Progress in Euclidean relativistic few-body quantum mechanics *

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Abstract

We discuss recent progress in the Euclidean formulation of relativistic few-body quantum mechanics.

1 Introduction

Euclidean relativistic quantum mechanics is a formalism for constructing quantum-mechanical models that have a unitary representation of the Poincaré group on a model Hilbert space[1]. The advantage of this approach is that it is straightforward to formulate relativistic few-body models that satisfy cluster separability along with a spectral condition. A distinctive feature is that the dynamics is formulated in terms of truncated Euclidean Green functions rather than a Hamiltonian with few-body interactions. We discus the construction of Green functions that satisfy reflection positivity, which is a sufficient condition on the truncated Green functions to ensure that the Hamiltonian satisfies a spectral condition and for the positivity of quantum probabilities. We also discuss conditions for establishing the existence of scattering wave operators and discuss the computation of S-matrix observables.

Below we briefly summarize the structure of the theory. A dense set of vectors in the model Hilbert space, $\mathcal{H}$, is represented by a collection of functions of Euclidean space-time variables, x_i, that have support for positive relative Euclidean times:

$$f \to (f_0, f_1(\mathsf{x}), f_2(\mathsf{x}_1, \mathsf{x}_2), \cdots)$$

satisfying

$$\textbf{support of } f_k(\mathsf{x}_1, \cdots, \mathsf{x}_k) = \{\mathsf{x}_1, \cdots, \mathsf{x}_k \,|\, 0 < \mathsf{x}_1^0 < \mathsf{x}_2^0 < \mathsf{x}_3^0 < \cdots < \mathsf{x}_k^0\}.$$

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The Euclidean time reflection operator, θ, is defined by $\theta(\mathsf{x}^0,\mathbf{x}) := (-\mathsf{x}^0,\mathbf{x})$. The quantum mechanical inner product is expressed in terms of a collection of Euclidean invariant Green functions

$$\{G_{m:n}(\mathsf{x}_m,\cdots,\mathsf{x}_1;\mathsf{y}_1,\cdots,\mathsf{y}_n)\}$$

by

$$\langle f|g\rangle = (f,\theta G g)_E :=$$

$$\sum_{m,n}\int f_m^*(\mathsf{x}_1,\cdots,\mathsf{x}_m)G_{m;n}(\theta\mathsf{x}_m,\cdots,\theta\mathsf{x}_1;\mathsf{y}_1,\cdots,\mathsf{y}_n)g(\mathsf{y}_1,\cdots,\mathsf{y}_n)d^{4m}\mathsf{x}d^{4n}\mathsf{y},$$

where the x_i variables are final variables and the y_i variables are initial variables. The collection of Green functions are called reflection positive when $\langle f|f\rangle \geq 0$ for all functions satisfying the positive relative-time support condition.

The collection of Green functions satisfy cluster properties if

$$\lim_{|\mathbf{a}|\to\infty} G_{m:n}(\mathsf{x}_m+\mathbf{a},\cdots,\mathsf{x}_{k+1}+\mathbf{a},\mathsf{x}_k,\cdots,\mathsf{x}_1;\mathsf{y}_1+\mathbf{a},\cdots,\mathsf{y}_l+\mathbf{a},\mathsf{y}_{l+1},\cdots,\mathsf{y}_n) =$$

$$G_{k:l}(\mathsf{x}_k,\cdots,\mathsf{x}_1;\mathsf{y}_1,\cdots,\mathsf{y}_l)G_{m-k,n-l}(\mathsf{x}_m,\cdots,\mathsf{x}_{k+1};\mathsf{y}_{l+1},\cdots,\mathsf{y}_n)$$

Poincaré generators $\{H,\mathbf{P},\mathbf{J},\mathbf{K}\}$ on $\mathcal{H}$ and defined by

$$\langle\mathsf{x}|H|\mathbf{f}\rangle := \{0,\frac{\partial}{\partial\mathsf{x}_{11}^0}f_1(\mathsf{x}_{11}),\left(\frac{\partial}{\partial\mathsf{x}_{21}^0}+\frac{\partial}{\partial\mathsf{x}_{22}^0}\right)f_2(\mathsf{x}_{21},\mathsf{x}_{22}),\cdots\}$$

$$\langle\mathsf{x}|\mathbf{P}|\mathbf{f}\rangle := \{0,-i\frac{\partial}{\partial\vec{\mathsf{x}}_{11}}f_1(\mathsf{x}_{11}),-i\left(\frac{\partial}{\partial\vec{\mathsf{x}}_{21}}+\frac{\partial}{\partial\vec{\mathsf{x}}_{22}}\right)f_2(\mathsf{x}_{21},\mathsf{x}_{22}),\cdots\}$$

$$\langle\mathsf{x}|\mathbf{J}|\mathbf{f}\rangle := \{0,-i\vec{\mathsf{x}}_{11}\times\frac{\partial}{\partial\vec{\mathsf{x}}_{11}}f_1(\mathsf{x}_{11}),$$

$$-i\left(\vec{\mathsf{x}}_{21}\times\frac{\partial}{\partial\vec{\mathsf{x}}_{21}}+\vec{\mathsf{x}}_{22}\times\frac{\partial}{\partial\vec{\mathsf{x}}_{22}}\right)f_2(\mathsf{x}_{21},\mathsf{x}_{22}),\cdots\}$$

$$\langle\mathsf{x}|\mathbf{K}|\mathbf{f}\rangle := \{0,\left(\vec{\mathsf{x}}_{11}\frac{\partial}{\partial\mathsf{x}_{11}^0}-\mathsf{x}_{11}^0\frac{\partial}{\partial\vec{\mathsf{x}}_{11}}\right)f_1(\mathsf{x}_{11}),$$

$$\left(\vec{\mathsf{x}}_{21}\frac{\partial}{\partial\mathsf{x}_{21}^0}-\mathsf{x}_{21}^0\frac{\partial}{\partial\vec{\mathsf{x}}_{21}}+\vec{\mathsf{x}}_{22}\frac{\partial}{\partial\mathsf{x}_{22}^0}-\mathsf{x}_{22}^0\frac{\partial}{\partial\vec{\mathsf{x}}_{22}}\right)f_2(\mathsf{x}_{21},\mathsf{x}_{22}),\cdots\}.$$

These operators are Hermetian and satisfy the Poincaré commutation relations on $\mathcal{H}$.

The invariant mass and transfer matrix, which are dynamical operators, are easily computed in this representation:

$$\langle\mathsf{x}|e^{-\beta H}|f\rangle =$$

$$(f_0,f_1(\mathsf{x}^0-\beta,\mathbf{x}),f_2(\mathsf{x}_1^0-\beta,\mathbf{x}_1,\mathsf{x}_2^0-\beta,\mathbf{x}_2),\cdots)\to$$

$$M^2 = (\frac{\partial^2}{\partial\beta^2} + \frac{\partial}{\partial\mathbf{a}}\cdot\frac{\partial}{\partial\mathbf{a}})\langle \mathsf{x}|e^{-\beta H - i\mathbf{a}\cdot\mathbf{P}}|f\rangle_{|\beta=0,\mathbf{a}=\mathbf{0}} =$$

$$(\frac{\partial^2}{\partial\beta^2} + \frac{\partial}{\partial\mathbf{a}}\cdot\frac{\partial}{\partial\mathbf{a}})(f_0, f_1(\mathsf{x}^0-\beta,\mathbf{x}-\mathbf{a}),$$

$$f_2(\mathsf{x}_1^0-\beta,\mathbf{x}_1-\mathbf{a},\mathsf{x}_2^0-\beta,\mathbf{x}_2-\mathbf{a}),\cdots)_{|\beta=0,\mathbf{a}=\mathbf{0}}.$$

These properties are motivated by the Osterwalder-Schrader reconstruction theorem of local field theory[2]. The difference between the Green functions of a local field theory and few-body quantum mechanics is that in the quantum mechanical case we only retain a finite number of these functions. In addition, in local field theory there is only one N-point Green function; while in the quantum-mechanical case there may be different N-point Green functions corresponding different designations of the initial and final Euclidean space-time coordinates. The full symmetry in the local field theory case leads to crossing symmetry, which may be violated in the quantum mechanical case.

The product, $\langle f|f\rangle$, is related to the standard Minkowski-space inner product. This is illustrated in the one-body case by the well-known [3] calculation

$$\langle f|f\rangle = \int f^*(x)G_{1:1}(\Theta x; y)f(y)d^4xd^4y$$

$$= \frac{1}{(2\pi)^4}\int d^4\mathsf{x}d^4\mathsf{y}d^4\mathsf{p}dm f^*(\mathsf{x})\frac{e^{i\mathsf{p}\cdot(\theta\mathsf{x}-\mathsf{y})}\rho(m)}{\mathsf{p}^2+m^2}f(\mathsf{y})$$

$$= \int\frac{d^3pdm\rho(m)}{2\omega_m(\mathbf{p})}|g(\mathbf{p},m)|^2 \geq 0$$

where the Euclidean and Minkowski wave functions $f(\mathsf{x})$ and $g(\mathbf{p},m)$ are related by

$$g(\mathbf{p},m) = \frac{1}{(2\pi)^{3/2}}\int f(\mathsf{x}_0,\mathbf{x})e^{-\omega_m(\mathbf{p})\mathsf{x}_0 - i\mathbf{x}\cdot\mathbf{p}}d^4\mathsf{x}$$

and the Lorentz invariant measure. $d^3p/\omega_m(\mathbf{p})$, appears naturally.

One of the challenges of constructing models based on the Euclidean formulation of relativistic quantum mechanics is the problem of finding a robust class of reflection-positive model Green functions. Two-body truncated Green functions with standard Källén-Lehmann representations are reflection positive, as illustrated above. One difficulty with multipoint Green functions is that reflection positivity is not stable[4] with respect to small Euclidean invariant perturbations. For example, if one starts with a product of reflection-positive free Green function, and solves the Bethe-Salpeter equation with a small Euclidean-invariant kernel for the four-point Green function, the resulting Green function is not automatically reflection positive[4]. On the other hand, Widder[5] demonstrated that the most general solution in 1 dimension to

$$\int_0^\infty f^*(t)g(t+t')f(t')dtdt' > 0$$

has the form

$$g(t) = \int e^{-\lambda t} \rho(\lambda) d\lambda = \int \frac{\lambda}{\pi} \frac{e^{itp}}{\lambda^2 + p^2} \rho(\lambda) dp d\lambda$$

which has a structure similar to the Källén-Lehmann representation of the Euclidean Green function. This observations suggest considering integral representations of connected four point Green functions of the form

$$G^c_{2:2}(\mathsf{x}_2, \mathsf{x}_1; \mathsf{y}_1, \mathsf{y}_2) =$$

$$\int e^{i\mathsf{p}_1 \cdot (\mathsf{x}_2 - \mathsf{x}_1)} e^{i\mathsf{p}_2 \cdot (\mathsf{x}_1 - \mathsf{y}_1)} e^{i\mathsf{p}_3 \cdot (\mathsf{y}_1 - \mathsf{y}_2)} \times$$

$$\frac{g(\mathsf{p}_1, \mathsf{p}_2, \mathsf{p}_3, m_2)}{(\mathsf{p}_1^2 + m^2)(\mathsf{p}_2^2 + m_2^2)(\mathsf{p}_3^2 + m^2)} d^4\mathsf{p}_1 d^4\mathsf{p}_2 d^4\mathsf{p}_3 dm_2. \tag{1}$$

Calculations show that this class of Green functions are reflection positive subject to mild conditions on $g(\mathsf{p}_1, \mathsf{p}_2, \mathsf{p}_3, m_2)$. It is straightforward to generalize this to higher order connected Green functions. What simplifies the reflection-positivity constraint, compared to the field theory case, is that the truncated four point Green functions $G_{1:3}(\mathsf{x}_1 : \mathsf{x}_2, \mathsf{x}_3, \mathsf{x}_4)$, $G_{2:2}(\mathsf{x}_1, \mathsf{x}_2 : \mathsf{x}_3, \mathsf{x}_4)$, and $G_{3:1}(\mathsf{x}_1, \mathsf{x}_2, \mathsf{x}_3; \mathsf{x}_4)$ do not have to be related. In the local field theory case they must be identified, which leads to additional restrictions on $g(\mathsf{p}_1, \mathsf{p}_2, \mathsf{p}_3, m_2)$.

Another complication is the formulation of scattering theory. This is because the dynamics enters in the structure of the Hilbert space inner product, so there is no asymptotic dynamics. In addition, the real-time evolution operator is difficult to construct in this formalism, while the transfer matrix involves a simple quadrature. The absence of an asymptotic dynamics can be treated using the two-Hilbert space formulation of scattering[6]. This requires solving the one-body problem for subsystems. In this form, time-dependent methods can be used to define scattering wave operators and a simple generalization of Cook's method can be used to test the existence of the wave operators.

The first step is to solve the mass eigenvalue problem

$$\langle \mathsf{x} | (M^2 - \lambda^2) | \lambda \rangle = 0$$

for eigenfunctions in the pure point spectrum of M^2 associated with a subsystem Green function. Here $\langle \mathsf{x}|$ is a shorthand notation for $\langle \mathsf{x}_1 \cdots \mathsf{x}_m|$ which are the initial or final variables of the subsystem Green function.

Next translations and rotations are used to extract sharp momentum and spin eigenstates of the same mass

$$\langle \mathsf{x} | \lambda, \mathbf{p} \rangle = \int \frac{d^3a}{(2\pi)^{3/2}} e^{-i\mathbf{p}\cdot\mathbf{a}} \langle \mathsf{x} - \mathbf{a} | \lambda \rangle$$

$$\langle \mathsf{x} | \lambda, j, \mathbf{p}, \mu \rangle = \int_{SU(2)} dR \sum_{\nu=-j}^{j} \langle \mathsf{x} | \lambda, R^{-1}\mathbf{p} \rangle D^{j*}_{\mu\nu}(R).$$

These one-particle solutions are used to construct a map from an asymptotic Hilbert space to the physical Hilbert space by taking symmetrized products of the "one-particle" plane-wave eigenstates

$$\langle \mathsf{x}|\Phi|\mathbf{p}_1,\mu_1,\cdots \mathbf{p}_k,\mu_k\rangle = \prod_i \langle \mathsf{x}_{i_1}\cdots \mathsf{x}_{i_{n_i}}|\lambda_i,j_i,\mathbf{p}_i,\mu_i\rangle .$$

Wave operators are defined by

$$|\Psi_\pm(g_1,\cdots g_n)\rangle := \lim_{t\to\infty} e^{iHt}\Phi e^{-iH_0 t}|\mathbf{g}\rangle = \Omega_\pm|\mathbf{g}\rangle$$

where $|\mathbf{g}\rangle$ represents wave packets in the asymptotic particles' momenta and spin and $H_0 = \sum_i \omega_{m_i}(\mathbf{p}_i)$.

A sufficient condition for the existence of this limit is the Cook condition[7]

$$\int_0^{\pm\infty} \|(H\Phi - \Phi H_0)e^{-iH_0 t}|\mathbf{g}\rangle\| dt < \infty. \tag{2}$$

For $N=2$ with $G_4 = G_2G_2 + G_4^c$ the G_2G_2 contribution to $\|(H\Phi - \Phi H_0)e^{-iH_0 t}|\mathbf{g}\rangle\|$ vanishes. What remains is a regularity condition that depends only on the truncated four-point function, G_4^c. It is interesting note that the truncated reflection-positive four point Euclidean Green functions in 1 are distributions rather than short-ranged kernels; however when one computes the integrand in (2), it becomes a localized kernel after integrating over the relative p^0 energy variables.

To calculate S-matrix elements the invariance principle [8] can be used, which allows us to make the replacement

$$H \to w(H) \qquad w(H) = -e^{-\beta H} \qquad \beta > 0$$

in the limits used to define the S matrix elements

$$S = \lim_{n\to\infty} \langle \mathbf{g}_f| e^{-ine^{-\beta H_0}} \Phi^\dagger e^{2ine^{-\beta H}} \Phi e^{-ine^{-\beta H_0}} |\mathbf{g}_i\rangle$$

Because the spectrum of $e^{-\beta H}$ is compact, for any fixed n, $e^{2ine^{-\beta H}}$ can be uniformly approximated by a polynomial in $e^{-\beta H}$. Recall that these matrix elements are related to the transfer matrix, $\langle f|T(0,n\beta)|g\rangle = \langle f|e^{-n\beta H}|g\rangle$, which can be calculated using only quadratures. Test calculations [1] demonstrate that this method can be used to accurately calculate GeV-scale scattering cross sections.

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