Schwinger functions

Moments of a Euclidean path integral

$$S_n(x_1, \cdots, x_n) := \int D[\phi] e^{-A[\phi]} \phi(x_1) \cdots \phi(x_n)$$

Euclidean invariance

$$S_n(x_1, \cdots x_n) = S_n(Ox_1 - a, \cdots Ox_n - a)$$

$OO^t = I;$ $\quad a \quad$ **constant Euclidean 4 vector**

Relativity and Euclidean invariance

$$\det(A) = \det(B) = 1$$

$$X = \begin{pmatrix} t+z & x-iy \\ x+iy & t-z \end{pmatrix} \qquad \mathcal{X} = \begin{pmatrix} i\tau+z & x-iy \\ x+iy & i\tau-z \end{pmatrix}$$

$$\det(X) = \det(AXB^t) = t^2 - \mathbf{x}^2$$

$$\det(\mathcal{X}) = \det(A\mathcal{X}B^t) = -(\tau^2 + \mathbf{x}^2)$$

$$(A,B) \in SL(2,\mathbb{C}) \times SL(2,\mathbb{C})$$

Complex Lorentz group = Complex O(4)

Real O(4) = Subgroup of Complex Lorentz group

Analytic properties - spectral condition $p^0 > 0$

$$\langle 0|\phi(x_n)\cdots\phi(x_1)|0\rangle_c =$$

Insert positive energy intermediate states (no vacuum in truncated functions)

$$\sum\!\!\!\!\!\!\int_c \langle 0|\phi(x_1)\cdots\phi(0)|\mathbf{p}_n\rangle e^{ip_n\cdot(x_k-x_{k-1})} d\mathbf{p}_n \langle \mathbf{p}_n|\phi(0)\cdots\phi(x_n)|0\rangle_c$$

$$ip_n\cdot(x_k-x_{k-1}) = -ip_n^0(t_k-t_{k-1}) + i\mathbf{p}_n\cdot(\mathbf{x}_k-\mathbf{x}_{k-1})$$

$p_n^0 > 0$ **analytic for** $t_k - t_{k-1} - i\tau \qquad \tau > 0$

Analytic continuation

Tube: $\mathrm{Im}(p \cdot z) > 0$

Covariance w.r.t. complex Lorentz group

Extended Tube: $z_k \in$ **tube** $z'_k = \Lambda_c z_k$

$$\langle 0|\phi(z'_n)\cdots\phi(z'_1)|0\rangle_c := \langle 0|\phi(\Lambda_c z_n)\cdots\phi(\Lambda_c z_1)|0\rangle_c$$

Extended permuted tube: $\{x_n\} \in$ **extended tube** $x_k - x_l$ **spacelike**

Use locality to permute variables

$$\langle 0|\phi(z_1)\cdots\phi(z_n)|0\rangle_c = \langle 0|\phi(z_{\sigma(1)})\cdots\phi(z_{\sigma(n)})|0\rangle_c$$

Schwinger functions symmetric!

Schwinger continuation

$$G_n(x_1, \cdots, x_n) = \lim_{\theta \to \pi/2} S_n(x_1(\phi), \cdots, x_n(\phi))$$

$$x_k(\phi) = (\tau_k e^{i\phi}, \mathbf{x}_k) \qquad 0 \le \phi < \pi/2$$

Euclidean time-order = Minkowski time-order

Osterwalder - Schrader continuation

$$\langle 0|(\phi_1(x_1) \cdots \phi_n(x_n))|0\rangle = \lim_{0<\tau_1<\cdots<\tau_n \to 0} S_n(\tau_1 + it_1, \mathbf{x}_1, \cdots \tau_n + it_n, \mathbf{x}_n)$$

$n!$ **different possible time orderings** $\rightarrow$ $n!$ **field orderings**

- **Locality: Single Schwinger function gives time-ordered Green function and $n!$ Wightman functions.**

- **Analytic continuation in x-space has no obstructions.**

- **Euclidean Green functions satisfy Schwinger-Dyson equations.**

- **Euclidean Green functions can be used to construct quantum mechanics $(\mathcal{H}, U(\Lambda, a) : \mathcal{H} \to \mathcal{H})$**

- **Euclidean Green functions related to Lagrangian via path integral.**

Quantum Mechanics: Hilbert space vectors

$$f := (f_1(x_{11}), f_2(x_{21}, x_{22}), f_3(x_{31}, x_{32}, x_{33}), \cdots) \in \mathcal{S}_+$$

$$\mathcal{S}_+ : \qquad f = 0 \qquad \textbf{unless} \qquad 0 < x^0_{n1} < x^0_{n2} < \cdots < x^0_{nn}$$

$$\Theta f := (f_1(\theta x_{11}), f_2(\theta x_{21}, \theta x_{22}), f_3(\theta x_{31}, \theta x_{32}, \theta x_{33}), \cdots)$$

$$\theta x = \theta(x^0, \mathbf{x}) = (-x^0, \mathbf{x})$$

Quantum mechanics: Hilbert space inner product

$$\langle f|g\rangle = (\Theta f, Sg)_e = (f, \Theta Sg)_e =$$

$$\sum_{mn} \int d^{4n}x \, d^{4m}y \, f_n^*(\mathrm{x}_{n1}, \mathrm{x}_{n2}, \cdots, \mathrm{x}_{nn}) \times$$

$$S_{m+n}(\theta \mathrm{x}_{nn}, \cdots, \theta \mathrm{x}_{1n}, \mathrm{y}_{1m}, \cdots, \mathrm{y}_{mm}) \times$$

$$g_m(\mathrm{y}_{m1}, \mathrm{y}_{m2}, \cdots, \mathrm{y}_{mm}).$$

Reflection positivity

$$f \in \mathcal{S}_+ \qquad \langle f|f\rangle \geq 0$$

No Analytic continuation!

Comments on reflection positivity

The support conditions are necessary

$$\Theta f = -f \quad \rightarrow \quad \langle f|f\rangle < 0$$

Euclidean time support must be disjoint -

the order is not important (locality)

$$S_n(x_1, \cdots, x_n) = S_n(x_{\sigma(1)}, \cdots, x_{\sigma(n)})$$

Sequential support needed for cluster properties

Not easy to verify,
not stable with respect to small perturbations,
can be satisfied on lattice
preserved under limits

$\langle f|g\rangle =$ **Minkowski inner product !**

Example: $\langle f|g\rangle = (f, \Theta S_2 g)$

General S_2: Källen-Lehmann representation

$$S_2(x-y) = \frac{1}{(2\pi)^4}\int \frac{e^{ip\cdot(x-y)}\rho(m)}{p^2+m^2} d^4p dm$$

$\rho(m) =$ **Lehmann weight**

$$\rho(m) = \sum_n z_n \delta(m-m_n) + \rho_c(m)$$

$$\langle f|g\rangle = \int f^*(\mathrm{x}) S_2(\theta\mathrm{x}-\mathrm{y}) g(\mathrm{y}) d^4\mathrm{x} d^4\mathrm{y} =$$

$$\frac{1}{(2\pi)^4}\int f^*(\mathrm{x}) \frac{e^{ip^0(-x^0-\mathrm{y}^0)+i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})}\rho(m)}{(p^0)^2+\mathbf{p}^2+m^2} g(\mathrm{y}) dm d^4p d^4\mathrm{x} d^4\mathrm{y} =$$

Close contour LHP

$$\frac{1}{(2\pi)^3}\int f^*(\mathrm{x}) \frac{e^{-\omega_m(\mathbf{p})(x^0+\mathrm{y}^0)+i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})}\rho(m)}{2\omega_m(\mathbf{p})} g(\mathrm{y}) dm d\mathbf{p} d^4\mathrm{x} d^4\mathrm{y} =$$

$$\int \chi^*(\mathbf{p}) \frac{\rho(m)}{2\omega_m(\mathbf{p})} dm d\mathbf{p} \psi(\mathbf{p})$$

Lorentz invariant measure / no analytic continuation!

The momentum-space wave functions are

$$\chi(\mathbf{p}) = \int \frac{d\mathbf{x}}{(2\pi)^{3/2}} f(\mathrm{x}^0, \mathbf{x}) e^{-\omega_m(\mathbf{p})\mathrm{x}^0 - i\mathbf{p}\cdot\mathbf{x}}$$

$$\phi(\mathbf{p}) = \int \frac{d\mathbf{y}}{(2\pi)^{3/2}} g(\mathrm{x}^0, \mathbf{x}) e^{-\omega_m(\mathbf{p})\mathrm{x}^0 - i\mathbf{p}\cdot\mathbf{y}}$$

The Hamiltonian and (mass)^2 operators are

$$H = \frac{\partial}{\partial \tau} \qquad M^2 = \frac{\partial^2}{\partial \tau^2} + \nabla_{\mathbf{x}}^2 = \nabla_x^2$$

Widder's theorem/reflection positivity

most general $g(t)$ satisfying:

$$g(t) : \int f(t) g(t+t') f(t') dt' dt \geq 0$$

$$\Downarrow$$

$$g(t) = \int dm \rho(m) e^{-mt}$$

Lehmann version

$$\frac{1}{\pi} \int_{-\infty}^{\infty} d\lambda \int_{0}^{\infty} dm \frac{e^{i\lambda t} m \rho(m)}{\lambda^2 + m^2} =$$

$$\theta(t) g(t) + \theta(-t) g(-t)$$

For positive-time support

$$g(t) = \int_0^\infty dm \frac{m\rho(m)}{\pi} e^{-mt} = \frac{1}{\pi} \int_{-\infty}^{\infty} d\lambda \int_0^\infty dm \frac{e^{i\lambda t} m\rho(m)}{\lambda^2 + m^2}.$$

Widder's theorem implies that the most general reflection positive operator in 1 dimension is essentially given by a Lehmann representation.

Relation to positive self-adjoint operators, P:

$$(f', f) = \int_0^\infty f^*(p) g(p) d\mu(p)$$

Spectral decomposition

$$f(p) = \int_0^\infty e^{-ps} k(s) ds = \int_0^\infty ds \int_{-\infty}^\infty d\lambda e^{-ps+i\lambda s} h(\lambda) =$$

$$\int_{-\infty}^\infty \frac{1}{p - i\lambda} h(\lambda) d\lambda$$

$$(f', f) = \int_{-\infty}^\infty d\lambda \int_0^\infty dp k'(\lambda) \frac{p d\mu(p)}{\pi} \frac{d\lambda}{\lambda^2 + p^2} k(\lambda) =$$

$$\int_{-\infty}^\infty d\lambda \int_0^\infty h'(\lambda) \frac{p d\mu(p)}{\pi} \frac{1}{\lambda^2 + p^2} h(\lambda)$$

Spectral decomposition of positive operator (energy) has the same form as most general reflection positive operator!

$$m\rho(m) \leftrightarrow p d\mu(p)/\pi$$

Energy poles appear in the Fourier transform variables.

Sequential time support because positive energy intermediate states between each adjacent pair of field operators?

Positive energy colorless intermediate states - reflection positivity for singlets.

Quantum mechanics: Poincaré generators

Euclidean group = complex subgroup of Poincaré group

$$Hf_n(\mathsf{x}_{n1},\mathsf{x}_{n2},\cdots,\mathsf{x}_{nn}) = \sum_{k=1}^{n} \frac{\partial}{\partial \mathsf{x}_{nk}^0} f_n(\mathsf{x}_{n1},\mathsf{x}_{n2},\cdots,\mathsf{x}_{nn})$$

$$\mathbf{P} f_n(\mathsf{x}_{n1},\mathsf{x}_{n2},\cdots,\mathsf{x}_{nn}) = -i \sum_{k=1}^{n} \frac{\partial}{\partial \mathbf{x}_{nk}} f_n(\mathsf{x}_{n1},\mathsf{x}_{n2},\cdots,\mathsf{x}_{nn})$$

$$\mathbf{J} f_n(\mathsf{x}_{n1},\mathsf{x}_{n2},\cdots,\mathsf{x}_{nn}) = -i \sum_{k=1}^{n} \mathbf{x}_{nk} \times \frac{\partial}{\partial \mathbf{x}_{nk}} f_n(\mathsf{x}_{n1},\mathsf{x}_{n2},\cdots,\mathsf{x}_{nn})$$

$$\mathbf{K} f_n(\mathsf{x}_{n1},\mathsf{x}_{n2},\cdots,\mathsf{x}_{nn}) = \sum_{k=1}^{n} (\mathbf{x}_{nk} \times \frac{\partial}{\partial x_{nk}^0} - x_{nk}^0 \frac{\partial}{\partial \mathbf{x}_{nk}}) f_n(\mathsf{x}_{n1},\mathsf{x}_{n2},\cdots,\mathsf{x}_{nn}).$$

Hermiticity of H on $\mathcal{H}$

$$H = \frac{\partial}{\partial \tau}$$

$$\int f^*(\tau')S_2(\theta(\tau') - \tau)\frac{\partial}{\partial \tau}f(\tau)d\tau d\tau' =$$

$$-\int f^*(\tau')\frac{\partial}{\partial \tau}S_2(-\tau' - \tau)f(\tau)d\tau d\tau' =$$

$$-\int f^*(\tau')\frac{\partial}{\partial \tau'}S_2(-\tau' - \tau)f(\tau)d\tau d\tau' =$$

$$\int ((\frac{\partial}{\partial \tau'})f(\tau'))^* S_2(\theta\tau' - \tau)f(\tau)d\tau d\tau'$$

$$\langle f|Hf\rangle = \langle Hf|f\rangle \qquad H = H^\dagger$$

Similarly $\mathbf{K} = \mathbf{K}^\dagger$ on $\mathcal{H}$

Self-adjointness of generators

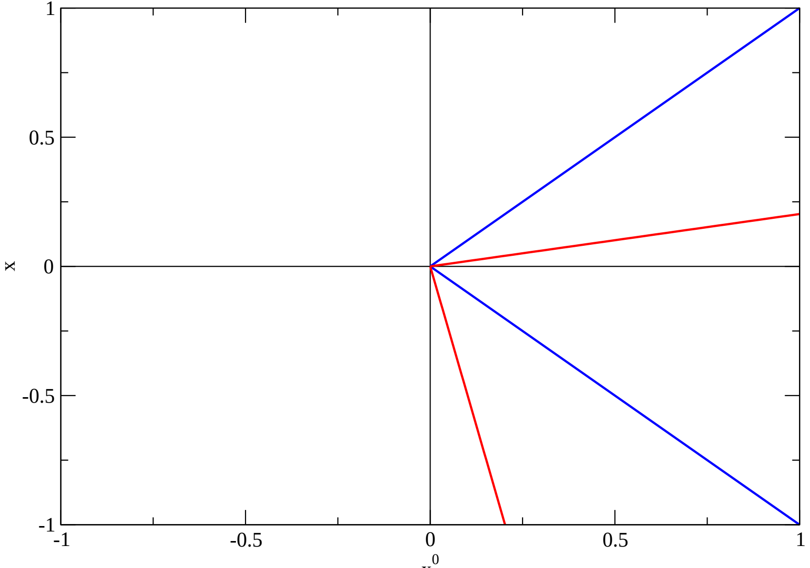
H generator of contractive semigroup on $\mathcal{H}$

P,J generators of unitary one-parameter groups on $\mathcal{H}$

K generator of local symmetric seimgroups

In all cases generators are self adjoint!

Generators satisfy Poincaré Lie algebra


1
0.5
0
-0.5
-1
x
-1
-0.5
0
0.5
1
x0

Spectral condition

$$\|e^{-\beta H}|f\rangle\|^2 = \langle e^{-\beta H} f | e^{-\beta H} f\rangle =$$

$$\langle f | e^{-2\beta H} f\rangle \leq \|e^{-2\beta H}|f\rangle\| \||f\rangle\|$$

$$\|e^{-\beta H}|f\rangle\| \leq \|e^{-2^n \beta H}|f\rangle\|^{\frac{1}{2^n}} \||f\rangle\|^{1-\frac{1}{2^n}} \leq (f, Sf)_e^{\frac{1}{2^{n+1}}} \||f\rangle\|^{1-\frac{1}{2^n}}$$

$$(f, Sf)_e < \infty$$

$$\Downarrow$$

$$\|e^{-\beta H}|f\rangle\| \leq \||f\rangle\|$$

$$\Downarrow$$

$$H \geq 0$$

Scattering - general considerations

$$P = |\langle f_+ | f_- \rangle|^2 = |\langle f_{0+} | S | f_{0-} \rangle|^2$$

where

$$\lim_{t \to \pm\infty} \| |f_\pm(t)\rangle - J|f_{0\pm}(t)\rangle \| = 0$$

$$\lim_{t \to \pm\infty} \| |e^{-iHt} f_\pm(0)\rangle - J e^{-iH_f t} |f_{0\pm}(0)\rangle \| = 0$$

$$\lim_{t \to \pm\infty} \| |f_\pm(0)\rangle - e^{iHt} J e^{-iH_f t} |f_{0\pm}(0)\rangle \| = 0$$

$$S = \lim_{t \to \infty} e^{iH_f t} J_f^\dagger e^{-i2Ht} J_i e^{iH_f t}$$

$$J = \int |(m_1, j_1)\mathbf{p}_1, \mu_1\rangle \times \cdots \times |(m_n, j_n)\mathbf{p}_n, \mu_n\rangle$$

$$H_f = \sum_n \sqrt{\mathbf{p}_n^2 + m_n^2}$$

J includes internal structure of composite particles and contributions due to self interactions

J separates observable properties of asymptotic particles (mass, momentum, spin) from internal structure

J maps tensor products of irreducible representations of Poincaré group (particles) into $\mathcal{H}$

$$\lim_{t\to\pm\infty} e^{iHt} J e^{-iH_f t} |f_{0\pm}(0)\rangle =$$

$$|f_{0\pm}(0)\rangle + \int_0^{\pm\infty} \frac{d}{dt} e^{iHt} J e^{-iH_f t} |f_{0\pm}(0)\rangle dt$$

sufficient condition for convergence (Cook condition)

$$\int_c^{\infty} \|(HJ - JH_f) e^{\mp iH_f t} |f_{0\pm}(0)\rangle\| dt < \infty$$

Scattering - Euclidean case $(2 \to 2)$

$$S_4(\mathrm{x}_1, \mathrm{x}_2, \mathrm{y}_2, \mathrm{y}_1) = S_2(\mathrm{x}_1, \mathrm{y}_1) S_2(\mathrm{x}_2, \mathrm{y}_2) + S_c(\mathrm{x}_1, \mathrm{x}_2, \mathrm{y}_2, \mathrm{y}_1)$$

$$S_2(\mathrm{x}_1, \mathrm{y}_1) = \frac{1}{(2\pi)^4} \int \frac{e^{ip\cdot(x-y} \rho(m)}{m^2 + p^2} d^4 p dm$$

$$\rho(m) = \sum_i z_i \delta(m - m_i) + \rho_c(m)$$

$$\langle x_1, x_2 | J | \mathbf{p}_1, \mathbf{p}_2 \rangle = h_1(\nabla_1^2) h_2(\nabla_2^2) \delta(\mathrm{x}_1^0 - \tau_1) \delta(\mathrm{x}_2^0 - \tau_2) \frac{1}{(2\pi)^3} e^{i\mathbf{p}_1 \cdot \mathbf{x}_1 + i\mathbf{p}_2 \cdot \mathbf{x}_2}$$

$$h(m_i^2) = 1 \qquad \rho(m) h(m^2) = z_i \delta(m - m_i)$$

Cluster properties of Euclidean Green functions essential for scattering.

$g(x) = f(\mathbf{x})\delta(x^0 - \tau)$ **is square integrable on** $\mathcal{H}$ **in spite of delta function!**

Simple way to satisfy the support condition.

We use the fact that m^2 **on the two-point function is represented by Laplacian.**

h **selects mass of scattering asymptote.**

We need to be sure that $h(\nabla^2)$ **preserves the relative Euclidean time support condition.**

Existence (Cook condition - Euclidean representation)

$$\int_0^\infty \|(HJ - JH_f)e^{-iH_f t}|f_{0\pm}(0)\rangle\| dt < \infty$$

$$\|(HJ - JH_f)e^{-iH_f t}|f_{0\pm}(0)\rangle\|^2 =$$

$$((HJ^\dagger - J^\dagger H_f)e^{-iH_f t} f \Theta(S_2 S_2 + S_c)(HJ - JH_f)e^{-iH_f t} f)$$

$$\|(HJ - JH_f)e^{iH_f t}|f_0\rangle\|^2 =$$

$$\int f_1^*(\mathbf{p}_1) f_2^*(\mathbf{p}_2) e^{i(\omega_{m_1}(\mathbf{p}_1)+\omega_{m_2}(\mathbf{p}_2))t} d\mathbf{p}_1 d\mathbf{p}_2$$

$$(\frac{\partial}{\partial x_1^0} + \frac{\partial}{\partial x_2^0} - \omega_{m_1}(\mathbf{p}_1) - \omega_{m_2}(\mathbf{p}_2))\langle \mathbf{p}_1, \mathbf{p}_2 | J^\dagger | \mathrm{x}_1, \mathrm{x}_2\rangle \times$$

$$d^4 x_1 d^4 x_2 S_2(\theta \mathrm{x}_1, \mathrm{y}_1) S_2(\theta \mathrm{x}_2, \mathrm{y}_2) + S_{4c}(\theta \mathrm{x}_1, \theta \mathrm{x}_2, \mathrm{y}_2, \mathrm{y}_1) d^4 y_1 d^4 y_2 \times$$

$$(\frac{\partial}{\partial y_1^0} + \frac{\partial}{\partial y_2^0} - \omega_{m_1}(\mathbf{p}_1') - \omega_{m_2}(\mathbf{p}_2'))\langle \mathrm{y}_1, \mathrm{y}_2 | J | \mathbf{p}_1', \mathbf{p}_2'\rangle$$

$$e^{-i(\omega_{m_1}(\mathbf{p}_1')+\omega_{m_2}(\mathbf{p}_2'))t} f_1(\mathbf{p}_1') f_2(\mathbf{p}_2') d\mathbf{p}_1' d\mathbf{p}_2'$$

$$\frac{\partial}{\partial x^0} \to \omega_m(\mathbf{p})$$

$h(m^2)$ in J picks out single m_i so the coefficients of the S_2 terms vanish.

This eliminates the Maiani Testa problem!

The strong limit would not exist without the $h(\nabla^2)$ factors.

Since $h(m^2) = h(\nabla^2)$ it is not automatic that hf and f have the same support condition (0 for $\tau < 0$):

$$e^{a\frac{\partial}{\partial \tau}} f(\tau, \mathbf{x}) = f(\tau + a, \mathbf{x})$$

$$\|(HJ - JH_f)e^{iH_f t}|f_0\rangle\|^2 =$$

$$\int f_1^*(\mathbf{p}_1) f_2^*(\mathbf{p}_2) e^{i(\omega_{m_1}(\mathbf{p}_1)+\omega_{m_2}(\mathbf{p}_2))t} d\mathbf{p}_1 d\mathbf{p}_2$$

$$\left(\frac{\partial}{\partial x_1^0} + \frac{\partial}{\partial x_2^0} - \omega_{m_1}(\mathbf{p}_1) - \omega_{m_2}(\mathbf{p}_2)\right)\langle \mathbf{p}_1, \mathbf{p}_2 | J^\dagger | \mathrm{x}_1, \mathrm{x}_2\rangle \times$$

$$d^4 x_1 d^4 x_2 S_{4c}(\theta \mathrm{x}_1, \theta \mathrm{x}_2, \mathrm{y}_2, \mathrm{y}_1) d^4 y_1 d^4 y_2 \times$$

$$\left(\frac{\partial}{\partial y_1^0} + \frac{\partial}{\partial y_2^0} - \omega_{m_1}(\mathbf{p}'_1) - \omega_{m_2}(\mathbf{p}'_2)\right)\langle \mathrm{y}_1, \mathrm{y}_2 | J | \mathbf{p}'_1, \mathbf{p}'_2\rangle$$

$$e^{-i(\omega_{m_1}(\mathbf{p}'_1)+\omega_{m_2}(\mathbf{p}'_2))t} f_1(\mathbf{p}'_1) f_2(\mathbf{p}'_2) d\mathbf{p}'_1 d\mathbf{p}'_2$$

Integral converges for good S_{4c}

Lorentz invariance of S-matrix

$$\lim_{t\to\pm\infty} \|(\mathbf{K}J - J\mathbf{K}_f)e^{iH_f t}|f_0\rangle\| = 0$$

$$\int_c^{\infty} \|(\mathbf{K}J - J\mathbf{K}_f)e^{\mp iH_f t}|f_{0\pm}(0)\rangle\| dt < \infty$$

Does $h(\nabla^2)$ preserve the support condition?

$$f(x) = 0 \quad x^0 < 0 \rightarrow h(\nabla_4^2) f(x) = 0 \quad x^0 < 0?$$

Yes if polynomials in m^2 are complete on $\mathcal{H}$,
$h(m^2) \approx p(m^2)$
Sufficient condition - Carleman

$$\sum_{n=1}^{\infty} \gamma_n^{-\frac{1}{2n}} > \infty$$

$$\gamma_n := \langle \psi | \Theta S_2 (\nabla^2)^n | \psi \rangle =$$

$$\int_0^{\infty} \psi(\mathbf{p}, x_0) \frac{e^{-\omega_m(\mathbf{p})(x_0+y_0)}}{2\omega_m(\mathbf{p}^2)} \rho(m^2) m^{2n} \psi(\mathbf{p}, y_0) d\mathbf{p} dx_0 dy_0 dm.$$

Satisfied if $\rho_c(m)$ is polynomially bounded.

Discrete Lehmann weight (not physical)

$$h_j(x) = \prod_{i \neq j}^{N} \frac{x - m_i^2}{m_j^2 - m_i^2}$$

Continuous part of Lehmann weight has compact support (not physical)

$$h_j(x) \approx p_j(m^2)$$

Continuous part of Lehmann weight semi infinite (physical case)

Completeness depends on growth of moments

$$\gamma_n := \int_0^\infty \frac{e^{-\sqrt{m^2+\mathbf{p}^2}\tau}}{2\sqrt{m^2+\mathbf{p}^2}}\rho(m)m^{2n}dm$$

$$\gamma_n \to \gamma_n' = \int_0^\infty \frac{e^{-\sqrt{m^2+\mathbf{p}^2}\tau}}{2\sqrt{m^2+\mathbf{p}^2}}m^{2n+k}dm$$

$$\frac{1}{2}\int_0^\infty \frac{e^{-p\tau\cosh(\eta)}}{\cosh(\eta)}(p\sinh(\eta))^{2n+k}\cosh(\eta)d\eta \leq \frac{1}{2}\tau^{-2n-k}\Gamma(2n+k-2)$$

$$\Gamma(x+1) = \sqrt{2\pi}x^{x+1/2}e^{-x+\theta/12x}$$

$$(\frac{1}{\gamma_n})^{\frac{1}{2n}} \geq \sqrt{\frac{2}{\pi}}^{\frac{1}{2n}}\tau^{1+\frac{k}{2n}}(2n+k-2)^{-1-(k-3/2)/2n}e^{(1+(k-2)/2n)-\frac{\theta}{1+(k-2)/2n}}$$

$$\sum_{n=1}^{\infty} \gamma_n^{-\frac{1}{2n}} > \sum_{n=0}^{\infty} \frac{c}{2n+k-2} > \infty$$

The inequality shows that $h(m^2)$ can be approximated by a polynomial with controlled error.

We still need a computational strategy to compute S-matrix elements.

Computations

$$\lim_{t\to\infty}(e^{iH_0 t} f_{0+} \Pi \Theta S_4 e^{-2iHt} \Pi e^{iH_0 t} f_{0-})$$

Invariance principle: $H \to f(H) = e^{-\beta H}$

$$= \lim_{t\to\infty}(e^{-ine^{-\beta H_0}} f_{0+} \Pi \Theta S_4 e^{i2ne^{-\beta H}} \Pi e^{-ine^{-\beta H_0}} f_{0-})$$

Leads to following expression for S-matrix elements

$$\langle f_+^0|S|f_-^0\rangle = \langle f_1' f_2'|S|f_1 f_2\rangle =$$

$$\lim_{s\to\infty}\int f_1'^*(\mathbf{p}_1')f_2'^*(\mathbf{p}_2')h_1'(-p_1'^2)h_2'(-p_2'^2)d4p_1'd4p_2'd^4p_1d^4p_2\times$$

$$e^{-ise^{-\beta(\omega_1'(\mathbf{p}_1')+\omega_2'(\mathbf{p}_2'))}}e^{2ise^{-i\beta(p_{10}+p_{20})}}e^{-ise^{-\beta(\omega_1(\mathbf{p}_1)+\omega_2(\mathbf{p}_4))}}\times$$

$$e^{-i((p_{10}+p_{10}')\tau_1+(p_{20}+p_{20}')\chi\tau_2)}\times$$

$$\tilde{S}_4(-p_2',-p_1',p_1,p_2)h_1(-p_1^2)h_2(-p_2^2)f_1(\mathbf{p}_1)f_2(\mathbf{p}_2)$$

No analytic continuation !
existence of limit requires smearing

To calculate $\langle \mathbf{p}_1', \mathbf{p}_2' | T | \mathbf{p}_1, \mathbf{p}_2 \rangle$ assume that it varies slowly on the width of the momentum-space wave packets.

$$\Downarrow$$

$$\langle \mathbf{p}_1', \mathbf{p}_2' | T | \mathbf{p}_1, \mathbf{p}_2 \rangle \approx \frac{i}{2\pi} \frac{\langle f_1' f_2' | (S - I) f_1 f_2 \rangle}{\langle f_1' f_2' \delta(E - E') f_1 f_2 \rangle}$$

for sharply peaked wave packets

Entire calculation can be performed without analytic continuation.

Straightforward generaliztion to arbitrary initial and final states.

Needs Schwinger functions as input.

Explicit formula - no analytic continuation.

Reflection positivity essential.

QCD only need reflection positivity for singlets.

Calculations? How can we perform 12 dimensional interals accurately?

Reflection positivity; structure theorem ?

Implementation on lattice?

Current matrix elements, final state interactions?

Timelike momentum transfers?