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A Poincaré invariant treatment of the three-nucleon problem.

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I summarize recent progress in the treatment of the Poincaré three-nucleon problem at intermediate energies

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The essential properties of mathematical models of few-nucleon systems that are applicable at energy scales up to a few GeV are (1) the model is a quantum theory (2) Poincaré invariance is an exact symmetry of the model (3) the energy spectrum is bounded from below and (4) the model satisfies space-like cluster properties. We discuss models of the three-nucleon system with these properties and demonstrate that these models can provide a realistic description of three-nucleon observables at these energy scales.

The mathematical setting for a quantum theory is a Hilbert space. Probability amplitudes are represented by Hilbert space scalar products of unit normalized rays. The Hilbert space in these models is the tensor product of mass m spin $1/2$ irreducible representation spaces of the Poincaré group, where m is the nucleon mass.

The Poincaré group has ten infinitesimal generators. From these generators it is possible to construct two Casimir invariants $(m^2, w^2 = j^2/m^2)$, four additional mutually commuting observables, $\mathbf{h}$, and 4 operators, $\Delta\mathbf{h}$, that are conjugate to the commuting observables $\mathbf{h}$. The spectrum of the commuting observables $\mathbf{h}$ is determined by the conjugate operators $\Delta\mathbf{h}$ and the spectrum of the Casimir operators. For particles, the eigenvalues of the Casimir operators m and j are the mass and spin of the particle. The Hilbert space of square integrable functions of the eigenvalues of the operators $\mathbf{h}$ over their joint spectrum is a mass m spin j irreducible representation space for the Poincaré group. The most common choice for $\mathbf{h}$ is the three components of the linear momentum and one component of the canonical spin; however, the canonical spin could be replaced by the light-front spin, helicity, or one component of the Pauli-Lubanski vector; the linear momentum could be replaced by light-front components of the four momenta, four velocity, or the Newton-Winger position operator. We denote the single nucleon Hilbert space by $\mathscr{H}_1$.

There is a natural unitary irreducible representation of the Poincaré group on $\mathscr{H}_1$. The Poincaré group Wigner $\mathscr{D}$-functions,

$$\mathscr{D}^{jm}_{\mathbf{h}';\mathbf{h}}[\Lambda, a] := \langle (m,j)\mathbf{h}'|U(\Lambda,a)|(m,j)\mathbf{h}\rangle,$$

in the irreducible basis $\{|(m,j)\mathbf{h}\rangle\}$ are known [1].

Few-nucleon Hilbert spaces are tensor products of single nucleon spaces, $\mathscr{H}_1$. The tensor product of the single nucleon irreducible representations of the Poincaré group defines the kinematic representation of the Poincaré group on the few-nucleon Hilbert space. These kinematic representations are reducible; they can be decomposed into a direct integral of irreducible representations of the Poincaré group using the Poincaré group Clebsch-Gordan coefficients. The basis-dependent Poincaré group Clebsch-Gordan coefficients,

$$C(1,2:3) = \langle (m_1,j_1)\mathbf{h}_1, (m_2,j_2)\mathbf{h}_2|(m_3,j_3)\mathbf{h}_3, \eta\rangle,$$

are known[1]. The parameter η represents invariant degeneracy quantum numbers that separate multiple copies of irreducible representations with the same mass and spin. These Clebsch-Gordan coefficients satisfy

$$\sum\!\!\!\!\!\!\int \mathscr{D}^{j_1 m_1}_{\mathbf{h}_1;\mathbf{h}'_1}[\Lambda,a]\mathscr{D}^{j_2 m_2}_{\mathbf{h}_2;\mathbf{h}'_2}[\Lambda,a]d\mathbf{h}'_1 d\mathbf{h}'_2 C(1',2':3) = \sum\!\!\!\!\!\!\int C(1,2:3')d\mathbf{h}'_3 \mathscr{D}^{j_3 m_3}_{\mathbf{h}'_3;\mathbf{h}_3}[\Lambda,a].$$

Dynamical representations of the Poincaré group are constructed by adding interactions to the non-interacting mass operator, M_0, that commute with the four operators $\mathbf{h}$ that label distinct

vectors in irreducible subspaces, the four conjugate operators $\Delta\mathbf{h}$, and the spin. The interacting mass operator in the non-interacting irreducible basis has a kernel of the form

$$\langle (m,j)\mathbf{h},\eta|M|(m',j')\mathbf{h}',\eta'\rangle = \delta(\mathbf{h};\mathbf{h}')\delta_{jj'}\langle m,\eta|M^j|m',\eta'\rangle$$

where $\delta(\mathbf{h};\mathbf{h}')$ is a product of Dirac delta functions in the continuous variables and Kronecker delta functions in the discrete variables. Diagonalizing M in this basis

$$\sum_j\!\!\!\!\!\!\int \langle m,\eta|M^j|m',\eta'\rangle dm' d\eta' \langle m',\eta'|\psi\rangle = \lambda\langle m,\eta|\psi\rangle$$

gives simultaneous eigenstates of M, j, and $\mathbf{h}$. The resulting eigenfunctions

$$\langle (m,j)\mathbf{h},\eta|(\lambda,j')\mathbf{h}'\rangle = \delta(\mathbf{h};\mathbf{h}')\delta_{jj'}\langle m,\eta|\psi\rangle$$

are complete, and because $\{M,j,\mathbf{h},\Delta\mathbf{h}\}$ have the same commutation relations as $\{M_0,j,\mathbf{h},\Delta\mathbf{h}\}$, the eigenstates transform irreducibly. The matrix elements of the irreducible representation in this basis are identical to the Poincaré Wigner $\mathscr{D}$ functions for the free particle representation of the same spin with the particle mass m replaced by the eigenvalue λ of M. These Poincaré-Wigner $\mathscr{D}$ functions define all of the matrix elements of a dynamical representation of the Poincaré group in this basis of irreducible eigenstates of M. The spectral condition is satisfied if the binding energy is less than the sum of the masses of the bound particles.

The problem of constructing realistic two-body interactions as input to the relativistic three-nucleon problem is solved by using existing realistic nucleon-nucleon interactions [2][3] as input. If we define the relative momentum operator $\mathbf{k}^2$ as a function of the kinematic two-body mass operator by

$$M_0(\mathbf{k}^2) := \sqrt{\mathbf{k}^2+m_1^2}+\sqrt{\mathbf{k}^2+m_2^2},$$

then a dynamical two-body mass operator is defined by

$$M = M_0(\mathbf{k}^2+2\mu V)$$

where μ is the two-nucleon reduced mass and $\langle (m',j')\mathbf{h}',\eta'|V|(m,j)\mathbf{h},\eta\rangle = \delta_{j'j}\delta(\mathbf{h};\mathbf{h}')\langle k,\eta|V^j|k,\eta\rangle$, where $\langle k,\eta|V^j|k,\eta\rangle$ is the kernel of a non-relativistic potential that is fit to differential scattering cross section data correctly transformed to the two-body center of momentum frame. Time-dependent scattering theory, along with the Kato-Birman invariance principle, can be used to show that the Møller wave operators for the Hamiltonian associated with the mass operator, M, and the non-relativistic Hamiltonian associated with the interaction, V, are identical functions of $\mathbf{k}^2$ [1]. It follows that the relativistic and non-relativistic S-matrices are identical functions of $\mathbf{k}^2,\eta$ and ,j. For nucleon-nucleon applications the Clebsch-Gordan coefficients can be designed so the degeneracy parameters η have the same spectrum as the non-relativistic l^2 and s^2 that appear as variables in typical nucleon-nucleon interactions. This means that high-precision interactions fit to scattering data can be used directly in Poincaré invariant two-nucleon models without modification.

Cluster properties require that multiparticle representations of the Poincaré group $U(\Lambda,a)$ approach tensor products of the subsystem representations, $U_i(\Lambda,a)$, when the subsystems are asymptotically separated by large space-like displacements:

$$\lim_{(b_i-b_j)^2\to\infty} ||(U(\Lambda,a)-\otimes U_k(\Lambda,a))\prod U_l(I,b_l)|\psi\rangle|| = 0. \qquad (1)$$

For a system consisting of an interacting pair of nucleons and a spectator nucleon it is possible to first decompose the tensor product of three irreducible representations of the Poincaré group into a superposition of irreducible representations and then add the two-body interaction. Alternatively, it is also possible to add an interaction directly to the two-body irreducible representation to construct an interacting two-body representation and then decompose the tensor product of that representation and the spectator representation into a three-body irreducible representation. In both cases the result is a three-body irreducible representation of the Poincaré group with an interacting pair of particles. If the non-trivial part of the interaction kernel, $\langle k',\eta'|V^j|k,\eta\rangle$, is the same in both constructions then both constructions give identical scattering matrices. It is also easy to demonstrate that the first construction fails to satisfy the cluster property (1) while the second one satisfies property (1) by construction.

The advantage of the first approach is that two-body interactions for different pairs can be combined in a manner that preserves the Poincaré symmetry. This is because the two-body interactions for each pair commute with the $j,\mathbf{h},\Delta\mathbf{h}$ of the non-interacting three-body irreducible representation. This construction gives a unitary representation of the Poincaré group for three interacting particles that unfortunately fails to satisfy cluster properties. The identity of the S-matrices for the separate 2+1 body problems implies [4] that the two $2+1$-body representations are related by a unitary operator, $A_{(ij)(k)}$, called a scattering equivalence. Cluster properties can be restored without breaking the Poincaré invariance by multiplying the three-body representation that fails to satisfy cluster properties by the (symmetrized) [5][1] product of the unitary operators $A_{(ij)(k)}$ for all three pairs. Because the product of scattering equivalences is a scattering equivalence, this transformation does not change the on-shell three-body S matrix. It also shows that even though the untransformed representation of the Poincaré group fails to satisfy cluster properties, the resulting S matrix satisfies cluster properties. These scattering equivalences are only needed when the three-body solutions are used in four- or more-body models or when they are used to compute matrix elements of electroweak current operators.

Note that the input relativistic two-body interactions fit to experiment and cluster properties fix the few-body dynamical operators up to three- or more-body interactions. Kinematic subgroups normally associated with Dirac's forms of dynamics correspond to different choices of $\mathbf{h}$ and $\Delta\mathbf{h}$. Models with a given kinematic subgroup are scattering equivalent to models with any other kinematic subgroup, but the scattering equivalences generate many-body interactions under change of representation.

The operator form of the Faddeev equation for the three-body problem is identical to the corresponding non-relativistic equation. The differences are (1) how the two-body interactions are embedded in the Faddeev kernel (2) the structure of the recoupling coefficients, which in the relativistic case are Racah coefficients of the Poincaré group that change the order of the pairwise coupling of irreducible representations (3) and kinematic factors. Relativistic effects are measured by comparing the difference between the relativistic and non-relativistic three-body calculations with identical two-body input. Since in both cases the two-body interactions are designed to fit experiment, there are no relativistic corrections at the two-body level.

Technical differences in the relativistic and non-relativistic Faddeev equation appear when the kernel and driving terms are evaluated in a basis[6][7]. In the relativistic case the two-body interactions appear inside of square root operators, the Racah coefficients for the Poincaré group

have a non-trivial spin and momentum dependence, and for systems of more than three particles it is necessary to compute the scattering equivalences that restore cluster properties. While these properties make the relativistic model more complicated from a computational point of view, it is possible to treat all of these complications without approximation [8][9][10][7].

For realistic interactions the Faddeev equations have been solved for energies up to 250 MeV in a representation with a Euclidean kinematic subgroup using a partial wave expansion[11] [12]. To extend these calculations to reactions with energies above 250 MeV we have demonstrated that the Faddeev equations can also be accurately solved by direct integration without partial waves [14][15][16]. Converged results, using direct integration, are obtained for energies up to 2 GeV.

At low energies, calculations with realistic interactions exhibit only small differences with the corresponding non-relativistic results [11] . Corrections to the Triton binding energy depend on the two-body interaction, but generally lead to a small decrease in the binding energy[17][18], typically less than a tenth of an MeV. The low-energy calculations also show nontrivial contributions to the observable A_y at energies (5-13 MeV) [12] due to relativistic spin rotations, confirming the scale, but not the sign of an effect observed by Miller and Schwenk[13] in a simpler model. While these relativistic contributions move the calculations away from the experimental results, they imply that these effects need to be accounted for in final resolution of the A_y puzzle.

The higher-energy calculations based on direct integration used the spin-independent Malfliet-Tjon interaction. These calculations demonstrated that approximations that include only relativistic kinematics can lead to large "effects" that are almost completely canceled by dynamical relativistic effects. These calculations [15][16] also demonstrated the non-uniformity of the convergence of the multiple scattering series for both elastic scattering and breakup reactions. For breakup reactions away from the quasielastic peak at least one iteration of the series was required to obtain converged results, even above 1 GeV. There were also departures from the non-relativistic calculations based on the same two-body interaction in the neighborhood of the quasielastic peak. Relativistic and non-relativistic exclusive breakup calculations at 500 MeV show a different energy dependence [15] for different pairs of proton angles. The behavior of the relativistic calculations with the Malfilet-Tjon interaction seem to explain differences between experiment and the corresponding non-relativistic calculation [15] for a large number of angle pairs.

To test the effects of ignoring the scattering equivalences that restore cluster properties in four-body calculations we (1) turned off the interaction with one of the nucleons in the three-body construction outlined above and (2) alternatively took a tensor product of a two-body calculation and a spectator nucleon. The two representations are scattering equivalent. We assumed that the interacting nucleons were bound in a deuteron state. Then we calculated the charge form factor for the electron scattering off of the spectator proton in the presence of the free deuteron. In the tensor product representation the momentum of the deuteron does not affect the form factor, however in the representation where the two-body interaction is added to the non-interacting three-body irreducible representation, there is a clear dependence of the form factor on the momentum of the deuteron. Figure 1 illustrates the difference between the these two calculations as a function of momentum transfer and the component of the momentum of the deuteron parallel to the momentum transfer. This unphysical effect can be as large as 6% and is a consequence of not transforming to a representation where both the current and dynamics clusters. This suggests that the three-nucleon wave functions should be transformed to a representation where the dynamics satisfies

cluster properties before they can be used in the computation of electromagnetic observables.

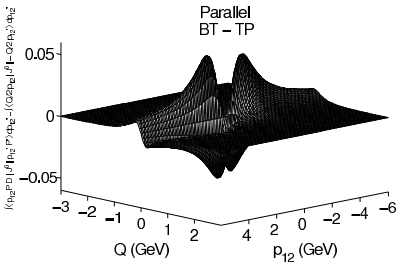

fig. 1

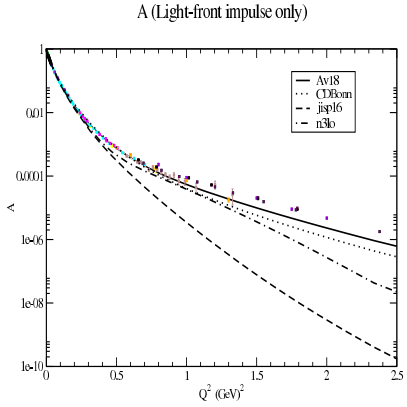

fig. 2

We also tested some modern soft interactions [19][20] that are precisely fit to the same scattering data as realistic meson-exchange interactions. These interactions are designed to be used in low-energy many-body calculations; the transformations that reduce the contribution of the high-momentum components to the two-body interactions also generate many-body interactions and exchange currents. It is interesting to investigate the extent to which these kinds of interactions can be used in models that study intermediate energy dynamics without introducing the generated three-body interactions and exchange currents. In this case we considered the problem of elastic electron-deuteron scattering in a Poincaré covariant impulse approximation in a model with a light-front kinematic subgroup. We performed calculations using AV18 [3] and CD-Bonn [2] interactions with calculations using soft interactions fit to the same two-body data. All calculations used the same impulse current as input. While each calculation requires different exchange current contributions, calculations of all three electromagnetic observables for momentum transfers in the GeV range were well described using standard meson exchange potential like Argonne V18 or CD Bonn. For these interactions the small discrepancies with experiment in all three observables are due to well-understood two-body exchange currents. On the other hand, for the calculations using the N3LO or the JISP 16 wave functions the exchange current contributions that are needed to explain the discrepancies between calculations and experiment become significant in the observable "A" for momentum transfers between .5 and 1 GeV^2, in "B" for momentum transfers above .5 GeV^2 and in the tensor polarization for momentum transfers above .15 GeV^2. Figure 2. shows the result of the four calculations of A up to $Q^2 = 2.5$ (GeV)^2. So while these soft interactions are clearly useful at low energy, the additional exchange currents that are required in a relativistic calculation limit the benefits of having a softer potential.

The research summarized above indicates that it is now possible to extend the few-nucleon physics program that has been successful at low energies to treat few-nucleon problems at interme-

diate energy scales in a manner that respects all fundamental principles of physics that are relevant at these scales.

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