# Charge Form Factors of Quark-Model Pions

F. Coester*†
*Physics Division, Argonne National Laboratory Argonne, IL 60439, USA*

W.N. Polyzou‡
*Department of Physics and Astronomy University of Iowa, Iowa City, IA 52242 USA*

Experimental data of the pion charge form factor are well represented by Poincaré invariant constituent-quark phenomenology depending on two parameters, a confinement scale and an effective quark mass. Pion states are represented by eigenfunctions of mass and spin operators, and of the light-front momenta. An effective current density is generated by the dynamics from a null-plane impulse current density. A simple shape of the wave function depending only on the confinement scale is sufficient. The range of quark-masses and confinement scales consistent with both low and high $Q^2$ data depends on the shape of the wave function.

The unitary Poincaré representations on the Hilbert space of confined quark states may be specified by mass and spin operators together with the choice of a kinematic subgroup [1, 2]. The four-momentum operator is a function of the mass operator $\mathcal{M}$ and three kinematic quantities which depend on the choice of a kinematic subgroup.[3] Single-hadron eigenstates of the mass operator may be represented by functions, $\phi_n$, of constituent quark momenta, spin, flavor and color variables. The mass operator can be defined by the spectral representation

$$\mathcal{M} := \sum_n \phi_n M_n \phi_n^\dagger \;, \tag{1}$$

using empirical hadron masses or by the conventional assumption that the operator $\mathcal{M}$ (or $\mathcal{M}^2$) is expressed as the sum of a kinetic term $\mathcal{M}_0$ (or $\mathcal{M}_0^2$), and a confining potential. However, mass operators of confined quark need not involve this conventional dependence on constituent-quark masses and the formal structure of the dynamics is simpler if it does not.[4, 5] The wave functions representing hadron states are not observable. The observable form factors are determined by both the wave functions and associated representations of the Poincaré covariant, conserved, current density operators $I^\mu(x)$,

$$\begin{aligned} U(\Lambda,a) I^\mu(x) U^\dagger(\Lambda,a) &= I^\nu(\Lambda x + a)\Lambda_\nu{}^\mu \;, \\ \imath [P_\mu, I^\mu(x)] &= \partial_\mu I^\mu(x) = 0 \;. \end{aligned} \tag{2}$$

Poincaré covariant conserved current operators may be generated by the dynamics from any input that is covariant under the kinematic subgroup. The form factors are invariant under unitary changes of the representations of both the state vectors and the current operators. Such transformations may preserve the kinematic subgroup [1, 2].

*corresponding author
†coester@anl.gov
‡polyzou@uiowa.edu

Here we investigate the compatibility of simple shapes of the wave function with current operators generated by the dynamics from impulse currents. Quark masses of confined quark enter as scale parameters of the impulse currents. Their values differ with different forms of kinematics [3]. They are not masses of potentially free constituent particles.

The null-plane kinematic subgroup leaves invariant the null-plane $n \cdot x = 0$, with $n^2 = 0$. With a convenient choice of the axes the components of $n$ are given by $n = \{1, 0, 0, 1\}$. With the null vector perpendicular to the momentum transfer, $n \cdot Q = 0$, the invariant $Q^2 = Q_\perp^2$ is kinematic.

It is an important consequence that the form factors do not depend on the hadron mass. This feature is essential for the quark phenomenology of pion form factors. With point-form kinematics [6] form factors obtain as functions of $\eta := Q^2/4m_\pi^2$, which is large for moderate values of $Q^2$. Thus, with simple wave functions, form factors are much too small for a realistic representation [7].

Previously [8] we demonstrated that the simple quark phenomenology sketched above yielded charge form factors of the pion in agreement with available data for both low $Q^2$ [9] and $Q^2 > 1\mathrm{GeV}^2$. Recently new measurements [10] provided more precise data for $Q^2 < 2$ GeV$^2$ and new data at higher values of $Q^2$ are expected. Assuming a simple shape of the wave function the model depends on two scales, the confinement scale of the wave function and the quark mass. The purpose of this paper is to investigate the implications of precise data for $Q^2 > 1$ GeV$^2$. Recent data constrain acceptable values of the quark mass and confinement scale to a narrow range of values, which depends on the the shape of the wave function. For $Q^2 > 2$ GeV$^2$ form factors consistent with existing data decrease at different rates depending on the shape of the wave function.

There is no intent to approximate features of quantum field theory in the construction of such quark models. In particular the representations of constituent-quark states are not meant to approximate Fock-space amplitudes and/or satisfy features of perturbative QCD [11].

As in ref. [8] the models are specified by input currents

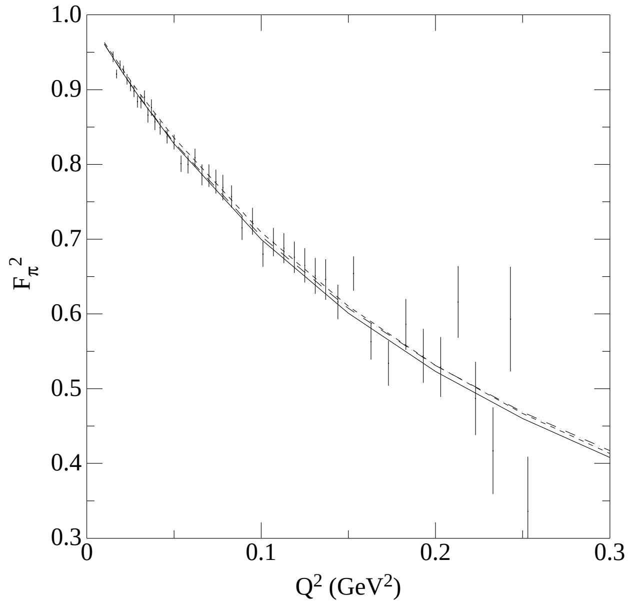


FIG. 1: Wave function dependence of pion form factors at low momentum transfer.

with the representation

$$\langle \xi', k'_\perp, \mu', \bar\mu' | n \cdot I(0) | \xi, k_\perp, \mu, \bar\mu \rangle =$$

$$\delta_{\mu',\mu}\delta_{\bar\mu',\bar\mu}\delta(\xi' - \xi)\delta(k'_\perp - k_\perp - (1-\xi)Q_\perp) \ , \quad (3)$$

and a representation of the pion state by wave functions $\phi(\xi, k_\perp, \mu, \bar\mu)$ which is proportional to a radial wave function $u(k^2)$ and Melosh rotation matrices,

$$\phi(\xi, k_\perp, \mu, \bar\mu) := \sum_{\mu',\bar\mu'} \langle \mu | \mathcal{R}^\dagger(\xi, k_\perp) | \mu' \rangle$$

$$\times \ \langle \bar\mu | \mathcal{R}^\dagger(1-\xi, -k_\perp) | \bar\mu' \rangle (\tfrac{1}{2}, \tfrac{1}{2}, \mu', \bar\mu' | 0, 0) u(k^2) \ . \quad (4)$$

The argument $k^2$ is related to the null-plane momenta by

$$k^2 + m_q^2 = \frac{k_\perp^2 + m_q^2}{4\xi(1-\xi)} \ . \quad (5)$$

The pion charge form factor is a functional of the radial wave function, $u(k^2)$,

$$F_\pi(Q^2) = \frac{1}{16\pi}\int_0^1 d\xi \int \frac{d^2\bar k_\perp}{\xi(1-\xi)} \mathcal{W}(\xi, \bar k_\perp) u(k'^2) u(k^2) \quad (6)$$

with

$$\bar k_\perp = k_\perp - \tfrac{1}{2}(1-\xi)Q_\perp = k'_\perp + \tfrac{1}{2}(1-\xi)Q_\perp \quad (7)$$

and

$$\mathcal{W} := \sqrt{\frac{\xi(1-\xi)}{\sqrt{(m_q^2 + k_\perp^2)(m_q^2 + {k'_\perp}^2)}}} \frac{m_q^2 + \bar k_\perp^2 - \frac{1}{4}(1-\xi)^2 Q^2}{\xi(1-\xi)} \ . \quad (8)$$

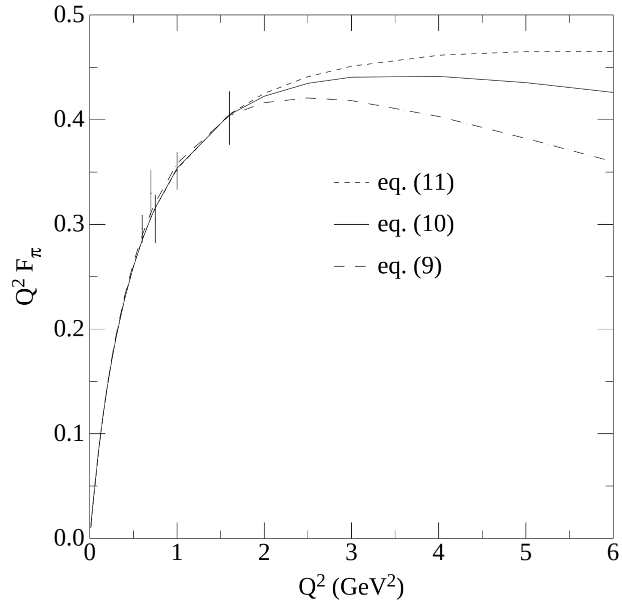


FIG. 2: Wave-function dependence of pion form factors at high momentum transfer

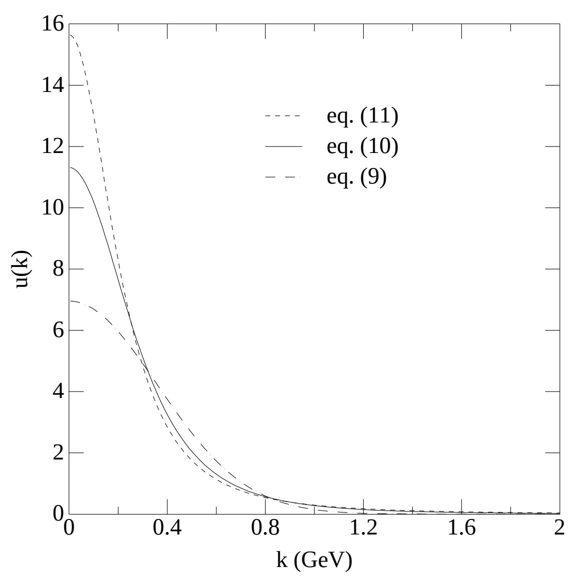


FIG. 3: Shapes of the three radial wave functions

In [8] a Gaussian shape

$$u(k^2) = \sqrt{\frac{4}{\sqrt{\pi} b^3}} \exp(-k^2/2b^2) \quad (9)$$

was used for numerical convenience. We expect that a rational shape, e.g.

$$u(k^2) = \sqrt{\frac{32}{\pi b^3}} \left(\frac{1}{1 + k^2/b^2}\right)^2 \quad (10)$$

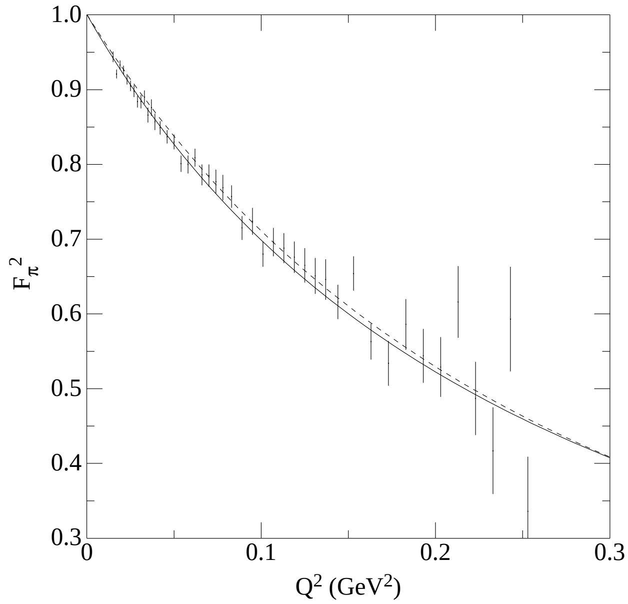


FIG. 4: Wave-function dependence of pion form factors at low momentum transfer.

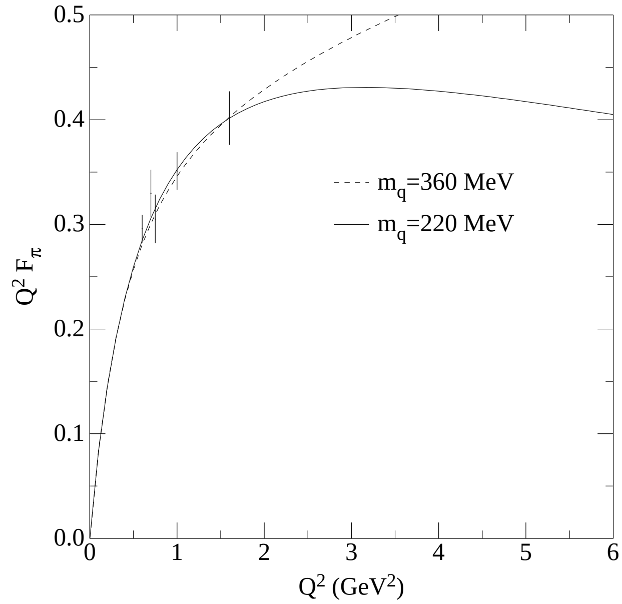


FIG. 5: Wave function dependence of pion form factors at high momentum transfer

may specify a better model. For each shape of the wave function a precise fit to both low and high $Q^2$ data tightly constrains acceptable values of the quark mass $m_q$ and the confinement scale $b$. A larger quark mass, requires a different shape, illustrated by the function

$$u(k^2) = \sqrt{\frac{16}{\pi b^3}} \left( \frac{1}{1+k^2/b^2} \right)^{\frac{3}{2}} . \tag{11}$$

The form factors obtained with the parameters listed in Table I are shown in Figs. 1 and 2 .

The pion decay constant $f_\pi$ is related to the quark axial coupling $g_A^q$ by [12, 13]

$$\begin{aligned} f_\pi &= \frac{\sqrt{3} g_A^q}{8\pi^2} \int_0^1 d\xi \int \frac{d^2 k_\perp}{\xi(1-\xi)} \frac{m_q}{\sqrt{M_0}} u(k^2) , \\ M_0^2 &:= \frac{m_q^2 + k_\perp^2}{\xi(1-\xi)} . \end{aligned} \tag{12}$$

The empirical value $f_\pi = 92.4$ MeV [14] implies the values of $g_A^q$ listed in Table I.

The shapes of the three wave functions are compared in Fig. 3. All three parameterizations are in agreement with

TABLE I: Dependence of quark masses, confinement scales and quark axial couplings on the shape of the wave functions

| wave function | $m_q$ (MeV) | $b$ (MeV) | $g_A^q$ |
|---|---|---|---|
| eq. (9 ) | 200 | 360 | 1.07 |
| eq. (10 ) | 230 | 430 | .91 |
| eq. (11) | 300 | 275 | .76 |

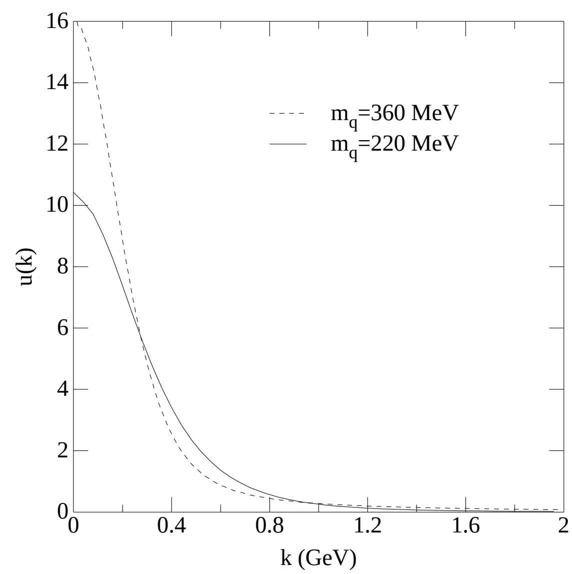


FIG. 6: Shapes of the two radial wave functions

existing data. For larger values of $Q^2$ the form factors decrease at different rates depending on the quark mass and the shape of the wave function. For $Q^2 > 2$ GeV$^2$ the form factors obtained with eq. (10) are in agreement with the QCD approximations of Maris and Tandy [15].

Conventional QCD motivated mass operators [16, 17],

$$\left( 2\sqrt{m_q^2 + k^2} + \alpha r - \frac{\beta}{r} + \gamma\, \vec{s}_q \cdot \vec{s}_{\bar{q}}\, e^{-\lambda^2 r^2} - m_0 \right) u(r) = 0 \tag{13}$$

designed to fit meson spectra, produce wave functions that require substantial modification of the input current [12]. The application of unitary "scattering equivalences" [18], which preserve the quark masses, produce wave functions compatible with impulse currents. These transformed wave functions may be realized adjusting the parameters in the potential.[18, 19] In Figs. 4-6 this is illustrated for $m_q = 220$ MeV [16] and $m_q = 360$ MeV [17]. The associated potential parameters and quark axial couplings are listed in Table II.

Minimal phenomenological models of confined quark dynamics, illustrated by eqs. (9) -(11), with null-plane impulse currents and quark masses between 200 and 300 MeV can easily accommodate existing experimental values of pion form factors, as well QCD based predictions for $Q^2$ in the range of several GeV$^2$ [15]. Sufficiently precise data for larger values of $Q^2$ will limit acceptable shapes of the wave functions and associated mass scales of the impulse currents.

### Acknowledgments

Research supported in part by the U.S. Department of Energy, Office of Nuclear Physics contracts W-31-109-ENG-38 and DE-FG02-86ER40286 .

TABLE II: Dependence of quark masses, confinement scales and quark axial couplings on the shape of the wave functions

| $m_q$ (MeV) | $\alpha$ GeV$^2$ | $\beta$ | $\gamma$ GeV | $\lambda$ GeV$^2$ | $g_A^q$ |
|---|---|---|---|---|---|
| 220 | .10 | .40 | .167 | .299 | .98 |
| 360 | .03 | .155 | 2.94 | 1.48 | .68 |

[1] B. D. Keister and W. N. Polyzou, Adv. Nucl. Phys. **20**, 225 (1991)
[2] F. Coester, Prog. Part. Nucl. Phys. **29**, 1 (1992)
[3] F. Coester and D.O. Riska, Nucl. Phys. A. **728**, 439 (2003)
[4] F. Feynman. M Kislinger, F. Ravndal Phys. Rev. D **3**. 2706 (1971)
[5] F. Coester, K. Dannbom, D.O. Riska, Nucl. Phys. A **634**, 335 (1998)
[6] T. W. Allen and W. H. Klink, Phys. Rev. C **58** , 3670 (1998)
[7] A. Amghar, B. Desplanques and L.Theussl, Phys. Lett. B **574** , 201 (2003)
[8] P. L. Chung, F. Coester and W. N. Polyzou, Phys. Lett. B **205**, 545 (1988)
[9] S. R. Amendolia et al., Nucl. Phys. B **277**, 168 (1986)
[10] J. Volmer et al., Phys. Rev. Lett. **86** , 1713 (2001)
[11] S. J. Brodsky et al., hep-ph/0311218
[12] F. Cardarelli et al., Phys. Lett. B **332**, 1 (1994); **357**, 267 (1995)
[13] W. Jaus, Phys. Rev. D **44**, 2851 (1991)
[14] B. Holstein, Phys. Lett. B **244**, 83 (1990)
[15] P. Maris and P. C. Tandy, Phys. Rev. C **62** , 055204 (2001)
[16] S. Godfrey and N. Isgur, Phys. Rev. D **32**, 189
[17] J. Carlson, J.B. Kogut and V.R. Pandharipande, Phys. Rev. D **28**, 2807 (1983).
[18] Evan Sengbusch and W.N. Polyzou Phys. Rev . C**70** 058201 (2004)
[19] B. D. Keister and W. N. Polyzou, J. Comp. Phys. **134** 231 (1997).