Euclidean Relativistic Quantum Mechanics

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Abstract

We discuss a formulation of exactly Poincaré invariant quantum mechanics where the input is model Euclidean Green functions or their generating functional. We discuss the structure of the models, the construction of the Hilbert space, the construction and transformation properties of single-particle states, and the construction of GeV scale transition matrix elements. A simple model is utilized to demonstrate the feasibility of this approach.

1 Introduction

The motivation for this work is to construct mathematically well-defined quantum mechanical models of few-body systems at the GeV energy scale that have a direct relation to an underlying quantum field theory. The goal is to use experience gained from the field theory to constrain the structure of the models.

We do this by starting with the quantum mechanical interpretation of the field theory. Normally this is given in terms of vacuum expectation values of products of fields (Wightman functions), which represent the kernel of the Hilbert space inner product of the field theory [2]; however the Wightman functions do not have a simple connection with the Lagrangian formulation of the field theory. The Euclidean Green functions have the advantage that they can be directly related to Lagrangian field theory and at the same time can be used to reconstruct the underlying quantum theory [3][4][5].

With this connection in mind we consider a class of models that are ideally expressed in terms of Euclidean-invariant reflection-positive Green functions or their generating functional. The generating functional is formally the functional Fourier transform of the path measure:

$$Z[f] := \int D_e[\phi] e^{-A[\phi]+i\phi(f)} = \sum_n \frac{(i)^n}{n!} G_n \underbrace{(f, \cdots, f)}_{\text{n times}} = \prod_n \exp\left(\frac{i^n}{n!} C_n(f, \cdots, f)\right). \qquad (1)$$

This provides the formal relation to the field-theoretic Lagrangian. For the purpose of illustration we restrict our considerations to generating functionals for scalar fields. The G_n are the n-point Euclidean Green functions smeared over test functions in Euclidean space-time variables and the C_n are the corresponding connected Green functions.

The generating functionals are assumed to be Euclidean invariant, reflection positive, and satisfy space-like cluster properties. The test functions are assumed to be Schwartz functions with support for positive Euclidean time. We denote this space by $\mathcal{S}^+$. Euclidean transforms on the test functions are denoted by $f(\mathsf{x}) \to f_{O,\mathsf{a}}(\mathsf{x}) := f(O\mathsf{x}+\mathsf{a}))$ and Euclidean time reflection is denoted by $\Theta f(\tau,\mathbf{x}) := f(-\tau,\mathbf{x})$. The requirements on the generating functional are [5][6]:

$$Z[f] = Z[f_{O,\mathsf{a}}] \qquad \textbf{Euclidean invariance} \tag{2}$$

$$\{f_i\}_{i=1}^N \in \mathcal{S}^+ \qquad M_{ij} = Z[f_i - \Theta f_j] \geq 0 \qquad \textbf{reflection positivity} \tag{3}$$

$$\lim_{|\mathbf{a}|\to\infty} (Z[f+g_{I,\mathbf{a}}] - Z[f]Z[g]) \to 0 \qquad \textbf{cluster properties}. \tag{4}$$

Models can be constructed by specifying a set of model connected n-point functions, C_n in (1). A sufficient condition for reflection positivity is that each term in the product (1) is reflection positive.

A dense set of vectors in the model Hilbert space is given by functionals of the form:

$$B[\phi] = \sum_{j=1}^{N_b} b_j e^{i\phi(f_j)} \qquad C[\phi] = \sum_{k=1}^{N_c} c_k e^{i\phi(g_k)} \tag{5}$$

where $b_j, c_k \in \mathbb{C}$, $f_j, g_k \in \mathcal{S}_+$ and $N_b, N_c < \infty$. The model Hilbert inner product of two such vectors is

$$\langle B|C\rangle := \sum_{j=1}^{N_b}\sum_{k=1}^{N_c} b_j^* c_k Z[g_k - \Theta f_j] = \int D_e[\phi] e^{-A[\phi]} B^*[\phi\circ\theta] C[\phi]. \tag{6}$$

The representation at the end of eq. (6) suggests that we can think of the vectors as wave functionals, however the computation of the inner product only requires the generating functional. The reflection positivity condition ensures that vectors have positive length:

$$\langle B|B\rangle \geq 0. \tag{7}$$

To understand how Poincaré invariance is realized observe that the determinants of the following matrices are $(-)$ the squares of the Lorentz and Euclidean lengths respectively:

$$X = \begin{pmatrix} t+z & x-iy \\ x+iy & t-z \end{pmatrix} \qquad \mathsf{X} = \begin{pmatrix} i\tau+z & x-iy \\ x+iy & i\tau-z \end{pmatrix}. \tag{8}$$

The group of linear transformation that preserves both of these determinants is $SL(2,\mathbb{C}) \times SL(2,\mathbb{C})$:

$$\mathsf{X}' = A\mathsf{X}B^t \qquad X = AXB^t \qquad \det(A) = \det(B) = 1. \tag{9}$$

These represent complex Lorentz or complex orthogonal transformations. Real Lorentz transformations have $B = A^*$ while real orthogonal transformations have $A, B \in SU(2)$. The group of real orthogonal transformations form a subgroup of the complex Lorentz group in the inner product (6). When one accounts for the support condition on the test functions, Euclidean time evolution becomes a contractive semigroup, rotations in Euclidean space-time planes become local symmetric semigroups [7][8] [9], and translations in a fixed direction and rotations about a fixed axis become unitary one-parameter groups. The generators of these transformations are self-adjoint operators on the physical Hilbert space. The one-parameter groups (semigroups) can be expressed in terms of their infinitesimal generators as

$$e^{-\beta H} \qquad \beta > 0 \qquad e^{i\mathbf{a}\cdot\mathbf{P}} \qquad e^{i\mathbf{J}\cdot\hat{\mathbf{n}}\psi} \qquad e^{\mathbf{K}\cdot\hat{\mathbf{n}}\psi}. \tag{10}$$

It is straightforward to show that the generators $\{H, \mathbf{P}, \mathbf{J}, \mathbf{K}\}$ satisfy the commutation relations of the Poincaré Lie algebra.

In this framework particles are point spectrum eigenstates of the square of the mass operator: $M^2 := H^2 - \mathbf{P}^2$. Normalizable mass eigenstates can be represented as wave functionals

$$B_\lambda[\phi] = \sum_n b_n e^{i\phi(f_n)}. \tag{11}$$

Simultaneous eigenfunctionals of mass, linear momentum and angular momentum can be extracted from these mass eigenstates using rotations and translations:

$$B_\lambda(\mathbf{p})[\phi] = \int \frac{d^3a}{(2\pi)^{3/2}} e^{-i\mathbf{p}\cdot\mathbf{a}} B_{\lambda,I,\mathbf{a}}[\phi] \tag{12}$$

$$B_{\lambda,j}(\mathbf{p},\mu)[\phi] := \int_{SU(2)} dR \sum_{\nu=-j}^{j} B_{\lambda,R,0}(R^{-1}\mathbf{p})[\phi] D^{j*}_{\mu\nu}(R). \tag{13}$$

The single-particle wave functionals can be interpreted as multiplication operators. These single-particle wave functionals can be used to construct the two Hilbert space injection operators that define the asymptotic conditions in the two Hilbert space [10] formulation of Haag-Ruelle Scattering theory [11][12][13][14]. The wave operators and injection operator have the form

$$|\Psi_\pm(f_1,\cdots f_n)\rangle := \lim_{t\to\infty} e^{iHt}\Phi e^{-iH_0t}|\mathbf{f}\rangle = \Omega_\pm|\mathbf{f}\rangle \tag{14}$$

$$\Phi|\mathbf{f}\rangle[\phi] =$$

$$\int \sum \prod_k \left(\omega_{\lambda_k}(\mathbf{p}_k) B_{\lambda_k,j_k}(\mathbf{p_k},\mu_k)[\phi] - [H, B_{\lambda_k,j_k}(\mathbf{p}_k,\mu_k)[\phi]]\right) \tilde{f}_k(\mathbf{p}_k,\mu_k) d\mathbf{p}_k. \tag{15}$$

The asymptotic Hilbert space is the tensor product of one-particle irreducible representation spaces of the Poincaré group. Existence of the wave operators can be checked in a given model by verifying the finiteness of the integral [15]:

$$\int_0^{\pm\infty} \|(H\Phi - \Phi H_0)e^{-iH_0t}|\mathbf{f}\rangle\| dt < \infty, \tag{16}$$

while Poincaré covariance of the wave operators,

$$U(\Lambda,a)\Omega_\pm = \Omega_\pm U_0(\Lambda,a) \tag{17}$$

can be checked by verifying the asymptotic condition in this representation of the Hilbert space

$$\lim_{t\to\pm\infty} \|(\mathbf{K}\Phi - \Phi\mathbf{K}_0)e^{-iH_0t}|\mathbf{f}\rangle\| = 0. \tag{18}$$

Approximate sharp-momentum transition matrix elements can be computed from S matrix elements in narrow wave packets using [16]

$$\langle \mathbf{p}_1', \mu_1', \cdots, \mathbf{p}_n', \mu_n'|T|\mathbf{p}_1,\mu_1,\mathbf{p}_2,\mu_2\rangle \approx \frac{\langle \mathbf{f}_f|S|\mathbf{f}_i\rangle - \delta_{ab}\langle \mathbf{f}_f|\mathbf{f}_i\rangle}{2\pi i\langle \mathbf{f}_f|\delta(E_+ - E_-)|\mathbf{f}_i\rangle}. \tag{19}$$

Using the Kato-Birman invariance principle [17] [18][13][14] the expression for the wave operators can be replaced by the limits

$$\Omega_\pm := \lim_{t\to\pm\infty} e^{-iHt}\Phi e^{iH_0t} = \lim_{n\to\pm\infty} e^{ine^{-\beta H}}\Phi e^{-ine^{-\beta H_0}}. \tag{20}$$

which for large enough n gives the approximate expression for the S-matrix elements in normalizable states:

$$\langle \mathbf{f}_f|S|\mathbf{f}_i\rangle = \langle \mathbf{f}_f|\Omega_+^\dagger\Omega_-|\mathbf{f}_i\rangle \approx \langle \mathbf{f}_f|e^{-ine^{-\beta H_f}}\Phi^\dagger e^{2ine^{-\beta H}}\Phi e^{-ine^{-\beta H_f}}|\mathbf{f}_i\rangle. \tag{21}$$

The compactness of the spectrum of $\exp(-\beta H)$ means that for large but fixed n that $e^{2ine^{-\beta H}}$ can be uniformly approximated by polynomial in $\exp(-\beta H)$:

$$e^{2ine^{-\beta H}} \approx \sum c_m(n)(e^{-\beta m H}). \tag{22}$$

Chebyshev expansions provide an accurate approximation [19] for large values of n:

$$f(e^{-\beta H}) \approx \frac{1}{2}c_0 T_0(e^{-\beta H}) + \sum_{k=1}^{N} c_k T_k(e^{-\beta H}) \tag{23}$$

$$c_j = \frac{2}{N+1}\sum_{k=1}^{N} f(\cos(\frac{2k-1}{N+1}\frac{\pi}{2})\cos(j\frac{2k-1}{N+1}\frac{\pi}{2}). \tag{24}$$

We demonstrate the feasibiliy of this computational method using an exactly solvable relativistic model with a mass square operator given by

$$M^2 = 4(\mathbf{k}^2 + m^2) - |g\rangle\lambda\langle g| \tag{25}$$

$$\langle \mathbf{k}|g\rangle = \frac{1}{m_\pi^2 + \mathbf{k}^2} \tag{26}$$

where m is mass of a nucleon and λ is chosen to give a bound state with the mass of a deuteron. First we test the approximation in equation (19). We use Gaussian wave packets of the form $e^{-\alpha(k-k_0)^2}$ and find that to get sharp-momentum transition matrix elements to a 0.1% accuracy the width of the wave packet, $k_w = 1/\sqrt{\alpha}$, needs to be about 3% of the initial momentum, k_0. This works at least up to 2 GeV. The results are illustrated in table 1:

Table 1

k_0	α	k_w	% error	k_w/k_0
[GeV]	[GeV^{-2}]	[GeV]		
0.1	105000	0.00308607	0.1	0.030
0.3	10500	0.009759	0.1	0.032
0.5	3000	0.0182574	0.1	0.036
0.7	1350	0.0272166	0.1	0.038
0.9	750	0.0365148	0.1	0.040
1.1	475	0.0458831	0.1	0.041
1.3	330	0.0550482	0.1	0.042
1.5	250	0.0632456	0.1	0.042
1.7	190	0.0725476	0.1	0.042
1.9	150	0.0816497	0.1	0.042

Next we test the approximation in (21) for the wave packet widths in table 1. We choose β so β times the center of momentum (CM) energy is a number of order unity. Table 2 shows that for $n = 300$ we get ten figure accuracy in the real and imaginary parts of the S-matrix elements for a 2GeV incident CM momentum. Similar results are obtained for all momentum scales between 100 MeV and 1.9 GeV.

Table 2: $k_0 = 2.0$**[GeV],** $\alpha = 135$**[GeV^{-2}]**

n	Re $\langle\phi\vert(S_n - I)\vert\phi\rangle$	Im $\langle\phi\vert(S_n - I)\vert\phi\rangle$
50	-2.60094316473225e-6	1.94120750171791e-3
100	-2.82916859895010e-6	2.35553585404449e-3
150	-2.83171624670953e-6	2.37471383801820e-3
200	-2.83165946257657e-6	2.37492460997990e-3
250	-2.83165905312632e-6	2.37492527186858e-3
300	-2.83165905257121e-6	2.37492527262432e-3
350	-2.83165905190508e-6	2.37492527262493e-3
400	-2.83165905234917e-6	2.37492527262540e-3
ex	-2.83165905227843e-6	2.37492527259701e-3

Finally we test the Cheybshev approximation for the wave packet widths in table 1 and the n-values in table 2. Table 3 shows that for polynomials of degree slightly larger than n one obtains a 10-13 figure accuracy uniformly for spectrum of $\exp(-\beta H)$.

Table 3: Convergence with respect to Polynomial degree e^{inx}

x	n	deg	poly error %
0.1	200	200	3.276e+00
0.1	200	250	1.925e-11
0.1	200	300	4.903e-13
0.1	630	630	2.069e+00
0.1	630	680	5.015e-08
0.1	630	700	7.456e-11
0.5	200	200	1.627e-13
0.5	200	250	3.266e-13
0.5	630	580	1.430e-14
0.5	630	680	9.330e-13
0.9	200	200	3.276e+00
0.9	200	250	1.950e-11
0.9	200	300	9.828e-13
0.9	630	630	2.069e+00
0.9	630	680	5.015e-08
0.9	630	700	7.230e-11

Table 4 shows the final approximation for the real and imaginary parts of the sharp-momentum transition matrix elements for CM momenta up to 1.9 GeV. The results are all within less than 0.1% of the exact results in this model.

Table 4: Approximate transition matrix elements

k_0	Real T	Im T	% error
0.1	-2.30337e-1	-4.09325e-1	0.0956
0.3	-3.46973e-2	-6.97209e-3	0.0966
0.5	-6.44255e-3	-3.86459e-4	0.0986
0.7	-1.88847e-3	-4.63489e-5	0.0977
0.9	-7.28609e-4	-8.86653e-6	0.0982
1.1	-3.35731e-4	-2.30067e-6	0.0987
1.3	-1.74947e-4	-7.38285e-7	0.0985
1.5	-9.97346e-5	-2.76849e-7	0.0956
1.7	-6.08794e-5	-1.16909e-7	0.0964
1.9	-3.92110e-5	-5.42037e-8	0.0967

2 Conclusion

We presented a formulation of relativistic quantum mechanics [1] that uses Euclidean generating functionals or Green functions as input. In applications these have to be modeled. One virtue of this representation is that all calculations can be performed without analytic continuation.

The expression in equation (1) suggests that the generating functionals can be modeled using a finite collection of model connected Green functions. While it is easy to maintain Euclidean covariance and cluster properties of the models in this representation, reflection positivity is a non-trivial constraint that will be the subject of future investigations. While it holds for free field generating functionals, it is not stable with respect to small perturbations [20]. Failure of reflection positivity points to violations of the spectral condition or the positivity of the Hilbert space norms.

The model calculations presented suggest that for models based on reflection positive generating functionals this framework can be used to accurately compute both bound state and scattering observables.

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References

[1] E. P. Wigner, Annals Math. **40**, 149 (1939).

[2] R. F. Streater and A. S. Wightman, *PCT, Spin and Statistics, and All That*, (W. A. Benjamin, 1980).

[3] K. Osterwalder and R. Schrader, Commun. Math. Phys. **31**, 83 (1973).

[4] K. Osterwalder and R. Schrader, Commun. Math. Phys. **42**, 281 (1975).

[5] J. Frohlich, Helv. Phys. Acta. **47**, 265 (1974).

[6] J. Glimm and A. Jaffe, *Quantum Physics; A functional Integral Point of View* (Springer-Verlag, 1981).

[7] A. Klein and L. Landau, J. Functional Anal. **44**, 121 (1981).

[8] A. Klein and L. Landau, Comm. Math. Phys **87**, 469 (1983).

[9] J. Frohlich, K. Osterwalder, and E. Seiler, Annals Math. **118**, 461 (1983).

[10] F. Coester, Helv. Phys. Acta **38**, 7 (1965).

[11] R. Haag, Phys. Rev. **112**, 669 (1958).

[12] D. Ruelle, Helv. Phys. Acta. **35**, 147 (1962).

[13] M. Reed and B. Simon, *Methods of Modern mathematical Physics*, vol. III Scattering Theory (Academic Press, 1979).

[14] H. Baumgärtel and M. Wollenberg, *Mathematical Scattering Theory* (Spinger-Verlag, Berlin, 1983).

[15] J. M. Cook, J. Math. Phys. **36**, 1011 (1957).

[16] W. Brenig and R. Haag, Fort. der Physik **7**, 183 (1959).

[17] T. Kato, *Perturbation theory for linear operators* (Spinger-Verlag, Berlin, 1966).

[18] C. Chandler and A. Gibson, Indiana Journal of Mathematics. **25**, 443 (1976).

[19] E. Zeidler, *Oxford User's Guide to Mathematics* (Oxford University Press, Oxford, 2004).

[20] V. Wessels and W. Polyzou, Few Body Syst. **35**, 51 (2004), `nucl-th/0312004`.