Model Tests of Cluster Separability In Relativistic Quantum Mechanics

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A physical theory should have both the properties of relativistic invariance and of cluster separability. A relativistically invariant quantum theory is defined by a dynamical unitary representation of the Poincaré group. Cluster separability means that symmetries and conservation laws that hold for a system of particles also hold for isolated subsystems. A standard construction of dynamical unitary representations of the Poincaré group solves the problem of adding interactions that preserve the Poincaré commutation relations by including kinematically invariant interactions in the Casimir mass operator. The resulting unitary representation of the Poincaré group fails to satisfy cluster properties for systems of three or more particles. Cluster separability can be restored by means of a recursive construction using unitary transformations, but implementation is difficult in practice. We examine a simple model of a current operator in a three-particle system in which the required unitary transformations are approximated by the identity operator. The difference between these unitary transformations and the identity provides a measure of the size of corrections needed to restore cluster properties. Our estimates suggest that in models based on nucleon degrees of freedom that the corrections that restore cluster properties are too small to affect calculations of observables.

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I. INTRODUCTION

There are two distinct requirements for describing quantum mechanical systems of particles under the requirements of special relativity. The first requirement is Poincaré invariance: probabilities, expectation values and ensemble averages for equivalent experiments performed in different inertial frames are identical. Formally, Poincaré invariance requires that the dynamics be described by a unitary ray representation of the Poincaré group [1]. The second requirement is cluster separability: isolated subsystems must have the same observable properties as they would in a framework in which the other “spectator” particles are absent entirely. This requirement applies both to systems of particles interacting among themselves (e.g. via the strong interaction) and to the interaction of such systems with external fields via a current operator.

In local quantum field theory, both of these properties of the theory follow as a consequence of the covariance, spectral properties and the local commutation relations of the field operators. Formal solutions of the field theory satisfy both requirements. Approximate solutions may or may not, and must be checked explicitly.

For quantum mechanical models of systems of a finite number of degrees of freedom the requirements of Poincaré invariance, cluster properties, current conservation and current covariance constrain the structure of dynamical models.

The general problem of constructing dynamical representations of the Poincaré group was studied by Dirac [2] from an algebraic perspective, using Poisson brackets. He concluded that at least three of the infinitesimal generators of the Poincaré group must include interactions, although he did not solve the problem of how to add interactions that preserve the Poincaré commutation relations. Bakamjian and Thomas [3] provided the first solution to this problem that did not require the assumption of local fields. They constructed all of the Poincaré generators for a class of models consisting or two interacting particles. The Bakamjian-Thomas construction has been successfully generalized to treat systems of arbitrary numbers of particles and is not limited to models that conserve particle number. It has been successfully applied to construct realistic relativistic quantum mechanical models of few-body systems.

The advantage of the Bakamjian Thomas construction is that the framework allows one to add interactions involving different pairs of particles in a manner that exactly preserves the underlying Poincaré symmetry. For a two-body interaction in an N-body system the Bakamjian-Thomas interactions give the same S-matrix elements as a pure two-body model. The limitation of the class of models discovered by Bakamjian and Thomas is that for systems of more than two particles the models fail to satisfy the cluster separability requirement discussed above. Unfortunately, interactions that are consistent with cluster separability cannot be added without violating the Poincaré symmetry. This explains why the Bakamjian-Thomas construction is used in many applications.

It turns out that the resulting formal lack of cluster separability has no effect upon two- and three-particle observables (binding energies, scattering cross sections), but the observable effects set in with four-particle systems, in which three-particle mass eigenstates are embedded in the four-particle Hilbert space, as well as current matrix elements of three-particle systems. Both of these latter cases require evaluating three-particle wave functions with different three-body total momenta.

Sokolov [4] discovered an inductive construction that starts with Bakamjian-Thomas two-body models and builds many-body dynamical models consistent with both Poincaré invariance and cluster properties. A key ingredient in Sokolov's construction is unitary operators that transform tensor product representations of the Poincaré group to representations with a non-interacting spin. These transformations can be designed to preserve the S matrix, but they do not always preserve cluster properties. Sokolov used these operators to inductively restore cluster properties. For systems of four or more particles the inductive construction does not preserve the S matrix. The size of the corrections introduced by these unitary operators provides a measure of the size of the violation of cluster properties in the Bakamjian-Thomas construction.

While it is desirable to utilize Sokolov's inductive construction to formulate models of few-body systems, that construction is complicated. Three-body Faddeev-like equations need to be solved just to construct the three-body Hamiltonian. The physical consequences of these corrections first appear in the four-body problem. As a result of these technical complications, Sokolov's inductive construction has never been used in realistic applications. Thus, the question remains how important these corrections are in practical calculations. In this paper we approach the question by constructing a simple model for which the physical result with cluster separability is known, and for which a corresponding result can be calculated using the Bakamjian-Thomas construction. The difference between these provides a direct measure of the necessary correction to restore cluster separability. We provide model results for typical kinematics and interaction parameters found in nuclear physics, as well as for higher momenta and stronger binding scales as found in quark models.

II. BAKAMJIAN-THOMAS MODELS

Poincaré invariance and the Dirac forms of dynamics for Bakamjian-Thomas (BT) constructions are discussed extensively in Ref. [5]. The framework provided here makes use of that discussion.

We consider a system of three distinct, spinless particles, each with mass m, in which particles 1 and 2 interact only with each other and can form a bound state, and particle 3 interacts with a field via a current operator. Within this model, the physics of the (123) system interacting with the field is dictated solely by the matrix element of the current operator in the particle 3 subspace, with no dependence whatever upon the kinematics of the (12) spectator system. This can be illustrated as a disconnected graph as shown in Fig. 1.

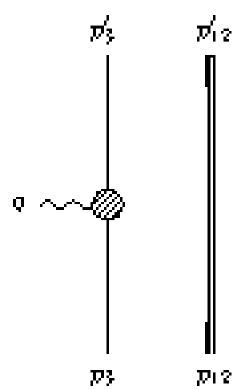

FIG. 1: Graph of one-body current plus bound two-body spectator.

The central element of a BT model is the invariant mass operator M. It replaces the center of mass Hamiltonian in the non-relativistic treatment. The mass operator for the (12) system in the absence of interactions is

$$M_{12}^{(0)} = 2\sqrt{m^2 + \mathbf{k}^2}, \tag{2.1}$$

where $\pm\mathbf{k}$ is the three-momentum of particle 1 (2) in the center-of-momentum frame. The BT method adds an interaction, U, to this mass operator, or its square. In the latter case, we express this as follows:

$$M_{12}^{(0)2} \to M_{12}^2 = M_{12}^{(0)2} + 4mU = 4m(\mathbf{k}^2/m + U) + 4m^2. \tag{2.2}$$

This is a simple function of the non-relativistic two-body Hamiltonian, in which case phenomenological fits based upon observed phase shifts and binding energies can immediately be applied.

The interaction U must commute with a set of nine independent functions of the non-interacting Poincaré generators. Dynamical Poincaré generators are functions of the interacting M and the nine-non-interacting operators the commute with U.

Using Eq. 2.2, one can construct a few-particle model whose input is an interaction U that has been fit to two-body binding and scattering data, where the overall problem satisfies Poincaré invariance.

III. CURRENT MATRIX ELEMENTS

We now consider matrix elements of a scalar current operator $j(x)$ that is the tensor product of a current for particle 3 with the identity for the 1-2 pair, which is bound with mass eigenvalue $\lambda = \lambda_{12}$.

A. Tensor-Product (TP) Representation

The relevant state vectors are tensor products of (12) and 3 eigenstates:

$$|\lambda_{12}, m; \mathbf{p}_{12}, \mathbf{p}_3\rangle = |\lambda_{12}; \mathbf{p}_{12}\rangle \otimes |m, \mathbf{p}_3\rangle. \tag{3.1}$$

Since the current $j(x)$ acts only in the space of particle 3, the matrix element has the form

$$\langle\lambda_{12} m; \mathbf{p}'_{12}, \mathbf{p}'_3|j(0)|\lambda_{12} m; \mathbf{p}_{12}, \mathbf{p}_3\rangle = \delta(\mathbf{p}'_{12} - \mathbf{p}_{12}) f(\mathbf{q}^2), \tag{3.2}$$

where

$$\langle m; \mathbf{p}'_3|j(0)|m; \mathbf{p}_3\rangle = f(\mathbf{q}^2); \quad \mathbf{q} = \mathbf{p}'_3 - \mathbf{p}_3, \tag{3.3}$$

λ_{12} is a bound-state eigenvalue of M_{12}, and $\mathbf{p}_{12}$ is its three-momentum.

If we specify initial momenta $\mathbf{p}_3$ and $\mathbf{p}_{12}$ and momentum transfer $\mathbf{q}$, and integrate the matrix element over the final spectator momentum $\mathbf{p}'_{12}$, the integral collapses due to the spectator momentum delta function, and we have

$$\mathcal{F}_{TP} := \int d\mathbf{p}'_{12} \langle\lambda_{12}, m; \mathbf{p}'_{12}, \mathbf{p}'_3|j(0)|\lambda_{12}, m; \mathbf{p}_{12}, \mathbf{p}_3\rangle = f(\mathbf{q}^2). \tag{3.4}$$

Equation 3.4 is the Tensor-Product (TP) result. It has no dependence upon the momentum $\mathbf{p}_{12}$ of the bound-state spectator, as expected from the physical requirement of cluster separability, and it does not depend upon the specific values of $\mathbf{p}_3$ and $\mathbf{p}'_3$, as long as $\mathbf{p}'_3 - \mathbf{p} = \mathbf{q}$.

In the following sections, we evaluate the same matrix element using BT representations of the spectator-nucleon system corresponding to different forms of dynamics [2]. The different forms of dynamics are associated with different subgroups that commute with the interaction U discussed earlier.

These current matrix elements will each depend upon the structure of the bound state, in violation of cluster separability, and our goal is to examine the magnitude of that violation.

In all cases, we specify the momentum transfer $\mathbf{q}$ and spectator momentum $\mathbf{p}_{12}$ for a given calculation and then vary these momenta for sensitivity tests.

B. BT Representation: Instant Form

We now evaluate current matrix element in Eq. 3.2 using a BT representation in an instant-form model. First, we change variables: We replace the spectator and nucleon momenta by the total momentum of the system and the momentum of the nucleon Lorentz transformed to the spectator-nucleon rest frame:

$$(\mathbf{p}_{12}, \mathbf{p}_3) \to (\mathbf{P}, \mathbf{p}), \tag{3.5}$$

where

$$\mathbf{P} = \mathbf{p}_{12} + \mathbf{p}_3; \quad \mathbf{p} = \mathbf{p}_3 + \Phi(\mathbf{p}_3, \mathbf{P}, M_\lambda)\mathbf{P}, \tag{3.6}$$

and

$$\Phi(\mathbf{p}_3, \mathbf{P}, M_\lambda) = \frac{1}{M_\lambda}\left[\frac{\mathbf{P}\cdot\mathbf{p}_3}{E_\lambda + M_\lambda} - \omega_m(\mathbf{p}_1)\right]; \quad M_\lambda = \sqrt{E_\lambda^2 - \mathbf{P}^2}; \quad E_\lambda = \sqrt{m^2 + \mathbf{p}_3^2} + \sqrt{\lambda^2 + \mathbf{p}_{12}^2}. \tag{3.7}$$

With this variable change the relation between the bound pair and spectator in these bases is:

$$|\lambda, m; \mathbf{p}_{12}, \mathbf{p}_3\rangle = \left|\frac{\partial(\mathbf{P}, \mathbf{p})}{\partial(\mathbf{p}_{12}, \mathbf{p}_3)}\right|_{M_\lambda}^{\frac{1}{2}} |\lambda, m; \mathbf{P}, \mathbf{p}\rangle. \tag{3.8}$$

We also introduce single-nucleon momenta $\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3$. We change variables to the total momentum of the 12 pair and the momentum of particle 1 Lorentz transformed to the rest frame of the 1-2 pair using:

$$\mathbf{k} = \mathbf{b}_1 + \Phi(\mathbf{b}_1, \mathbf{b}_{12}, M_{12}^0)\mathbf{b}_{12}; \quad \mathbf{b}_{12} = \mathbf{b}_1 + \mathbf{b}_2; \quad M_{12} = 2\sqrt{m^2 + \mathbf{k}^2}. \tag{3.9}$$

We define the three-nucleon invariant mass

$$M_k := \sqrt{(M_{12}^0)^2 + \mathbf{p}^2} + \sqrt{m^2 + \mathbf{p}^2} \tag{3.10}$$

and the momentum of particle 3 Lorentz boosted to the rest frame or the three-body system:

$$\mathbf{p} = \mathbf{p}_3 + \Phi(\mathbf{p}_3, \mathbf{P}, M_k)\mathbf{P}, \tag{3.11}$$

where

$$\Phi(\mathbf{p}_3, \mathbf{P}, M_k)\mathbf{P} = \frac{1}{M_k}\left[\frac{\mathbf{P}\cdot\mathbf{p}_3}{E_k + M_k} - \omega_m(\mathbf{p}_1)\right]; \quad E_k = \sqrt{m^2 + \mathbf{p}_3^2} + \sqrt{M_{12}^2 + \mathbf{p}_{12}^2}. \tag{3.12}$$

In an instant-form Bakamjian Thomas dynamics, the connection between a state with three free particles and that with particle plus bound spectator is

$$\langle\lambda; \mathbf{P}', \mathbf{p}'|\mathbf{P}, \mathbf{p}, \mathbf{k}\rangle = \delta(\mathbf{P}' - \mathbf{P})\delta(\mathbf{p}' - \mathbf{p})\phi_\lambda(\mathbf{k}), \tag{3.13}$$

Note that $\mathbf{p}$ depends upon the free two-particle mass $M_{12}^{(0)}$ via Eq. 3.11, while $\mathbf{p}'$ depends upon the bound-state mass eigenvalue via λ Eq. 3.6. The association of these two three-momenta in the delta function of Eq. 3.13 is therefore not fully consistent. There is no observable consequence for the two-body S matrix or mass eigenvalues, but this association will lead to a result that violates cluster separability.

We now make use of Eq. 3.13 to compute the BT counterpart to $\mathcal{F}_{TP}$ that was defined by Eq. 3.4:

$$\begin{aligned}\mathcal{F}_{BT} := & \int d\mathbf{p}'_{12} d\mathbf{b}'_{12} d\mathbf{b}'_3 d\mathbf{k}' d\mathbf{b}_{12} d\mathbf{b}_3 d\mathbf{k} \\ & \times \left|\frac{\partial(\mathbf{P}', \mathbf{p}')}{\partial(\mathbf{p}'_{12}, \mathbf{p}'_3)}\right|_{M_\lambda}^{\frac{1}{2}} \left|\frac{\partial(\mathbf{P}, \mathbf{p})}{\partial(\mathbf{p}_{12}, \mathbf{p}_3)}\right|_{M_\lambda}^{\frac{1}{2}} \left|\frac{\partial(\mathbf{b}'_{12}, \mathbf{b}'_3)}{\partial(\mathbf{P}', \mathbf{p}')}\right|_{M_k}^{\frac{1}{2}} \left|\frac{\partial(\mathbf{b}_{12}, \mathbf{b}_3)}{\partial(\mathbf{P}, \mathbf{p})}\right|_{M_k}^{\frac{1}{2}} \\ & \times \phi_\lambda^*(\mathbf{k}')\langle\mathbf{b}'_{12}, \mathbf{b}'_3, \mathbf{k}'|j(0)|\mathbf{b}_{12}, \mathbf{b}_3, \mathbf{k}\rangle.\phi_\lambda(\mathbf{k})\end{aligned} \tag{3.14}$$

Since the current operator j operates only in the space of particle 3, we have

$$\langle\mathbf{b}'_{12}, \mathbf{b}'_3, \mathbf{k}'|j(0)|\mathbf{b}_{12}, \mathbf{b}_3, \mathbf{k}\rangle = \delta(\mathbf{k}' - \mathbf{k})\delta(\mathbf{b}'_{12} - \mathbf{b}_{12})f[(\mathbf{b}'_3 - \mathbf{b}_3)^2]. \tag{3.15}$$

The integral over $\mathbf{p}'_{12}$ can be converted to an integral over $\mathbf{b}'_{12}$ by means of Jacobians:

$$\int d\mathbf{p}'_{12} = \int d\mathbf{b}'_{12} \left|\frac{\partial(\mathbf{P}', \mathbf{p}')}{\partial(\mathbf{b}_{12},' \mathbf{b}'_3)}\right|_{M_k} \left|\frac{\partial(\mathbf{p}'_{12}, \mathbf{p}'_3)}{\partial(\mathbf{P}', \mathbf{p}')}\right|_{M_\lambda}. \tag{3.16}$$

The final result is

$$\mathcal{F}_{BT}^{\text{instant}} := \int d\mathbf{k} \left|\frac{\partial(\mathbf{p}'_{12}, \mathbf{p}'_3)}{\partial(\mathbf{P}', \mathbf{p}')}\right|_{M_\lambda}^{\frac{1}{2}} \left|\frac{\partial(\mathbf{P}, \mathbf{p})}{\partial(\mathbf{p}_{12}, \mathbf{p}_3)}\right|_{M_\lambda}^{\frac{1}{2}} \left|\frac{\partial(\mathbf{P}', \mathbf{p}')}{\partial(\mathbf{b}'_{12}, \mathbf{b}'_3)}\right|_{M_k}^{\frac{1}{2}} \left|\frac{\partial(\mathbf{b}_{12}, \mathbf{b}_3)}{\partial(\mathbf{P}, \mathbf{p})}\right|_{M_k}^{\frac{1}{2}} |\phi_\lambda(\mathbf{k})|^2 f[(\mathbf{b}'_3 - \mathbf{b}_3)^2]. \tag{3.17}$$

The interesting observation is that this integral has a non-trivial dependence on $\mathbf{p}_{12}$, in contrast to the $\mathbf{p}_{12}$ independence of the tensor product result. In this model the BT and tensor product representations are related one of Sokolov's unitary transformations, $A_{(12)(3)}$, that preserve the three-body S-matrix:

$$|\lambda, m; \mathbf{p}_{12}, \mathbf{p}_3\rangle_{TP} = A_{(12)(3)} |\lambda, m; \mathbf{p}_{12}, \mathbf{p}_3\rangle_{BT}. \tag{3.18}$$

The scale of the $\mathbf{p}_{12}$ dependence in (3.17) provides a measure of the size of the violation of cluster properties that results from ignoring $A_{(12)(3)}$ by replacing it with the unit operator. Since we know both states in eq 3.18 we do not have to calculate $A_{(12)(3)}$ explicitly to determine its impact.

For this calculation, we vary the three-momentum transfer $\mathbf{q}$ and the spectator momentum $\mathbf{p}_{12}$. The initial momentum of particle 3 and the final momentum of the system are fixed in terms of these variables:

- $\mathbf{p}_3 = -\frac{1}{2}\mathbf{q}$;
- $\mathbf{P}' = \mathbf{P} + \mathbf{q}$.

Since

$$\mathbf{P} = \mathbf{p}_{12} + \mathbf{p}_3 = \mathbf{b}_{12} + \mathbf{b}_3; \quad \mathbf{P}' = \mathbf{p}'_{12} + \mathbf{p}'_3 = \mathbf{b}'_{12} + \mathbf{b}'_3, \tag{3.19}$$

and the current matrix element constrains $\mathbf{b}'_{12} = \mathbf{b}_{12}$, we find that $\mathbf{b}'_3 = \mathbf{b}_3 + \mathbf{q}$, in which case the argument of the form factor is $\mathbf{q}^2$ as it is in the TP case. However, the final momenta $\mathbf{p}'_{12}$ and $\mathbf{p}'_3$ are not constrained, and in general there are non-vanishing contributions to this matrix element for $\mathbf{p}'_3 \neq \mathbf{p}_3 + \mathbf{q}$ and $\mathbf{p}'_{12} \neq \mathbf{p}_{12} + \mathbf{q}$.

In the nonrelativistic limit, where $\mathbf{k}$, $\mathbf{q}$ and $\mathbf{p}_{12}$ are all small with respect to the relevant masses, the Jacobians are approximately unity and can be factored out of the integral, leaving a unit wave function normalization and a result identical to the TP case. The quantitative level of disagreement with the TP result is therefore linked to the extent to which the model goes beyond the nonrelativistic limit.

C. BT Representation: Front Form

Dirac's front-form dynamics is described in detail in Ref. [5]. We provide a summary here.

Basis states in the front form are described by light-front momenta

$$\tilde{\mathbf{p}} = (\mathbf{p}_\perp, p^+); \quad p^+ = p^0 + p^3. \tag{3.20}$$

These generate translations in a hyperplane $x^+ = 0$ tangent to the light cone.

In the front form the Lorentz transformations used to define the nucleon momentum in the rest frame of the spectator-nucleon or three nucleon system in the instant form are replaced by boosts that leave the light front invariant. While these boosts are interaction independent, there is an interaction dependence in the boost parameters. The Bakamjian Thomas model again identifies the momentum of the nucleon in the rest frame of the spectator-nucleon and three-nucleon systems. As in the instant form case this is an incorrect identification that has no consequences for the two-body S matrix. With this modification the front-form result has the same structure as the instant-form result:

$$\mathcal{F}_{BT}^{\text{front}} := \int d\mathbf{k} \left|\frac{\partial(\tilde{\mathbf{p}}'_{12}, \tilde{\mathbf{p}}'_3)}{\partial(\tilde{\mathbf{P}}', \tilde{\mathbf{p}}')}\right|_{M_\lambda}^{\frac{1}{2}} \left|\frac{\partial(\tilde{\mathbf{P}}, \tilde{\mathbf{p}})}{\partial(\tilde{\mathbf{p}}_{12}, \tilde{\mathbf{p}}_3)}\right|_{M_\lambda}^{\frac{1}{2}} \left|\frac{\partial(\tilde{\mathbf{P}}', \tilde{\mathbf{p}}')}{\partial(\tilde{\mathbf{b}}'_{12}, \tilde{\mathbf{b}}'_3)}\right|_{M_k}^{\frac{1}{2}} \left|\frac{\partial(\tilde{\mathbf{b}}_{12}, \tilde{\mathbf{b}}_3)}{\partial(\tilde{\mathbf{P}}, \tilde{\mathbf{p}})}\right|_{M_k}^{\frac{1}{2}} |\phi_\lambda(\mathbf{k})|^2 \, f[(\tilde{\mathbf{b}}'_3 - \tilde{\mathbf{b}}_3)^2]. \tag{3.21}$$

This quantity has an unphysical dependence on $\tilde{\mathbf{b}}_{12}$ that does not occur in the tensor product.

For this calculation, we vary $\tilde{\mathbf{q}}$ and $\tilde{\mathbf{p}}_{12}$, with coordinate axes chosen such that $q^+ = 0$ and $p^+_{12} = 0$. We define $\mathbf{p}_\perp$ and $\tilde{\mathbf{P}}'$ in terms of these quantities:

- $\mathbf{p}_\perp = -\frac{1}{2}\mathbf{q}_\perp$;
- $\tilde{\mathbf{P}}' = \tilde{\mathbf{P}} + \tilde{\mathbf{q}}$.

Analogous to the discussion following Eq. 3.19, we have that

$$\tilde{\mathbf{P}} = \tilde{\mathbf{p}}_{12} + \tilde{\mathbf{p}}_3 = \tilde{\mathbf{b}}_{12} + \tilde{\mathbf{b}}_3; \quad \mathbf{P}' = \tilde{\mathbf{p}}'_{12} + \tilde{\mathbf{p}}'_3 = \tilde{\mathbf{b}}'_{12} + \tilde{\mathbf{b}}'_3, \tag{3.22}$$

The current matrix element constrains $\tilde{\mathbf{b}}'_{12} = \tilde{\mathbf{b}}_{12}$, and therefore $\tilde{\mathbf{b}}'_3 = \tilde{\mathbf{b}}_3 + \tilde{\mathbf{q}}$, in which case the argument of the form factor is $\mathbf{q}^2$ as it is in the TP case. However, the final momenta $\tilde{\mathbf{p}}'_{12}$ and $\tilde{\mathbf{p}}'_3$ are not constrained, and in general the integral has non-zero contributions from $\tilde{\mathbf{p}}'_3 \neq \tilde{\mathbf{p}}_3 + \tilde{\mathbf{q}}$ and $\tilde{\mathbf{p}}'_{12} \neq \tilde{\mathbf{p}}_{12} + \tilde{\mathbf{q}}$.

D. BT Representation: Point Form

Dirac's point-form dynamics are also described in detail in Ref. [5]. We provide a summary here.

Basis states in the point form are described by four-velocity vectors $\mathbf{v}$. Momenta are obtained by multiplying the four-velocities by (interacting or non-interacting) masses. Thus, we seek to evaluate matrix elements of the current operator $j(x)$ between three-particle states with initial four-velocity $\mathbf{V}$ and final four-velocity $\mathbf{V}'$.

The derivations proceed in a fashion similar to the instant form except that we must take care to use velocities in cases where the associated masses can acquire interactions.

With these conventions, the point-form $\mathcal{F}_{BT}$ has the structure

$$\mathcal{F}_{BT}^{\text{point}} := \int d\mathbf{k} \left|\frac{\partial(\mathbf{p}'_{12},\mathbf{p}'_3)}{\partial(\mathbf{V}',\mathbf{p}')}\right|_{M_\lambda}^{\frac{1}{2}} \left|\frac{\partial(\mathbf{V},\mathbf{p})}{\partial(\mathbf{p}_{12},\mathbf{p}_3)}\right|_{M_\lambda}^{\frac{1}{2}} \left|\frac{\partial(\mathbf{V}',\mathbf{p}')}{\partial(\mathbf{b}'_{12},\mathbf{b}'_3)}\right|_{M_k}^{\frac{1}{2}} \left|\frac{\partial(\mathbf{b}_{12},\mathbf{b}_3)}{\partial(\mathbf{V},\mathbf{p})}\right|_{M_k}^{\frac{1}{2}} |\phi_\lambda(\mathbf{k})|^2 \, f[(\mathbf{b}'_3-\mathbf{b}_3)^2]. \tag{3.23}$$

For the instant- and front-form calculations, we chose to fix three three-momenta in a way that kept the physical momentum transfer in the particle-3 current matrix element. For the point form, we provide two different constraint choices, each analogous to the previous calculations, but leading to different results.

A. Constrain the momentum transfer to particle 3:

- $\mathbf{p}_3 = -\frac{1}{2}\mathbf{q}$;
- $\mathbf{p}'_3 = \mathbf{p}_3 + \mathbf{q}$.

Note in this case that the velocities $\mathbf{V}$ and $\mathbf{V}'$ are the same for the kinematics using the interacting mass λ or the free mass M_{12}. This implies that

$$\mathbf{b}_3 = \mathbf{p}'_3; \quad \mathbf{b}'_3 = \mathbf{p}'_3, \tag{3.24}$$

and therefore that $\mathbf{b}'_3 - \mathbf{b}_3 = \mathbf{q}$. One consequence is that in general, $M_\lambda(\mathbf{V}'-\mathbf{V}) \neq \mathbf{q}$.

B. Constrain the final velocity to match momentum transfer $\mathbf{q}$.

- $\mathbf{p}_3 = -\frac{1}{2}\mathbf{q}$;
- $\mathbf{V}' = \mathbf{V}_{\text{calc}}$.

To obtain $\mathbf{V}_{\text{calc}}$, we use temporary quantities

$$\mathbf{P}'_{\text{temp}} = \mathbf{P} + \mathbf{q} \tag{3.25}$$

and

$$P'^{0}_{\text{temp}} = P^0. \tag{3.26}$$

These are the results that one expects in the tensor-product representation. From these relations we obtain

$$M'_{\text{temp}} = \sqrt{(P'^{0}_{\text{temp}})^2 - \mathbf{P}'^{2}_{\text{temp}}}, \tag{3.27}$$

and then define

$$\mathbf{V}_{\text{calc}} = \frac{\mathbf{P}'_{\text{temp}}}{M'_{\text{temp}}}. \tag{3.28}$$

Substituting, we get

$$\mathbf{V}_{\text{calc}} = \frac{\mathbf{P}+\mathbf{q}}{\sqrt{(P^0)^2 - (\mathbf{P}+\mathbf{q})^2}}. \tag{3.29}$$

Thus, the three-vector $\mathbf{V}'$ is constrained, but the actual values of M' and $\mathbf{P}'$ may be different than the temporary values used to obtain $\mathbf{V}_{\text{calc}}$. In this form,

$$\begin{aligned} \mathbf{V}_{\text{int}} &= \mathbf{V}_{\text{ext}}; \\ \mathbf{V}'_{\text{int}} &= \mathbf{V}'_{\text{ext}}, \end{aligned} \tag{3.30}$$

which also implies that

$$\begin{aligned} \mathbf{b}_3 &= \mathbf{p}_3; \\ \mathbf{b}'_3 &= \mathbf{p}'_3, \end{aligned} \tag{3.31}$$

IV. RESULTS AND DISCUSSION

In this section we discuss results of calculations of the form factor $\mathcal{F}_{BT}$ for instant, front and two point-form kinematic choices. Initially we consider scales that are relevant for systems of nucleons interacting with two-body interactions. Then we turn to examples more appropriate to hadron models.

In all figures we show the figure of merit:

$$\frac{(\mathcal{F}_{BT} - \mathcal{F}_{TP})}{\mathcal{F}_{TP}}, \tag{4.1}$$

which represents the relative error induced by ignoring the unitary transformation eq. 3.18 that restores cluster separability.

A. Malfliet-Tjon Deuteron Wave Function

To model realistic conditions for nuclear physics, we use a deuteron spectator constructed with Malfliet-Tjon [6] potential IV, which contains both attractive and repulsive forces.

We first consider the figure of merit as a function of the momentum transfer and momentum of the deuteron in each of Dirac's forms of dynamics and for the deuteron momentum perpendicular and parallel to the momentum transfer. The results are shown in the six figures below. In all cases the expected results are given by the flat planes.

The fractional deviation of this BT model calculation from the TP result that satisfies cluster separability is very small for the instant- and front-form cases, typically of order 10^{-3} or smaller at the highest values of the three-momenta $\mathbf{q}$ and $\mathbf{p}_{12}$. The deviations for the point-form Option A are somewhat larger, and larger still for Option B.

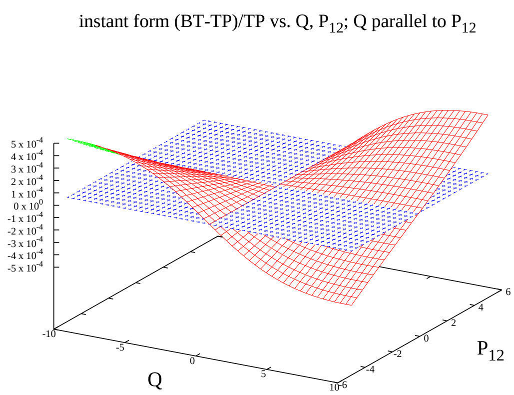

FIG. 2: Model differences for instant-form BT calculation, $\mathbf{q} \| \mathbf{p}_{12}$.

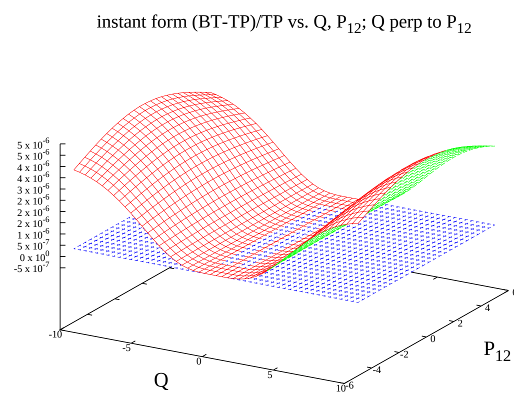

FIG. 3: Model differences for instant-form BT calculation, $\mathbf{q} \perp \mathbf{p}_{12}$.

B. Binding Energy Variation

The calculations above assumed a bound deuteron with a wave function having a standard dependence on the relative momentum of the constituent nucleons. The next set of curves illustrates the figure of merit for fixed values of $\mathbf{q}$ and $\mathbf{p}_{12}$ as we vary the binding energy and momentum scale of the wave function. The variations that we consider are still scales that are relevant to nuclear systems. Figures 10 and 11 show the results of calculations vary the binding energy with the Malfliet-Tjon wave function.

As with the earlier cases that employed the deuteron binding energy, the fractional deviation of the BT results from the TP benchmark is quite small, of order 10^{-3} or less, for all cases except point-form Option B.

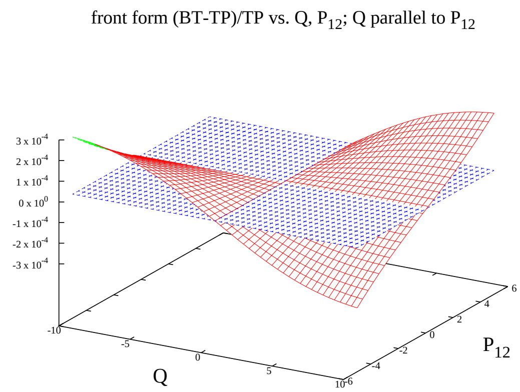

FIG. 4: Model differences for front-form BT calculation, $\mathbf{q} \| \mathbf{p}_{12}$.

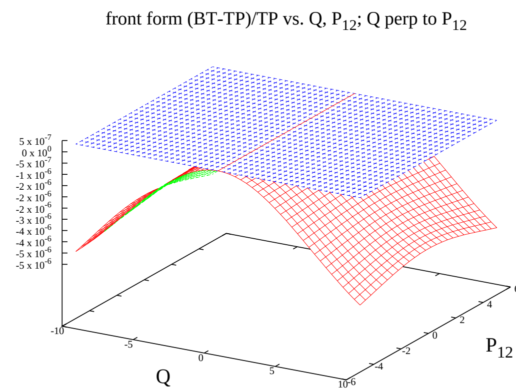

FIG. 5: Model differences for front-form BT calculation, $\mathbf{q} \perp \mathbf{p}_{12}$.

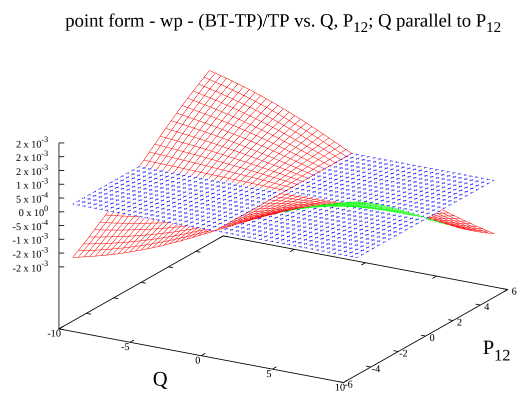

FIG. 6: Model differences for point-form BT calculation A, $\mathbf{q} \| \mathbf{p}_{12}$.

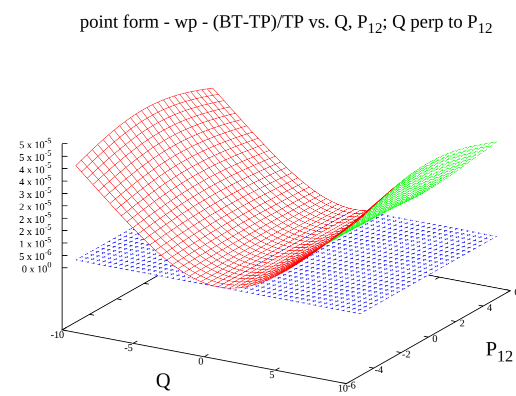

FIG. 7: Model differences for point-form BT calculation A, $\mathbf{q} \perp \mathbf{p}_{12}$.

C. Wave Function Scale Variation

We also examined sensitivity to the scale of the wave function by replacing the Malfliet-Tjon function with a Gaussian form:

$$\phi(\mathbf{k}) = \frac{1}{\sqrt{N}} e^{-(k/k_0)^2}. \tag{4.2}$$

Figures 12 and 13 show the results for the spectator momentum perpendicular and parallel of the momentum transfer in all three forms of dynamics.

These results mirror those discussed above: the fractional deviation of the BT results from the TP benchmark is quite small, of order 10^{-3} or less, for all cases except point-form Option B.

D. Implications for Nuclear Theory

Our results indicate that Bakamjian-Thomas models, which explicitly satisfy the requirements of Poincaré invariance, can be utilized in typical nuclear physics problems with minimal quantitative error due to the lack of cluster separability using Dirac's instant- or front-form dynamics. For the point form, particularly Option B, one must exercise care, since the effects can be large enough to play a role when comparing results to experimental data.

We believe that there is a physical basis behind the distinctive results for the point form, which center on the use of velocities, whereas a current matrix element depends upon momentum transfer. In all of the calculations discussed

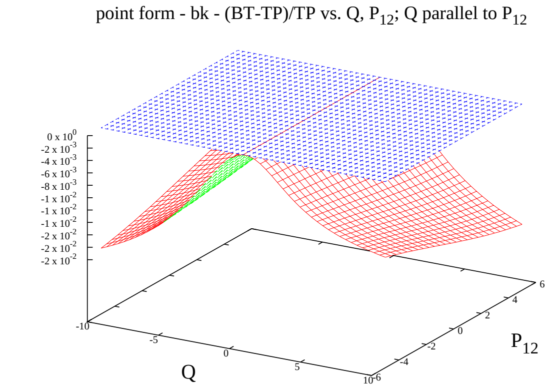

FIG. 8: Model differences for point-form BT calculation B, $\mathbf{q} \| \mathbf{p}_{12}$.

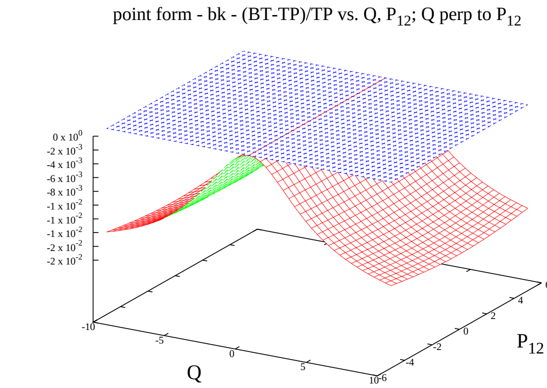

FIG. 9: Model differences for point-form BT calculation B, $\mathbf{q} \perp \mathbf{p}_{12}$.

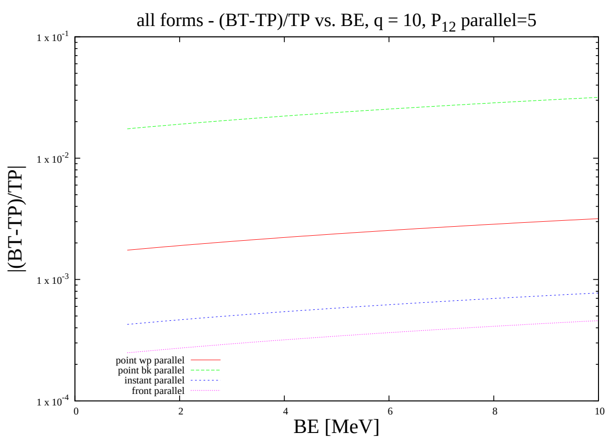

FIG. 10: Model differences for BT calculations as a function of two-body spectator binding energy, $\mathbf{q} = 10\ \mathrm{fm}^{-1} \| \mathbf{p}_{12}$.

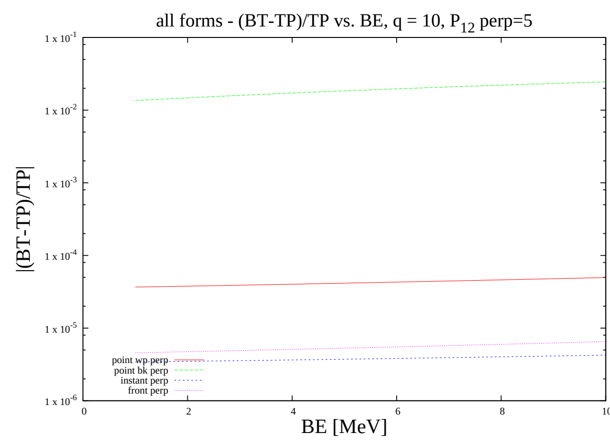

FIG. 11: Model differences for BT calculations as a function of two-body spectator binding energy, $\mathbf{q} = 10\ \mathrm{fm}^{-1} \perp \mathbf{p}_{12}$.

above, the BT approach links "external variables" such as $\mathbf{p}_3$ and $\mathbf{p}'_3$ that describe the observable kinematics to "internal variables" such as $\mathbf{b}_3$ and $\mathbf{b}'_3$ that are used to calculate such quantities as the particle-3 current matrix element. The momentum transfer $\mathbf{q}$ plays an explicit role in the calculation of the "inner" quantities for the instant and front forms, and even point-form Option A. However, with Option B, the effect of $\mathbf{q}$ is buried in a calculated vector $\mathbf{V}$ rather than a momentum that enters the internal kinematics directly. This option thus has a weaker connection to the physical kinematics dictated by the TP benchmark.

E. Implications for Hadron Models

The final set of figures show the results of calculations with scales that are more appropriate models of hadrons based on sub-nucleonic degrees of freedom.

We note here that QCD confinement precludes separating arbitrary subsystems by large distance scales, so the general requirement of cluster separability is irrelevant for models of hadrons. The issue may be relevant, however, for systems of hadrons described by subnucleonic degrees of freedom.

In these cases, in order to understand the relevant scales, we replace the nucleon masses in the above calculations by constituent quark masses of 220 MeV, and consider two-body masses ranging from 200 to 600 MeV for Gaussian wave functions with a 1 fm^{-1} scale, and Gaussian wave functions with scale ranging from 0.5 fm^{-1} to 10 fm^{-1}, with a diquark mass of 600 MeV.

The results are similar to those appropriate to nuclear physics discussed above, except that the scale of deviation

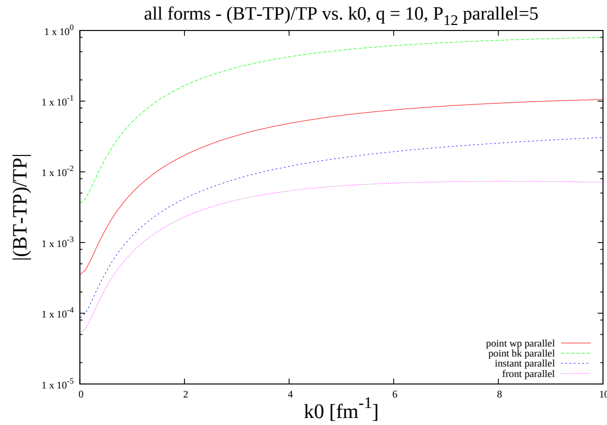

FIG. 12: Model differences for BT calculations as a function of two-body Gaussian wave function scale, $\mathbf{q} = 10\ \mathrm{fm}^{-1} \| \mathbf{p}_{12}$.

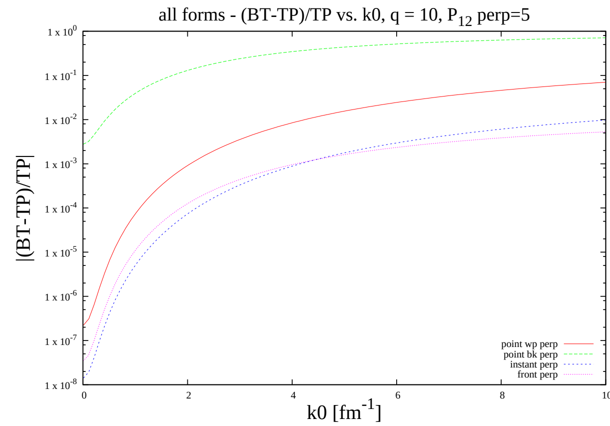

FIG. 13: Model differences for BT calculations as a function of two-body Gaussian wave function scale, $\mathbf{q} = 10\ \mathrm{fm}^{-1} \perp \mathbf{p}_{12}$.

from the TP benchmark is somewhat larger, as high as 1% for instant and front forms, and of order unity for the point form options. This not surprising given that the mass/momentum scale variation for these calculations is much higher than for typical cases in nuclear physics with nucleons.

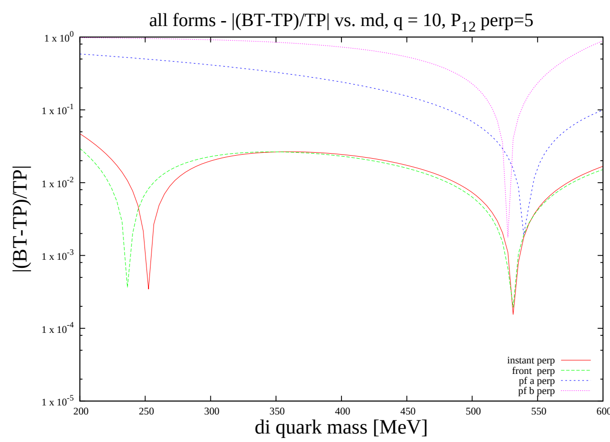

FIG. 14: Model differences for BT calculations as a function of two-body spectator mass, $\mathbf{q} = 10\ \mathrm{fm}^{-1} \| \mathbf{p}_{12}$.

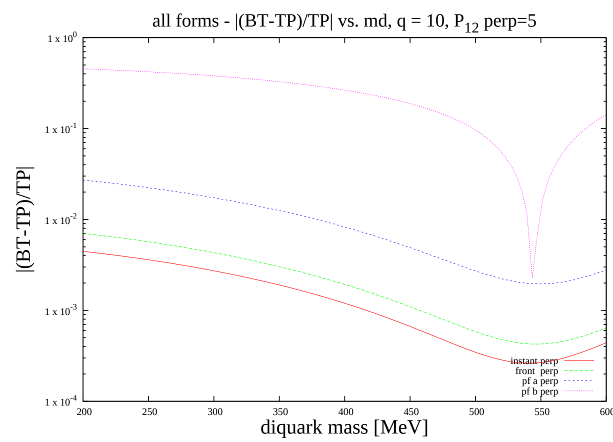

FIG. 15: Model differences for BT calculations as a function of two-body spectator mass, $\mathbf{q} = 10\ \mathrm{fm}^{-1} \perp \mathbf{p}_{12}$.

V. SUMMARY

Bakamjian-Thomas formulations, which explicitly satisfy the requirements of Poincaré invariance, do not satisfy cluster separability above the three-particle level, i.e. in systems that involve three-body systems whose total momentum must vary. The cluster properties can be restored via a hierarchy of unitary transformations. These transformations depend upon the full solution of a three-body problem, and are difficult to implement in practice.

Rather than attempt to calculate directly the size of these unitary transformations (e.g. the difference of matrix elements from those of the unit operator), we have developed a simple model in which the exact result consistent with cluster separability is known, and then compare to it the results of Bakamjian-Thomas calculations.

We conclude from these model studies that Bakamjian-Thomas models, which explicitly satisfy the requirements of Poincaré invariance, can be utilized in typical nuclear physics problems with minimal quantitative error due to the lack of cluster separability using Dirac's instant- or front-form dynamics. For the point form, one must exercise care, since the effects can be large enough to play a role when comparing results to experimental data.

We also examined models utilizing mass/momentum scales appropriate for quark models. QCD confinement precludes separating arbitrary subsystems by large distance scales, so the general requirement of cluster separability is

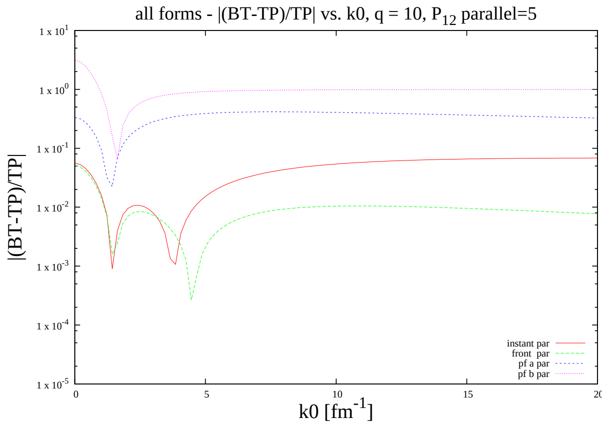

FIG. 16: Model differences for BT calculations as a function of Gaussian wave function scale, $\mathbf{q} = 10\ \mathrm{fm}^{-1} \| \mathbf{p}_{12}$.

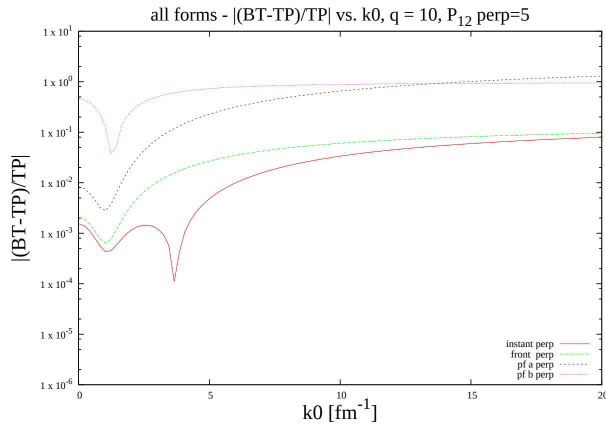

FIG. 17: Model differences for BT calculations as a function of Gaussian wave function scale, $\mathbf{q} = 10\ \mathrm{fm}^{-1} \perp \mathbf{p}_{12}$.

irrelevant for models of hadrons. The issue may be relevant, however, for systems of hadrons described by subnucleonic degrees of freedom. In such cases, the deviations from the model benchmark are larger than those for typical nuclear physics calculations with nucleons, though they are still manageable for the instant and front forms.

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