Relativistic Quantum Theories and Neutrino Oscillations

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Abstract

Neutrino oscillations are examined under the broad requirements of Poincaré-invariant scattering theory in an S-matrix formulation. This approach can be consistently applied to theories with either field or particle degrees of freedom. The goal of this paper is to use this general framework to identify all of the unique physical properties of this problem that lead to a simple oscillation formula. We discuss what is in principle observable, and how many factors that are important in principle end up being negligible in practice.

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I. INTRODUCTION

There is now strong experimental evidence that neutrino flavor eigenstates are mixed via a non-diagonal matrix that connects them to neutrino mass eigenstates [1, 2]. The fact that space-time propagation is governed by mass rather than flavor eigenstates gives rise to the possibility of oscillation between flavor states over space and time.

There is a large body of literature addressing various aspects of the quantum mechanics of neutrino oscillation. Many of the primary issues were set forth by Kayser [3] in 1981. Since then, this subject has been examined from a variety of perspectives [4–13].

The phenemenon of neutrino oscillations seems simple enough from the perspective of introductory quantum mechanics, for which there are countless examples of oscillations in two- and three-level systems. There are, however, differences in the neutrino case that tend to work against the intuition of the standard examples: the initial conditions for neutrino production in reactors or the Sun are generally not controlled; the coherence length over which states are virtual can be kilometers rather than subatomic distances, and this affects considerations such as what is large or small in a calculation; one of the weak interactions that determine the oscillation interval takes place inside the volume of the detector rather than far from it; the neutrino kinematics are ultrarelativistic, but not fully so. There is also the distinction between what is measured in an experiment and features of a specific theoretical approach. These issues have led to a variety of approaches as cited above. Some of these approaches have differing perspectives [10, 11], yet all lead to the same simple oscillation formula.

In this paper, we provide an approach to neutrino oscillations from the perspective of scattering theory in relativistic quantum mechanics. The virtue of this approach is its generality and its focus on what can actually be measured in oscillation experiments. The scattering theory approach advocated in this paper has been examined [14] using quantum field theory, but the conclusions that follow from our approach are not limited to a field theoretic treatment. There are many papers on neutrino oscillations, and many of them have significant overlap with this work. Our goal here is to provide a very general framework utilizing the S matrix.

The S matrix for a neutrino oscillation experiment is the probability amplitude for transitions between states where both initial and final particles are localized near two macroscopically separated space-time points; the points of neutrino creation and neutrino absorption. We discuss the additional physical conditions that must be satisfied for the transition amplitude which, when squared, leads to a neutrino oscillation formula. There is also a freedom in the choice of variables used to label the single-particle intermediate states, the choice of scattering equivalent Hamiltonian, as well as the choice of kinematic symmetries of the interaction. Different choices lead to the same scattering matrix, but will normally lead to different formulae in specific implementations and different interpretations of the dynamics. We find that the conditions that lead to a simple oscillation formula also combine to render as insignificant many factors that might otherwise serve to distinguish among various theoretical approaches.

II. FORMAL BASIS: POINCARÉ INVARIANT QUANTUM MECHANICS

We present here an overview of the basic ingredients of quantum mechanical systems that satisfy relativistic invariance. Further details can be found in Ref. [15].

Relativistic invariance in a quantum theory means that a change of inertial coordinate system is a symmetry of the theory. In special relativity, inertial coordinate systems are related by space-time translations and proper orthochronous Lorentz transformations. The group generated by these transformations is the Poincaré group. Proper orthochronous Lorentz transformations do not include the discrete transformations associated space reflection and time reversal, which are broken by the weak interaction. Wigner proved [16] that the existence of a unitary representation, $U(\Lambda, a)$ of the Poincaré group is both a necessary and a sufficient condition for a quantum theory to be relativistically invariant.

The dynamics are given by the time-translation subgroup of the Poincaré group, which is generated by the Hamiltonian, H. Consistency of the initial value problem requires that interactions must also appear in additional Poincaré generators. This means that at most a subgroup of the Poincaré group can be independent of interactions. When such a subgroup exists, it is called a kinematic subgroup. While the existence of a kinematic subgroup is a choice of representation, the choice of representation may imply kinematic symmetries of the interaction that have no impact on the scattering matrix elements; this is an example of a representation-dependent feature of an interaction that is not experimentally observable.

The largest kinematic subgroups were classified by Dirac [17]; the classification was completed by Patera and Winternitz [18]. The three largest kinematic subgroups are the three-dimensional Euclidean group (instant-form dynamics), the Lorentz group (point-form dynamics) and the subgroup that leaves a three-dimensional hyper-plane tangent to the light cone invariant (null-plane dynamics).

Interactions in a dynamical model where the kinematic subgroup is the three-dimensional Euclidean group are translationally invariant and thus conserve three-momentum; interactions in a dynamical model with a null-plane kinematic symmetry are invariant with respect to translations in the null plane and thus conserve components of the four momentum that the generate translations on the null plane; while interactions in a dynamical model where the kinematic symmetry is the Lorentz group are Lorentz invariant.

Scattering theory in a Poincaré invariant quantum theory can be formulated using the same time-dependent methods, based on dynamical and asymptotic Hamiltonians, H and H_0. that are used in both non-relativistic quantum mechanics and quantum field theory.

Scattering states $|\Psi(t)\rangle$ are solutions of the time-evolution equation (Schrödinger equation)

$$|\Psi_\pm(t)\rangle = e^{-iHt}|\Psi_\pm(0)\rangle, \tag{1}$$

where the initial condition, $|\Psi_\pm(0)\rangle$, is determined by an asymptotic condition. There are two natural scattering asymptotic conditions; they require that the state $|\Psi_\pm(t)\rangle$ approach a state of non-interacting particles, $|\Phi_\pm(t)\rangle$, in the asymptotic future or past:

$$\lim_{t\to\pm\infty} \| |\Psi_\pm(t)\rangle - |\Phi_\pm(t)\rangle \| = 0 \tag{2}$$

where

$$|\Phi_\pm(t)\rangle = e^{-iH_0t}|\Phi_\pm(0)\rangle \tag{3}$$

is the non-interacting state. In the field theoretic case the H_0 must include self-interactions. The appropriate generalization of the asymptotic condition (2), first formulated by Haag and Ruelle, is discussed in [19][20][21].

Equations (1-3) imply that the initial condition for the scattering states are related to the initial conditions for the non-interacting states by

$$|\Psi_\pm(0)\rangle = \lim_{t\to\pm\infty} e^{iHt}e^{-iH_0t}|\Phi_\pm(0)\rangle := \Omega_\pm|\Phi_\pm(0)\rangle. \tag{4}$$

The probability amplitude that a state prepared to become $|\Phi_-(t)\rangle$ in the asymptotic past scatters into a state that becomes $|\Phi_+(t)\rangle$ in the asymptotic future is

$$\langle S\rangle = \langle\Psi_+(t)|\Psi_-(t)\rangle = \langle\Psi_+(0)|\Psi_-(0)\rangle = \langle\Phi_+(0)|S|\Phi_-(0)\rangle \tag{5}$$

where

$$S := \Omega_+^\dagger \Omega_- \tag{6}$$

is the scattering operator, and we have exploited the invariance of the probability amplitudes under time translation to emphasize that the probability amplitude can be computed using states at any common time. We will use this in formulating scattering involving neutrino intermediate states.

We note that if H is transformed with a unitary transformation A such that $H' = AHA^\dagger$, and A satisfies

$$\lim_{t\to\pm\infty} \|(I-A)e^{-iH_0t}|\Psi\rangle\| = 0 \tag{7}$$

for *both* time limits then

$$S = S(H, H_0) = S(H', H_0). \tag{8}$$

Note that H_0 is not transformed, so the representation of the asymptotically free particles remains unchanged. This means $V = H - H_0$ and $V' = H' - H_0$ are distinct interactions that cannot be experimentally distinguished. Operators A satisfying the asymptotic property (7) are called scattering equivalences. They preserve the scattering matrix without changing H_0. Ekstein [22] proved that the existence of such operators is a necessary and a sufficient condition for two Hamiltonians to be scattering equivalent.

Any experiment that only measures scattering matrix elements cannot distinguish different scattering equivalent Hamiltonians. If a dynamical representation of the Poincaré group has a given kinematic subgroup, it is possible to construct unitary scattering equivalences [23] that change the kinematic subgroup to any other kinematic subgroup, so any properties attributed to a particular kinematic symmetry are not observable.

III. ASYMPTOTIC STATES

The probability amplitude for a transition from the prepared initial state to the measured final state can be expressed in terms of a scattering operator S. As discussed in the previous section, the scattering probability amplitude is the inner product of two solutions of the time-evolution equation with past and future asymptotic conditions evaluated at any common time. We can describe neutrino scattering reactions using an S-matrix approach. This was first done by Cardall [14] within a field-theoretic framework.

As an illustrative example we consider a reaction where an initial proton and electron interact to produce a linear combination of neutrino mass eigenstates which propagate over a macroscopic distance until they are absorbed by a second interaction that produces a final proton and electron. In the language of second-order perturbation theory this will involve two successive reactions, e.g.,

$$
\begin{aligned}
p + e &\to n' + \nu \, (a) \\
n + \nu &\to p' + e' \, (b).
\end{aligned} \tag{9}
$$

This example can be generalized to other cases of interest without affecting the overall conclusions, since the role of the intermediate neutrino mass eigenstates is the same. In this example, the asymptotic states are given by $\{p, n, e, p', n', e'\}$ and not the neutrino mass eigenstates, which are virtual.

For the cases of interest, reaction (b) takes place inside a neutrino detector volume, and another device records that event via the emerging charged lepton. This is distinct from more typical applications in which the S matrix describes one or more events that are distinctly separated from any detection equipment.

The asymptotic free-particle state before the reaction is represented by a localized wave packet describing an electron e moving toward the proton p; in addition, there is also a free neutron n that is traveling toward the point where it will eventually interact with the neutrino mass eigenstates.

Similarly, the asymptotic free-particle state after the reaction is represented by a localized wave packet describing a neutron n' traveling away from the region of the initial interaction, and an electron e' and proton p' traveling away from the point where the neutrino mass eigenstate was absorbed by the initial neutron n.

There is a definite probability amplitude for a transition from the initial state containing $\{p, e, n\}$ to the final state containing $\{p', e', n'\}$. The unusual feature is that there is not a single localized space-time region where the initial and final states overlap. Because we only need to construct the scattering states, $|\Psi_{\pm}(t)\rangle$, whose inner product can be evaluated at any common time, the S matrix can be computed in the same manner that is used with more traditional asymptotic states.

For the example discussed above, the reaction is characterized by two disjoint space-time regions localized about x_a, where the coherent neutrino mass eigenstate superposition is created, and x_b, where it is absorbed. In our example the initial proton p and electron e and final neutron n' are localized near x_a at a common time t_a, and the initial neutron n and final proton p' and electron e' are localized near x_b at a later common time t_b.

In what follows we generalize the expressions to include all lepton flavors (e, μ, τ), labeled by Greek indices α, β.

The construction of asymptotic states corresponding to this reaction starts with normalizable single-particle states that localize the particles near $\mathbf{x}_a$ at time t_a or $\mathbf{x}_b$ at time t_b with the appropriate expectation value for the initial or final momenta of the observed particles. In order to localize these states at the different points $\mathbf{x}_a$ and $\mathbf{x}_b$ we initially localize them at the origin and use single particle spatial translations, $T_j(-\mathbf{x}_c)|\phi_j\rangle$, $c = a$ or b, to localize the j-th particle near $\mathbf{x}_a$ or $\mathbf{x}_b$. The $-$ sign is consistent with

$$\mathbf{x}_c = \langle\phi|(\mathbf{x} + \mathbf{x_c})|\phi_j\rangle = \langle\phi_j|T_j(\mathbf{x_c})\mathbf{x}T_j^\dagger(\mathbf{x_c})|\phi_j\rangle = \langle\phi_j|T_j^\dagger(-\mathbf{x}_c)\mathbf{x}T_j(-\mathbf{x}_c)|\phi\rangle. \tag{10}$$

From Eq. 4, we have that

$$|\Psi_\pm(t)\rangle = \Omega_\pm|\Phi_\pm(t)\rangle. \tag{11}$$

This means that the initial and final scattering states can be transformed to a common time by transforming the non-interacting asymptotic states to the common time. To construct the non-interacting multi-particle asymptotic states at a common time, all of the particles that are near $\mathbf{x}_b$ at time t_b are time-translated to where they would be at time $t = t_a$ using single-particle time evolution:

$$T_j(t_a - t_b)T_j(-\mathbf{x}_b)|\phi_j\rangle = T_j(t_a - t_b, -\mathbf{x}_b)|\phi_j\rangle. \tag{12}$$

Choosing $t_a = 0$ gives initial and final time-zero asymptotic states of the form

$$|\Phi_\alpha(0)\rangle = \prod_i T(-x_a)|\phi_{i\alpha}\rangle \prod_j T(-x_b)|\phi_{j\alpha}\rangle. \tag{13}$$

Here the single-particle states $|\phi_{i\alpha}\rangle$ are initially localized at the origin, and all of the "reaction geometry" appears in the one-body space-time translation operators, $T_i(-x_i)$. In the absence of interactions these non-interacting asymptotic states, $|\Phi_\alpha(0)\rangle$, put the initial or final reaction products that interact at time $t = t_a = 0$ near $\mathbf{x}_a$ and the initial and final reaction products that interact at time $t = t_b$ near $\mathbf{x}_b$. Because all of the "reaction geometry" is in the structure of $|\Phi_\alpha(0)\rangle$, the formal scattering operator, S, is unchanged and can be calculated using standard methods of quantum mechanics or quantum field theory.

In what follows we use the notation $S_{fi}(x_b, x_a)$ for S-matrix elements with asymptotic states describing neutrinos created at space-time point x_a and absorbed at space-time point x_b.

In this formulation of the scattering problem, the space-time displacement, $x_{ba} := x_b - x_a$, transforms like a four-vector under Lorentz transformations. To show this, note that the scattering matrix element is Lorentz invariant:

$$|\Psi_\alpha^\pm\rangle \to |\Psi_\alpha^{\pm\prime}\rangle := U(\Lambda, 0)|\Psi_\alpha^\pm\rangle \qquad \to \qquad S_{fi} = S'_{fi}. \tag{14}$$

The effect of this transformation on the asymptotic states follows from the Poincaré invariance of the wave operators

$$U(\Lambda, a)\Omega_\pm = \Omega_\pm U_0(\Lambda, a) \tag{15}$$

where $U_0(\Lambda, a)$ is a product of single-particle unitary representations of the Poincaré group, $U_j(\Lambda, a)$. The Poincaré invariance of the wave operators ensures that the representations of the Poincaré group for the free and interacting systems agree when the particles are asymptotically separated. Because $T_j(x_c) = U_j(I, x_c)$, the group representation property implies

$$U_j(\Lambda, 0)T_j(-x_c) = T_j(-\Lambda x_c)U_j(\Lambda, 0). \tag{16}$$

It follows from (15) and (16) that

$$S_{fi}(x_b, x_a) = S_{f'i'}(\Lambda x_a, \Lambda x_b) \tag{17}$$

where the single particle states localized at the origin are replaced by the transformed states localized at the origin

$$|\phi'_{j\alpha}\rangle = U_j(\Lambda, 0)|\phi_{j\alpha}\rangle. \tag{18}$$

Thus, in this formalism, dynamical Poincaré transformations imply the expected transformation properties of the parameters that describe the geometry of the interaction region. This will be relevant for the oscillation formula, which in this formalism is determined by considering how the scattering operator behaves when these parameters are varied.

IV. SECOND-ORDER S MATRIX

In this section we examine the conditions that lead to neutrino oscillations in the S-matrix approach. We assume that the weak interaction can be treated perturbatively. In our example the leading contribution to the scattering matrix is of second order in the weak interaction. As noted above, the initial weak interaction at space-time point x_a produces a superposition of neutrino (or anti-neutrino) mass eigenstates, which propagate to the space-time point x_b of the final weak interaction. We assume that the initial particle(s) collide or decay near the space-time point x_a producing a neutrino or anti-neutrino and final particle(s). The neutrino or anti-neutrino travels and interacts with initial particle(s) near space-time point x_b to produce final particles. The superposition of neutrino mass eigenstates is never directly observed; it appears only as an intermediate state. The second-order calculation can formally be done in the interaction representation, where the H_0 includes all interactions except for the interactions with the neutrinos.

For the purpose of illustration, we will assume that the kinematic subgroup of the Poincaré group is the three-dimensional Euclidean group, generated by spatial translations and rotations. This is called an "instant form" dynamics. In an instant-form dynamics it is natural to label single-particle basis vectors, $|\mathbf{p}\rangle$, by the particle's three momentum $\mathbf{p}$, while the fourth component p^0 satisfies the mass-shell condition $p^0 = (m^2 + \mathbf{p}^2)^{\frac{1}{2}}$. Spin labels are suppressed. The weak interaction, V, that couples to the neutrinos creates flavor eigenstates that can be decomposed into mass eigenstates that propagate. In an instant-form dynamics the neutrino production interaction is invariant with respect to kinematic translations and thus conserves the sum of the single-particle momenta. As mentioned earlier, this conservation law is related to the freedom to choose among scattering-equivalent interactions and does not affect the S matrix. There are many other scattering-equivalent interactions that do not have this kinematic symmetry.

The second-order contribution to the kernel of scattering operator, S, for our example reaction is

$$\langle \mathbf{p}'_{\beta b}, \mathbf{p}'_{pb}, \mathbf{p}'_{na} | S | \mathbf{p}_{\alpha a}, \mathbf{p}_{pa}, \mathbf{p}_{nb} \rangle =$$

$$-2\pi i \delta^4(p_a + p_b - p'_a - p'_b) \times$$

$$\sum_{j=1}^{3} \frac{\langle (\mathbf{p}'_{\beta b}, \mathbf{p}'_{pb})^+ \| V^\dagger_{j\beta} \| \mathbf{p}_{nb}, \mathbf{p}_{\nu_j} \rangle \langle \mathbf{p}'_{na}, \mathbf{p}_{\nu_j} \| V_{j\alpha} \| (\mathbf{p}_{\alpha a}, \mathbf{p}_{pa})^- \rangle}{E_a - E_{a'} - E_{\nu_j} + i0^+}, \tag{19}$$

where p_a, p_b, p'_a, p'_b are the initial and final four momenta at points a and b. The Roman index j labels the mass eigenstate, and the neutrino energy is

$$E_{\nu_j} = \sqrt{m_{\nu_j} - \mathbf{p}_{\nu_j}^2}. \tag{20}$$

In addition, the following relations are specific to an interaction with the three-dimensional Euclidean group as a kinematic subgroup:

$$\langle \mathbf{p}'_{na}, \mathbf{p}_{\nu_j} | V_{j\alpha} | (\mathbf{p}_{\alpha a}, \mathbf{p}_{pa})^- \rangle = \delta(\mathbf{p}'_{na} + \mathbf{p}_{\nu_j} - \mathbf{p}_{\alpha a} - \mathbf{p}_{pa}) \langle \mathbf{p}'_{na}, \mathbf{p}_{\nu_j} \| V_{j\alpha} \| (\mathbf{p}_{\alpha a}, \mathbf{p}_{pa})^- \rangle. \tag{21}$$

The sum in (19) is over neutrino mass eigenstates. The $()^{\pm}$ states are electron-proton scattering eigenstates. Equation (19) is a generalization of the standard "two-potential" formula of Gell-Mann and Goldberger [24] where cluster properties have been used to factorize the incoming and outgoing scattering states into products of independent scattering states associated with the reactions at x_a and x_b.

The neutrinos enter in the matrix elements

$$\langle (\mathbf{p}'_{\beta b}, \mathbf{p}'_{pb})^+ \| V^{\dagger}_{j\beta} \| \mathbf{p}_{nb}, \mathbf{p}_{\nu_j} \rangle \tag{22}$$

and

$$\langle \mathbf{p}'_{na}, \mathbf{p}_{\nu_j} \| V_{j\alpha} \| (\mathbf{p}_{\alpha a}, \mathbf{p}_{pa})^- \rangle. \tag{23}$$

The three-momentum delta function in Eq. (21) implies that in this representation the virtual neutrino three-momentum is constrained by the external kinematics:

$$\mathbf{p}_{\nu_j} = \mathbf{p}_{\alpha a} - \mathbf{p}_{pa} - \mathbf{p}_{na}. \tag{24}$$

The flavor of the initial and final states determines the flavor of the created/detected neutrinos. These matrix elements are constructed from an elementary vertex that involves the neutrino mixing matrix.

It is useful to include explicitly the one-body space-time translation operators used in the construction of the non-interacting asymptotic states with the kernel of the scattering operator, S. Using the four-momentum conservation of the second-order contribution to the S operator gives the expression

$$e^{i(p'_{\beta b} + p'_{pb} - p_{nb}) \cdot (x_b - x_a)} \langle \mathbf{p}'_{\beta b}, \mathbf{p}'_{pb}, \mathbf{p}'_{na} | S | \mathbf{p}_{\alpha a}, \mathbf{p}_{pa}, \mathbf{p}_{nb} \rangle =$$

$$-2\pi i \delta^4 (p_{\alpha a} + p_{pa} + p_{nb} - p'_{\beta b} - p'_{pb} - p'_{na}) e^{i(p'_{\beta b} + p'_{pb} - p_{nb}) \cdot (x_b - x_a)} \times$$

$$\sum_j \frac{\langle(\mathbf{p}'_{\beta b},\mathbf{p}'_{pb})^+\|V^\dagger_{j\beta}\|\mathbf{p}_{nb},\mathbf{p}_{\nu_j}\rangle\langle\mathbf{p}'_{na},\mathbf{p}_{\nu_j}\|V_{j\alpha}\|(\mathbf{p}_{\alpha a},\mathbf{p}_{pa})^-\rangle}{E_{\alpha a}+E_{pa}-E'_{na}-E_{\nu_j}+i0^+}. \tag{25}$$

A probability amplitude is obtained by integrating this kernel over initial and final wave packets associated with states localized at the origin. The relevant matrix elements have the form

$$\begin{gathered}
S_{fi}(x_b,x_a) = \\
\int \phi^*_{\beta f}(\mathbf{p}_{\beta b})\phi^*_{pf}(\mathbf{p}_{pb})\phi^*_{nf}(\mathbf{p}_{na})d\mathbf{p}_{\beta b}d\mathbf{p}_{pf}d\mathbf{p}_{na}\times \\
e^{i(p'_{\beta b}+p'_{pb}-p_{nb})\cdot(x_b-x_a)}\langle\mathbf{p}'_{\beta b},\mathbf{p}'_{pb},\mathbf{p}'_{na}|S|\mathbf{p}_{\alpha a}\mathbf{p}_{pa},\mathbf{p}_{nb}\rangle\times \\
d\mathbf{p}_{\alpha a}d\mathbf{p}_{pa}d\mathbf{p}_{nb}\phi_{\alpha i}(\mathbf{p}_{\alpha a})\phi_{pi}(\mathbf{p}_{pa})\phi_{ni}(\mathbf{p}_{nb}) = \\
-2\pi i\int \phi^*_{\beta f}(\mathbf{p}_{\beta b})\phi^*_{pf}(\mathbf{p}_{pb})\phi^*_{nf}(\mathbf{p}_{na})d\mathbf{p}_{\beta b}d\mathbf{p}_{pf}d\mathbf{p}_{na}\times \\
\delta^4(p_{\alpha a}+p_{pa}+p_{nb}-p'_{\beta b}-p'_{pb}-p'_{na})e^{i(p'_{\beta b}+p'_{pb}-p_{nb})(x_b-x_a)}\times \\
\sum_j \frac{\langle(\mathbf{p}'_{\beta b},\mathbf{p}'_{pb})^+\|V^\dagger_{j\beta}\|\mathbf{p}_{nb},\mathbf{p}_{\nu_j}\rangle\langle\mathbf{p}'_{na},\mathbf{p}_{\nu_j}\|V_{j\alpha}\|(\mathbf{p}_{\alpha a},\mathbf{p}_{pa})^-\rangle}{E_{\alpha a}+E_{pa}-E'_{na}-E_{\nu_j}+i0^+}\times \\
d\mathbf{p}_{\alpha a}d\mathbf{p}_{pa}d\mathbf{p}_{nb}\phi_{\alpha i}(\mathbf{p}_{\alpha a})\phi_{pi}(\mathbf{p}_{pa})\phi_{ni}(\mathbf{p}_{nb})+\cdots .
\end{gathered} \tag{26}$$

Equation (26) assumes that the weak interaction can be treated using perturbation theory. It was derived in a representation in which the kinematic subgroup is the three-dimensional Euclidean group. Equivalent calculations using different kinematic subgroups will have a similar form, but will have different "off-shell" neutrino variables. This equation provides the starting point for derivations of the oscillation formula.

V. ESSENTIAL INGREDIENTS FOR OSCILLATIONS

Equation (26) describes the scattering matrix element for production and absorption of a superposition of neutrino mass eigenstates as a second-order perturbation in the weak interaction. Oscillations require that this expression can be approximated by a sum of non-vanishing neutrino mass combinations multiplied by different phase factors. We now examine the essential assumptions that are needed to derive the standard simple oscillation formula in the literature from Eq. (26).

A. (almost) ultrarelativistic

Experiments focused on neutrino mixing lead to constraints upon the squared differences among neutrino masses, with a scale much less than that of the virtual neutrino momenta, which make the neutrinos very relativistic.

An important property of very relativistic neutrinos is that neutrinos with different masses propagate with approximately the same velocity, near the speed of light. This ensures that the wave packets associated different mass neutrinos continue to overlap over macroscopic distances where oscillations can be observed.

For very relativistic neutrinos the energy of a mass eigenstate with three-momentum $p_j = |\mathbf{p}_j|$ can be approximated by:

$$E_j \approx p_j + \frac{m_j^2}{2p_j}. \tag{27}$$

In a representation like Eq. (26), where the interactions are chosen to be kinematically translationally invariant, the neutrino momenta, p_j, are the same for all j. Oscillations are sensitive to the difference of these energies for different mass neutrinos. Contributions from corrections beyond this expression are very small in the relativistic limit.

The leading phase differences are thus proportional to $(m_i^2 - m_j^2) = (m_i - m_j)(m_i + m_j)$. This needs to be small enough so variations in the phase are small over the scale of the size of the single particle wave packets.

It is not clear whether neutrinos with larger mass differences would lead to oscillations. It has been shown, for example, that a non-relativistic massive neutrino cannot oscillate against much lighter partners [4].

B. interactions and factorization

The operator $V_{j\alpha}$ in Eq. (26) connects the leptonic and hadronic spaces, but factors into separate components for each space. This is the case whether one uses the contact Fermi interaction or a diagram with W exchange. The hadronic contribution will involve vector and axial vector current matrix elements that are measurable; they represent a separate factor in the S matrix and contribute to the normalization of S, but not the oscillation formula.

The leptonic contribution for our example is a matrix element of the charge current interaction of the form $\langle\beta\|J^{\mu}_{V-A}\|\nu_j\rangle$. This matrix depends in principle on the masses of the neutrino j and the charged lepton β. However, the leading contribution to the neutrino mass comes from the oscillation phase described above, and so we can substitute an approximate form for the matrix element by substituting $\nu_j \to \nu_0$, where ν_0 here and henceforth denotes a massless neutrino of arbitrary flavor. The lepton matrix element does, however, depend upon the mass of the charged lepton. Put another way, the response of the detector to a specific charged lepton flavor depends upon the kinematics of this matrix element.

In the absence of mixing, the weak interaction could be expressed as an operator V_α for each lepton flavor α. With mixing, the weak interaction couples each lepton flavor α to each of the neutrino mass eigenstates j via an operator $V_{j\alpha}$.

These points permit us to factor the weak interaction in the following schematic way:

$$V_{j\alpha} = U_{j\alpha} V_{0\alpha}, \tag{28}$$

where U is a unitary mixing matrix associated with the lepton-neutrino side of the interaction, and $V_{0\alpha}$ depends upon the mass of the charged lepton but employs a massless neutrino of flavor α. This approximation places all of the neutrino mass and mixing information into the matrix U.

Different representations of the Poincaré group for this process will in general involve different V_α that have different kinematic factors. However, the ability to factorize the interaction via Eq. (28) will lead to the same $U_{j\alpha}$. That is, the interaction could have a representation-dependent form, but very small neutrino masses lead to the ability to factorize and thereby extract a mixing matrix independent of the representation [25].

To the extent that particle kinematics can deviate from the mass or energy shell, contributions from the interactions can enter. This means that in general, approaches using different forms of dynamics will yield different results at the level of perturbation theory, although the exact expression to all orders will agree. For the problem at hand, off-shell neutrino kinematics corresponds to higher order weak contributions, which are very small in magnitude; as noted below, the deviations from mass/energy shell kinematics are also very small.

Henceforth we also omit the $\pm$ labels in Eq. 26 and use plane-wave lepton and hadron states.

C. delta functions and packets

Equation (26) reflects the use of time-ordered perturbation theory, in which the intermediate neutrinos are on their mass shells and the intermediate state can have an energy different from that of the external initial or final state. The energy denominator in Eq. (26) can be written as

$$\frac{1}{E_{\alpha a}+E_{pa}-E'_{na}-E_{\nu_j}+i0^+} = -i\pi\delta(E_{\alpha a}+E_{pa}-E'_{na}-E_{\nu_j}) + \mathcal{P}\frac{1}{E_{\alpha a}+E_{pa}-E'_{na}-E_{\nu_j}}. \tag{29}$$

The energies of the initial and final particles that appear in this denominator depend on the three-momenta of these particles. When these momenta are integrated over the factors in the numerator of Eq. (26), the delta function term only gives a contribution for momenta that give an on-shell neutrino energy, which leads to the phase factor $\exp(-iE_\nu T)$, where the time difference $T = t_b - t_a$ is the macroscopic time difference used to construct the asymptotic states, which is very large on microscopic scales.

The principal-value term depends upon an integral that involves the phase factor $\exp(-iET)$ where T is large and E depends on the momenta. It is not hard to show that if the remaining part of the integrand is a smooth function of E, then as T becomes large the principal value contribution will be exponentially suppressed relative to the delta function in Eq. (29). In practice the integrand includes Jacobians involving square roots and wave packets of unknown smoothness. Less restrictive assumptions on the smoothness of the integrand lead to an algebraic suppression of the principal value term for large T, but the contribution is still very small in practice. The general case was carefully examined by Grimus and Stockinger [6].

The dominance of the energy delta function in Eq. (29) means that all four components of the neutrino four-momentum are constrained:

$$p^\mu_{\nu_j} = p^\mu_{\alpha a} - p^\mu_{pa} - p^\mu_{na}. \tag{30}$$

If we use this result to evaluate Eq. (26) using plane-wave momentum states, then there is no oscillation. To see this, we note that oscillations require the superposition of two or more neutrino mass eigenstates. Any particular combination of external particle four-momenta will yield *at most one* neutrino four-momentum p_{ν_j} satisfying Eq. (30) [12, 13], and satisfying the mass condition $p^2_{\nu_j} = m^2_j$ of only one mass eigenstate. For plane-wave external particle states, there can be no superposition of neutrino mass eigenstates that would set up an interference.

Thus, the oscillation effect necessarily entails momentum distributions, or packets, of the external particles. Under such distributions, Eq. (26) will in general give rise to a superposition of neutrino mass eigenstates, with each term carrying slightly different four-momenta. Since the energy and three-momentum values will vary inside the external wave packets, so too the energy and three-momentum components of the neutrino mass eigenstates will differ from each other. The most that we can say is that the contributions from different mass eigenstates will have quantitatively *similar* components of their four-momenta as dictated by the momentum ranges of the external wave packets.

It is important to note here that the term *packet* refers to the momentum distributions of the external particles. The neutrino mass eigenstates also have a momentum distribution that will depend upon the external packet shapes via the four-momentum condition of Eq. 30, but this distribution is not a wave packet in the usual sense of preparation of observable states.

We have seen that external wave packets are essential to obtaining an oscillation formula. At the same time, the relevant distributions must be narrow enough that the phase factor in Eq. (26) can be factored out of the integrals over these distributions. When the conditions for oscillation are satisfied, the phase is approximately stationary. The external distributions must therefore be narrow enough to maintain this property.

A somewhat curious conclusion to this train of reasoning is that, once the phases in the oscillation formula are fixed by the kinematics, they depend, to lowest order, only on the neutrino eigenstate masses and the *average* neutrino energy as determined by the external particle kinematics. One can therefore make the following replacement:

$$\delta(E_{\alpha a} + E_{pa} - E'_{na} - E_{\nu_j}) \rightarrow \delta(E_{\alpha a} + E_{pa} - E'_{na} - E_{\nu_0}), \tag{31}$$

where, as noted above, ν_0 denotes a massless neutrino of arbitrary flavor.

D. a matter of time

The expression (26) depends upon the space-time separation between the points $(\mathbf{x}_a,\, t_a)$ and $(\mathbf{x}_b, t_b)$. These points could be determined by events that can in principle be measured.

However, in a typical oscillation experiment (e.g. a reactor and a detector located a few kilometers away), the spatial components $\mathbf{x}_a$ and $\mathbf{x}_b$ (and thereby the displacement $\mathbf{L}$) are essentially known (at least to within a scale characteristic of the reactor size), but the time difference $T = t_b - t_a$ is not: the detector could record the time of a neutrino detection event, but there is no corresponding record of the initial weak interaction event. Thus, the detector in principle is sensitive to a range of possible times T. As we shall see, the range of values is in practice quite restricted, to the point where it is possible to use any single value of T within this range and obtain the usual oscillation formula.

We now examine the variation of the S matrix with the time T in Eq. (26). Equivalently, we can consider the variation as a function of a velocity $v = |\mathbf{L}|/T$. Since the neutrino mass eigenstates are almost ultrarelativistic, we could first assume that $v = c = 1$. Alternatively, we could assume that v corresponds to the velocity of one of the neutrinos:

$$\mathbf{v} = \mathbf{v}_j = \frac{\mathbf{p}_j}{E_j}. \tag{32}$$

These seem to be reasonable assumptions, given the kinematics. For oscillations it should not matter which assumption is made. Furthermore, conditions should be such that the oscillating phase approximately factors out the integrals in (26).

To study these conditions consider the interference phase ϕ_{12} from mass eigenstates 1 and 2:

$$\phi_{12} = (p_1 - p_2) \cdot x = L \left[\frac{E_1 - E_2}{v} - (|\mathbf{p}_1| - |\mathbf{p}_2|) \right], \tag{33}$$

where the time T has an associated velocity $v = L/T$. If the neutrino masses m_i are small compared to their energies, then

$$E_1 \approx |\mathbf{p}_1| + \frac{m_1^2}{2|\mathbf{p}_1|}; \quad E_2 \approx |\mathbf{p}_2| + \frac{m_2^2}{2|\mathbf{p}_2|}, \tag{34}$$

Now define $p, \delta p$ such that $|\mathbf{p}_{1,2}| = p \pm \delta p$. The momentum difference δp has a maximum value determined by the momentum distributions of the external particles. We also express the velocity parameter v in terms of an average velocity $\bar{v}$ as $v = \bar{v} + \delta v$, where

$$\bar{\mathbf{v}} = \frac{1}{2}\left(\frac{\mathbf{p}_1}{E_1} + \frac{\mathbf{p}_2}{E_2} \right); \quad \bar{v} \approx 1 - \frac{m_1^2}{4|\mathbf{p}_1|^2} - \frac{m_2^2}{4|\mathbf{p}_2|^2}. \tag{35}$$

The phase is then

$$\phi_{12} \approx L \left[\frac{m_1^2}{2|\mathbf{p}_1|} - \frac{m_2^2}{2|\mathbf{p}_2|} + (|\mathbf{p}_1| - |\mathbf{p}_2|) \left(\frac{m_1^2}{4|\mathbf{p}_1|^2} + \frac{m_2^2}{4|\mathbf{p}_2|^2} - \delta v \right) \right]. \tag{36}$$

Now define $p, \delta p$ such that $|\mathbf{p}_{1,2}| = p \pm \delta p$. If we further assume that $\delta p \ll p$, then

$$\phi_{12} \approx L\frac{m_1^2 - m_2^2}{2\bar{p}} - 2\delta p \delta v + O(\delta p^2). \tag{37}$$

For the specific case that $\delta v = 0$,we are left with

$$\phi_{12} \approx L\frac{m_1^2 - m_2^2}{2p} + O(\delta p^2). \tag{38}$$

In principle, δv is arbitrary (as is the time), but its contribution to the phase is limited by a factor involving δp. In fact, δv is further limited by the range of the p_i values that determine it. For example, if, at one extreme, $v = v_1$, then

$$\phi_{12} \approx L\left[\frac{m_1^2 - m_2^2}{2\bar{p}} + \frac{(m_1^2 - m_2^2)\delta p}{2\bar{p}^2}\right] + O(\delta p^2), \tag{39}$$

that is, it generates a phase contribution that is suppressed by order $\delta p/p$. Thus, we find that $\delta p \ll 1$ in order to have interfering neutrino mass eigenstate contributions, and to fix the time T to within correction factors that can be neglected.

VI. RESULT AND DISCUSSION

Combining all of the simplifying results of the previous section, we find that the space-time phase factors and the mixing matrices factor out of the integrals over external wave packets, leaving a scattering matrix element of the form

$$S_{fi}(x_b, x_a) \approx \sum_j e^{-i\bar{p}_{\nu_j}\cdot(x_b - x_a)} U_{j\alpha} U^*_{j\beta} s_{fi}(\beta, \alpha, j) \tag{40}$$

where

$$\begin{gathered} s_{fi}(\beta, \alpha, j) = \\ -2\pi^2 \int d\mathbf{p}_{af} d\mathbf{p}_{bf} \phi^*_{bf}(\mathbf{p}_{bf}) \phi^*_{af}(\mathbf{p}_{af}) \delta^4(p_a + p_b - p'_a - p'_b) \times \\ \langle \mathbf{p}'_{\beta b}, \mathbf{p}'_{pb} \| V^\dagger_{0\beta} \| \mathbf{p}_{nb}, \mathbf{p}_{\nu_0}\rangle \delta(E_a - E_{a'} - E_{\nu_0}) \langle \mathbf{p}'_{na}, \mathbf{p}_{\nu_0} \| V_{0\alpha} \| (\mathbf{p}_{\alpha a}, \mathbf{p}_{pa})\rangle \times \\ d\mathbf{p}_{ai} d\mathbf{p}_{bi} \phi_{bi}(\mathbf{p}_{bi}) \phi_{ai}(\mathbf{p}_{ai}). \end{gathered} \tag{41}$$

Integrating over wave packets satisfying the conditions described above, we obtain a measurement probability for neutrino production via initial flavor α and neutrino absorption via final flavor β:

$$P^{\text{rel}}_{\beta\alpha} = \frac{|S_{fi}(x_b, x_a)|^2}{|s_{fi}|^2} \approx \sum_{jk} U^*_{j\beta} e^{-i\bar{p}_{\nu_j}\cdot(x_b - x_a)} U_{j\alpha} U^*_{k\alpha} e^{-i\bar{p}_{\nu_k}\cdot(x_b - x_a)} U_{k\alpha} = \tag{42}$$

$$\delta_{\beta\alpha} + 2\Re \sum_{j>k} U^*_{j\beta} U_{j\alpha} U^*_{k\alpha} U_{k\beta} \exp\left(i \frac{\Delta m^2_{jk} L}{2E_\nu} \right) \tag{43}$$

where

$$\Delta m^2_{jk} = m^2_j - m^2_k. \tag{44}$$

Equation (43) is the standard oscillation formula found in the literature. Its simplicity rests upon all of the elements described in the previous section.

We conclude this section with a brief discussion of what is observable among the elements of neutrino oscillations.

In principle, the mixing matrix U is really part of the interaction Hamiltonian, whose matrix elements describe the connection of charged leptons (identified by flavor) with neutrino mass eigenstates, since it is the latter (rather than neutrino flavor eigenstates) that must form the basis of a the Poincaré group with the correct transformation properties. An interaction Hamiltonian in general is not observable, because it is always possible to generate scattering equivalent Hamiltonians that leave the observables unchanged. One can constrain the problem by completely specifying the Hilbert space, but the extraction will depend upon this specification. That said, all of this uncertainty is eliminated if the interaction can be treated perturbatively, as it can for many electromagnetic and weak processes. One can still perform interaction-dependent unitary transformations that lead to different Hamiltonians, but the effect on extracting parameters such as mixing angles will always be of higher order.

A perturbative S-matrix calculation that exhibits dependence upon the choice of kinematic symmetry group in relativistic dynamics is incomplete: the full calculation will be unitarily equivalent to other full calculations based upon other symmetry groups. We note here that our derivation of the oscillation formula makes use of the symmetry of the three-dimensional Euclidean group. This leads to delta functions in three-momenta in the interaction matrix elements. However, the three-momentum delta functions end up being paired (modulo very small corrections) with an energy delta function. As noted above, what matters in the end is the resulting *four-momentum* delta function together with external wave packets that determine the kinematics of the neutrino mass eigenstates. One could also formulate the derivation using front-form dynamics, which exhibit the symmetry of the null plane. The interaction matrix elements contain a delta function in front-form three-momenta $(\mathbf{p}_\perp, p^+ = p^0 + p^3)$. In turn, it is paired with a denominator in $p^- = p^0 - p^3$, which, for the geometry in question, is again approximately a p^- delta function. This again leads to a four-momentum delta function, modulated by external wave packets.

VII. CONCLUSION

In this paper we used an S-matrix approach to study the problem of neutrino oscillations. This approach is exact and only involves initial and final states that can in principle be prepared and detected in laboratory experiments. From this exact formalism we investigated the features that are combined to arrive at the standard neutrino oscillation formula, Eq. (43). We summarize these briefly below:

1. The perturbative nature of the weak interaction means that the oscillation formula can be derived from a second-order expression via the two-potential equation (26).

2. All three neutrino masses are small. The leading contribution to the oscillation formula comes from the first correction to ultrarelativistic kinematics in the space-time phase. All remaining dependence upon neutrino mass can be neglected.

3. Wave packets are essential for oscillations. The spread of external particle four-momenta makes it possible for all intermediate mass eigenstates to contribute coherently to the S matrix with slightly differeing four-momenta.

4. Explicit off-shell effects can be neglected. Oscillations can be computed in a variety of approaches that differ by unitary transformations that depend upon the interaction. These approaches will all have the same leading contribution to the oscillation formula, with all differences (which can be written as off-shell effects) being of higher order in the weak interaction.

5. The weak Hamiltonians in the second-order calculation can be factored into a product of terms, one containing only mixing information and the other describing a weak interaction involving a fictitious massless, flavored neutrino. Corrections to factorization are of higher order in the interaction and/or the neutrino mass.

6. The oscillation formula depends upon the space-time separation between production and disappearance of the neutrino mass eigenstates. The spatial separation is known, but the time is not measured. Nevertheless, for reasonably small sizes of the external

four-momentum distributions (e.g. detector resolution), the range of relevant times is restricted to values that produce the standard oscillation formula.

We conclude that the existence of a simple neutrino oscillation formula that can be derived from a variety of theoretical perspectives represents a remarkable convergence of several specific physical properties of neutrinos and their interactions. However, to improve on this simple formula will in principle bring back all of the interrelated effects that have so far been neglected, and "corrected" oscillation formulae may acquire theoretical dependencies that the lowest-order expression does not carry.

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