of exclusive proton deuteron breakup scattering at non-symmetric angles.

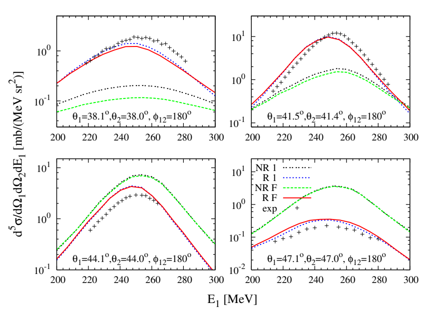


Fig. 3 Comparison of relativistic and non-relativistic calculations of exclusive proton deuteron breakup scattering at symmetric angles.

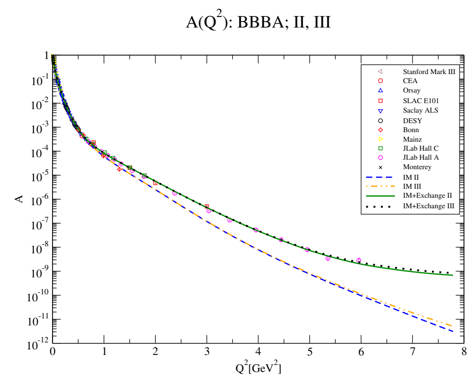
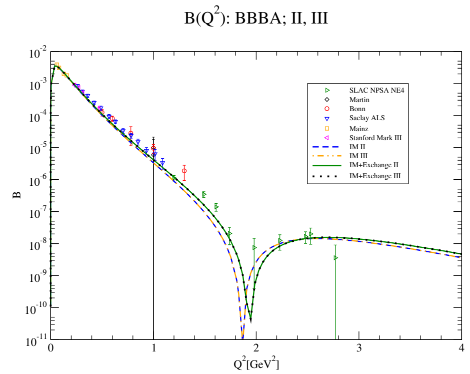


Fig. 4 Comparison of a front-form calculation of the elastic electron-deuteron scatting observable $A(Q^2)$ with and without "pair current" contributions.



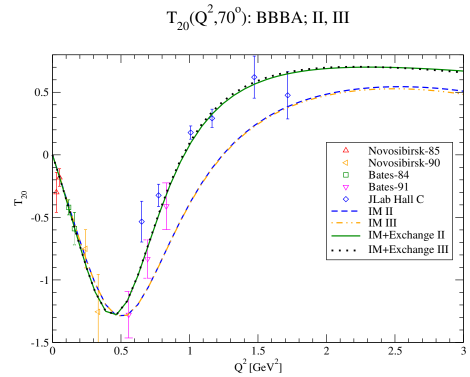


Fig. 5 Comparison of a front-form calculation of the elastic electron-deuteron scatting observable $B(Q^2)$ with and without "pair current" contributions.

Fig. 6 Comparison of a front-form calculation of the elastic electron-deuteron tensor polarization with and without "pair current" contributions.

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