Relativistic Formulation of Reaction Theory

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A relativistic formulation of reaction theory for nuclei with a dynamics given by a unitary representations of the Poincaré group is developed. Relativistic dynamics is introduced by starting from a relativistic theory of free particles to which rotationally invariant interactions are added to the invariant mass operator. Poincaré invariance is realized by requiring that simultaneous eigenstates of the mass and spin transform as irreducible representations of the Poincaré group. A relativistic formulation of scattering theory is presented and approximations emphasizing dominant degrees of freedom that preserve unitarity, exact Poincaré invariance and exchange symmetry are discussed. A Poincaré invariant formulation of a (d,p) reaction as a three-body problem is given as an explicit example.

I. INTRODUCTION

The physics of exotic nuclei has become a major subject within nuclear physics. A new generation of radioactive beam facilties such as RIBF at RIKEN in Japan, FAIR at GSI in Germany, SPIRAL2 at GANIL in France, and FRIB at MSU in the USA have been or will be soon in operation. With the access to exotic nuclei at the very limits of nuclear stability, the physics of neutron and proton driplines has become a focus of interest. Nuclei close to these driplines exhibit phenomena different from the known stable ones, like the normal shell closures may disappear and be replaced by new magic numbers, or threshold phenomena like nuclear halo states may occur (for reviews see e.g. [1, 2]).

Ongoing and planned technical developments in beam production as well as in detection systems allow not only experiments with a larger variety of nuclei, but more importantly allow measurements of reactions which were traditionally carried out with stable beams like knock-out or transfer reactions at a variety of energies. Even first polarization experiments with radioactive beams are now possible [3].

Theoretical developments have been moving at a somewhat slower pace. Here one should note, that at first the light exotic nuclei received most of the theoretical attention, and approaches describing their reactions were developed for a higher energy regime, where it was believed only a few degrees of freedom determine the reaction and thus approximations are justified. These include the eikonal approximation, or the adiabatic approximation in which degrees of freedom are frozen. Many theoretical advances made over the last decade however concentrate on the lower energy regime (lower than roughly 50 MeV per nucleon) to energies relevant for astrophysical processes. This energy regime is the realm of non-relativistic quantum mechanics, in which e.g. coupled discretized continuum channel (CDCC) methods are applied to direct reactions. A review of selected methods is given in Ref. [4]. Direct reactions also lend themselves to adoptions of few-body techniques, which are well established in the non-relativistic regime. Well-defined examples here are the (d,p) reaction on light nuclei, which can be successfully described within a modified Faddeev approach [5].

One essential goal in the study of rare isotopes is to obtain information about their specific structure from their reactions with either light or heavy targets. Therefore, if one wants to concentrate on extracting structure information, the description of reaction mechanism needs to be understood. Within the non-relativistic framework, the formulation of multiple scattering theories [6, 7] has a long tradition, and over the years, efforts have been made in practical implementations.

As the practical implementation of nuclear reaction theory in the lower energy regime becomes more sophisticated, it is worthwhile and necessary to re-examine approaches that were and still are used in the higher energy regime. Though fewer degrees of freedom of the nucleus may be sufficient to describe the reaction, sub-nuclear degrees of freedom may become relevant. For example proton-deuteron elastic scattering at 250 MeV underpredicts the experimental cross section at back angles even after enhancements due to both three-body forces and relativistic effects are included [8, 9]. Given that back angles are more sensitive to short distance physics, this suggests additional degrees of freedom may be relevant[10]. Relativistic reaction theory models are also important for determining observables that are sensitive to relativistic effects [11]. In addition it becomes relevant to investigate if a Galilei invariant reaction theory needs to be replaced by a Poincaré invariant one.

Relativistic invariance of a quantum theory requires the invariance of quantum probabilities with respect to changes in the inertial coordinate system. This requires that the dynamics is given by a unitary representation of the Poincaré group [12]. Here the Poincaré group refers to the transformations continuously connected to the identity; invariance with respect to space reflections and time reversal is not required and is not satisfied by the weak interaction. Approximations that emphasize dominant degrees of freedom that preserve both unitarity and exact Poincaré invariance are discussed below.

The simplest way to construct a relativistic dynamics is to start with a relativistic theory of free particles. Interactions can be added in a manner that preserves the overall Poincaré invariance. We do this in three steps. First we boost the non-interacting N-body system to the N-body rest frame. Second, we add rotationally invariant interactions to the non-interacting rest Hamiltonian, which in the relativistic case is the N-particle invariant mass operator. Third, we solve for simultaneous eigenstates of the interacting mass and spin, which can be done because of the rotational invariance of the interaction. The mass and spin are the invariant labels for irreducible representations of the Poincaré group. The relativistic dynamics is defined by requiring that in all other frames these mass-spin eigenstate transform irreducibly with respect to the Poincaré group. Once we have these operators the treatment of the reaction theory is similar to the non-relativistic treatment.

In the first section we derive the transformation properties for a single relativistic particle, then we consider the case of N non-interacting relativistic particles. We discuss how to decompose products of irreducible representation of the Poincaré group into direct integrals of irreducible representations. In the third section we add interactions to the mass Casimir operator of the non-interacting irreducible representations to construct dynamical unitary representations of the Poincaré group. Then we focus on reaction theory models and relativistic scattering theory. After addressing the treatment of identical particles, we explicitly consider (d,p) reactions as illustration of the general formulation.

II. RELATIVISTIC KINEMATICS

In this section we discuss the first step, which is the relativistic description of a single particle. The state of a single particle of mass m and spin j is characterized by its momentum and the projection of its spin on a given axis. These are a complete set of commuting observables for a structureless particle. Simultaneous eigenstates of these observables, denoted by $|(m,j)\mathbf{p},\mu\rangle$, are a basis for a single-particle Hilbert space, $\mathcal{H}_{mj}$.

A unitary representation of the Poincaré group on $\mathcal{H}_{mj}$ can be constructed by considering the transformation properties of rotations, translations, and Lorentz boosts on rest ($\mathbf{0}$-momentum) eigenstates.

A particle at rest remains at rest under rotations. On the other hand the spins undergo rotations. If the particle has spin j then the rest eigenstates transform under a $2j+1$ dimensional unitary representation of the rotation group. These elementary transformations are

$$U(R)|(m,j)\mathbf{0},\mu\rangle = \sum_{\nu=-j}^{j} |(m,j)\mathbf{0},\nu\rangle D^j_{\nu\mu}(R) \tag{1}$$

where $D^j_{\nu\mu}(R)$ is an ordinary Wigner D-function, which is a $2j+1$ dimensional unitary representation of the rotation group.

Since these states are rest eigenstates of the 4 momentum, it also follows that under translations

$$T(a)|(m,j)\mathbf{0},\mu\rangle = e^{-ima^0}|(m,j)\mathbf{0},\mu\rangle. \tag{2}$$

Because sequences of Lorentz boosts can generate rotations, we need an unambiguous definition of a spin observable in frames moving with momentum $\mathbf{p}$ relative to the rest frame. There are many possible definitions.

We define the spin observable in a general frame by the requirement that it does not Wigner rotate when it is transformed to the particle's rest frame by a rotationless Lorentz transformation

$$U(B(\mathbf{p}/m))|(m,j)\mathbf{0},\mu\rangle := |(m,j)\mathbf{p},\mu\rangle\sqrt{\frac{\omega_m(\mathbf{p})}{m}}. \tag{3}$$

This is normally referred to as the canonical spin.

The rotationless Lorentz boost $B(\mathbf{p}/m)$ is the usual textbook Lorentz boost that is normally expressed in terms of hyperbolic sines and cosines of a rapidity. The rotationless boost from the particle's rest frame to a frame where it has momentum $\mathbf{p}$ is

$$B(\mathbf{p}/m) := \begin{pmatrix} \omega_m(\mathbf{p})/m & \mathbf{p}/m \\ \mathbf{p}/m & I+\frac{p_i p_j}{m(m+\omega_m(\mathbf{p}))} \end{pmatrix}. \tag{4}$$

In (3) and (4) $\omega_m(\mathbf{p}) = \sqrt{m^2+\mathbf{p}^2}$ is the energy of a particle of mass m and momentum $\mathbf{p}$.

The energy factors make (3) unitary if the states, $|(m,j)\mathbf{p},\mu\rangle$, are given a delta function normalization,

$$\langle (m,j)\mathbf{p}',\mu'|(m,j)\mathbf{p},\mu\rangle = \delta_{\mu'\mu}\delta(\mathbf{p}'-\mathbf{p}). \tag{5}$$

From (4) it follows that

$$p = B(\mathbf{p}/m)(m,0,0,0) = (\omega_m(\mathbf{p}),\mathbf{p}). \tag{6}$$

A general Poincaré transformation on a single-particle state, $|(m,j)\mathbf{p},\mu\rangle$, can be decomposed into a product of the three elementary unitary transformations (1),(2) and (3) using the group representation property

$$U(\Lambda,a) = U(B(\mathbf{\Lambda}p/m)\,T(B^{-1}(\mathbf{\Lambda}p/m)a)\,U(R_w(\Lambda,\mathbf{p}/m))\,U(B^{-1}(\mathbf{p}/m)) \tag{7}$$

where

$$R_w(\Lambda,\mathbf{p}/m) := B^{-1}(\mathbf{\Lambda}p/m)\Lambda B(\mathbf{p}/m) \tag{8}$$

is a Wigner rotation.

The decomposition (7) is an inverse boost from a state with momentum $\mathbf{p}$ to the rest state, followed by a rotation of the rest state, followed by a translation of the rest state, and finishing with a boost from the rest state to a state with the Lorentz transformed momentum.

When the sequence of transformations (7) is applied to $|(m,j)\mathbf{p},\mu\rangle$ the result is

$$U_{mj}(\Lambda,a)|(m,j)\mathbf{p},\mu\rangle := \sum_{\nu=-j}^{j} |(m,j)\mathbf{\Lambda}p,\nu\rangle e^{i\Lambda p\cdot a}\sqrt{\frac{\omega_m(\mathbf{\Lambda}p)}{\omega_m(\mathbf{p})}} D^j_{\nu\mu}\left[R_w(\Lambda,\mathbf{p}/m)\right] \tag{9}$$

where the subscript mj indicates that this is the a unitary representation of the Poincaré group for a particle of mass m and spin j. Eq. (9) defines mass m spin j unitary irreducible representation of the Poincaré group.

It acts irreducibly on the Hilbert space $\mathcal{H}_{mj}$ spanned by the single particle states $|(m,j)\mathbf{p},\mu\rangle$. The irreducibility means that $\mathcal{H}_{mj}$ can be generated from any fixed vector in $\mathcal{H}_{mj}$ by Poincaré transformations.

The construction used above to construct single-particle irreducible representations will be used to construct N-particle irreducible representations, which will be used in the construction of dynamical irreducible representations.

III. N NON-INTERACTING PARTICLES

The Hilbert space for a system of N non-interacting particles is the N-fold tensor product of the single-particle Hilbert spaces

$$\mathcal{H} := \otimes_{i=1}^{N}\mathcal{H}_{m_i,j_i}. \tag{10}$$

For identical particles the physical Hilbert space is the projection on the appropriately symmetrized or antisymmetrized subspace of $\mathcal{H}$.

The non-interacting (kinematic) unitary representation of the Poincaré group on $\mathcal{H}$ is the tensor product of the single-particle unitary representations of the Poincaré group

$$U_0(\Lambda,a) = \otimes_{i=1}^{N} U_{m_ij_i}(\Lambda,a). \tag{11}$$

A basis for the N-particle system is the direct product of the N one-particle basis vectors

$$|\mathbf{p}_1,\mu_1,\cdots,\mathbf{p}_N,\mu_N\rangle := \prod_l |(m_l,j_l)\mathbf{p}_l,\mu_l\rangle, \tag{12}$$

where we have suppressed all of the single-particle mass and spin quantum numbers on the left.

Following what we did in the single-particle case, we consider a basis for the N-particle system in the rest frame of the N-particle system. We let $\mathbf{q}_i$ denote the momentum of the i^{th} particle in the N-body rest frame. The variables $\mathbf{q}_i$ are constrained so

$$\sum \mathbf{q}_i = \mathbf{0}. \tag{13}$$

We write the rest eigenstates as

$$|\mathbf{q}_1,\mu_1,\cdots,\mathbf{q}_N,\mu_N\rangle \tag{14}$$

where it is understood that $\mathbf{q}_N = -\sum_{i\neq N}\mathbf{q}_i$. Following what we did for the single-particle states we examine the rotational properties the rest eigenstates.

Using the transformation properties of the single-particle states (9) and the expression for N-particle Poincaré transformations in terms of the single-particle transformations (11) give the following transformation properties for the N-particle rest eigenstates under rotations:

$$U_0(R,0)|\mathbf{q}_1,\mu_1,\cdots,\mathbf{q}_N,\mu_N\rangle = \sum |R\mathbf{q}_1,\nu_1,\cdots,R\mathbf{q}_N,\nu_N\rangle \prod_l D^{j_l}_{\nu_l\mu_l}\left(B^{-1}(R\mathbf{q}_l/m_l)RB(\mathbf{q}_l/m_l)\right). \tag{15}$$

The rotationless boosts have the distinguishing property that

$$B^{-1}(R\mathbf{q}/m)RB(\mathbf{q}/m) = R \tag{16}$$

for any $\mathbf{q}$. This implies that "the Wigner rotation of a rotation is the rotation". It is a special property that is not shared by other types of boosts.

As a consequence of this property (15) becomes

$$U_0(R,0)|\mathbf{q}_1,\mu_1,\cdots,\mathbf{q}_N,\mu_N\rangle = \sum |R\mathbf{q}_1,\nu_1,\cdots,R\mathbf{q}_N,\nu_N\rangle \prod_l D^{j_l}_{\nu_l\mu_l}(R). \tag{17}$$

This is exactly how a non-relativistic N-particle state transforms under rotations. It follows that all of the spins and orbital angular momenta can be added with ordinary SU(2) Clebsch-Gordan coefficients and spherical harmonics. The primary difference with the single-particle case is that there can be many orthogonal rotationally invariant subspaces with the same j. They are distinguished by internal spins, orbital angular momenta and sub-energies.

The result is that the rest state can be decomposed into an orthogonal direct sum of states with different total spin. Since there are many possible orders of coupling we denote these states by

$$|(M_0,j)\mathbf{0},\mu;\mathbf{d}\rangle, \tag{18}$$

where

$$M_0 = \sum_l \sqrt{\mathbf{q}_l^2 + m_l^2} \tag{19}$$

is the invariant mass (rest energy) of this system and $\mathbf{d}$ are invariant degeneracy quantum numbers that distinguish different subspaces with the same value of j.

For a two-body system with spin j typical degeneracy parameters would be $\mathbf{d}=\{l,s\}$. For a three-particle system we could have $l_{ij},s_{ij},j_{ij},k_{ij}$ for the ij pair, where k_{ij} is the magnitude of the rest momentum of the ij pair, and $L_{ij,k},S_{ij,k}$ representing the orbital and spin quantum numbers associated with the pair and third particle. In this case $\mathbf{d}=\{l_{ij},s_{ij},j_{ij},k_{ij},L_{ij,k},S_{ij,k}\}$.

The choice of degeneracy parameters is normally made for convenience; for example the three-body choice above would be useful for constructing matrix elements of an interaction between particles i and j. The important observation is that they are all rotationally invariant quantum numbers. In general $\mathbf{d}$ includes both discrete quantum numbers like $l_{ij},s_{ij},L_{ij,k},S_{ij,k}$ and continuous ones like k_{ij}. Different choices of $\mathbf{d}$ are related by unitary transformations whose coefficients are Racah coefficients for the Poincaré group.

The result of coupling the spins means that in this basis (17) has the same form as (1):

$$U_0(R,0)|\mathbf{0},M_0,j,\mu;\mathbf{d}\rangle = \sum_{\nu=-j}^{j} |\mathbf{0},M_0,j,\nu;\mathbf{d}\rangle D^j_{\nu\mu}(R). \tag{20}$$

The differences are the presence of the invariant degeneracy parameters $\mathbf{d}$ and the fact that the invariant mass M_0 has a continuous spectrum that runs from the sum of the individual masses to infinity.

The states (18) are rest states. We can define states with a non-zero total momentum and the same spin by analogy with (3)

$$|(M_0,j)\mathbf{P},\mu,\mathbf{d}\rangle := U_0(B(\mathbf{P}/M_0))|(M_0,j)\mathbf{0},\mu;\mathbf{d}\rangle\sqrt{\frac{M_0}{\omega_{M_0}(\mathbf{P})}}. \tag{21}$$

The difference between this equation and (3) is that (3) was used to define the unitary representation of the rotationless boost, while in this case the representation of the rotationless boost is given by (11) so (21) defines the momentum-spin eigenstate. This definition implies a delta function normalization in $\mathbf{P}$. It redefines the magnetic quantum numbers so they agree with the single-particle magnetic quantum numbers when boosted to the rest frame of the N-particle system with a rotationless boost.

Unitarity gives the normalization

$$\langle (M_0',j')\mathbf{P}',\mu';\mathbf{d}'|(M_0,j)\mathbf{P},\mu;\mathbf{d}\rangle = \delta(M_0'-M_0)\;\delta(\mathbf{P}-\mathbf{P}')\;\delta_{j'j}\delta_{\mu'\mu}\delta_{\mathbf{d}':\mathbf{d}} \tag{22}$$

where $\delta_{\mathbf{d}':\mathbf{d}}$ is a product of Dirac delta functions in the continuous degeneracy quantum numbers and Kronecker delta functions in the discrete degeneracy quantum numbers.

It is not hard to show that (21) is an eigenstate of the total momentum. The same steps used in (9) lead to the following unitary representation of the Poincaré group for the non-interacting system,

$$U_0(\Lambda,a)|(M_0,j)\mathbf{P},\mu;\mathbf{d}\rangle = \sum_{\nu=-j}^{j} |(M_0,j)\mathbf{\Lambda}P,\nu;\mathbf{d}\rangle e^{i\Lambda P\cdot a}\sqrt{\frac{\omega_{M_0}(\mathbf{\Lambda}P)}{\omega_{M_0}(\mathbf{P})}} D^j_{\nu\mu}\left[R_w(\Lambda,\mathbf{P}/M_0)\right]. \tag{23}$$

In constructing this basis we have decomposed products of irreducible representations of the Poincaré group into orthogonal direct integrals of irreducible representations. The coefficients of this transformation are the Clebsch-Gordan coefficients for the Poincaré group.

It is instructive to see the form of these coefficients in a specific example. We consider the case of coupling two particles. In that case the two-body rest state (14) is

$$|\mathbf{q}_1,\mu_1,-\mathbf{q}_1,\mu_2\rangle \tag{24}$$

where we have used the constraint $\mathbf{q}_1+\mathbf{q}_2=0$. The decomposition of (18) into irreducible representations of the rotation group is

$$|(M_0,j)\mathbf{0},\mu;l.s\rangle := \sum\int d\hat{\mathbf{q}}_1|\mathbf{q}_1,\mu_1,-\mathbf{q}_1,\mu_2\rangle\; Y_{lm}(\hat{\mathbf{q}}_1)\langle j_1,\mu_1,j_2,\mu_2,|s,\mu_s\rangle\langle s,\mu_2,l,m|j,\mu\rangle, \tag{25}$$

where

$$M_0=\sqrt{\mathbf{q}_1^2+m_1^2}+\sqrt{\mathbf{q}_1^2+m_2^2}. \tag{26}$$

The $Y_{lm}(\hat{\mathbf{q}}_1)$ are spherical harmonics and $\langle j_1,\mu_1,j_2,\mu_2|j_3,\mu_3\rangle$ are $SU(2)$ Clebsch-Gordan coefficients. Applying a rotationless boost to both side of equation (25), using (9) and (11) on the right and (21) on the left gives

$$\begin{aligned}&|(M_0,j)\mathbf{P},\mu;l.s\rangle := \\ &\sum\int \hat{\mathbf{q}}_1|\mathbf{p}_1,\nu_1,\mathbf{p}_2,\nu_2\rangle\sqrt{\frac{\omega_{m_1}(\mathbf{p}_1)}{\omega_{m_1}(\mathbf{q}_1)}}\sqrt{\frac{\omega_{m_2}(\mathbf{p}_2)}{\omega_{m_2}(\mathbf{q}_1)}}\; D^{j_1}_{\nu_1\mu_1}[B^{-1}(\mathbf{p}_1/m_1)B(\mathbf{P}/M_0)B(\mathbf{q}_1/m_1)]\times \\ &D^{j_2}_{\nu_2\mu_2}\left[B^{-1}(\mathbf{p}_2/m_2)B(\mathbf{P}/M_0)B(-\mathbf{q}_1/m_2)\right]Y_{lm}(\hat{\mathbf{q}}_1)\;\langle j_1,\mu_1,j_2,\mu_2,|s,\mu_s\rangle\langle s,\mu_2,l,m|j,\mu\rangle\sqrt{\frac{M_0}{\omega_{M_0}(\mathbf{P})}}\end{aligned} \tag{27}$$

where $\mathbf{q}_i$ and $\mathbf{p}_i$ are related by

$$q_i = B^{-1}(\mathbf{P}/M_0)p_i \tag{28}$$

which can be expressed in terms of the three-vector components using (4) as

$$\mathbf{q}_i=\mathbf{p}_i+\frac{\mathbf{P}}{M_0}\left(\frac{\mathbf{P}\cdot\mathbf{p}_i}{M_0+\omega_{M_0}(\mathbf{P})}-\omega_{m_i}(\mathbf{p}_i)\right). \tag{29}$$

The Poincaré group Clebsch-Gordan coefficients are the coefficients of the unitary transformation (27).

Returning to the N-particle case, note that the boost acts on the left side of equation (21) while the transformation between $\{M_0,j,\mu,\mathbf{d}\}$ and $\{\mathbf{q}_1,\mu_1,\cdots,\mathbf{q}_N,\mu_N\}$ acts on the right. The result of transforming the variables on right side of (21) leads to

$$|\mathbf{P};\mathbf{q}_1,\mu_1,\cdots,\mathbf{q}_N,\mu_N\rangle := U(B(\mathbf{P}/M_0)|\mathbf{q}_1,\mu_1,\cdots,\mathbf{q}_N,\mu_N\rangle\sqrt{\frac{M_0}{\omega_{M_0}(\mathbf{P})}}. \tag{30}$$

The relation of these states to the original single-particle states follows from (9),(11) and (30):

$$\begin{aligned}&|\mathbf{P};\mathbf{q}_1,\mu_1,\cdots,\mathbf{q}_N,\mu_N\rangle := \\ &\sum|\mathbf{p}_1,\nu_1,\cdots,\mathbf{p}_N,\nu_N\rangle\sqrt{\frac{M_0}{\omega_{M_0}(\mathbf{P})}}\;\prod_k D^{j_k}_{\nu_k\mu_k}[B^{-1}(\mathbf{p}_k/m_k)B(\mathbf{P}/M_0)B(\mathbf{q}_k/m_k)]\sqrt{\frac{\omega_{m_k}(\mathbf{p}_k)}{\omega_{m_k}(\mathbf{q}_k)}}\end{aligned} \tag{31}$$

where the p_i are related to the q_i by (29). There is a corresponding relation between the spins implied by (31). We refer to the spins on the left side of (31) as constituent single-particle spins and the spins on the right as single-particle spins. The corresponding spin operators are related by Wigner rotations

$$(0,\mathbf{j}_{ic})=B^{-1}(\mathbf{q}_i/m_i)B^{-1}(\mathbf{P}/M_0)B(\mathbf{p}_i/m_i)(0,\mathbf{j}_i). \tag{32}$$

These spins become identical in the N-particle rest frame. The advantage of using a basis with constituent spins is that they can be added like non-relativistic spins.

IV. N INTERACTING PARTICLES

In this section we construct a dynamical unitary representation of the Poincaré group. We use two equivalent constructions - one is designed to provide an explicit representation of the dynamical unitary representation while the other is more appropriate for N-particle applications. We start with the construction of the explicit representation of the dynamical unitary representation.

The simplest way to construct a relativistic N-particle dynamics is to start with the non-interacting N-particle irreducible basis (23) constructed in the previous section

$$|(M_0, j)\mathbf{P}, \mu, \mathbf{d}\rangle. \tag{33}$$

In order to construct an interacting unitary irreducible representation of the Poincaré group we add an interaction V to M_0 that commutes with the non-interacting constituent spin, $\mathbf{j}_c$,

$$M = M_0 + V. \tag{34}$$

We also assume that V is translationally invariant and is independent of the total momentum.

A general interaction of this form has matrix elements in the N free-particle irreducible basis (23) of the form

$$\langle(M_0', j')\mathbf{P}', \mu'; \mathbf{d}'|V|(M_0, j)\mathbf{P}, \mu; \mathbf{d}\rangle =$$

$$\delta(\mathbf{P}' - \mathbf{P})\delta_{j'j}\delta_{\mu'\mu}\langle M_0', \mathbf{d}'\|V^j\|M_0, \mathbf{d}\rangle. \tag{35}$$

For two particles $M_0 = \sqrt{\mathbf{q}^2 + m_1^2} + \sqrt{\mathbf{q}^2 + m_2^2}$ where $\mathbf{q}$ is the rest-frame momentum of particle 1 and the degeneracy parameters, l^2 and s^2, are orbital and spin angular momenta, so with a suitable change of variables (35) looks like a standard two-body interaction in a partial-wave representation.

Simultaneous eigenstates of M, $\mathbf{P}$, $\mathbf{j}^2$ and $\hat{\mathbf{z}} \cdot \mathbf{j}$ can be constructed by diagonalizing M in the basis of eigenstates of M_0, $\mathbf{P}$, $\mathbf{j}^2$ and $\hat{\mathbf{z}} \cdot \mathbf{j}$.

The symmetry properties of the interaction (35) imply that eigenfunctions have the form

$$\langle(M_0, j)\mathbf{P}, \mu; \mathbf{d}|(\lambda, j'), \mathbf{P}', \mu'\rangle = \delta(\mathbf{P} - \mathbf{P}')\delta_{jj'}\delta_{\mu\mu'}\psi_{\lambda,j}(\mathbf{d}, M_0), \tag{36}$$

where the wave functions, $\psi_{\lambda,j}(\mathbf{d}, M_0)$, are solutions to the relativistic mass eigenvalue problem

$$(\lambda - M_0)\psi_{\lambda,j}(\mathbf{d}, M_0) = \sum{}' \int dM_0' d\mathbf{d}' \langle M_0, \mathbf{d}|V^j|M_0', \mathbf{d}'\rangle\psi_{\lambda,j}(\mathbf{d}', M_0') \tag{37}$$

and λ is the mass eigenvalue. This equation replaces the many-body Schrödinger equation for the center of mass Hamiltonian in non-relativistic quantum mechanics. The eigenstates

$$|(\lambda, j), \mathbf{P}, \mu\rangle \tag{38}$$

transform like (23) with the mass eigenvalue λ replacing M_0 in (23):

$$U(\Lambda, a)|(\lambda, j)\mathbf{P}, \mu\rangle =$$

$$\sum_{\nu=-j}^{j} |(\lambda, j)\mathbf{\Lambda}P, \nu\rangle e^{i\Lambda P\cdot a}\sqrt{\frac{\omega_\lambda(\mathbf{\Lambda}P_0)}{\omega_\lambda(\mathbf{P}_0)}} D^j_{\nu\mu}[R_w(\Lambda, \mathbf{P}/\lambda)], \tag{39}$$

where in this case the Wigner rotation depends on the mass eigenvalue,

$$R_w(\Lambda, \mathbf{P}/\lambda) = B^{-1}(\mathbf{\Lambda}P/\lambda).\Lambda B(P/\lambda) \qquad P^0 = \sqrt{\lambda^2 + \mathbf{P}^2}. \tag{40}$$

A complete set of irreducible eigenstates will have multiple copies of states with the same mass and spin that are distinguish by invariant degeneracy quantum numbers. Since the eigenstates (38) are complete, (39) defines the dynamical unitary representation of the Poincaré group on $\mathcal{H}$.

This shows that the construction of the dynamical representation of the Poincaré group can be reduced to solving the mass eigenvalue problem (37). This is analogous to constructing the unitary time evolution operator by diagonalizing the center of mass Hamiltonian in non-relativistic quantum mechanics.

This construction was first performed by Bakamjian and Thomas [13] for the two-particle system. For systems of more than two particles this construction fails to satisfy cluster properties [14, 15], which means that

$$U(\Lambda, a) \not\to U_I(\Lambda, a) \otimes U_{II}(\Lambda, a) \tag{41}$$

on states corresponding to asymptotically separated subsystems, I and II.

This deficiency can be systematically corrected[16]: the corrections appear in the form of additional many-body interactions that are functions of the input interactions. The interactions that restore cluster properties fall-off like powers of $(V/M)^{N-1}$ [14], where V is the two-body interaction. They appear to be small in nuclear physics applications [15]. Thus in the following these corrections will be ignored.

While the N free-particle irreducible basis is the most convenient for illustrating the construction of a dynamical unitary representation of the Poincaré group, like a partial-wave basis it is not an ideal basis for many-body problems. In addition, for relativistic problems partial-wave expansions can lead to numerical challenges [17].

Note that the rest states (14) and (18) only differ by an ordinary partial-wave expansion that acts on the right side of the these states, and states with arbitrary momentum are constructed by applying a unitary boost on the left, that leaves all of the quantum numbers on the right unchanged except the total momentum.

The implies that the N-body basis

$$|\mathbf{P}; \mathbf{q}_1, \mu_1, \cdots \mathbf{q}_N, \mu_N\rangle := U_0(B(\mathbf{P}/M_0))|\mathbf{q}_1, \mu_1, \cdots \mathbf{q}_n, \mu_n\rangle \sqrt{\frac{M_0}{\omega_{M_0}(\mathbf{P})}} \tag{42}$$

is related to (21) by SU(2) Clebsch-Gordan coefficients and spherical harmonics.

In the basis (42) the interaction can be expressed as

$$\langle \mathbf{P}, \mathbf{q}_1, \mu_1, \cdots, \mathbf{q}_N, \mu_N|V|\mathbf{P}', \mathbf{q}'_1, \mu'_1, \cdots, \mathbf{q}'_N, \mu'_N\rangle = \delta(\mathbf{P} - \mathbf{P}')\langle \mathbf{q}_1, \mu_1, \cdots, \mathbf{q}_N, \mu_N\|V\|\mathbf{q}'_1, \mu'_1, \cdots, \mathbf{q}'_N, \mu'_N\rangle, \tag{43}$$

where rotational invariance means that the reduced kernel satisfies

$$\begin{gathered}\langle \mathbf{q}_1, \mu_1, \cdots, \mathbf{q}_N, \mu_N\|V\|\mathbf{q}'_1, \mu'_1, \cdots, \mathbf{q}'_N, \mu'_N\rangle = \\ \sum \prod D^{j_i}_{\mu_i \nu_i}[R^{-1}]\langle R\mathbf{q}_1, \nu_1, \cdots, R\mathbf{q}_N, \nu_N\|V\|R\mathbf{q}'_1, \nu'_1, \cdots, R\mathbf{q}'_N, \nu'_N\rangle \prod D^{j_l}_{\nu'_l \mu'_l}[R]\end{gathered} \tag{44}$$

for any rotation R. The only other requirements on V are $V = V^\dagger$ and $M_0 + V > 0$.

In this representation a general interaction is a sum of $2, 3, 4 \cdots$-body interactions. The mass eigenfunctions (36) have the form

$$\langle \mathbf{P}, \mathbf{q}_1, \mu_1, \cdots, \mathbf{q}_N, \mu_N|(\lambda, j)\mathbf{P}', \mu'\rangle = \delta(\mathbf{P} - \mathbf{P}'), \langle \mathbf{q}_1, \mu_1, \cdots, \mathbf{q}_N, \mu_N|(\lambda, j)\mu'\rangle, \tag{45}$$

and the mass eigenvalue problem (37) has the form

$$\begin{gathered}\left(\lambda - \sum_i \sqrt{\mathbf{q}_i^2 + m_i^2}\right) \langle \mathbf{q}_1, \mu_1, \cdots, \mathbf{q}_N, \mu_N|(\lambda, j)\mu'\rangle = \\ \sum \int \langle \mathbf{q}_1, \mu_1, \cdots, \mathbf{q}_N, \mu_N\|V\|\mathbf{q}'_1, \mu'_1, \cdots, \mathbf{q}'_N, \mu'_N\rangle \, d\mathbf{q}'_1 \cdots \mathbf{q}'_N \, \delta\left(\sum \mathbf{q}'_i\right) \langle \mathbf{q}'_1, \mu'_1, \cdots, \mathbf{q}'_N, \mu'_N|(\lambda, j)\mu'\rangle.\end{gathered} \tag{46}$$

The relativistic transformation properties can be easily determined once M is diagonalized. These eigenstates transform like mass λ spin j irreducible representations (39).

V. REACTION THEORY MODELS

For most nuclear systems a direct solution of the quantum mechanical scattering problem is not feasible. Approximations that are dominated by a more limited number of degrees of freedom are often amenable to a numerical solution. Success depends on identifying the most important degrees of freedom. In addition the effective interactions need to be modeled. Nevertheless it is useful to have a formalism where this is the first step in a well-defined systematic approximation to the exact solution.

To formulate a relativistic reaction model the steps are (1) start with an exact relativistic quantum mechanical model, (2) identify the most important degrees of freedom and then (3) construct an approximate relativistic quantum mechanical model with those degrees of freedom. To do this we project the exact mass operator on a relativistically invariant coupled-channel subspace of the full Hilbert space that allows scattering in all of the chosen important reaction channels. The relativistic invariance is preserved by choosing the projection to have the same symmetries as the interaction. The relation to the full theory provides a means to systematically include additional degrees of freedom.

The starting point is a relativistic mass operator (34) (or rest energy operator) which in the basis (30) has the form

$$M = \sum_i \sqrt{\mathbf{q}_i^2 + m_i^2} + \sum V_{ij} + \sum V_{ijk} + \cdots \tag{47}$$

where the sum of the $\mathbf{q}_i$ add to zero and the interactions are rotationally invariant operators that depend on the $\mathbf{q}_i$ and the constituent spins (31-32).

For any partition a of the N-particle system into disjoint subsystems we construct the partition mass operator M_a by eliminating interactions that involve particles in different clusters of the partition a. We also define the residual interactions

$$V^a := M - M_a. \tag{48}$$

The operator M_a is a sum of operators M_{a_k} for each disjoint non-empty cluster, a_k of a:

$$M_a = \sum_k M_{a_k} \tag{49}$$

given by

$$M_{a_k} = \sum_{i \in a_k} \sqrt{\mathbf{q}_i^2 + m_i^2} + \sum_{ij \in a_k} V_{ij} + \sum_{ijl \in a_k} V_{ijl} + \cdots . \tag{50}$$

In these expressions the $\mathbf{q}_i$ are not constrained in the various subsystems, however the total momentum of the subsystems is constrained to add up to zero only in the N-body system. This means that the operators M_{a_k} represent the energy of the moving clusters.

M_{a_k} has the same form as (47) except the sum is only over the particles in the k^{th} cluster of a. The natural variables for the for solving the subsystem problem are the subsystem constituent spins and the subsystem rest momenta $\mathbf{k}_i$. These are related to the system constituent spins and rest momenta by a relation like (31)

$$\begin{aligned} &|\mathbf{q}_{ak1}, \mu_{ak1}, \cdots, \mathbf{q}_{akl}, \mu_{akl}\rangle = \sum |\mathbf{q}_{a_k}, \mathbf{k}_1, \nu_1, \cdots, \mathbf{k}_l, \nu_l\rangle \times \\ &\sqrt{\frac{\sum_{i \in a_k} \omega_{m_{aki}}(\mathbf{q}_i)}{\sum_{i \in a_k} \omega_{m_{aki}}(\mathbf{k}_i)}} \prod D_{\nu_i \mu_{aki}}[(B^{-1}(k_i/m_{aki}) B^{-1}(\mathbf{q}_a/M_{0a}) B(\mathbf{q}_{aki}/m_{aki})] \sqrt{\frac{\omega_{m_{aki}}(\mathbf{k}_i)}{\omega_{m_{aki}}(\mathbf{q}_i)}} \end{aligned} \tag{51}$$

where

$$\mathbf{q}_{a_k} = \sum_{i \in a_k} \mathbf{q}_i, \qquad k_i := B^{-1}(\mathbf{q}_{a_k}/M_{ak0}) q_i, \qquad \sum_{i \in a_k} \mathbf{k}_i = 0 \tag{52}$$

and M_{ak0} is the invariant mass of the non-interacting subsystem. When these are embedded in the full system the sum of the cluster momenta, $\sum_k \mathbf{q}_{a_k} = 0$, are constrained to add to zero.

Each of the M_{a_K} will have simultaneous eigenstates of $\mathbf{q}_{a_k}$ and subsystem mass λ_{a_k}. For the purpose of reaction theory we are interested only in the case that λ_{a_k} are point-spectrum eigenvalues corresponding to bound clusters. In the N-free particle basis these states have the form

$$\begin{aligned} &\langle \mathbf{q}_{k1}, \mu_{k1}, \cdots, \mathbf{q}_{kn_k}, \mu_{kn_k} | \lambda_{a_k}, j_{a_k}; \mathbf{q}_a; \mu_{a_k} \rangle = \\ &\delta\left(\sum \mathbf{q}_{ki} - \mathbf{q}_{a_k}\right) \langle \mathbf{q}_{k1}, \mu_{k1}, \cdots, \mathbf{q}_{kn_k}, \mu_{kn_k} | \lambda_{a_k}, j_{a_k}; \mathbf{q}_a, \mu_{a_k} \rangle \end{aligned} \tag{53}$$

where we have removed an overall momentum-conserving delta function. In this expression (51) is used to relate the subsystem variables to the variables of the basis (31).

Channel projection operators can be defined in terms of products of these eigenstates:

$$\Pi_\alpha = \prod_j \int \sum_{\mu_{a_j}} |(\lambda_{a_j}, j_{a_j}) \mathbf{q}_{a_j}, \mu_{a_j}\rangle d\mathbf{q}_{a_j} \, \delta\left(\sum_j \mathbf{q}_{a_j}\right) \langle (\lambda_{a_j}, j_{a_j}) \mathbf{q}_{a_j}, \mu_{a_j}| \, |\mathbf{p}\rangle d\mathbf{p} \langle \mathbf{p}| \tag{54}$$

where the product is over all subsystems a_j in a given partition a of the N-particle system and the additional index α indicates both the partition into bound subsystems as well as the specific collection of bound states associated with each subsystem.

These channel projectors are used to build a projection on the model space.

To construct a relativistic reaction theory we project the mass operator on a subspace of the Hilbert space using projection operators Π that commute with $\mathbf{P}$, are independent of $\mathbf{P}$ and commute with $\mathbf{j}$,

$$M_\pi := \Pi M \Pi. \tag{55}$$

Simultaneous eigenstates of the projected mass operator, M_π, and $\mathbf{P}$, $\mathbf{j}$ and $\hat{\mathbf{z}} \cdot \mathbf{j}$ transform like (39) with respect to the Poincaré group. This defines the relativistic model in terms of a unitary representation of the Poincaré group on the model space.

The projection operator is a relativistic version of the projection operators that appear in coupled-channel approximations. It is constructed from elementary projection operators that project on subspaces generated by disjoint subsystems, where particles in the same subsystem are bound and the bound subsystems are free to move like free particles. The subsystem bound states are solutions to relativistic eigenvalue problems of the form (37) with λ being a point-spectrum eigenvalue of the subsystem mass operator.

The first step in making a reaction model is usually to construct the projection operator $\Pi = \Pi_{\mathcal{C}}$ corresponding to a chosen set of dominant reaction channels, $\mathcal{C}$. Typically, if $\alpha \in \mathcal{C}$ then it is normal to also include all channels generated from the channel α by exchange of identical particles.

The sum $\Sigma_{\mathcal{C}}$ of the channel projectors over the subset $\mathcal{C}$ of scattering channels is the positive self-adjoint operator

$$\Sigma_{\mathcal{C}} := \sum_{\alpha \in \mathcal{C}} \Pi_\alpha. \tag{56}$$

The main ideas that underly the formalism below were developed in a series of papers by Chandler and Gibson [18]. Let $\Sigma^{\#}_{\mathcal{C}}$ be the Moore-Penrose generalized inverse of $\Sigma_{\mathcal{C}}$. It is the unique solution to the Penrose equations [19]:

$$\begin{aligned} (\Sigma^{\#}_{\mathcal{C}} \Sigma_{\mathcal{C}}) &= (\Sigma^{\#}_{\mathcal{C}} \Sigma_{\mathcal{C}})^\dagger \\ (\Sigma_{\mathcal{C}} \Sigma^{\#}_{\mathcal{C}}) &= (\Sigma_{\mathcal{C}} \Sigma^{\#}_{\mathcal{C}})^\dagger \\ \Sigma_{\mathcal{C}} \Sigma^{\#}_{\mathcal{C}} \Sigma_{\mathcal{C}} &= \Sigma_{\mathcal{C}} \\ \Sigma^{\#}_{\mathcal{C}} \Sigma_{\mathcal{C}} \Sigma^{\#}_{\mathcal{C}} &= \Sigma^{\#}_{\mathcal{C}}. \end{aligned} \tag{57}$$

Because $\Sigma_{\mathcal{C}} = \Sigma^\dagger_{\mathcal{C}}$ it follows that

$$[\Sigma_{\mathcal{C}}, \Sigma^{\#}_{\mathcal{C}}] = 0 \tag{58}$$

and

$$\Pi_{\mathcal{C}} = \Sigma_{\mathcal{C}} \Sigma^{\#}_{\mathcal{C}} = \Sigma^{\#}_{\mathcal{C}} \Sigma_{\mathcal{C}} \tag{59}$$

is an orthogonal projector on the subspace of the Hilbert space satisfying

$$\Pi_\alpha \Pi_{\mathcal{C}} = \Pi_{\mathcal{C}} \Pi_\alpha = \Pi_\alpha \tag{60}$$

and

$$\Pi_\alpha |x\rangle = 0 \tag{61}$$

for all $\alpha \in \mathcal{C}$ then

$$\Pi_{\mathcal{C}} |x\rangle = 0. \tag{62}$$

The results above follow because the range of $\Sigma_{\mathcal{C}}$ contains the range of Π_α. To show this assume that $|x\rangle$ is in the range of Π_α but $|x\rangle$ is not in the range of X. It follows that

$$0 = \langle x|\Sigma_{\mathcal{C}}|x\rangle = \langle x|x\rangle + \sum_{\alpha' \neq \alpha} \langle x|\Pi_{\alpha'}|x\rangle \geq \| |x\rangle \|^2 > 0 \tag{63}$$

which is a contradiction. This shows that $\Pi_{\mathcal{C}}$ is an orthogonal projector on the smallest subspace containing all of the channel subspaces in $\mathcal{C}$. Some methods to compute the Moore-Penrose generalized inverse are discussed in Appendix A

VI. RELATIVISTIC SCATTERING THEORY

This section derives the symmetrized S matrix for a relativistic mass operator projected on a subspace that allows scattering in a limited number of channels. Rather than working on the model Hilbert space defined on by range of $\Pi_{\mathcal{C}}$, it is useful to work on the asymptotic channel spaces. This has the advantage that the dynamical equations only involve transition matrix elements projected on the appropriate asymptotic states and interactions smeared with subsystem bound state wave functions. This leads to a slightly different type of coupled integral equations, where only the projected part to the transition operators appear in the equations. This is an important simplification for reaction models because the projection of the transition operator on unphysical subspaces do not appear in the equations.

The relativistic reaction theory is the approximate theory defined by replacing the exact mass operator by the projected mass operator

$$M \to M_{\Pi} = \Pi_{\mathcal{C}} M \Pi_{\mathcal{C}}. \tag{64}$$

The set of retained channels $\mathcal{C}$ is assumed to be invariant with respect to permutations. For this choice M_{Π} commutes with the symmetrizer (antisymmetrizer) A,

$$[M_{\Pi}, A] = 0. \tag{65}$$

In order to formulate scattering asymptotic conditions for each channel $\alpha \in \mathcal{C}$ there is a natural asymptotic Hilbert space defined as the tensor product of irreducible representation spaces associated with the mass and spin of each bound cluster in the channel α;

$$\mathcal{H}_{\alpha} := \otimes_{j\in\alpha} \mathcal{H}_{\lambda_j j_j}. \tag{66}$$

The product of the irreducible state vectors in the channel α defines a mapping from $\mathcal{H}_{\alpha}$ to the model Hilbert space $\mathcal{H}_{\Pi}$ (the range of $\Pi_{\mathcal{C}}$) by:

$$\Phi_{\alpha} : \mathcal{H}_{\alpha} \to \mathcal{H}_{\Pi} \tag{67}$$

given by

$$\Phi_{\alpha}|f_1 \cdots f_m\rangle = \int \prod_j \sum_{\mu_{a_j}} |(\lambda_{a_j}, j_{a_j})\mathbf{q}_{a_j}, \mu_{a_j}\rangle \, \delta\left(\sum \mathbf{q}_{a_l}\right) d\mathbf{q}_{a_j} f_j(\mathbf{q}_{a_j}, \mu_{a_j}) \tag{68}$$

where we have removed the total momentum conserving delta function. In this notation the channel projectors (54) can be expressed as

$$\Pi_{\alpha} = \Phi_{\alpha}\Phi_{\alpha}^{\dagger}. \tag{69}$$

The asymptotic Hilbert space for the reaction model is defined by

$$\mathcal{H}_{as,\mathcal{C}} := \oplus_{\alpha\in\mathcal{C}} \mathcal{H}_{\alpha}. \tag{70}$$

The sum of the Φ_{α} defines a mapping from the asymptotic Hilbert space to the model Hilbert space by

$$\Phi_{\mathcal{C}} := \sum_{\alpha\in\mathcal{C}} \Phi_{\alpha}, \tag{71}$$

were each Φ_{α} is understood to act on the corresponding channel subspace $\mathcal{H}_{\alpha}$. Note that because of (60) and (69) the range of $\Phi_{\mathcal{C}}$ and $\Pi_{\mathcal{C}}$ coincide.

Symmetrized scattering channel wave functions are defined by the strong limits

$$|\Psi_{\alpha}^{\pm}\rangle = \lim_{t\to\pm\infty} A e^{iM_{\pi}t}\Phi_{\alpha} e^{-iM_{\alpha}t}|f_{\alpha}\rangle = \lim_{t\to\pm\infty} e^{iM_{\pi}t} A\Phi_{\alpha} e^{-iM_{\alpha}t}|f_{\alpha}\rangle, \tag{72}$$

where M_{α} is the invariant mass of the asymptotic initial or final state

$$M_{\alpha} = \sum_{j\in\alpha} \omega_{\lambda_{a_j}}\left(\mathbf{q_{a_j}}^2\right) = \sum \sqrt{\lambda_{a_j}^2 + \mathbf{q}_{a_j}^2}, \tag{73}$$

and the normalization of $|f_\alpha\rangle$ is chosen so $\langle\Psi^\pm_\alpha|\Psi^\pm_\alpha\rangle = 1$. The replacement of the Hamiltonian by the mass operator in (72) is justified [14, 20] by the invariance principle [21, 22]. Formally it corresponds to calculating the Poincaré invariant S matrix in the zero-momentum frame.

The relativistic S matrix is defined for each initial and final channel $\beta, \alpha \in \mathcal{C}$ by

$$S_{\alpha\beta} := \langle\Psi^+_\alpha|\Psi^-_\beta\rangle = \lim_{t\to\infty}\langle f_\alpha|e^{iM_\alpha t}\Phi^\dagger_\alpha A e^{-2iM_\pi t} A\Phi_\beta e^{iM_\beta t}|f_\beta\rangle. \tag{74}$$

Since $[M_\pi, A] = 0$ and $A^2 = A$ one symmetrizer can be eliminated. It is convenient to replace the initial and final states $|f_{\alpha/\beta}\rangle$ by channel mass eigenstates with sharp momenta $|\alpha/\beta\rangle$ and insert an $e^{-\epsilon t}$ factor to control the integral

$$\begin{aligned}\langle\Psi^+_\alpha|\Psi^-_\beta\rangle &= \langle\alpha|\Phi^\dagger_\alpha A\Phi_\beta|\beta\rangle + \lim_{\epsilon\to 0^+}\int_0^\infty \frac{d}{dt}\langle\alpha|e^{iM_\alpha t}\Phi^\dagger_\alpha A e^{-2iM_\pi t}A\Phi_\beta e^{iM_\beta t}|\beta\rangle e^{-\epsilon t}\\ &= \langle\alpha|\Phi^\dagger_\alpha A\Phi_\beta|\beta\rangle - i\lim_{\epsilon\to 0^+}\int_0^\infty \langle\alpha|e^{im_\alpha t}(\Phi^\dagger_\alpha M_\pi - m_\alpha\Phi^\dagger_\alpha)e^{-2iM_\pi t}A\Phi_\beta e^{im_\beta t}|\beta\rangle e^{-\epsilon t}\\ &\quad -i\lim_{\epsilon\to 0^+}\int_0^\infty \langle\alpha|e^{im_\alpha t}\Phi^\dagger_\alpha A e^{-2iM_\pi t}(M_\pi\Phi_\beta - \Phi_\beta m_\beta)e^{im_\beta t}|\beta\rangle e^{-\epsilon t},\end{aligned} \tag{75}$$

where it is understood that the limit is to be taken after smearing with wave packets. The same result would be obtained without introducing the ϵ factor if the wave packets were retained.

The quantity m_α is the sharp momentum eigenvalue of M_α given by (73), similarly for m_β. It is useful to introduce the average of the initial and final invariant mass, defined by

$$\bar{m} = \frac{1}{2}(m_\alpha + m_\beta) \tag{76}$$

and note that

$$\int_0^\infty e^{-2i(M_\pi - \bar{m} - i\epsilon)} = \frac{i}{2}\frac{1}{\bar{m} - M_\pi + i\epsilon}. \tag{77}$$

Using (77) in (75) gives

$$\begin{aligned}&\langle\alpha|\Phi^\dagger_\alpha A\Phi_\beta|\beta\rangle + \frac{1}{2}\langle\alpha|(\Phi^\dagger_\alpha M_\pi - m_\alpha\Phi^\dagger_\alpha)\frac{1}{\bar{m} - M_\pi + i\epsilon^+}A\Phi_\beta|\beta\rangle\\ &\qquad + \frac{1}{2}\langle\alpha|\Phi^\dagger_\alpha A\frac{1}{\bar{m} - M_\pi + i\epsilon^+}(M_\pi\Phi_\beta - \Phi_\beta m_\beta)|\beta\rangle.\end{aligned} \tag{78}$$

Applying the second resolvent identities as outlined in Appendix B, the resulting expression for the approximate S matrix element is:

$$\begin{aligned}\langle\Psi^+_\alpha|\Psi^-_\beta\rangle &= \langle\alpha_r|\Phi^\dagger_\alpha A\Phi_\beta|\beta\rangle\delta_{\alpha\beta} - 2\pi i\delta(m_\alpha - m_\beta)\Big[\langle\alpha|\Phi^\dagger_\alpha A(M_\pi\Phi_\beta - m_\beta\Phi_\beta)|\beta\rangle\\ &\quad + \langle\alpha|\left(\Phi^\dagger_\alpha M_\pi - m_\alpha\Phi^\dagger_\alpha\right)\frac{1}{m_\beta - M_\pi + i0^+}S(M_\pi\Phi_\beta - m_\beta\Phi_\beta)\Big]\beta\rangle.\end{aligned} \tag{79}$$

Note that

$$(M_\pi - M_{b\pi})\Phi_\beta = (M_\pi - m_\beta)\Phi_\beta \tag{80}$$

because Φ_β is an eigenstate of $M_{b\pi}$ with eigenvalue m_β (here $M_{b\pi}\Phi_\beta := M_b\Phi_\beta$). Defining

$$M^b_\pi := M_\pi - M_{b\pi} \tag{81}$$

leads to the expression for the S-matrix elements in this approximation

$$\begin{aligned}\langle\Psi^+_\alpha|\Psi^-_\beta\rangle &= \langle\alpha|\Phi^\dagger_\alpha A\Phi_\beta|\beta\rangle\delta_{\alpha\beta}\\ &\quad - 2\pi i\delta(m_\alpha - m_\beta)\Big[\langle\alpha|\Phi^\dagger_\alpha A M^b_\pi\Phi_\beta|\beta\rangle + \langle\alpha|\Phi^\dagger_\alpha M^a_\pi\frac{1}{m_\beta - M_\pi + i0^+}AM^b_\pi\Phi_\beta|\beta\rangle\Big].\end{aligned} \tag{82}$$

The symmetrized approximate transition operator that acts on the open channel spaces is

$$T_{\alpha\beta} := \Phi^\dagger_\alpha A\, M^b_\pi\Phi_\beta + \Phi^\dagger_\alpha M^a_\pi\frac{1}{m_\beta - M_\pi + i0^+}AM^b_\pi\Phi_\beta. \tag{83}$$

Note that in this form all of the internal degrees of freedom do not appear in the transition matrix. This is because the operators Φ_β and $\Phi^\dagger_\alpha$ project the standard form of the transition operators on the asymptotic channels subspaces. The result is that the internal degrees of freedom associated with the bound clusters do not appear in $T_{\alpha\beta}$.

Both equation (82) and (83) contain an overall momentum-conserving delta function that can be factored out of both equations.

One would like to get integral equations directly for $T_{\alpha\beta}$, which avoid having to treat all of the unphysical degrees of freedom in the unprojected transition operators. In order to construct such equations use (59) and (69) to get the following identity

$$\Pi_{\mathcal{C}} = \sum_{\gamma\in\mathcal{C}} \Sigma^{\#}\Phi_\gamma\Phi^\dagger_\gamma. \tag{84}$$

Inserting (84) in the expression (83) for T gives the equations

$$T_{\alpha\beta} := \Phi^\dagger_\alpha A M^b_\pi \Phi_\beta + \sum_\gamma \Phi^\dagger_\alpha M^a_\pi A \Sigma^{\#}\Phi_\gamma\Phi^\dagger_\gamma \frac{1}{m_\beta - M_\pi + i0^+} A M^b_\pi \Phi_\beta. \tag{85}$$

Using the second resolvent identity from (B7) in in (85) gives

$$\begin{aligned} T_{\alpha\beta} &= \Phi^\dagger_\alpha A M^b_\pi \Phi_\beta \\ &+ \sum_\gamma \Phi^\dagger_\alpha M^a_\pi \Sigma^{\#} A \Phi_\gamma \frac{1}{m_\beta - m_\gamma + i0^+} \left[\Phi^\dagger_\gamma A M^b_\pi \Phi_\beta + \Phi^\dagger_\gamma M^c_\pi \frac{1}{m_\beta - M_\pi + i0^+} A M^b_\pi \Phi_\beta \right] \end{aligned} \tag{86}$$

or

$$T_{\alpha\beta} = \Phi^\dagger_\alpha A M^b_\pi \Phi_\beta + \sum_{\gamma\in\mathcal{C}} \Phi^\dagger_\alpha M^a_\pi \Sigma^{\#} A \Phi_\gamma \frac{1}{m_\beta - m_\gamma + i0^+} T_{\gamma\beta}. \tag{87}$$

Here the sum is over all retained channels.

In general equation (87) does not have a compact iterated kernel which allows one to compute uniformly convergent approximations. It can be recast into such a form where the iterated kernel is compact. The basic idea is simple in principle, but the operators are can be complicated depending on the reaction mechanism.

Abstractly equation (87) has the form

$$T_{\alpha\beta} = D_{\alpha\beta} + \sum_\gamma K_{\alpha\gamma} T_{\gamma\beta} \tag{88}$$

The kernel $K_{\alpha\gamma}$ has a cluster expansion. For each partition a of the N particle system into subsystems it can be expressed as

$$K_{\alpha\gamma} = K_{a\alpha\gamma} + K^a_{\alpha\gamma} \tag{89}$$

where

$$K_{a\alpha\gamma} \tag{90}$$

is the part of $K_{a\alpha\gamma}$ that commutes with the $\mathbf{q}_{ak}$ and $K^a_{\alpha\gamma}$ is the remainder.

For each partition a we can construct

$$(I - K_a)^{-1}_{\alpha\beta}. \tag{91}$$

With this, for each a the system of equations has the form

$$T_{\alpha\beta} = (I - K_a)^{-1}_{\alpha\delta} D_{\delta\beta} + \sum_\gamma (I - K_a)^{-1}_{\alpha\delta} K^a_{\delta\gamma} T_{\gamma\beta}. \tag{92}$$

The following equation

$$T_{\alpha\beta} = \sum_{n_a\geq 2} (-)^{n_a}(n_a - 1)!(I - K_a)^{-1}_{\alpha\delta} D_{\delta\beta} + \sum_\gamma \sum_{n_a\geq 2} (-)^{n_a}(n_a - 1)!(I - K_a)^{-1}_{\alpha\delta} K^a_{\delta\gamma} T_{\gamma\beta} \tag{93}$$

where n_a is the number of disjoint clusters in the partition a, has a connected iterated kernel [23]. All of the terms in these equations only involve degrees of freedom in the model Hilbert space.

In general the individual terms $(I - K_a)^{-1}_{\alpha\delta}$ have to be constructed recursively from subsystem equations, however for reaction theories these operators are generally modeled. Iterating these equations gives a generalization of the usual multiple scattering series [6, 7].

VII. IDENTICAL PARTICLES

For systems of identical particles the number of channel in the scattering equations can be significantly reduced. For identical particles note that for each channel a permutation operator either leaves the channel unchanged or transforms it to an equivalent channel. The permutations that leave the channel unchanged involve permutation of particles in each asymptotic bound state, or exchanges of identical asymptotic bound states. There are $n_{a_1}! \cdots n_{a_m}!$ permutations that leave each cluster of an m cluster channel γ unchanged. There are also $s!$ exchanges for s identical clusters with identical bound states.

Two channels that are related by permutation are called permutation equivalent. Those that are not are called permutation inequivalent. Let $[\gamma]$ be the equivalence class of channels equivalent to γ. Let $n_{[\gamma]}$ be the number of channels in $[\gamma]$,

$$n_{[\gamma]} = \frac{N!}{n_{a_1}! \cdots n_{a_m}! s_1! \cdot s_k!}. \tag{94}$$

For each channel γ the symmetrizer can be decomposed as follows

$$A = \frac{1}{N!} \sum P_\sigma = \frac{1}{n_{[\gamma]}} \sum_{\delta \in [\gamma]} P_{\delta\gamma} A_\gamma = \frac{1}{n_{[\delta]}} \sum_{\delta \in [\gamma]} A_\gamma P_{\gamma\delta} \tag{95}$$

where the permutation operator P_σ is defined to include a factor of $(-)^{|\sigma|}$ for identical fermions. The channel sum in the T-matrix equation can be decomposed into a sum over equivalence classes of channels and a sum over elements in each equivalence class

$$\sum_{\gamma \in \mathcal{C}} = \sum_{[\gamma] \in \mathcal{C}} \sum_{\gamma \in [\gamma]} \tag{96}$$

Using this in the integral equation (87) gives

$$T_{\alpha\beta} = \Phi_\alpha^\dagger \frac{1}{n_{[\alpha]}} \sum_{\delta \in [\alpha]} P_{\alpha\delta} M_\pi^b \Phi_\beta + \sum_{[\gamma] \in \mathcal{C}} \sum_{\gamma \in [\gamma]} \Phi_\alpha^\dagger M_\pi^a \Sigma^\# \frac{1}{n_{[\gamma]}} \sum_{\delta \in [\gamma]} P_{\delta\gamma} \Phi_\gamma \frac{1}{m_\beta - m_\gamma + i0^+} T_{\gamma\beta} \tag{97}$$

We note that

$$\sum_{\gamma \in [\gamma]} \frac{1}{n_{[\gamma]}} \sum_{\delta \in [\gamma]} P_{\delta\gamma} \Phi_\gamma \frac{1}{m_\beta - m_\gamma + i0^+} T_{\gamma\beta} =$$

$$\sum_{\delta \in [\gamma]} P_{\delta\gamma} \Phi_\gamma \frac{1}{m_\beta - m_\gamma + i0^+} T_{\gamma\beta} \tag{98}$$

which when used in (97) gives the symmetrized equation

$$T_{\alpha\beta} = \Phi_\alpha^\dagger \frac{1}{n_{[\alpha]}} \sum_{\delta \in [\alpha]} P_{\alpha\delta} M_\pi^b \Phi_\beta + \sum_{[\gamma] \in \mathcal{C}} \sum_{\delta \in [\gamma]} \Phi_\alpha^\dagger M_\pi^a \Sigma^\# P_{\delta\gamma} \Phi_\gamma \frac{1}{m_\beta - m_\gamma + i0^+} T_{\gamma\beta}. \tag{99}$$

In this equation γ, α and β are arbitrary but fixed elements of the classes $[\gamma]$, $[\alpha]$ and $[\beta]$.

The effective interactions for this symmetrized equation are

$$\sum_{\delta \in [\gamma]} \Phi_\alpha^\dagger M_\pi^a \Sigma^\# P_{\delta\gamma} \Phi_\gamma. \tag{100}$$

The kernel of this equation is only compact for models with only two cluster channels. When the reaction mechanism includes channels with three or more clusters then it is necessary to construct an equivalent compact kernel equation or to establish that there are no-non-trivial solutions to the homogeneous equations.

These equations give the approximate transition operator derived in section 6 however they do not include the effects of the eliminated channels. We could have replaced M_Π by

$$M_\Pi \to M_\Pi + \Pi M \Pi' (\lambda - \Pi' M \Pi' + i0)^{-1} \Pi' M \Pi \tag{101}$$

with $I = \Pi + \Pi'$, which would lead to equations of the same form with the interaction terms replaced by energy dependent optical potentials. Since this decomposition still preserves the rotational invariance, it will lead to irreducible representations of the Poincaré group.

VIII. (D,P) REACTIONS

To illustrate the formalism we consider the case of a (d,p) reaction. We choose the dominant reaction channels $\mathcal{C}$ to include the deuteron and an A-particle target nucleus, two nucleons and the target nucleus, a nucleon an $A+1$ particle nucleus, the deuteron and an A-particle excited nucleus, and all channels generated by exchange of identical nucleons. Here we treat the proton and neutrons as different isospin states of a system of $A+2$ identical nucleons. This leads to an effective three-body problem. For low energy (d,p) reactions this approach was pioneered in Ref. [5, 24–26] in the framework of the Faddeev AGS equations. Within a Poincaré invariant formulation the dynamical equations governing this system are formally given by (99). The channel injection operators are

$$\Phi_1 := |\mathbf{P}; (m_d, 1)\mathbf{q}_d, \mu_d, (m_A, j_A) - \mathbf{q}_d, \mu_A\rangle \tag{102}$$

$$\Phi_2 := |\mathbf{P}; (m_d, 1)\mathbf{q}_d, \mu_d, (m_B, j_B) - \mathbf{q}_d, \mu_B\rangle \tag{103}$$

$$\Phi_3 := |\mathbf{P}; (m_n, \frac{1}{2})\mathbf{q}_n, \mu_n, (m_n, \frac{1}{2})\mathbf{q}'_n, \mu'_n(m_A, j_A) - \mathbf{q}_n - \mathbf{q}'_n, \mu_{A-1}\rangle \tag{104}$$

$$\Phi_4 := |\mathbf{P}; (m_n, \frac{1}{2})\mathbf{q}_n, \mu_n, (m_{A+1}, j_{A+1}) - \mathbf{q}_N, \mu_{A+1}\rangle. \tag{105}$$

The full set of channels $\mathcal{C}$ is generated by applying permutations to these channels.

The operator Σ is given by

$$\Sigma := \sum_{\gamma\in[1]} P_{\gamma 1}\Phi_1\Phi_1^\dagger P_{1\gamma}^\dagger + \sum_{\gamma\in[2]} P_{\gamma 2}\Phi_2\Phi_2^\dagger P_{2\gamma}^\dagger + \sum_{\gamma\in[3]} P_{\gamma 3}\Phi_3\Phi_3^\dagger P_{3\gamma}^\dagger + \sum_{\gamma\in[4]} P_{\gamma 4}\Phi_4\Phi_4 P_{4\gamma}^\dagger \tag{106}$$

and

$$\Pi_{\mathcal{C}} := \Sigma^{\#}\Sigma. \tag{107}$$

The model mass operator is

$$M_\Pi = \Pi_{\mathcal{C}} M \Pi_{\mathcal{C}}. \tag{108}$$

The individual channel masses are

$$\begin{aligned} m_1 &= \sqrt{m_d^2 + \mathbf{q}_d^2} + \sqrt{m_A^2 + \mathbf{q}_d^2} \\ m_2 &= \sqrt{m_d^2 + \mathbf{q}_d^2} + \sqrt{m_B^2 + \mathbf{q}_d^2} \\ m_3 &= \sqrt{m_n^2 + \mathbf{q}_{n1}^2} + \sqrt{m_n^2 + \mathbf{q}_{n2}^2} + \sqrt{m_A^2 + (\mathbf{q}_{n1} + \mathbf{q}_{n2})^2} \\ m_4 &= \sqrt{m_n^2 + \mathbf{q}_n^2} + \sqrt{m_{A+1}^2 + \mathbf{q}_n^2} \end{aligned} \tag{109}$$

and

$$\begin{aligned} M_{\Pi i} &:= \Phi_i m_i \Phi_i^\dagger \\ M_\Pi^i &:= M_\Pi - M_{\Pi i}. \end{aligned} \tag{110}$$

The projected transition matrix elements are

$$\begin{aligned} T_{11} &= \langle \mathbf{P}(m_d, 1)\mathbf{q}_d, \mu_d, (m_A, j_A) - \mathbf{q}_n, \mu_A | T^{11}(z) | \mathbf{P}'(m_d, 1)\mathbf{q}'_d, \mu'_d, (m_A, j_A) - \mathbf{q}'_d, \mu'_A\rangle \\ &= \delta(\mathbf{P} - \mathbf{P}') t_{11}(\mathbf{q}_d, \mu_d, \mu_A; \mathbf{q}'_d, \mu'_d, \mu'_A, z) \end{aligned} \tag{111}$$

$$\begin{aligned} T_{21} &= \langle \mathbf{P}(m_d, 1)\mathbf{q}_d, \mu_d, (m_B, j_B) - \mathbf{q}_d, \mu_B | T^{21}(z) | \mathbf{P}'(m_d, 1)\mathbf{q}'_d, \mu'_d, (m_A, j_A) - \mathbf{q}'_d, \mu'_A\rangle \\ &= \delta(\mathbf{P} - \mathbf{P}') t_{21}(\mathbf{q}_d, \mu_d, \mu_B; \mathbf{q}'_d, \mu'_d, \mu'_A, z) \end{aligned} \tag{112}$$

$$\begin{aligned} T_{31} &= \langle \mathbf{P}(m_n, \frac{1}{2})\mathbf{q}_{n1}, \mu_{n1}, (m_n, \frac{1}{2})\mathbf{q}_{n2}, \mu_{n2}(m_A, j_A) - (\mathbf{q}_{n1} + \mathbf{q}_{n2}), \mu_A | T^{31}(z) \\ &\quad \times |\mathbf{P}'(m_d, 1)\mathbf{q}'_d, \mu'_d, (m_A, j_A) - \mathbf{q}'_d, \mu'_A\rangle \\ &= \delta(\mathbf{P} - \mathbf{P}') t_{31}(\mathbf{q}_{n1}, \mu_{n1}, \mathbf{q}_{n2}, \mu_{n2}, \mu_A; \mathbf{q}'_d, \mu'_d, \mu'_A, z) \end{aligned} \tag{113}$$

$$T_{41} = \langle \mathbf{P}(m_n, \frac{1}{2})\mathbf{q}_n, \mu_n, (m_{A+1}, j_{A+1}) - \mathbf{q}_n, \mu_{A+1} | T^{41}(z) | \mathbf{P}'(m_d, 1)\mathbf{q}'_d, \mu'_d, (m_A, j_A) - \mathbf{q}'_d, \mu'_A\rangle \tag{114}$$

$$= \delta(\mathbf{P} - \mathbf{P}') t_{41}(\mathbf{q}_n, \mu_n, \mu_{A+1}; \mathbf{q}'_d, \mu'_d, \mu'_A, z). \tag{115}$$

For a reasonable sized target nucleus the input to the equations, while well defined, must ultimately be treated phenomenologically. These elements are interactions and kernel terms. The 11 driving term is

$$\begin{aligned}V_{11} &= \frac{1}{n_{[1]}} \sum_{\gamma\in[1]} \langle \mathbf{P}(m_d,1)\mathbf{q}_d,\mu_d,(m_A,j_A)-\mathbf{q}_d,\mu_A|P_{1\gamma}M^1_\Pi|\mathbf{P}'(m_d,1)\mathbf{q}'_d,\mu'_d,(m_A,j_A)-\mathbf{q}'_d,\mu'_A\rangle \\ &= \delta(\mathbf{P}-\mathbf{P}')v_{11}(\mathbf{q}_d,\mu_d,\mu_A;\mathbf{q}'_d,\mu'_n,\mu'_A) \end{aligned} \tag{116}$$

where

$$v_{11}(\mathbf{q}_d,\mu_d,\mu_A;\mathbf{q}'_d,\mu'_d,\mu'_A) \tag{117}$$

which is a rotationally invariant functions of the $\mathbf{q}_i$ and constituent spins. There are three other driving terms, V_{21}, V_{31}, V_{41} associated with the three other final channels.

The interaction part of the kernel has 16 terms of the form K_{ij}. They have a form similar to K_{11}

$$\begin{aligned}K_{11} = \sum_{\gamma\in[1]} &\langle \mathbf{P}\,(m_n,1)\,\mathbf{q}_d,\mu_d,(m_A,j_A)-\mathbf{q}_d,\mu_A|M^1_\Pi\Sigma^\# P_{\gamma 1}|\mathbf{P}'\,(m_d,1)\,\mathbf{q}'_d,\mu'_d,(m_A,j_A)-\mathbf{q}'_d,\mu'_A\rangle \\ &\delta(\mathbf{P}-\mathbf{P}')k_{11}(\mathbf{q}_d,\mu_d,\mu_A;\mathbf{q}'_d,\mu'_d,\mu'_A)\end{aligned} \tag{118}$$

where

$$k_{11}(\mathbf{q}_d,\mu_d,\mu_A;\mathbf{q}'_d,\mu'_d,\mu'_A) \tag{119}$$

is a rotationally invariant kernel.

The integral equation is a four by four matrix of equations involving all four amplitudes. After factoring out the overall momentum conserving delta function we get

$$\begin{aligned}&t_{11}(\mathbf{q}_d,\mu_d,\mu_A;\mathbf{q}'_d,\mu'_d,\mu'_A,m_{A'}) \\ &= v_{11}(\mathbf{q}_d,\mu_d,\mu_A;\mathbf{q}'_d,\mu'_d,\mu'_A) \\ &+ \sum\int \frac{k_{11}(\mathbf{q}_d,\mu_d,\mu_A;\mathbf{q}''_d,\mu''_d,\mu''_A)d\mathbf{q}''_d t_{11}(\mathbf{q}''_d,\mu''_d,\mu''_A;\mathbf{q}'_d,\mu'_d,\mu'_A,m_{A'})}{m_{1'}-m''_1+i0^+} \\ &+ \sum\int \frac{k_{12}(\mathbf{q}_d,\mu_d,\mu_A;\mathbf{q}''_d,\mu''_d,\mu''_B)d\mathbf{q}''_d t_{21}(\mathbf{q}''_d,\mu''_d,\mu''_B;\mathbf{q}'_d,\mu'_d,\mu'_A,m_{A'})}{m_{1'}-m''_2+i0^+} \\ &+ \sum\int \frac{k_{13}(\mathbf{q}_d,\mu_d,\mu_A;\mathbf{q}''_{n1},\mu''_{n1},\mathbf{q}''_{n2},\mu''_{n2},\mu''_A)d\mathbf{q}''_{n_1}d\mathbf{q}''_{n_1}t_{31}(\mathbf{q}''_{n1},\mu''_{n1},\mathbf{q}''_{n2},\mu''_{n2},\mu''_A;\mathbf{q}'_d,\mu'_d,\mu'_A,m_{A'})}{m_{1'}-m''_3+i0^+} \\ &+ \sum\int \frac{k_{14}(\mathbf{q}_d,\mu_d,\mu_A;\mathbf{q}''_n,\mu''_n,\mu''_{A+1})d\mathbf{q}''_n t_{41}(\mathbf{q}''_n,\mu''_n,\mu''_{A+1};\mathbf{q}'_d,\mu'_d,\mu'_A,m_{A'})}{m_{1'}-m''_4+i0^+}.\end{aligned} \tag{120}$$

This is the first of four coupled equations, the other are for t_{21}, t_{31}, t_{41}. These equations have the same general structure.

These are a set of four coupled channel equations for the four symmetrized transition matrix elements. The kernel has disconnected terms which remain disconnected upon iteration. These can be replaced by equivalent connected kernel equations using the methods discussed at the end of section 6.

In this case a direct solution is easier. The starting point is equations (120) which have the abstract form:

$$t_{i1} = v_{i1} + \sum_{j=1}^4 K_{ij}t_{j1} \tag{121}$$

The first step is to eliminate breakup amplitude ($j = 3$) using

$$t_{31} = (1-K)^{-1}_{33}v_{31} + (1-K)^{-1}_{33}K_{31}t_{11} + (1-K)^{-1}_{33}K_{32}t_{21} + (1-K)^{-1}_{33}K_{34}t_{41}. \tag{122}$$

The second step is to insert this into the remaining three equations

$$\begin{aligned}t_{k1} &= v_{k1} + K_{k3}(1-K)^{-1}_{33}v_{31} \\ &\quad + (K_{k1}+K_{k3}(1-K)^{-1}_{33}K_{31})t_{11} + (K_{k2}+K_{k3}(1-K)^{-1}_{33}K_{32})t_{21} \\ &\quad + (K_{k4}+K_{k3}(1-K)^{-1}_{33}K_{34})t_{41}.\end{aligned} \tag{123}$$

The last step is to make the kernels connected upon iteration which gives the following three coupled equations for the two-cluster amplitudes:

$$t_{k1} = (I - K_{kk} + K_{k3}(1-K)^{-1}_{33}K_{3k})^{-1}v_{k1}K_{k3}(1-K)^{-1}_{33}v_{31}$$

$$+ \sum_{l \neq k,3} (I - K_{kk} + K_{k3}(1-K)^{-1}_{33} K_{3k})^{-1} (K_{kl} + K_{k3}(1-K)^{-1}_{33} K_{3l})^{-1} t_{l1}. \tag{124}$$

These equations can be solved using Faddeev methods. The breakup amplitude can be calculated these solutions using (122). The effective interactions are complicated many-body operators that, while precisely defined, have to be modeled in practice. The interactions include both effective two and three-body interactions. In this model the "three-body forces" will be important because they include effects from the exchange channels. If one wants to include corrections from some of the eliminated channels, then the interactions are replaced by energy dependent optical potentials.

The number of continuous variables is the same as one would get on a three-body Faddeev equation. Unlike the relativistic few-body problem, depending on the charge of the core, Coulomb effects may have to be included. This requires an additional analysis due to the long-range nature of the Coulomb interaction.

IX. DISCUSSION AND OUTLOOK

In the preceeding sections a formulation of a theory for nuclear reactions is given in a representation of Poincaré invariant quantum mechanics where the interactions are invariant with respect to kinematic translations and rotations. Though a practical implementation is not yet in sight, it is important to lay out the necessary ingredients for such a theory. Specifically, as experimental capabilities in investigating reaction with rare isotopes are continuously refined, assumptions and approximations applied to study reactions at higher energies may need to be re-examined.

One of the experimentally as well as theoretically most carefully studied system is the three-nucleon system. The Faddeev equations with two and three-nucleon forces can be exactly solved [27, 28] for neutron and proton-deuteron scattering. A Poincaré invariant formulation of the Faddeev equations was pioneered in model calculations up to 1 GeV for spin-independent forces [11, 17, 29] and then employed for realistic two and three-nucleon forces [8] for projectile energies up to 250 MeV. Both, the realistic as well as the model calculation indicate that differential cross sections exhibit some differences between a non-relativistic an relativistic treatment at large momentum transfer already as low as 250 MeV. However, those calculations clearly indicate, that breakup reactions are considerably more sensitive to a correct treatment of Poincaré invariance than elastic scattering. The model calculation in Refs. [11] indicate that the correct Poincaré invariant treatment of relativistic kinematics and dynamics in exclusive breakup reaction cross sections can differ up to an order of magnitude from the Galilei invariant calculation already at projectile energies around 500 MeV, while at the same energy elastic scattering cross sections only differ by about 15%. The calculations with realistic forces [8] obtained similar conclusions when considering projectile energies of 250 MeV.

These findings may have consequences for reactions with unstable ions at higher energies. Light probes e.g. allow to explore shell evolution. Among the light probes, deuteron induced reactions are attractive from an experimental point of view, since deuterated targets are readily available. From a theoretical perspective they are attractive, because the scattering problem can be reduced to an effective three-body problem. At higher energies, target excitations may be less important than at lower energies. However, since (d,p) reactions are breakup reactions, it will need to be explored if they are as sensitive to a consistent Poincaré invariant treatment as proton-deuteron breakup reactions.

As mentioned in the introduction, an often and successfully employed approximation to study states at higher energies is the eikonal approximations. One- and two-nucleon removal reactions at higher energies as well as pickup reactions are used in an eikonal formulation to study the structure of exotic nuclei [30] by calculating e.g. inclusive one- and two-nucleon removal. The emphasis here is the description of the structure entering the reaction calculation. In this way data have been described quite successfully [31].

Efforts to compare a reaction calculation carried out in the eikonal approximation with a corresponding exact calculation have up now only been possible in the three-nucleon problem. The most comprehensive study of the quality of a refined eikonal model is carried out for proton-deuteron (pd) elastic scattering at 250 MeV and 1 GeV [10]. Here the Glauber model was refined to take into account the full spin structure of the proton-neutron and proton-proton interactions in the single and double scattering term of the Glauber diffraction model. At 250 MeV different cross sections and polarization are compared to non-relativistic Faddeev calculations [32] for all possible values of the momentum transfer. This comparison reveals that in the pd system the re-scattering term in the Glauber model is essential to be close to the Faddeev calculation (and the experiment) for up to about half of the total allowed momentum transfer. In contrast the first order Glauber term is only valid for very small momentum transfers. This finding holds even at 1 GeV. Calculations in similar spirit comparing a Glauber ansatz including the rescattering term with a Poincaré invariant Faddeev calculation, but based a simple spin-independent model for the nucleon-nucleon interaction were carried out up to 2 GeV projectile energy [33]. Here the main finding was that a Glauber type calculation can describe cross section relatively well in the GeV regime, differential cross sections were not as well captured for large momentum transfers. Thus, when studying nuclear reactions at higher energies, it will be necessary to develop a reaction theory based on relativistic quantum mechanics.

Appendix A: Moore-Penrose generalized inverse

In this appendix we discuss methods for computing the Moore Penrose generalized inverse. The definition

$$\Sigma_\alpha^{\#} := \Pi_\alpha \Sigma_{\mathcal{C}}^{\#} \tag{A1}$$

implies

$$\Pi_{\mathcal{C}} = \sum_{\alpha\in\mathcal{C}} \Sigma_\alpha^{\#}. \tag{A2}$$

Multiplying both sides of (A1) by Π_α and rearranging terms gives

$$\Sigma_\alpha^{\#} = \Pi_\alpha - \sum_{\beta\neq\alpha\in\mathcal{C}} \Pi_\alpha \Sigma_\beta^{\#}. \tag{A3}$$

For two cluster channels this set of equations, after factoring our the total momentum conserving delta functions, has a non-singular compact iterated kernel, which can be uniformly approximated by a finite-dimensional matrix. This gives a straightforward means to construct the solution to these equations using uniform approximations.

The solution of (A3) can be used to calculate

$$\Pi_{\mathcal{C}} = \sum_{\alpha\in\mathcal{C}} \Sigma_\alpha^{\#}. \tag{A4}$$

When the projectors in $\Sigma_{\mathcal{C}}$ include more than two clusters channels the series and the non-zero eigenvalues of $\Sigma_{\mathcal{C}}$ are bounded above zero then

$$\Pi_{\mathcal{C}} = \sum_{n=0}^{\infty}(1-\gamma\Sigma_{\mathcal{C}})^n\gamma\Sigma_{\mathcal{C}} = \gamma\Sigma_{\mathcal{C}}\sum_{n=0}^{\infty}(1-\gamma\Sigma_{\mathcal{C}})^n \tag{A5}$$

will converge uniformly for γ less that 1/(number of channels). The relevant iteration is

$$\Pi(0) := \gamma\Sigma_{\mathcal{C}} \tag{A6}$$

$$\Pi(n+1) = \Pi(n)(1-\gamma\Sigma_{\mathcal{C}}) \tag{A7}$$

$$\Pi_{\mathcal{C}} = \lim_{n\to\infty} \Pi(n). \tag{A8}$$

The rate of convergence depends on both the choice of γ and the size of the smallest non-zero eigenvalue of $\Sigma_{\mathcal{C}}$. There is also a similar series for

$$\Sigma_\alpha^{\#} = \gamma\Pi_\alpha\sum_{n=0}^{\infty}(1-\gamma\Sigma_{\mathcal{C}})^n. \tag{A9}$$

Cluster expansions for $\Sigma_\alpha^{\#}$ can be developed from this representation.

An alternative way to calculate $\Sigma^{\#}\Phi_\alpha$, which uses connected kernel equations, is based on the observation that the resolvent of X satisfies the Weinberg-Van Winter equations[23]

$$\frac{1}{z-\Sigma} = \sum_{a,n_a\geq 2} C_a \frac{1}{z-\Sigma_a} + \sum_{a,n_a\geq 2} C_a(\Sigma-\Sigma_a)\frac{1}{z-\Sigma}$$

where Σ_a is the sum of all projectors that commute with translations of the cluster of the partition a. The coefficients C_a are

$$C_a = (-)^{n_a}(n_a-1)! \tag{A10}$$

where n_a is the number of non-empty clusters in the partition a. These equations always have compact kernels. They can be solved recursively (n the number of particles) to build up the $\frac{1}{z-\Sigma_a}$ that are the input to these equations. The starting point corresponds to the finest partitions where the resolvents have the trivial form

$$\frac{1}{z-\Pi_\alpha} = \Pi_\alpha\frac{1}{z-1} + \Pi\frac{1}{z} \tag{A11}$$

This gives a Faddeev type of construction to find Π. It requires that the Moore-Penrose generalized inverse is bounded or equivalently that the spectrum of Σ has a gap between 0 and its first non-zero eigenvalue.

The operator $\Sigma^{\#}\Phi_\alpha$ which appears in the integral equation can be calculated using

$$\Sigma^{\#}\Phi_\alpha = \lim_{z\to 0} \frac{1}{X-z}\Phi_\alpha$$

This limit makes sense because the range of Φ_α is in the range of Σ. The Weinberg-Van Winter equation can be replaced by

$$\frac{1}{z-\Sigma}\Phi_\alpha = \sum_{a,n_a\geq 2} C_a \frac{1}{z-\Sigma_a}\Phi_\alpha + \sum_{a,n_a\geq 2} C_a(\Sigma-\Sigma_a)\frac{1}{z-\Sigma}\Phi_\alpha \tag{A12}$$

Appendix B: Formulations with Resolvent Identities

The second resolvent identities are used in (78) to obtain

$$\begin{aligned}
&\langle\alpha|\Phi_\alpha^\dagger A\Phi_\beta|\beta\rangle \\
&+\frac{1}{2}\langle\alpha|(\Phi_\alpha^\dagger M_\pi - m_\alpha\Phi_\alpha^\dagger)\ A(\Phi_\beta + \frac{1}{\bar{m}-M_\pi+i\epsilon^+}(M_\pi\Phi_\beta - \Phi_\beta m_\beta))\frac{1}{\bar{m}-m_\beta+i\epsilon^+}|\beta\rangle \\
&+\frac{1}{2}\langle\alpha|\frac{1}{\bar{m}-m_\alpha+i\epsilon^+}(\Phi_\alpha^\dagger + (\Phi_\alpha^\dagger M_\pi - m_\alpha\Phi_\alpha^\dagger)\frac{1}{\bar{m}-M_\pi+i\epsilon^+})A(M_\pi\Phi_\beta - \Phi_\beta m_\beta)|\beta\rangle.
\end{aligned} \tag{B1}$$

Separating the kinematical and dynamical terms gives

$$\begin{aligned}
&\langle\alpha|\Phi_\alpha^\dagger A\Phi_\beta|\beta\rangle \\
&+\frac{1}{m_\alpha-m_\beta+i\epsilon}\langle\alpha_r|(\Phi_\alpha^\dagger M_\pi - (m_\alpha-m_\beta+m_\beta)\Phi_\alpha^\dagger)A\Phi_\beta|\beta\rangle \\
&-\frac{1}{m_\alpha-m_\beta-i\epsilon}\langle\alpha_r|\Phi_\alpha^\dagger A(M_\pi\Phi_\beta - m_\beta\Phi_\beta)|\beta\rangle \\
&+\langle\alpha|(\Phi_\alpha^\dagger M_\pi - m_\alpha\Phi_\alpha^\dagger)A\frac{1}{\bar{m}-M_\pi+i0^+}(M_\pi\Phi_\beta - m_\beta\Phi_\beta)|\beta\rangle \times \left[\frac{1}{m_\alpha-m_\beta+i\epsilon} - \frac{1}{m_\alpha-m_\beta-i\epsilon}\right].
\end{aligned} \tag{B2}$$

This becomes

$$\begin{aligned}
&\langle\alpha|\Phi_\alpha^\dagger A\Phi_\beta|\beta_r\rangle \\
&+\langle\alpha|\Phi_\alpha^\dagger A(M_\pi\Phi_\beta - \Phi_\beta m_\beta)|\beta\rangle\left[\frac{1}{m_\alpha-m_\beta+i\epsilon} - \frac{1}{m_\alpha-m_\beta-i\epsilon}\right] \\
&+\langle\alpha|\Phi_\alpha^\dagger A\Phi_\beta|\beta\rangle\frac{m_\beta-m_\alpha}{m_\alpha-m_\beta+i\epsilon}\times \\
&\quad\langle\alpha|(\Phi_\alpha^\dagger M_\pi - m_\alpha\Phi_\alpha^\dagger)A\frac{1}{\bar{m}-M_\pi+i0^+}(M_\pi\Phi_\beta - m_\beta\Phi_\beta)|\beta\rangle \times \left[\frac{1}{m_\alpha-m_\beta+i\epsilon} - \frac{1}{m_\alpha-m_\beta-i\epsilon}\right]
\end{aligned} \tag{B3}$$

which is equal to

$$\begin{aligned}
&\langle\alpha|\Phi_\alpha^\dagger A\Phi_\beta|\beta\rangle\frac{i\epsilon}{m_\alpha-m_\beta+i\epsilon^+} \\
&+\langle\alpha|\Phi_\alpha^\dagger A(M_\pi\Phi_\beta - \Phi_\beta m_\beta)|\beta\rangle\left[\frac{1}{m_\alpha-m_\beta+i\epsilon} - \frac{1}{m_\alpha-m_\beta-i\epsilon}\right] \\
&+\langle\alpha|(\Phi_\alpha^\dagger M_\pi - m_\alpha\Phi_\alpha^\dagger)A\frac{1}{\bar{m}-M_\pi+i0^+}(M_\pi\Phi_\beta - m_\beta\Phi_\beta)|\beta\rangle\left[\frac{1}{m_\alpha-m_\beta+i\epsilon} - \frac{1}{m_\alpha-m_\beta-i\epsilon}\right].
\end{aligned} \tag{B4}$$

The ϵ factors become

$$\frac{i\epsilon}{m_\alpha-m_\beta+i\epsilon} = \delta_{\alpha\beta} \tag{B5}$$

and

$$\left[\frac{1}{m_\alpha-m_\beta+i\epsilon} - \frac{1}{m_\alpha-m_\beta-i\epsilon}\right] = \frac{-2i\epsilon}{(m_\alpha-m_\beta)^2+\epsilon^2} \to -2\pi i\delta(m_\alpha-m_\beta). \tag{B6}$$

The first term vanishes if $m_\alpha \neq m_\beta$ as $\epsilon \to 0$; it becomes 1 when the channels are the same - as a Kronecker delta.

In order to obtain (86) we note that $A^2 = A$ has been used to put A in two places separated by operators that commute with A. Next the second resolvent equations are used to arrive at

$$\begin{aligned} &\frac{1}{m_\beta - M_\pi + i0^+} = \\ \frac{1}{m_\beta - M_{c\pi} + i0^+} + &\frac{1}{m_\beta - M_{c\pi} + i0^+} M_\pi^c \frac{1}{m_\beta - M_\pi + i0^+} \\ &\frac{1}{m_\beta - M_{c\pi} + i0^+}(1 + M_\pi^c \frac{1}{m_\beta - M_\pi + i0^+}). \end{aligned} \tag{B7}$$

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