Model Tests of Cluster Separability In Relativistic Quantum Mechanics

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A relativistically invariant quantum theory first advanced by Bakamjian and Thomas has proven very useful in modeling few-body systems. For three particles or more, this approach is known formally to fail the constraint of cluster separability, whereby symmetries and conservation laws that hold for a system of particles also hold for isolated subsystems. Cluster separability can be restored by means of a recursive construction using unitary transformations, but implementation is difficult in practice, and the quantitative extent to which the Bakamjian-Thomas approach violates cluster separability has never been tested. This paper provides such a test by means of a model of a current operator in a three-particle system for which (1) it is simple enough that there is a straightforward solution that satisfies Poincaré invariance and cluster separability, and (2) one can also apply the Bakamjian-Thomas approach. The difference between these calculations provides a measure of the size of the corrections from the Sokolov construction that are needed to restore cluster properties. Our estimates suggest that, in models based on nucleon degrees of freedom, the corrections that restore cluster properties are too small to effect calculations of observables.

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I. INTRODUCTION

There are two distinct requirements for describing quantum mechanical systems of particles under the requirements of special relativity. The first requirement is Poincaré invariance: probabilities, expectation values and ensemble averages for equivalent experiments performed in different inertial frames are identical. A necessary and sufficient condition for a quantum theory to be Poincaré invariant is that the dynamics is described by a unitary representation of the Poincaré group [1]. The second requirement is cluster separability: isolated subsystems must have the same observable properties as they would in a framework in which the other "spectator" particles are absent entirely. This requirement justifies tests of special relativity on isolated subsystems. It applies both to systems of particles interacting among themselves (e.g. via the strong interaction) and to the current operators that allow them to interact with electroweak fields.

In local quantum field theory, both of these properties of the theory follow as a consequence of the covariance, spectral properties and the local commutation relations of the field operators. Formal solutions of field theories satisfy both requirements. Truncations or approximations may or may not be consistent with these requirements, which must be verified explicitly.

These physical requirements also apply to Poincaré invariant quantum mechanical models of systems with a finite number of degrees of freedom. The problem of constructing dynamical representations of the Poincaré group was studied by Dirac [2] from an algebraic perspective, using Poisson brackets. He concluded that at least three of the infinitesimal generators of the Poincaré group must include interactions, although he did not solve the non-linear problem of how to add interactions that preserve the Poincaré commutation relations. Bakamjian and Thomas [3] provided a solution to this problem for a system of two interacting particles.

Many realistic, Poincaré invariant, quantum mechanical models of strongly interacting few-body systems are based on a generalization of the Bakamjian-Thomas construction [4]. This construction can be used to formulate models of systems of arbitrary numbers of particles and systems that do not conserve particle number. A representative sample of few-body applications of Poincaré invariant quantum mechanics based on the Bakamjian-Thomas construction include relativistic constituent quark models [5–9], relativistic few-nucleon models [10–14] relativistic models involving electromagnetic probes [15–17], and relativistic models with particle production [18, 19].

The virtue of the Bakamjian-Thomas construction is that it provides a means for constructing Poincaré generators for systems of interacting particles. The limitation of this construction is that for systems of more than two particles, the dynamical representation of the Poincaré group does not become a tensor product on states representing asymptotically separated subsystems. While this limitation does not lead to observable consequences in the three-body S-matrix or bound-state observables, there are observable consequences when the three-body system is embedded in the four-particle Hilbert space, as it is in four-body problems or when an electroweak probe interacts with a three-body system.

The most ambitious applications of the Bakamjian-Thomas construction are relativistic Faddeev calculations [10–14] of three-nucleon scattering which have been performed using realistic nucleon-nucleon forces and three-nucleon forces. The next step in developing a useful relativistic few-body theory would be to model four-nucleon systems or electron scattering from a three-nucleon system. In both of these applications a three-nucleon subsystem is embedded in a four-particle Hilbert space, and, for the first time, there is the possibility of observable consequences of violations of cluster properties.

The restoration of cluster properties can be achieved through a recursive construction due to Sokolov [20, 21]. The Sokolov construction consists of a hierarchy of unitary transformations, each of which preserves the S matrix for systems consisting of isolated subsystems. The Bakamjian-Thomas representation is retrieved by setting all of the unitary transformations to the identity operator. The Sokolov construction is non-trivial and interaction dependent. An isolated three-body system is a special case where the cluster property can be achieved by means of a single overall unitary transformation that preserves the S matrix. For systems of more than three particles there are observable differences between the two representations. The interested reader can find a complete discussion in [21].

Actual construction of the Sokolov hierarchy of unitary transformation is sufficiently complicated that the technique has never been used in realistic calculations. Before undertaking computationally intensive four-body calculations, one would like to know the importance of the corrections required by cluster properties. While Ref. [21] argued that these corrections should be small in nuclear physics applications, this was never quantified in any model calculations. An additional investigation is needed to determine if these corrections can be ignored, can be treated perturbatively, or must be treated exactly. To address this question we construct a model involving a current matrix element for a system of three particles. The model is simple enough that Poincaré invariance and cluster separability are easily satisfied; this unusual pair of features is due to the simplicity of the test model. At the same time, the model permits a Bakamjian-Thomas construction that illustrates the quantitative impact of the breakdown of cluster separability. The conclusion of our preliminary analysis suggests that the corrections required by cluster properties are too small to be observable in nuclear physics applications. The corrections are more important for models based on sub-nucleon degrees of freedom.

II. TEST MODEL

Poincaré invariance and the Dirac forms of dynamics for Bakamian-Thomas (BT) constructions are discussed extensively in Ref. [4]. The framework provided here makes use of that discussion. We approach the model in a heuristic way in order to illustrate the issue of cluster separability combined with Poincaré invariance with a minimum of formal development.

The simplest four-body model where we might observe the breakdown of cluster separability consists of a three-body system interacting with an electromagnetic probe. In the one-photon exchange approximation the relevant dynamical quantities are matrix elements of a current density between eigenstates of the three-body four-momentum operator.

~~In~~ qIn our model we make the following simplifications. First, we assume that all of the particles are spinless with mass, m. Second, we replace the four-vector current density by a scalar current density. We will see that, even with these simplifications, there are differences in the current matrix elements calculated using three-body eigenstates of a BT four-momentum operator compared to current matrix elements calculated using three-body eigenstates of a four momentum operator that clusters. Our simplifying assumptions show that the violations of cluster properties has nothing to do with spin or choices of the orientation of the current.

The scalar current operator, $j(x)$, is assumed to be a one-body operator. The three-particle system consists of a bound pair (which we label the 1-2 subsystem) and a third particle (3) that does not interact with particles 1 or 2. Only particle 3 has a "charge" with respect to $j(x)$. Therefore, the (12) subsystem acts as a spectator with respect to the action of the current operator. This can be illustrated as a disconnected graph as shown in Fig. 1.

The three-body eigenstates that appear in the current matrix element can be treated as eigenstates of the four-momentum of the tensor product of a one-body and two-body Bakamjian-Thomas unitary representation of the Poincaré group. Alternatively they can be treated as eigenstates of the four-momentum of a three-body Bakamjian-Thomas unitary representation of the Poincaré group. These representation are related by ~~a~~ an S-matrix preserving unitary transformation of the type that appears in the Sokolov construction. Comparing the results of the two

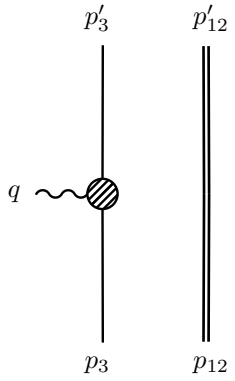

FIG. 1: Graph of one-body current plus bound two-body spectator.

calculations provides a measurement of the impact of the unitary transformations that restore cluster properties.

A. Tensor-Product (TP) Model:

In this model the three-body eigenstates are tensor products of (12) and 3 eigenstates:

$$|\lambda_{12}, m; \mathbf{p}_{12}, \mathbf{p}_3\rangle = |\lambda_{12}; \mathbf{p}_{12}\rangle \otimes |m, \mathbf{p}_3\rangle. \tag{2.1}$$

Since the current $j(x)$ acts only in the space of particle 3, the matrix element has the form

$$\langle \lambda_{12}, m; \mathbf{p}'_{12}, \mathbf{p}'_3|j(0)|\lambda_{12}, m; \mathbf{p}_{12}, \mathbf{p}_3\rangle = \delta(\mathbf{p}'_{12} - \mathbf{p}_{12}) f(^2) \langle m; \mathbf{p}'_3|j(0)|m; \mathbf{p}_3\rangle, \tag{2.2}$$

where

$$\langle m; \mathbf{p}'_3|j(0)|m; \mathbf{p}_3\rangle = f(\mathbf{q}^2); \quad \mathbf{q} = \mathbf{p}'_3 - \mathbf{p}_3, \tag{2.3}$$

λ_{12} is a bound-state eigenvalue of M_{12}, and $\mathbf{p}_{12}$ is its three-momentum.

If we specify initial momenta $\mathbf{p}_3$ and $\mathbf{p}_{12}$ and momentum transfer $\mathbf{q}$, and integrate the matrix element over the final spectator momentum $\mathbf{p}'_{12}$, the integral collapses due to the spectator momentum delta function, and we have

$$\mathcal{F}_{TP} := \int d\mathbf{p}'_{12} \langle \lambda_{12}, m; \mathbf{p}'_{12}, \mathbf{p}'_3|j(0)|\lambda_{12}, m; \mathbf{p}_{12}, \mathbf{p}_3\rangle = f(\mathbf{q}^2). \tag{2.4}$$

Equation 2.4 is the Tensor-Product (TP) result. It has no dependence upon the momentum $\mathbf{p}_{12}$ of the bound-state spectator, as expected from the physical requirement of cluster separability, and it does not depend upon the specific values of $\mathbf{p}_3$ and $\mathbf{p}'_3$, as long as $\mathbf{p}'_3 - \mathbf{p}_3 = \mathbf{q}$. This result also has the correct properties of relativistic invariance.

B. Discussion

The model is a simple yet representative test because it has a straightforward solution satisfying Poincaré invariance and cluster separability (Eq. 2.4) and a related Bakamjian Thomas solution, without the complications of ~~the full three-body~~ solving the Faddeev equations with and without the full Sokolov construction. ~~Note that a full few-body system does not have a simple solution that exhibits both of these properties.~~ At the same time, the model is sufficiently non-trivial that it also permits ~~an analysis using the~~ a comparison of the solution satisfying Poincaré invariance and cluster separability with the BT ~~scheme~~ solution, that provides a quantitative measure of the violation of cluster separability that one would expect in more complicated and realistic systems where a BT approach would be essential. The violations of cluster properties are due to the size of the difference between the unitary transform, $A_{(12)(3)}$, that relates the $2+1$ BT representation to the TP representation just described.

In the following sections, we evaluate the same current matrix element using BT representations of the three-particle system corresponding to different forms of dynamics [2]. The purpose of this is to examine the quantitative sensitivity of the size of violations of cluster properties in models based on each of Dirac's three forms of dynamics. These current matrix elements will each depend upon the structure of the bound state, in violation of cluster separability, and our goal is to examine the magnitude of that violation. In all cases, we specify the momentum transfer $\mathbf{q}$ and ~~bound state~~ bound-state momentum $\mathbf{p}_{12}$ for a given calculation and then vary these momenta for sensitivity tests.

C. BT Representation: Instant Form

We now evaluate current matrix element in Eq. 2.2 using an instant-form BT model. Instant-form models have no interactions in the generators of rotations and space translations.

We provide first an observation about frames that will be relevant to each of the BT representations. Calculations of current-matrix observables typically use a Breit frame (in which the energy transfer is zero), a lab frame (initial target three-momentum is zero), or an anti-lab frame (final target momentum is zero). There is no frame in which both initial and final target three-momenta can be zero, so the calculation will require non-trivial transformations from non-zero to zero target momentum. These are provided explicitly where needed below.

First, we change variables, replacing the bound state and particle 3 momenta by the total momentum of the system and the momentum of particle 3 Lorentz transformed to the frame in which $\mathbf{P} = 0$:

$$(\mathbf{p}_{12}, \mathbf{p}_3) \to (\mathbf{P}, \mathbf{p}), \tag{2.5}$$

where

$$\mathbf{P} = \mathbf{p}_{12} + \mathbf{p}_3; \quad \mathbf{p} = \mathbf{p}_3 + \Phi(\mathbf{p}_3, \mathbf{P}, M_\lambda)\mathbf{P}, \tag{2.6}$$

and

$$\begin{aligned} \Phi(\mathbf{p}_3, \mathbf{P}, M_\lambda) &= \frac{1}{M_\lambda}\left[\frac{\mathbf{P}\cdot\mathbf{p}_3}{E_\lambda + M_\lambda} - \omega_m(\mathbf{p}_3)\right]; \\ M_\lambda &= \sqrt{E_\lambda^2 - \mathbf{P}^2}; \quad E_\lambda = \sqrt{m^2 + \mathbf{p}_3^2} + \sqrt{\lambda^2 + \mathbf{p}_{12}^2}; \\ \omega_m(\mathbf{p}) &= \sqrt{m^2 + \mathbf{p}^2}. \end{aligned} \tag{2.7}$$

With this variable change the relation between the bound state and particle 3 in these bases is:

$$|\lambda, m; \mathbf{p}_{12}, \mathbf{p}_3\rangle = \left|\frac{\partial(\mathbf{P}, \mathbf{p})}{\partial(\mathbf{p}_{12}, \mathbf{p}_3)}\right|_\lambda^{\frac{1}{2}} |\lambda, m; \mathbf{P}, \mathbf{p}\rangle, \tag{2.8}$$

where the suffix λ in the Jacobian indicates that the interacting two-body mass eigenvalue was used in the transformation.

We also introduce single-particle momenta $\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3$. We change variables to the total momentum of the two-particle pair (12) and the momentum of particle 1 Lorentz transformed to the ~~rest~~ $\mathbf{b}_{12} = 0$ frame of the two-particle subsystem using:

$$\mathbf{k} = \mathbf{b}_1 + \Phi(\mathbf{b}_1, \mathbf{b}_{12}, M_{12}^0)\mathbf{b}_{12}; \quad \mathbf{b}_{12} = \mathbf{b}_1 + \mathbf{b}_2; \quad M_{12} = 2\sqrt{m^2 + \mathbf{k}^2}. \tag{2.9}$$

The three-particle invariant mass is

$$M_k := \sqrt{(M_{12}^0)^2 + \mathbf{p}^2} + \sqrt{m^2 + \mathbf{p}^2} \tag{2.10}$$

and the momentum of particle 3 in a frame where the three-particle system has total momentum $\mathbf{P}$ is:

$$\mathbf{b}_3 = \mathbf{p} + \Phi(\mathbf{p}, \mathbf{P}, M_k)\mathbf{P}, \tag{2.11}$$

where

$$\Phi(\mathbf{p}, \mathbf{P}, M_k)\mathbf{P} = \frac{1}{M_k}\left[\frac{\mathbf{P}\cdot\mathbf{p}}{E_k + M_k} + \omega_m(\mathbf{p})\right]; \quad E_k = \sqrt{M_k^2 + \mathbf{P}^2} \tag{2.12}$$

In an instant-form Bakamjian-Thomas dynamics, the ~~connection between a state with three free particles and that with particle 3 plus bound state~~ $\mathbf{p}$ in 2.6 is identified with the $\mathbf{p}$ in 2.12. The BT four-momentum eigenstate with particles 1 and 2 bound in the three-particle basis is

$$\langle\lambda; \mathbf{P}', \mathbf{p}'|\mathbf{P}, \mathbf{p}, \mathbf{k}\rangle = \delta(\mathbf{P}' - \mathbf{P})\delta(\mathbf{p}' - \mathbf{p})\phi_\lambda(\mathbf{k}), \tag{2.13}$$

where $\phi_\lambda(\mathbf{k})$ is the bound state-wave function. Note that $\mathbf{p}$ depends upon the free two-particle mass $M_{12}^{(0)}$ via Eq. 2.11, while $\mathbf{p}'$ depends upon the bound state mass eigenvalue via λ Eq. 2.6. The association of these two three-momenta in

the delta function of Eq. 2.13 is therefore not fully consistent. There is no observable consequence for the two-body S matrix or mass eigenvalues, but this association will lead to a result that violates cluster separability.

We now make use of Eq. 2.13 to compute the BT counterpart to $\mathcal{F}_{TP}$ that was defined by Eq. 2.4:

$$\begin{aligned}\mathcal{F}_{BT} \;:=\; & \int d\mathbf{p}'_{12} d\mathbf{b}'_{12} d\mathbf{b}'_3 d\mathbf{k}' d\mathbf{b}_{12} d\mathbf{b}_3 d\mathbf{k} \\ & \times \left|\frac{\partial(\mathbf{P}',\mathbf{p}')}{\partial(\mathbf{p}'_{12},\mathbf{p}'_3)}\right|_\lambda^{\frac{1}{2}} \left|\frac{\partial(\mathbf{P},\mathbf{p})}{\partial(\mathbf{p}_{12},\mathbf{p}_3)}\right|_\lambda^{\frac{1}{2}} \left|\frac{\partial(\mathbf{b}'_{12},\mathbf{b}'_3)}{\partial(\mathbf{P}',\mathbf{p}')}\right|_k^{\frac{1}{2}} \left|\frac{\partial(\mathbf{b}_{12},\mathbf{b}_3)}{\partial(\mathbf{P},\mathbf{p})}\right|_k^{\frac{1}{2}} \\ & \times \phi^*_\lambda(\mathbf{k}') \langle \mathbf{b}'_{12}, \mathbf{b}'_3, \mathbf{k}' | j(0) | \mathbf{b}_{12}, \mathbf{b}_3, \mathbf{k}\rangle \phi_\lambda(\mathbf{k})\end{aligned} \tag{2.14}$$

The suffix k in the Jacobian indicates that the non-interacting two-body mass was used in the transformation of Eq. 2.11. Since the current operator $j(x)$ operates only in the space of particle 3, we have

$$\langle \mathbf{b}'_{12}, \mathbf{b}'_3, \mathbf{k}' | j(0) | \mathbf{b}_{12}, \mathbf{b}_3, \mathbf{k}\rangle = \delta(\mathbf{k}'-\mathbf{k})\delta(\mathbf{b}'_{12}-\mathbf{b}_{12}) f[(\mathbf{b}'_3-\mathbf{b}_3)^2]. \tag{2.15}$$

The integral over $\mathbf{p}'_{12}$ can be converted to an integral over $\mathbf{b}'_{12}$ by means of Jacobians:

$$\int d\mathbf{p}'_{12} = \int d\mathbf{b}'_{12} \left|\frac{\partial(\mathbf{P}',\mathbf{p}')}{\partial(\mathbf{b}_{12},' \mathbf{b}'_3)}\right|_{\underline{M'_k k}} \left|\frac{\partial(\mathbf{p}'_{12},\mathbf{p}'_3)}{\partial(\mathbf{P}',\mathbf{p}')}\right|_\lambda . \tag{2.16}$$

The final result is

$$\mathcal{F}^{\rm instant}_{BT} := \int d\mathbf{k} \left|\frac{\partial(\mathbf{p}'_{12},\mathbf{p}'_3)}{\partial(\mathbf{P}',\mathbf{p}')}\right|_\lambda^{\frac{1}{2}} \left|\frac{\partial(\mathbf{P},\mathbf{p})}{\partial(\mathbf{p}_{12},\mathbf{p}_3)}\right|_\lambda^{\frac{1}{2}} \left|\frac{\partial(\mathbf{P}',\mathbf{p}')}{\partial(\mathbf{b}'_{12},\mathbf{b}'_3)}\right|_{\underline{M_k k}}^{\frac{1}{2}} \left|\frac{\partial(\mathbf{b}_{12},\mathbf{b}_3)}{\partial(\mathbf{P},\mathbf{p})}\right|_{\underline{M_k k}}^{\frac{1}{2}} |\phi_\lambda(\mathbf{k})|^2 f[(\mathbf{b}'_3-\mathbf{b}_3)^2]. \tag{2.17}$$

The interesting observation is that this integral has a non-trivial dependence on momentum, $\mathbf{p}_{12}$, of the bound state, in contrast to the $\mathbf{p}_{12}$ independence of the tensor product result. In this model the BT and tensor product representations are related one of Sokolov's unitary transformations, $A_{(12)(3)}$, that preserve the three-body S-matrix:

$$|\lambda, m; \mathbf{p}_{12}, \mathbf{p}_3\rangle_{TP} = A_{(12)(3)} |\lambda, m; \mathbf{p}_{12}, \mathbf{p}_3\rangle_{BT}. \tag{2.18}$$

The scale of the $\mathbf{p}_{12}$ dependence in 2.17 provides a measure of the size of the violation of cluster properties that results from ignoring $A_{(12)(3)}$ by replacing it with the unit operator. Since we know both states in eq 2.18 we do not have to calculate $A_{(12)(3)}$ explicitly to determine its impact.

For this calculation, we vary the three-momentum transfer $\mathbf{q}$ and the spectator momentum $\mathbf{p}_{12}$. The initial momentum of particle 3 and the final momentum of the system are fixed in terms of these variables:

- $\mathbf{p}_3 = -\frac{1}{2}\mathbf{q}$;
- $\mathbf{P}' = \mathbf{P} + \mathbf{q}$.

Since in the instant form

$$\mathbf{P} = \mathbf{p}_{12} + \mathbf{p}_3 = \mathbf{b}_{12} + \mathbf{b}_3; \quad \mathbf{P}' = \mathbf{p}'_{12} + \mathbf{p}'_3 = \mathbf{b}'_{12} + \mathbf{b}'_3, \tag{2.19}$$

and the current matrix element constrains $\mathbf{b}'_{12} = \mathbf{b}_{12}$, we find that $\mathbf{b}'_3 = \mathbf{b}_3 + \mathbf{q}$, in which case the argument of the form factor is $\mathbf{q}^2$ as it is in the TP case. However, the final momenta $\mathbf{p}'_{12}$ and $\mathbf{p}'_3$ are not constrained, and in general there are non-vanishing contributions to this matrix element for $\mathbf{p}'_3 \neq \mathbf{p}_3 + \mathbf{q}$ and $\mathbf{p}'_{12} \neq \mathbf{p}_{12}$.

In the nonrelativistic limit, where $\mathbf{k}$, $\mathbf{q}$ and $\mathbf{p}_{12}$ are all small with respect to the relevant masses, the Jacobians are approximately unity and can be factored out of the integral, leaving a unit wave function normalization and a result identical to the TP case. The quantitative level of disagreement with the TP result is therefore linked to the extent to which the model goes beyond the nonrelativistic limit.

D. BT Representation: Front Form

Dirac's front-form dynamics is described in detail in Ref. [4]. In a front-form dynamics the generators of transformations that leave a plane tangent to the light cone invariant are free of interactions. We provide a summary here.

Basis states in the front form are eigenstates of the light-front momenta

$$\tilde{\mathbf{p}} = (\mathbf{p}_\perp, p^+); \quad p^+ = p^0 + p^3. \tag{2.20}$$

These generate translations in a hyperplane $x^+ = 0$ tangent to the light cone.

In the front form the Lorentz transformations used to define a particle momentum in the rest frame of the three-particle system in the instant form are replaced by boosts that leave the light front invariant. While these boosts are interaction independent, there is an interaction dependence in the boost parameters. The ~~Bakamjian Thomas~~ Bakamjian-Thomas model again identifies the momentum of particle 3 in the rest frame of the three-particle system. As in the instant-form case this is an incorrect identification that has no consequences for the two-body S matrix. With this modification the front-form result has the same structure as the instant-form result:

$$\mathcal{F}_{BT}^{\text{front}} := \int d\mathbf{k} \left|\frac{\partial(\tilde{\mathbf{p}}'_{12}, \tilde{\mathbf{p}}'_3)}{\partial(\tilde{\mathbf{P}}', \tilde{\mathbf{p}}')}\right|_\lambda^{\frac{1}{2}} \left|\frac{\partial(\tilde{\mathbf{P}}, \tilde{\mathbf{p}})}{\partial(\tilde{\mathbf{p}}_{12}, \tilde{\mathbf{p}}_3)}\right|_\lambda^{\frac{1}{2}} \left|\frac{\partial(\tilde{\mathbf{P}}', \tilde{\mathbf{p}}')}{\partial(\tilde{\mathbf{b}}'_{12}, \tilde{\mathbf{b}}'_3)}\right|_{M'_k k}^{\frac{1}{2}} \left|\frac{\partial(\tilde{\mathbf{b}}_{12}, \tilde{\mathbf{b}}_3)}{\partial(\tilde{\mathbf{P}}, \tilde{\mathbf{p}})}\right|_{M_k k}^{\frac{1}{2}} |\phi_\lambda(\mathbf{k})|^2 \, f[(\tilde{\mathbf{b}}'_3 - \tilde{\mathbf{b}}_3)^2]. \tag{2.21}$$

This quantity has an unphysical dependence on $\tilde{\mathbf{b}}_{12}$ that does not occur in the tensor product.

For this calculation, we vary $\tilde{\mathbf{q}}$ and $\tilde{\mathbf{p}}_{12}$, with coordinate axes chosen such that $q^+ = 0$ and $p^+_{12} = 0$. We define $\mathbf{p}_\perp$ and $\tilde{\mathbf{P}}'$ in terms of these quantities:

- $\mathbf{p}_\perp = -\frac{1}{2}\mathbf{q}_\perp$;
- $\tilde{\mathbf{P}}' = \tilde{\mathbf{P}} + \tilde{\mathbf{q}}$.

Analogous to the discussion following Eq. 2.19, we have that

$$\tilde{\mathbf{P}} = \tilde{\mathbf{p}}_{12} + \tilde{\mathbf{p}}_3 = \tilde{\mathbf{b}}_{12} + \tilde{\mathbf{b}}_3; \quad \mathbf{P}' = \tilde{\mathbf{p}}'_{12} + \tilde{\mathbf{p}}'_3 = \tilde{\mathbf{b}}'_{12} + \tilde{\mathbf{b}}'_3, \tag{2.22}$$

The current matrix element constrains $\tilde{\mathbf{b}}'_{12} = \tilde{\mathbf{b}}_{12}$, and therefore $\tilde{\mathbf{b}}'_3 = \tilde{\mathbf{b}}_3 + \tilde{\mathbf{q}}$, in which case the argument of the form factor is $\mathbf{q}^2$ as it is in the TP case. However, the final momenta $\tilde{\mathbf{p}}'_{12}$ and $\tilde{\mathbf{p}}'_3$ are not constrained, and in general the integral has non-zero contributions from $\tilde{\mathbf{p}}'_3 \neq \tilde{\mathbf{p}}_3 + \tilde{\mathbf{q}}$ and $\tilde{\mathbf{p}}'_{12} \neq \tilde{\mathbf{p}}_{12}$.

E. BT Representation: Point Form

Dirac's point-form dynamics are also described in detail in Ref. [4]. In this case the Lorentz group is non-interacting. We provide a summary here.

Basis states in the point form are described by four-velocity vectors $\mathbf{v}$. Momenta are obtained by multiplying the four-velocities by (interacting or non-interacting) masses. Thus, we seek to evaluate matrix elements of the current operator $j(x)$ between states of particle 3 and the (12) bound state with initial four-velocity $\mathbf{V}$ and final four-velocity $\mathbf{V}'$.

The derivations proceed in a fashion similar to the instant form except that we must take care to use velocities in cases where the associated masses can acquire interactions.

With these conventions, the point-form $\mathcal{F}_{BT}$ has the structure

$$\mathcal{F}_{BT}^{\text{point}} := \int d\mathbf{k} \left|\frac{\partial(\mathbf{p}'_{12}, \mathbf{p}'_3)}{\partial(\mathbf{V}', \mathbf{p}')}\right|_\lambda^{\frac{1}{2}} \left|\frac{\partial(\mathbf{V}, \mathbf{p})}{\partial(\mathbf{p}_{12}, \mathbf{p}_3)}\right|_\lambda^{\frac{1}{2}} \left|\frac{\partial(\mathbf{V}', \mathbf{p}')}{\partial(\mathbf{b}'_{12}, \mathbf{b}'_3)}\right|_k^{\frac{1}{2}} \left|\frac{\partial(\mathbf{b}_{12}, \mathbf{b}_3)}{\partial(\mathbf{V}, \mathbf{p})}\right|_k^{\frac{1}{2}} |\phi_\lambda(\mathbf{k})|^2 \, f[(\mathbf{b}'_3 - \mathbf{b}_3)^2]. \tag{2.23}$$

For the instant- and front-form calculations, we chose to fix three three-momenta in a way that kept the physical momentum transfer in the particle-3 current matrix element. For the point form, we provide two different constraint choices, each analogous to the previous calculations, but leading to different results.

A. Constrain the momentum transfer to particle 3:

- $\mathbf{p}_3 = -\frac{1}{2}\mathbf{q}$;
- $\mathbf{p}'_3 = \mathbf{p}_3 + \mathbf{q}$.

Note in this case that the velocities $\mathbf{V}$ and $\mathbf{V}'$ are the same for the kinematics using the interacting mass λ or the free mass M_{12}. This implies that

$$\mathbf{b}_3 = \mathbf{p}'_3; \quad \mathbf{b}'_3 = \mathbf{p}'_3, \tag{2.24}$$

and therefore that $\mathbf{b}'_3 - \mathbf{b}_3 = \mathbf{q}$. One consequence is that in general, $M'_\lambda \mathbf{V}' - M_\lambda \mathbf{V} \neq \mathbf{q}$.

B. Constrain the final velocity to match momentum transfer $\mathbf{q}$.

- $\mathbf{p}_3 = -\frac{1}{2}\mathbf{q}$;
- $\mathbf{V}' = \mathbf{V}_{\rm calc}$.

To obtain $\mathbf{V}_{\rm calc}$, we use temporary quantities

$$\mathbf{P}'_{\rm temp} = \mathbf{P} + \mathbf{q} \tag{2.25}$$

and

$$P'^0_{\rm temp} = P^0. \tag{2.26}$$

These are the results that one expects in the tensor-product representation. From these relations we obtain

$$M'_{\rm temp} = \sqrt{(P'^0_{\rm temp})^2 - \mathbf{P}'^2_{\rm temp}}, \tag{2.27}$$

and then define

$$\mathbf{V}_{\rm calc} := \frac{\mathbf{P}'_{\rm temp}}{M'_{\rm temp}}. \tag{2.28}$$

Substituting, we get

$$\mathbf{V}_{\rm calc} = \frac{\mathbf{P} + \mathbf{q}}{\sqrt{(P^0)^2 - (\mathbf{P} + \mathbf{q})^2}}. \tag{2.29}$$

Thus, the three-vector $\mathbf{V}'$ is constrained, but the actual values of M'_λ and $\mathbf{P}'$ may be different than the temporary values used to obtain $\mathbf{V}_{\rm calc}$. In this form,

$$\begin{aligned} \mathbf{V}_{\rm int} &= \mathbf{V}_{\rm ext}; \\ \mathbf{V}'_{\rm int} &= \mathbf{V}'_{\rm ext}, \end{aligned} \tag{2.30}$$

which also implies that

$$\begin{aligned} \mathbf{b}_3 &= \mathbf{p}_3; \\ \mathbf{b}'_3 &= \mathbf{p}'_3, \end{aligned} \tag{2.31}$$

III. RESULTS AND DISCUSSION

In this section we discuss results of calculations of the form factor $\mathcal{F}_{BT}$ for instant, front and two point-form kinematic choices. Initially we consider scales that are relevant for systems of nucleons interacting with two-body interactions. Then we turn to examples more appropriate to hadron models.

In all figures we show the figure of merit:

$$\frac{(\mathcal{F}_{BT} - \mathcal{F}_{TP})}{\mathcal{F}_{TP}}, \tag{3.1}$$

which represents the relative error induced by ignoring the unitary transformation eq. 2.18 that restores cluster separability.

A. Malfliet-Tjon Deuteron Wave Function

To model realistic conditions for nuclear physics, we use a bound-state spectator with deuteron properties constructed from Malfliet-Tjon [22] potential IV, which contains both attractive and repulsive forces. Ref. [4] discusses how to make a phase-equivalent relativistic model that has the same wave functions as the non-relativistic model, and our numerical results follow that approach.

We first consider the figure of merit as a function of the momentum transfer and momentum of the bound state in each of Dirac's forms of dynamics and for the bound state momentum perpendicular and parallel to the momentum transfer. The results are shown in the six figures below. In all cases the expected results are given by the flat planes.

The fractional deviation of this BT model calculation from the TP result that satisfies cluster separability is very small for the instant- and front-form cases, typically of order 10^{-3} or smaller at the highest values of the three-momenta $\mathbf{q}$ and $\mathbf{p}_{12}$. The deviations for the point-form Option A are somewhat larger, and larger still for Option B.

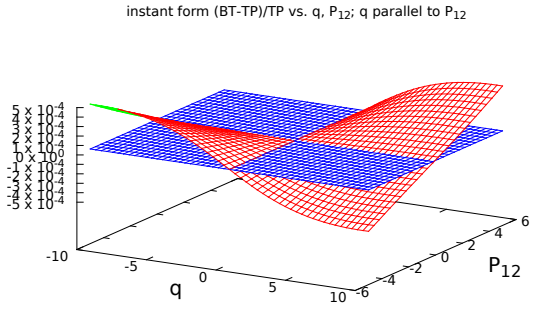

FIG. 2: Model differences for instant-form BT calculation, $\mathbf{q}\|\mathbf{p}_{12}$.

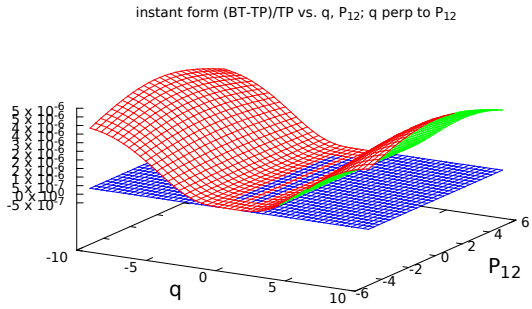

FIG. 3: Model differences for instant-form BT calculation, $\mathbf{q} \perp \mathbf{p}_{12}$.

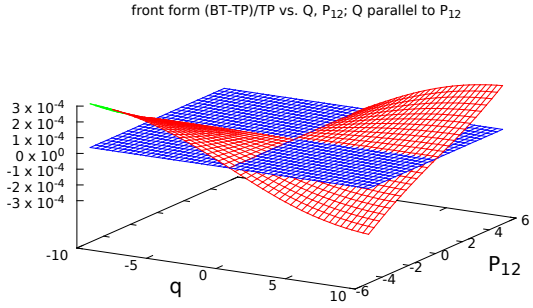

FIG. 4: Model differences for front-form BT calculation, $\mathbf{q}\|\mathbf{p}_{12}$.

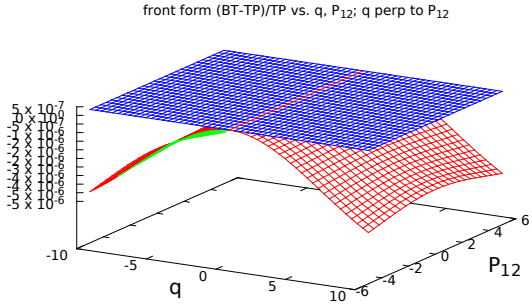

FIG. 5: Model differences for front-form BT calculation, $\mathbf{q} \perp \mathbf{p}_{12}$.

B. Binding Energy Variation

The calculations above assumed a bound state with a wave function having a standard dependence on the relative momentum of the constituent nucleons. The next set of curves illustrates the figure of merit for fixed values of $\mathbf{q}$ and $\mathbf{p}_{12}$ as we vary the binding energy and momentum scale of the wave function. The variations that we consider are still scales that are relevant to nuclear systems. Figures 10 and 11 show the results of calculations vary the binding energy with the Malfliet-Tjon wave function.

As with the earlier cases that employed the bound state binding energy, the fractional deviation of the BT results from the TP benchmark is quite small, of order 10^{-3} or less, for all cases except point-form Option B.

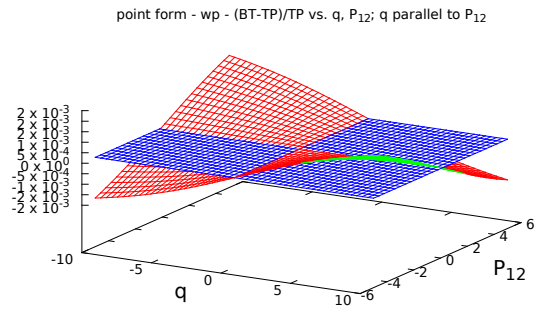

FIG. 6: Model differences for point-form BT calculation A, $\mathbf{q}\|\mathbf{p}_{12}$.

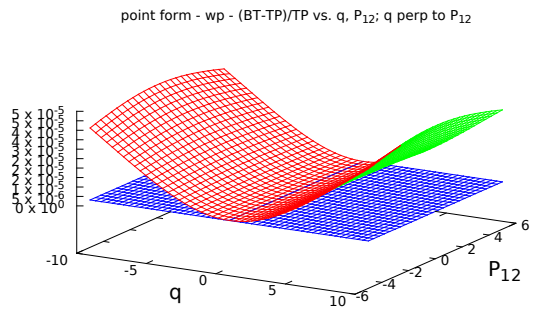

FIG. 7: Model differences for point-form BT calculation A, $\mathbf{q} \perp \mathbf{p}_{12}$.

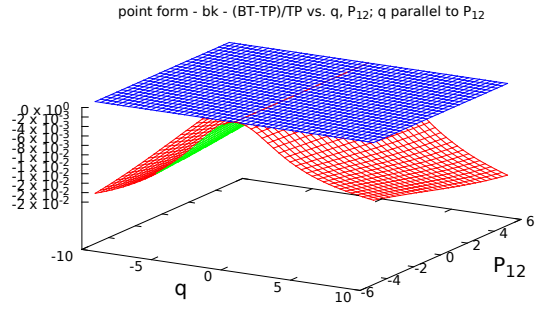

FIG. 8: Model differences for point-form BT calculation B, $\mathbf{q}\|\mathbf{p}_{12}$.

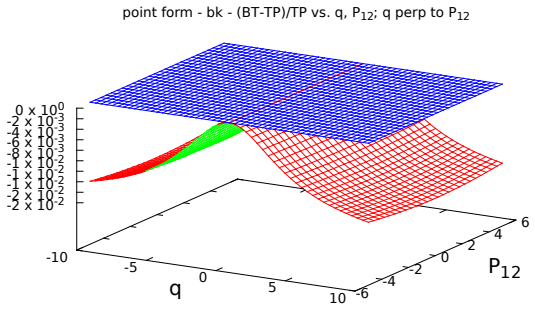

FIG. 9: Model differences for point-form BT calculation B, $\mathbf{q} \perp \mathbf{p}_{12}$.

C. Wave Function Scale Variation

We also examined sensitivity to the momentum scale, k_0, of the wave function by replacing the Malfliet-Tjon wave function with a Gaussian form:

$$\phi(\mathbf{k}) = \frac{1}{\sqrt{N}} e^{-(k/k_0)^2}. \tag{3.2}$$

Figures 12 and 13 show the results for the spectator momentum perpendicular and parallel of the momentum transfer in all three forms of dynamics.

These results mirror those discussed above: the fractional deviation of the BT results from the TP benchmark is quite small, of order 10^{-3} or less, for all cases except point-form Option B.

D. Implications for Nuclear Theory

Our results indicate that Bakamjian-Thomas models, which explicitly satisfy the requirements of Poincaré invariance, can be utilized in typical nuclear physics problems with minimal quantitative error due to the lack of cluster separability using Dirac's instant- or front-form dynamics. For the point form, particularly Option B, one must exercise care, since the effects can be large enough to play a role when comparing results to experimental data.

We believe that there is a physical basis behind the distinctive results for the point form, which center on the use of velocities, whereas a current matrix element depends upon momentum transfer. In all of the calculations discussed above, the BT approach links "external variables" such as $\mathbf{p}_3$ and $\mathbf{p}'_3$ that describe the observable kinematics to "internal variables" such as $\mathbf{b}_3$ and $\mathbf{b}'_3$ that are used to calculate such quantities as the particle-3 current matrix element. The momentum transfer $\mathbf{q}$ plays an explicit role in the calculation of the "inner" quantities for the instant and front forms, and even point-form Option A. However, with Option B, the effect of $\mathbf{q}$ is buried in a calculated vector

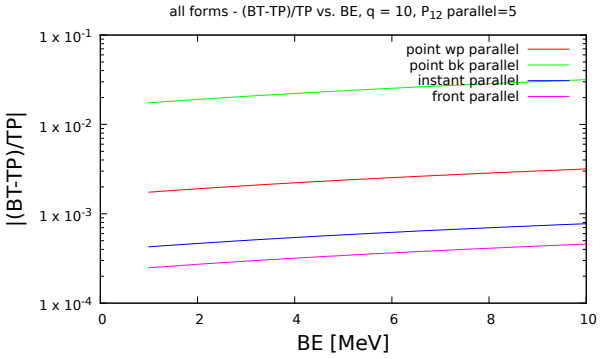

FIG. 10: Model differences for BT calculations as a function of two-body spectator binding energy, $\mathbf{q} = 10\ \mathrm{fm}^{-1} \| \mathbf{p}_{12}$.

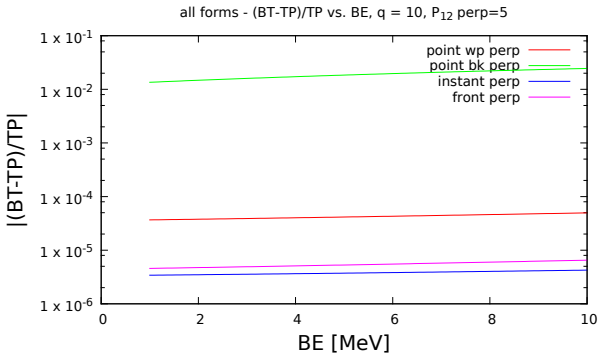

FIG. 11: Model differences for BT calculations as a function of two-body spectator binding energy, $\mathbf{q} = 10\ \mathrm{fm}^{-1} \perp \mathbf{p}_{12}$.

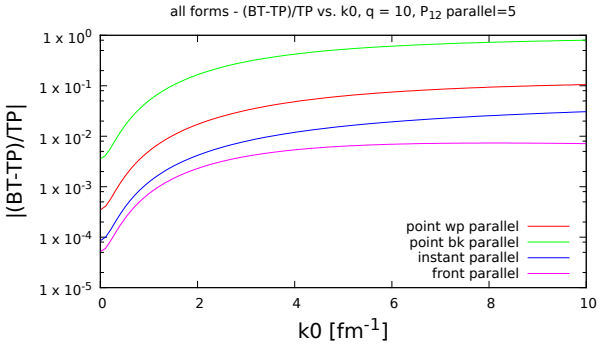

FIG. 12: Model differences for BT calculations as a function of two-body Gaussian wave function scale, $\mathbf{q} = 10\ \mathrm{fm}^{-1} \| \mathbf{p}_{12}$.

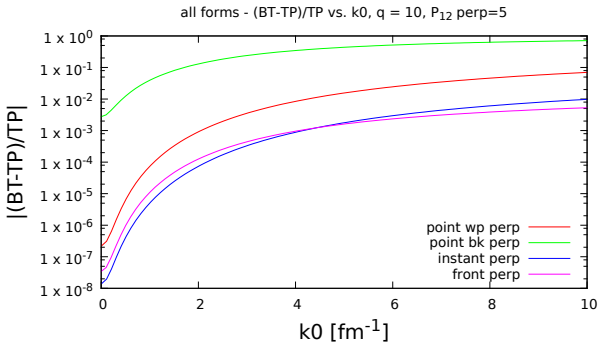

FIG. 13: Model differences for BT calculations as a function of two-body Gaussian wave function scale, $\mathbf{q} = 10\ \mathrm{fm}^{-1} \perp \mathbf{p}_{12}$.

$\mathbf{V}$ rather than a momentum that enters the internal kinematics directly. This option thus has a weaker connection to the physical kinematics dictated by the TP benchmark

While our model is considerably simpler than a realistic model of a three-nucleon system interacting with an electron, the difference between the Bakamjian-Thomas formulation of the model and the model that clusters properly is due entirely to the same unitary transformations discussed in our simplified model. This difference vanishes in the limit that these transformations become the identity. The analysis in this papers established that these operators, which implicitly depend on the interactions, are close to the identity for interactions with typical nuclear physics scales. This provides a strong justification for applying Bakamjian-Thomas construction of dynamical representations Poincaré group to system of more than three nucleons.

E. Implications for Hadron Models

The final set of figures show the results of calculations with scales that are more appropriate models of hadrons based on sub-nucleonic degrees of freedom.

We note here that QCD confinement precludes separating arbitrary subsystems by large distance scales, so the general requirement of cluster separability is irrelevant for models of hadrons. The issue may be relevant, however, for systems of hadrons described by subnucleonic degrees of freedom.

In these cases, in order to understand the relevant scales, we replace particle 3 masses in the above calculations by constituent quark masses of 220 MeV, and consider two-body ("diquark") masses ranging from 200 to 600 MeV for Gaussian wave functions with a 1 fm^{-1} scale, and Gaussian wave functions with scale ranging from 0.5 fm^{-1} to 10 fm^{-1}, with a diquark mass of 600 MeV.

The results are similar to those appropriate to nuclear physics discussed above, except that the scale of deviation

from the TP benchmark is somewhat larger, as high as 1% for instant and front forms, and of order unity for the point form options. This not surprising given that the mass/momentum scale variation for these calculations is much higher than for typical cases in nuclear physics with nucleons.

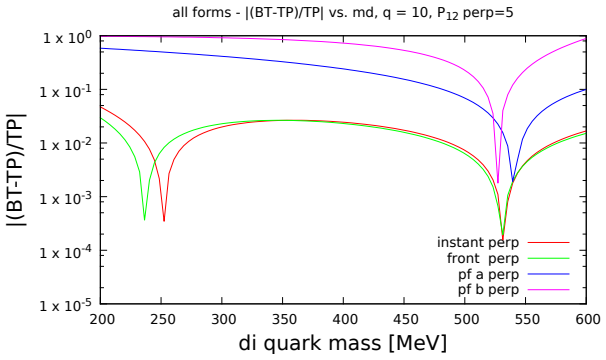

FIG. 14: Model differences for BT calculations as a function of two-body spectator mass, $\mathbf{q} = 10$ fm$^{-1} \| \mathbf{p}_{12}$.

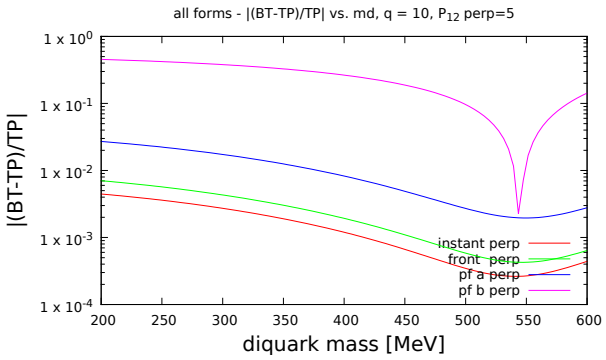

FIG. 15: Model differences for BT calculations as a function of two-body spectator mass, $\mathbf{q} = 10$ fm$^{-1} \perp \mathbf{p}_{12}$.

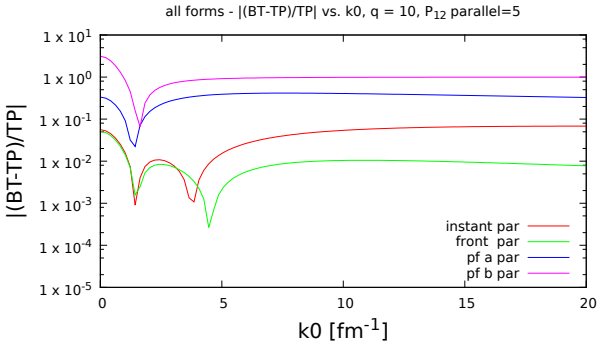

FIG. 16: Model differences for BT calculations as a function of Gaussian wave function scale, $\mathbf{q} = 10$ fm$^{-1} \| \mathbf{p}_{12}$.

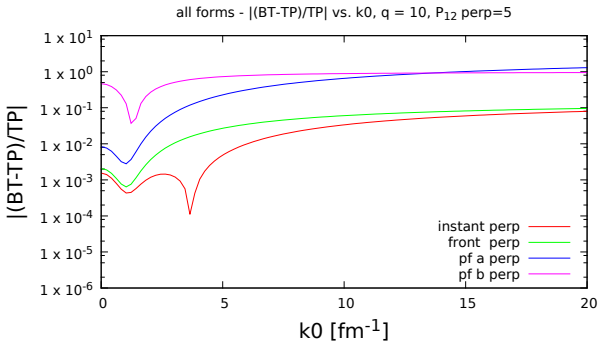

FIG. 17: Model differences for BT calculations as a function of Gaussian wave function scale, $\mathbf{q} = 10$ fm$^{-1} \perp \mathbf{p}_{12}$.

IV. SUMMARY

Bakamjian-Thomas formulations, which explicitly satisfy the requirements of Poincaré invariance, do not satisfy cluster separability above the three-particle level, i.e. in systems that involve three-body subsystems whose total momentum must vary. The cluster properties can be restored via a hierarchy of unitary transformations. These transformations depend upon the full solution of a three-body problem, and are difficult to implement in practice.

Rather than attempt to calculate directly the size of these unitary transformations (e.g. the difference of matrix elements from those of the unit operator), we have developed a simple model in which the exact result consistent with cluster separability is known, and then compare to it the results of Bakamjian-Thomas calculations.

We conclude from these model studies that Bakamjian-Thomas models, which explicitly satisfy the requirements of Poincaré invariance, can be utilized in typical nuclear physics problems with minimal quantitative error due to the lack of cluster separability using Dirac's instant- or front-form dynamics. For the point form, one must exercise care, since the effects can be large enough to play a role when comparing results to experimental data.

We also examined models utilizing mass/momentum scales appropriate for quark models. QCD confinement precludes separating arbitrary subsystems by large distance scales, so the general requirement of cluster separability is irrelevant for models of hadrons. The issue may be relevant, however, for systems of hadrons described by subnucleonic degrees of freedom. In such cases, the deviations from the model benchmark are larger than those for typical

nuclear physics calculations with nucleons, though they are still manageable for the instant and front forms.

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