Relativistic Hamiltonian Dynamics in Nuclear and Particle Physics

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ABSTRACT

This review is intended to provide an introduction to the formulation of relativistic quantum mechanical models, particularly for use in strong interaction problems, whose dynamics is given by a unitary representation of the inhomogeneous Lorentz group. In the first portion, an overview is given in which the properties of these models are defined and some analytically solvable examples are given. This is followed by a deductive construction of these models from physical principles. Particle production, electron scattering, macroscopic locality, and the relation to local quantum field theory are discussed in the second half.

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1. Introduction

Strong-interaction problems in nuclear and particle physics are often formulated in terms of phenomenological models because of the difficulties in formulating convergent approximations in local field theories such as QCD. Phenomenological models are designed to be simple enough that they can be solved accurately and, if they are suitably refined, they can lead to realistic descriptions of physical systems, as is the case in atomic and molecular physics and low energy nuclear physics. For many problems of current interest in nuclear physics these models must be consistent with the principle of special relativity. Relativity is needed to model reactions where particles are produced, reactions involving energy and momentum transfers that are comparable to the mass scales of the problem, bound systems where the binding energies are comparable to the masses of the constituent particles, and coordinate system independent treatments of problems in lepton-hadron scattering.

Relativistic quantum mechanics began with attempts to construct manifestly covariant extensions of the Schrödinger equation. Schrödinger (Sc 26) had already discovered and discarded the Klein-Gordon equation (Kl 26, Go 26a, Go 26b) in his original paper. It was realized early on by Heisenberg, Born and Jordan (Bo 26) that laws of quantum mechanics also should apply to the electromagnetic field, which transforms covariantly under Poincaré transformations. This led to the introduction of the quantum theory of the electromagnetic field (Di 27, He 29, Sc 58). The impressive agreement of the predictions of quantum electrodynamics with experiment, coupled with the realization that a quantized field provided a means for avoiding the concept of instantaneous action at a distance, led to the acceptance of local relativistic field theory as the correct way to model the fundamental interactions of nature at accessible energies.

For the strong interaction, however, *ab initio* calculations based on local field theories are difficult because the infinite number of degrees of freedom and the large coupling constants make it difficult to control the size of the error in any calculation. Field theoretic calculations involve manipulations of a finite number of renormalized Feynman diagrams, using ladder sums (Sa 51) or other techniques. These calculations ignore an infinite number of graphs with large coupling constants and they fail to address the extent to which the terms in the perturbation series define

the dynamics. In addition, most applications in nuclear physics involve composite systems, either of nuclei composed of nucleons, or of nucleons composed of quarks and gluons. The treatment of composite systems in quantum field theories is nonperturbative at the outset. For the case of nucleons as composites of quarks and gluons, the problem is more difficult because the quark and gluon fields do not correspond to observable particles. At present there are no known algorithms for constructing approximate solutions of dynamical problems in strongly interacting quantum field theories with arbitrary precision.

Integral formulations of field theory, such as lattice approximations, (Wi 74) may ultimately lead to computational methods where errors can be controlled. These methods are not developed to the point where they provide sufficient control of computational error to make many useful quantitative statements about nuclear or hadronic dynamics of realistic systems.

In spite of the acceptance of field theories as a matter of principle, most realistic dynamical calculations in nuclear physics, and many in particle physics, utilize the nonrelativistic Schrödinger equation. Nonrelativistic models can be solved using well defined computational algorithms (Fa 65, Ya 67) in which errors can be made as small as desired. In a nonrelativistic approach, one begins with a large class of models, most of which can be discarded upon comparison with experiment. In a field theoretic approach, one begins with a smaller class of models, but in most cases, it is impossible to perform a calculation with errors small enough to discard the model if it is not in agreement with experiment. Although the problem of putting error bounds on field theoretic calculations can be justifiably considered a technical problem, it has resisted solution for over 50 years.

Subsequent to the development of quantum field theory, Wigner (Wi 39) analyzed the mathematical formulation of the physical requirement of special relativity in quantum mechanics. Physical states in quantum mechanics are in one-to-one correspondence with one-dimensional subspaces, or "rays," of the Hilbert space. Wigner showed that a necessary and sufficient condition for quantum mechanical probabilities to have values that are independent of the choice of inertial coordinate system is the existence of a unitary ray representation of the inhomogeneous Lorentz group (Poincaré group) on the quantum mechanical Hilbert space. Wigner's analysis applies both to quantum field theories and to quantum theories of particles, although its application

to theories of particles was not vigorously pursued at the time.

Most of what follows in this review is motivated by five seminal papers that took the work of Wigner to its logical conclusion for systems of interacting particles. First, Dirac (Di 49) formulated the problem of including interactions in relativistic classical mechanics. This was done in Hamiltonian form, which has a natural canonical quantization. Although Dirac did not solve the classical problem, he simplified it to one of several simpler problems. These three different types of solutions to this problem are now called the "point," "instant" and "front" forms of the dynamics. Bakamjian and Thomas (Ba 53) successfully constructed the first relativistic quantum mechanical model of two interacting particles in Dirac's "instant" form of dynamics. Foldy (Fo 61) recognized the importance of macroscopic locality as an additional constraint on these models. This condition replaces the concept of Einstein causality or microscopic locality in local field theories. Coester (Co 65) then extended the work of Bakamjian and Thomas to systems of three particles, with a scattering operator consistent with the principle of macroscopic locality. Finally, Sokolov (So 77) provided the general construction for N particles in a manner consistent with macroscopic locality. These five papers define the scope of this review. Relativistic quantum mechanical models of directly interacting particles have the following features:

- consistency with requirements of relativity and quantum mechanics
- connection between few-body dynamics and the many-body problem
- possibility of composite particles
- large class of permissible interactions
- tractable few-body calculation

We believe that such models are very attractive for a wide variety of applications in nuclear and particle physics.

Relativistic direct interaction theories of particles lie between local field theoretic models and nonrelativistic quantum mechanical models. They are applicable to situations involving larger momentum transfers and binding energies than nonrelativistic models, and they permit the formulation of invariant calculations involving particle production, electromagnetic and weak probes (in the one-boson exchange approximation); none of the latter applications is possible

in nonrelativistic models. They replace the microscopic locality of field theories with a weaker condition, called macroscopic locality, but, unlike field theories, lead to mathematically well defined models where computational error can be controlled. Because of this, they should provide a useful framework for the construction of mathematical models of the dynamics of hadrons and nuclei at intermediate energies.

In comparing the contents of this review to other formulations of relativistic quantum mechanics, it is useful to keep in mind that there are (at least) two ways to consider the formulation of this problem. The most common is to begin with a local relativistic field theory, and truncate the dynamics in such a way that what remains is a closed system of dynamical equations involving a finite number of important degrees of freedom. A second approach is to assume that the system is governed by a finite number of degrees of freedom, and then to construct the most general class of dynamical models with these degrees of freedom, consistent with a set of general principles that include relativistic invariance. In some cases, equations obtained by these two different approaches may be identical, but the emphasis and subsequent application is usually different.

In the first approach, the connection to field theory is emphasized. In general, relations such as the Schwinger-Dyson equations involve an infinite number of coupled amplitudes. Models are constructed by retaining the coupling only among a finite number of amplitudes, or by replacing an unknown amplitude with a phenomenological amplitude. Ladder approximations to the Bethe-Salpeter equation (Sa 51) and approaches based upon mean field theory (Se 86) are typical examples. We refer to these procedures as truncations. In general, truncations are not controlled approximations, and the physical properties of the field theory (*i.e.,* the axioms of field theory) are not necessarily preserved on truncation. The models may violate Poincaré invariance, current covariance, or other symmetries; one must then believe that for a sensible truncation, the corrections needed to restore these symmetries are small.

In the second approach, basic principles are emphasized. The connection to field theory is of secondary concern. In this case, some principles have to be given up in passing from a local field theory to a particle theory, but this is accomplished directly by weakening specific axioms. In this paper the axiom of microscopic locality is replaced by a weaker requirement called

macroscopic locality, which simply means that observables associated with different spacetime regions commute in the limit of large spacelike separation, rather than for arbitrary spacelike separations. The second approach is used in atomic physics and low energy nuclear physics, where the underlying spacetime symmetry is governed by the Galilean group. For relativistic models Poincaré invariance is demanded to be an exact symmetry of the model.

The approach in this paper advocates the second point of view. This approach can be developed from physical principles, and the structure of the models constructed can be shown to follow, up to unitary transformation, as a consequence of these principles. The resulting symmetries relating to relativistic invariance are realized exactly. In Section 10, the connection between models based on this point of view and those based on local field theory is discussed. The first approach is well represented in the existing literature (Sa 51, Bl 66, Gr 82a, Gr 82b).

The purpose of this review is to discuss methods for constructing relativistic quantum mechanical models of particles by the explicit construction of a unitary representations of the Poincaré group on a model Hilbert space. These models are similar to models in nonrelativistic quantum mechanics. Like nonrelativistic models, there exist well defined algorithms for finding solutions of the dynamical equations to any desired precision. Such methods have proven their value in application to few-body systems in atomic and nuclear physics. Unlike the nonrelativistic models, the models constructed in this review are not limited to low energies, or systems that conserve particle number. While the literature on this subject is quite extensive, it is difficult to gain easy access to the necessary tools for doing practical calculations – certainly nowhere near the ease with which one can learn Feynman rules for perturbative field theory. The goal of this review is to provide these tools together in one place with a consistent set of conventions.

It is assumed that the reader is familiar with nonrelativistic quantum mechanics, and has some exposure to basic few-body and many-body quantum mechanics. It is also assumed that the reader is familiar with the language of quantum field theory. The approach in this review is to use ideas from elementary quantum mechanics where possible. The intent is to prepare the reader with sufficient background for digesting papers and properly formulating calculations.

In preparing this review, an attempt has been made to anticipate a variety of backgrounds and interests among the readers. Those with specific interests may find their desired material in

relatively self-contained sections or groups of sections. For example, those who wish to proceed quickly to a point where they can do simple calculations can read the introduction to relativistic quantum mechanics, together with the solvable two-body models, in Section 2. Sections 3–7 present a systematic development of relativistic direct interaction quantum mechanics based on physical principles. The mathematical realization of the physical requirement for relativistic invariance of quantum mechanical models is presented in Section 3, along with a comparison to the corresponding requirements in a Galilean invariant model. Sections 4–7 contain a discussion of the one-body problem, the two-body problem, macroscopic locality, and the three-body problem, respectively. There are separate discussions of models with particle production in Section 8, electromagnetic probes in Section 9, and the relationship between particle dynamics and local field theories in Section 10. We make some general concluding observations in Section 11. There are also three Appendices. The first provides a discussion of scattering theory which is relevant to the relativistic models. The second is a collection of useful formulas for use in a front-form quantum mechanics. The third contains expressions for Racah coefficients of the Poincaré group which are needed to formulate three-body equations as integral equations.

Following are some general comments about our notation:

- We use units such that $\hbar = c = 1$.
- We use the metric $-g^{00} = g^{11} = g^{22} = g^{33} = 1$. Repeated four-vector indices are summed without the presence of a summation sign.
- Expressions involving *definitions* use the symbol :=.
- The commutator of two operators A and B is written as $[A, B]_-$, and the anticommutator as $\{A, B\}_+$ to avoid confusion with other bracketed quantities.
- Summations over angular momentum indices will be denoted by a summation sign $\sum$ without indices, it being implicit that repeated indices are summed. Summations over bound-state spectral indices will be displayed explicitly.
- The energy of a particle of mass m and three-momentum $\mathbf{k}$ is represented as $\omega_m(k) := \sqrt{m^2 + k^2}$.

- We employ *non-covariant* normalization of state vectors:

$$\langle \mathbf{p}' | \mathbf{p} \rangle = \delta(\mathbf{p}' - \mathbf{p}).$$

This means that Lorentz transformations of the states are accompanied by square-root factors.

2. Relativistic Quantum Mechanics: Principles and Examples

The formulation of relativistic quantum mechanics differs from the formulation of nonrelativistic quantum mechanics by the replacement of invariance under Galilean transformations with invariance under Poincaré (inhomogeneous Lorentz) transformations. Poincaré invariant models can be formulated without the use of local quantum fields. A well defined initial value problem is achieved only after a non-trivial implementation of the invariance under Poincaré transformations. The equations that are derived are similar and sometimes even identical to those derived in the nonrelativistic case, but in these two cases the interpretation of the equations differ.

This section is intended as an intuitive introduction to the more formal developments in the sections which follow. First, we discuss the requirements that relativistic invariance imposes on quantum mechanical models. Then, after a brief historical review, we provide some examples which demonstrate how these requirements can be satisfied within the context of Hamiltonian particle dynamics.

2.1. **Relativistic Invariance**

In physical systems, it is observed that there are special coordinate systems in which the laws of physics have a simple form. These are *inertial coordinate systems*, in which the momentum of a non-interacting particle is constant.

Experimentally, it is found that there are many inertial coordinate systems. The principle of relativity states that the laws of physics do not distinguish different inertial coordinate systems. This statement holds in either the Galilean principle of relativity or Einstein's special principle of relativity. The difference between these two principles is the way in which different inertial coordinate systems are related. The Galilean principle of relativity assumes that the coordinate transformations that preserve the form of Newton's laws for a free particle are the coordinate transforms relating different inertial coordinate systems. The special principle assumes that the coordinate transformations that preserve the form of Maxwell's equations for a free electromag-

netic field are the coordinate transforms that relate different inertial coordinate systems. These two characterizations of inertial coordinate systems are not compatible.

The null result in the Michelson-Morley experiment supports the hypothesis of the special principle of relativity, which asserts that any two inertial coordinate systems are related by a point transformation that preserves Maxwell's equations for a free electromagnetic field. These point transformations preserve proper time τ_{AB} between events with spacetime coordinates x^μ_A and x^μ_B, where

$$\tau^2_{AB} := -g_{\mu\nu}(x_A - x_B)^\mu (x_A - x_B)^\nu; \tag{2.1}$$

$$x^\mu_A = (t_A, \mathbf{x}_A); \qquad x^\mu_B = (t_B, \mathbf{x}_B). \tag{2.2}$$

The most general point transformation that preserves τ^2 for all pairs of events has the form

$$x^\mu \to x'^\mu = \Lambda^\mu{}_\nu x^\nu + a^\mu, \tag{2.3}$$

where a^μ is a constant four-vector representing a space-time translation and $\Lambda^\mu{}_\nu$ is a constant matrix that defines a Lorentz transformation:

$$g^{\mu\nu} = \Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma g^{\rho\sigma}. \tag{2.4}$$

The set of transformations of the form (2.3) forms a group under composition called the inhomogeneous Lorentz group or the Poincaré group. The composition of two Poincaré transformations is given by

$$(\Lambda_2, b_2) \circ (\Lambda_1, b_1) = (\Lambda_2\Lambda_1, \Lambda_2 b_1 + b_2), \tag{2.5}$$

with inverse and identity

$$(\Lambda, b)^{-1} := (\Lambda^{-1}, -\Lambda^{-1}b) \qquad I := (I, 0). \tag{2.6}$$

The Poincaré group has four disconnected components that are related by the discrete transformations of space reflection and/or time reversal. The component containing the identity is a

subgroup which is distinguished by the conditions $\det|\Lambda| = 1$ and ${\Lambda^0}_0 \geq 1$. These are called proper ($\det|\Lambda| = 1$) orthochronous (${\Lambda^0}_0 \geq 1$) Lorentz transformations. The remaining components are obtained by applying a time reversal, space reflection, or both, to a proper, orthochronous Lorentz transformation. Although Maxwell's equations are invariant under the full Poincaré group, the weak interaction is not invariant under those Poincaré transformations involving the discrete symmetries of time reversal and/or space reflection. It is customary to consider these discrete symmetries separately from the continuous symmetries. In what follows, relativistic invariance will refer to invariance under Poincaré transformations associated with proper orthochronous Lorentz transformations, or, equivalently, the Poincaré transformations continuously connected to the identity. In all that follows, references to the Poincaré group will mean this subgroup, and it will be assumed that any two inertial coordinate systems are related by a transformation in this subgroup.

The principle of relativity is a statement that there is nothing in the laws of physics that distinguishes different inertial coordinated systems:

> A *system satisfies the principle of special relativity if the results of equivalent experiments done in different inertial coordinate systems are identical. A theory is consistent with the principle of special relativity if the measurable predictions of the theory for equivalent experiments done in different inertial coordinate systems are identical.*

In classical physics, the solutions of the dynamical equations are observable. A classical theory is relativistically invariant if the solution of a Poincaré transformed equation is identical the Poincaré transformed solution of the original equation. This will follow if the equation transforms covariantly under the action of the Poincaré group.

In quantum physics, some modifications are required, because the measurable quantities are not the solutions of the Schrödinger equation, but are instead probabilities constructed from scalar products of two solutions of the Schrödinger equation. In general, the solutions of the Schrödinger equation can be transformed by a large class of unitary transformations that change the form of the equations and solutions but leave the probabilities unchanged. Clearly, invariance under change of representation does not change the physics. Thus, in formulating the principle of special relativity in quantum mechanics, it is appropriate to demand that the physically measurable

quantities, *i.e.,* the probabilities associated with an isolated system, cannot be used to distinguish different inertial coordinate systems. This is the point of view taken by (Wi 39) and refined by Bargmann (Ba 54).

Before discussing Wigner's formulation of relativistic invariance in quantum theories, it is useful to give a brief historical review of relativistic quantum mechanics.

2.2. **Historical Perspective**

Almost as soon as Heisenberg and Schrödinger formulated nonrelativistic quantum mechanics, considerable effort was aimed at finding a suitable relativistic quantum theory. The relativistic Schrödinger equation (Su 63, Du 83, Fr 83) is obtained from the correspondence principle by replacing the nonrelativistic relation between energy and momentum with the corresponding relativistic relation. The result for a free particle is:

$$i\frac{\partial}{\partial t}\,\psi = \sqrt{-\nabla^2 + m^2}\,\psi. \tag{2.7}$$

Although this equation is acceptable in principle, it was discarded because the square root in the kinetic energy operator was difficult to utilize with interactions, and because of the non-symmetric treatment of space and time.

These objections can be overcome by squaring Eq. (2.7), which gives the Klein-Gordon equation (Sc 26, Kl 26, Go 26a, Go 26b, Fo 26a, Fo 26b, Ku 26, Do 26):

$$(\frac{\partial}{\partial x_\mu}\frac{\partial}{\partial x^\mu} - m^2)\psi = 0. \tag{2.8}$$

The Klein-Gordon equation for a particle in a Coulomb field originally appeared in Schrödinger's 1926 paper (Sc 26), but was discarded because it does not generate the experimentally observed fine structure splitting in the Hydrogen spectrum. The Klein-Gordon equation has well known problems. Because it is a second order equation in the time variable, probabilities constructed out of the wave functions are not conserved in time. In addition, the energy spectrum is not bounded from below. More recent investigations show that the predictions of the Klein-Gordon equation are in good agreement with the experimental spectrum of mesonic atoms (De 79, Wo 80, Fr 83).

The Dirac equation (Di 28):

$$(i\gamma^\mu \partial_\mu + m)\psi = 0, \tag{2.9}$$

was designed to give a symmetric treatment of the space and time derivatives (for covariance) and have probabilities that are conserved in time. Solutions to the Dirac equation for an electron in the Coulomb field of a proton are in good agreement with the experimentally measured fine structure splitting of the spectrum of the Hydrogen atom. The Dirac equation has an energy spectrum that is not bounded from below, but this was fixed with Dirac's "hole theory" (Di 30), which predicted the existence of positrons. In spite of the difficulties with the energy spectrum, following the discovery of positrons by Anderson in 1932 (An 32), the Dirac equation was believed to be the correct equation for treating relativistic quantum mechanics until Pauli and Weisskopf (Pa 34) reinterpreted the Klein-Gordon equation as an equation for a quantized field.

The quantization of the Electromagnetic field was motivated by physical consideration. Dirac 1927 (Di 27) gave the first quantum mechanical treatment of a particle interacting with a quantized electromagnetic field. Heisenberg and Pauli (He 29) were the first to attempt to quantize the full electromagnetic field.

The theory of quantized fields developed in the following years. The theory of quantized fields allowed one to eliminate the concept of instantaneous action at a distance. The mathematical difficulties with the theory made its acceptance slow. The acceptance of quantum electrodynamics is in part due to the agreement of the calculated magnetic moment of the electron (Sc 48) and the Lamb shift (Be 47) with the experimentally measured values (Fo 48, La 47). These calculations, and confidence in the theory, have steadily improved over the subsequent years. The experimental success of perturbative quantum electrodynamics has shown that quantum field theory with weak coupling can provide a quantitative description of dynamics at currently accessible energies. Although systematic expansions for physical observables can be found in perturbative quantum electrodynamics, it is still not known how to give the complete mathematical interpretation of the theory.

Today, especially with the discovery of asymptotically free (Po 73, Gr 73a, Gr 73b, Gr 74) non-Abelian gauge theories (Ya 56), most physicists believe that the laws of physics at currently

accessible energies are governed by local relativistic field theories. At the same time, there are no known non-trivial examples of field theories in 3+1 dimensions satisfying all of the physical properties (axioms) expected of a local relativistic quantum field. What this means in practice is that there are no known algorithms with *ab initio* error bounds that allow one to find a solution of the field equations to arbitrary accuracy. Even in quantum electrodynamics, our confidence comes from the comparison of theory with experiments, and not from a thorough understanding of the theory. Physical arguments suggest that the radius of convergence (in the coupling constant α) of the perturbation series in quantum electrodynamics is zero (Dy 52). For quantum electrodynamics, these deficiencies are largely academic; however, for models of the strong interactions, perturbative quantum field theory is manifestly inadequate.

Because the theoretical foundations of field theory have never been under complete control, there has always been activity involved with the formulation of alternative methods for constructing models of physical systems that combine relativity and quantum mechanics. Although one can take the point of view that these alternative models should be considered at a fundamental level, one must then explain any differences between the predictions of such models with those of quantum electrodynamics. Alternatively, one does not have to consider these models as fundamental; they can be considered as phenomenologies that properly combine the principles of quantum mechanics and relativity. It only needs to be demonstrated that such a phenomenology provides a good quantitative description of the physics for a sufficiently large class of systems under a sufficiently large class of conditions. For instance, one goal would be to find a model of nucleon-nucleus scattering for all target nuclei for all energy transfers below 1 GeV. Nonrelativistic quantum mechanics with phenomenological nucleon-nucleon interactions provides such a model that is valid for energy transfers below the threshold for pion production. Relativity must be included for higher energy transfers.

It is difficult to give a complete review of all of the alternative methods that have been proposed to construct relativistic quantum mechanical models. This is in part because there are so many different starting points and approaches which make it difficult to compare different models. One thing that can be said at the outset is that in the same way that there are many quantum theories with the same classical limit, there are also many relativistic models with the same nonrelativistic limit. This means that the concept of a relativistic correction is meaningful

only within the context of a given model. Instead of giving a historic account of the various attempts to construct quantum mechanical models, we give a brief discussion of some of the fundamental issues involved in the development of such models, and a discussion of some of the attempts used to deal with these issues. This will hopefully make it easier for the reader to make a critical analysis of various approaches.

1. **Quantum mechanics:** This requires a linear theory, to ensure the superposition principle, formulated on a Hilbert space. States of the system are represented by one dimensional subspaces, or "rays," of the Hilbert space. The square magnitude of the inner product between two normalized state vectors represents the probability that if the system is prepared in the state represented by one of the vectors that it will be measured to be in the state represented by the second vector. There are two relevant comments. One is that the underlying Hilbert space can be quite abstract, which turns out to be the case in local field theories (St 64) and covariant quantum mechanical models (Po 85a). The second is that the more general algebraic formulation of quantum mechanics (Vo 36) is relevant for treating systems with an infinite number of degree of freedom (Ha 64).

   One instance where questions about quantum mechanics become tricky occurs when approximate sets of equations for objects such as transition operators and Green functions are obtained by truncation. In applications, equations such as Bethe-Salpeter equations (Sa 51), Blankenbecler-Sugar (Bl 66) equations and the Gross equation (Gr 82a, Gr 82b), are normally formulated with a kernel which, although motivated by field theory, is phenomenological. Such phenomenological equations are called quasipotential equations, and they are used for realistic calculations of the energy levels in positronium (Lo 63a, Lo 63b, To 73). The solution of these equations is usually an observable of interest, and for such applications, no further analysis is required. If one wants to compare these models to other approaches, however, it is natural to want to be able to reconstruct the underlying quantum model, *i.e.,* the model Hilbert space and a unitary time translation operator, and to extract either sufficient or necessary conditions on the structure of the input to allow such a reconstruction. For the case of Green functions, the problem is to use time ordered Green functions to construct the non-ordered Green functions (Wightman functions), which can then be used to construct the quantum mechanics using the reconstruction theorem (St 64),

or the retarded Green functions, which can be used to reconstruct the field (Gl 57).

2. **Relativistic Invariance:** Relativity requires the existence of inertial coordinate systems and physical equivalence of coordinate systems related by Lorentz transformations and space-time translations. Wigner (Wi 39) analyzed this requirement for the case of quantum mechanics and found that it is equivalent to the existence of a unitary ray representation of the Poincaré group (inhomogeneous Lorentz group) on the quantum mechanical Hilbert space. Note that the Schrödinger equation and the existence of the Hamiltonian are consequences of applying Wigner's analysis to invariance under time translations. The beauty of Wigner's theorem is that it is an inescapable consequence of the invariance of probabilities under changes of inertial coordinate system.

   Wigner's theorem applies both to quantum theories of fields and of particles. It can be satisfied in a variety of ways. One way is to construct an explicit representation of a unitary representation of the Poincaré group on the quantum Hilbert space. A second way is to construct a representation of the infinitesimal generators of the Poincaré group, which are self-adjoint operators that satisfy commutation relations characteristic of the group. In canonical field theories, these expressions are normally generated by integrating the energy-momentum and angular momentum tensors over a suitable three dimensional surface (Sc 62, Ch 73). A third way is to require that Poincaré transformations are implemented by manifest covariance, and find a representation of the Hilbert space for which these Poincaré transformations are unitary. This approach is used in axiomatic field theory (St 64), covariant constraint dynamics (Lo 87, Po 85, Ri 85, Sa 86a, Sa 86b, Sa 88), and any non-trivial manifestly covariant quantum theory.

Beyond these minimal requirements, there are many other important issues which are listed below:

1. **The spectral condition:** This requires that the Hamiltonian of the theory has an energy spectrum bounded from below. The invariance of quantum mechanical probabilities in time implies the existence of a Hamiltonian. The spectral condition is essential for the theory to be stable against spontaneous decay. The spectral condition is relevant because it is clearly violated by the Klein-Gordon equation and the Dirac equation; although it is resolved in the

Dirac case by "hole theory."

2. **Einstein causality:** This is also referred to as microscopic locality. It assumes the existence of observables associated with arbitrarily small regions of spacetime, and the ability to make independent measurements of any two such observables in causally disconnected regions. It is this condition that requires an infinite number of degrees of freedom (*i.e.,* independent observables for every spacetime volume). This condition is independent both of relativistic invariance and of the existence of a well defined initial value problem. When combined with conditions imposed by relativity, quantum mechanics and the spectral condition, one obtains the core of the "axioms of local quantum field theory." Although there are many sets of axioms (St 64, Ha 64, Os 75a, Os75 b, Ne 73, Fr 74), these are the main physical assumptions that appear either directly or as consequences of any set of axioms. What is relevant is that except for the case of the free fields, there are no known non-trivial models in 3+1 dimensions that satisfy any of these sets of axioms. In this sense, Einstein causality requires a local field theory.

   Einstein causality is an idealization of a sensible macroscopic condition to arbitrarily small volumes of spacetime. It is interesting to question whether this condition can be tested by experiment. In order to make measurements in arbitrarily small laboratories, one needs to transfer larger and larger momenta to the system. This requires a knowledge of asymptotic properties of the theory at arbitrarily high energy and momenta. Tests of locality thus involve probing models at all energy scales, which in turn requires an infinite number of experiments. On the other hand, for each finite scale, one can argue that only a finite number of quantum mechanical degrees of freedom is relevant. Haag and Swieca (Ha 65) showed that for local fields with a particle interpretation, there is a finite number of quantum mechanical degrees of freedom associated with any finite volume of classical phase space. This can be interpreted to mean that experiments that probe any finite volume of classical phase space cannot distinguish a model with a finite number of degrees of freedom from a local field theory. These consideration suggest that Einstein causality cannot be tested by experiment, although the axioms of field theory imply (St 64) that if it is satisfied up to a certain minimal distance in all coordinated systems, then it must be valid for all distances.

It should be emphasized that there are many observed consequences of microscopic locality, such as crossing symmetry, PCT, existence of antiparticles, *etc.* Although these properties suggest an underlying local theory, all of them can be satisfied in nonlocal models of a finite number of degrees of freedom.

3. **Macroscopic locality:** This is sometimes called cluster separability. It is the experimentally relevant part of Einstein causality. It assumes that observables associated with regions of spacetime that have a sufficiently large (as opposed to arbitrarily small) spacelike separation commute. This condition can be realized independently of Einstein causality (Os 74a). The relevance of this condition was emphasized by Foldy (Fo 61, Fo 74). If the scattering operator is considered to be the fundamental observable of the model, it requires cluster properties of the scattering matrix (Co 65). This is essential in order to provide the connection between few- and many-body physics. For systems of more than two particles, it is known that in certain formulations of relativistic quantum mechanics, macroscopic locality and the existence of a non-trivial scattering theory are manifestly incompatible (Mu 78).

4. **Relativistic Scattering theory:** This means that the S matrix should be relativistically invariant. Fong and Sucher (Fo 64) exhibited a counterexample which shows that relativistic invariance of the model does not imply relativistic invariance of the scattering matrix. They also gave sufficient conditions for a model to have a Poincaré invariant scattering matrix. The problem is that the scattering asymptotic conditions must also be formulated in an invariant way.

5. **Connection to Classical Physics:** There are a number of issues here that have led to many difficulties. The simplest expectation is that the classical limit of a relativistic quantum mechanical model should be relativistic classical mechanics. The difficulties start with defining relativistic classical mechanics:

   5.1 **Canonical formulation:** Dirac (Di 49) outlined a canonical approach to relativistic classical mechanics, in which the Lie algebra of the Poincaré group is realized in terms of Poisson brackets. Poincaré transformations are then implemented, at least locally, in terms of canonical transformations.

   5.2 **World line condition:** One classical notion, known as the *world line condition,* is

that the spacetime coordinates of classical particles should transform as four-vectors. It turns out that this is incompatible with the existence of interactions in Dirac's canonical formulation of classical mechanics. This is the conclusion of the "no-go" theorem of Currie, Jordan and Sudarshan (Cu 63a, Cu 63b). The content of this theorem is that there are three sets of Poisson bracket relations: those required by relativity (Di 49), those involving the generalized coordinates and momenta, and those that define the world line condition (Cu 63a, Cu 63b). These relations can only be satisfied simultaneously if the model has no interactions. Because of this, there does not appear to be a classical limit of the quantum theory with all of the desired properties. This problem has led to an industry that has tried various ways to get around this problem in the classical case, and to construct quantum mechanical models with these limits. Classical many-time equations (Va 65) provide one approach, but these equations are not canonical, and consequently are difficult to quantize. Another approach, called covariant constraint dynamics, uses Dirac's generalized classical mechanics for constrained Hamiltonian system (Di 50), and is reviewed in (Lo 87). The quantum mechanical version can be realized in a Hilbert space setting by using the constraints to define a scalar product on the Hilbert space (Po 85a, Ri 85, Sa 86a, Sa 86b, Sa 88). In this case, the scalar product is described by a non-trivial kernel that plays the same role in quantum constraint dynamics as the Wightman functions (Wi 65a) of quantum field theory. The kernels should be consistent with the all of the axioms of the field theory except the locality axiom (Po 85a). The main difficulties occur in simultaneously demanding both macroscopic locality and the spectral condition.

5.3 **Covariance:** Covariance of classical wave equations is a consequence of relativistic invariance if the solutions are classical observables. For Maxwell's equations, the electric and magnetic fields must transform as a rank-two antisymmetric tensor density. The vector potential, which is not observable, may be subject to non-covariant gauge conditions. The same comment applies to quantum theories: if the solutions of a quantum mechanical equation are observable, then they should transform covariantly, and if they are not, then covariance is not required. This has been the source of some confusion. In Dirac's 1927 paper (Di 27), covariance was an essential element of the derivation,

although the solutions were interpreted as wave functions, which are not observable. It is clear that one can construct unitary operators that transform the solutions and the equation in a manner that destroys the covariance without changing the physics. In spite of this observation, the focus historically of much work on relativistic quantum mechanics has been one of maintaining covariance.

There are two different type of objects that transform covariantly in field theories. The first class of objects involve matrix elements of covariant field operators between physical states. This class includes current matrix elements, Bethe-Salpeter wave functions (Sa 51) , Blankenbecler-Cook wave functions (Bl 60), and N-quantum amplitudes (Gr 65a). These objects, which are sometime called wave functions, do not have the usual quantum mechanical interpretation of a wave function, in the sense that there is no scalar product for which these objects can be interpreted as vectors in a Hilbert space. Nevertheless, matrix elements of operators between the corresponding physical states can be expressed in terms of bilinear forms involving these matrix elements (Ma 55, Hu 75).

The second class of covariant objects involves covariant wave functions that are true wave functions, in the sense that there exists an inner product as discussed above. These objects are computed in covariant constraint dynamics (Lo 87, Ri 85, Sa 86a, Sa 86b, Sa 88). In this case, the covariance requires an interaction dependent scalar product (Po 85a).

5.4 **Retardation:** This is the classical statement of Einstein causality. In a quantum theory, it does not imply microscopic locality, and is not needed for macroscopic locality. The first of these statements appeals to the axioms of field theory, where it is possible to construct models with retardation that are Poincaré invariant and satisfy the spectral condition, but which violate some consequence of the full theory, such as crossing symmetry. In this case, one of the axioms must be violated, and microscopic locality is usually the troublesome axiom. Likewise, there exist models with instantaneous direct interactions (*i.e.,* that violate retardation) which are consistent with macroscopic locality. The coordinates that are retarded in interactions are free-particle coordinates;

this is simply a choice of representation in a quantum mechanical theory. All that is relevant about this representation is that these free particles have an asymptotic interpretation, so that they can be used to formulate asymptotic conditions in a scattering theory. The physics is not affected by unitary transformations that become the identity asymptotically, but destroy the retardation in the interaction region.

The one unifying principle in all relativistic formulations is Wigner's (1939) theorem. In a quantum mechanical setting it gives a precise mathematical formulation of the invariance of quantum probabilities under change of inertial coordinate systems. Wigner defined a quantum theory to be Poincaré invariant if and only if all quantum mechanical probabilities have values that are independent of the choice of inertial coordinate system. A precise mathematical characterization of this condition is given by the following theorem (Wi 39):

Theorem: (Wigner) *A quantum mechanical model formulated on a Hilbert space preserves probabilities in all inertial coordinate systems if and only if the correspondence between states in different inertial coordinate systems can be realized by a unitary ray representation* $U(\Lambda, a)$ *of the Poincaré group.*

This result applies both to quantum field theories and to particle theories. Quantum field theories have additional properties. Most notable of these is microscopic locality, which implies a model with an infinite number of degrees of freedom, since it requires the existence of independent observables associated with each arbitrarily small region of spacetime.

To understand the mathematical implementation of Wigner's theorem, let $\mathcal{H}$ be the model Hilbert space, and let X and X' be two inertial coordinate systems related by a Poincaré transformation (Λ, a). Consider an experiment viewed by an observer in X where a system is initially prepared in a state $|\psi\rangle$ and detectors are prepared to measure the probability that the system is in a state $|\phi\rangle$. For states normalized to unity, the probability that this system is measured to be in the state $|\phi\rangle$ is:

$$P = |\langle\psi|\phi\rangle|^2. \tag{2.10}$$

Consider an equivalent experiment done by an observer in X'. In this case, the system is initially prepared in the state $|\psi'\rangle$ and detectors are prepared to measure the probability that the system

is in the state $|\phi'\rangle$:

$$P' = |\langle\psi'|\phi'\rangle|^2. \tag{2.11}$$

Relativistic invariance demands that $P = P'$, that is, the probability of obtaining this result should be independent of where the laboratory is located, when the experiment is performed, which direction the apparatus in oriented, and how fast the laboratory is moving relative to another inertial coordinate system. Wigner's theorem implies that if the above applies to all pairs of (normalizable) vectors and all inertial coordinate systems, then the vectors $|\psi\rangle$ and $|\psi'\rangle$ (resp. $|\phi\rangle$ and $|\phi'\rangle$) are related by a correspondence of the form

$$|\psi'\rangle = e^{i\theta}U(\Lambda, a)|\psi\rangle, \tag{2.12}$$

where $U(\Lambda, a)$ is unitary and satisfies the group representation property:

$$U(\Lambda_2, a_2)U(\Lambda_1, a_1) = e^{i\phi(2,1)}U(\Lambda_2\Lambda_1, a_2 + \Lambda_2 \cdot a_1). \tag{2.13}$$

The phases appear because physical states are only defined up to phase. Bargmann gave a refinement of Wigner's theorem that removes the phase in (2.13). This is discussed in Section 3.

The specification of $U(\Lambda, a)$ on $\mathcal{H}$ defines a relativistic quantum mechanical model. Note that $U(\Lambda, a)$ contains the time evolution subgroup, and thus includes the dynamics. In relativistic quantum mechanics, the construction of $U(\Lambda, a)$ replaces the construction of the unitary time evolution operator in nonrelativistic quantum mechanics.

The construction of relativistic quantum mechanical models of interacting particles was not vigorously pursued after Wigner stated his theorem. On the other hand, as phenomenologies, they fill in a gap that exists between the nonrelativistic quantum mechanics and local relativistic field theories. The construction of $U(\Lambda, a)$ is necessarily more complicated than the construction of the time evolution operator in nonrelativistic quantum mechanics, as will be discussed below.

One non-trivial consequence of special relativity is that it involves the Poincaré group rather than the Lorentz group. The complicating feature is that the Poincaré group includes the time evolution subgroup. Because time is also involved in Lorentz transformations, this requires that

$U(\Lambda, a)$ involves the dynamics in a non-trivial way. The simplest way to appreciate the problem is to note that if a Lorentz transformation is applied to a four-vector, both the space and time coordinate will change. Any change in the spatial coordinates can be undone by a spatial translation. What remains is a pure time translation. Thus, time evolution can be expressed entirely using Lorentz transformations and space translations. If the particles are not free, then time evolution involves interactions which by consistency must be contained in the combined operations of Lorentz transformation and spatial translation. This is illustrated graphically in the figure below:

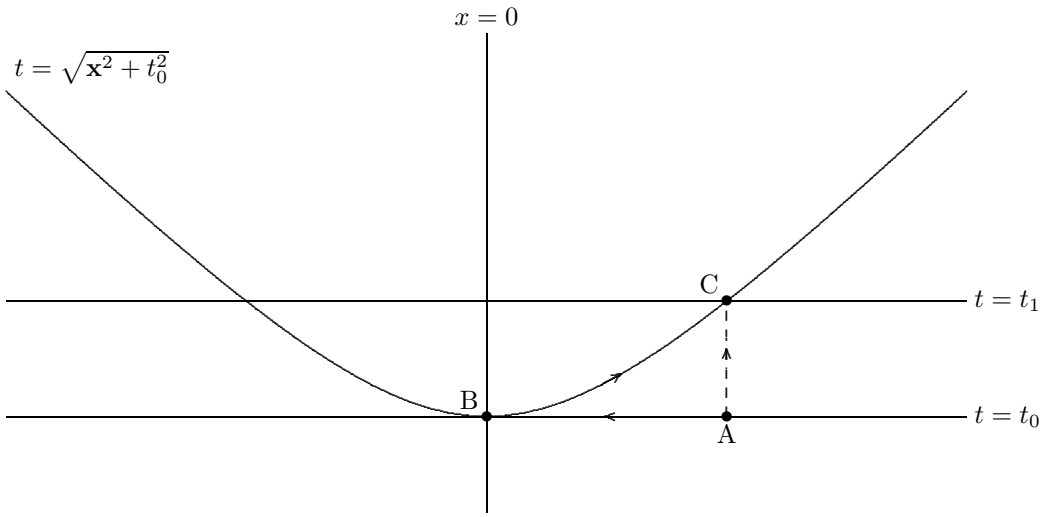

fig. 1

In this diagram, the surfaces $t = t_0$ and $t = t_1$ represent two different constant time surfaces. The surface defined by $t = \sqrt{\mathbf{x}^2 + t_0^2}$ is the surface generated by applying all possible Lorentz transformations to the point $(t_0, 0, 0, 0)$. This surface has the property that any two points on the surface are related by a Lorentz transformation. Time evolution connects point A on the initial surface to point C on the future surface $t = t_1$. It is also possible to get to point C on the future surface by first using space translations to get to point B, and then using a Lorentz transformation to go from B to C. Since the result of time translating from A to C depends on the interactions in the Hamiltonian, consistency of the initial value problem requires that the combined operations of space translation followed by a Lorentz transformation must also depend on the interaction. The problem of constructing a relativistic quantum model is to construct a

unitary representation of the Poincaré group consistent with time evolution in the sense discussed above.

Note also from Fig. 1 that there are many different possible formulations of the initial value problem in relativistic theories. In the diagram, both the fixed time surface and the Lorentz invariant surface have the property that all points on each surface have a relative space-like separation and each surface intersects every possible world line once and only once. Clearly, either one, but not both, of these surfaces would make a suitable initial value surface. In general there is an infinite number of possible initial value surfaces. The interaction dependence of the representation of the Poincaré group ensures the compatibility of these choices. These complications do not arise nonrelativistically because in that case, with the speed that defines the "light cone" unbounded, the constant time surface is the only surface on which all points have a relative space-like separation.

To summarize, experimentally it is found that there are coordinate systems where free particles move with constant linear momentum and that any two such systems are related by a Poincaré transformation. The laws of physics are blind to the choice of inertial coordinate systems. In quantum theories, it is found that this is true if and only if there exists a unitary ray representation of the Poincaré group on the quantum mechanical Hilbert space.

Direct construction of models based on these ideas was initiated by Dirac's work on the Hamiltonian formulation of the classical dynamics of particles (Di 49). Classical Hamiltonian systems have the advantage that they can be used to construct quantum systems by canonical quantization. In the Hamiltonian formulation of classical mechanics, the goal is to construct a representation of the Poincaré group as a group of canonical transformations on phase space. Dirac analyzed these requirements infinitesimally, and showed that the problem is equivalent to the construction of a representation of the Lie algebra of the Poincaré group in terms of Poisson brackets that includes interactions in a consistent way. Although Dirac did not solve the problem, he did show that it could be reduced to one of three simpler non-linear problems. These simpler realizations are based upon the observation that the Poincaré group has several subgroups (Pa 75, Le 78) that do not involve the Hamiltonian explicitly. The Lie algebra can be constructed consistently by assuming that the generators of one of these subgroups contains no interactions.

The general problem is non-linear, since interactions must be added to all 10 generators in a manner that preserves the Lie algebra.

These three methods of realizing the Lie algebra are called Dirac's forms of the dynamics. They go by the name "instant," "front" and "point" forms. The names are characteristic of the subgroups that are chosen to contain no interactions. In the "instant" form, it is the set of Poincaré transformations which leave the instant plane, $t = t_c$, invariant. In the "point" form, it is the subgroup that leaves the Lorentz invariant surface $t^2 - \mathbf{x}^2 = c^2$ invariant. In the "front" from, it is the subgroup which leaves the light front, $x^+ = x^0 + x^3 = 0$, invariant. A more complete discussion can be found in Dirac's original review and a paper by Leutwyler and Stern (Le 78).

The first construction of a relativistic quantum mechanical model of two interacting particles based on these ideas was given by Bakamjian and Thomas (Ba 53). The non-linear problem defined by Dirac is solved in the quantum mechanical case by realizing the Lie algebra of the Poincaré group in terms of commutators of Hermitian operators rather than Poisson brackets of functions on classical phase space. Bakamjian and Thomas used Dirac's "instant form" of the dynamics. Foldy (Fo 61) pointed out the importance of cluster separability or macroscopic locality in these models.

The work of Bakamjian and Thomas was extended to the case of three particles by Coester (Co 65). Sokolov (So 75, So 77, So 78a, So 78b) extended this work to the "point form" and "front form" of the dynamics, the N-body problem with cluster properties, and models involving particle production. Leutwyler and Stern (Le 78) formulated the two-body problem in the "front form." Coester and Polyzou (Co 82) formulated models in all three forms, with cluster properties for arbitrary numbers of particles, and a limited class of models with particle production. Polyzou (Po 89) has extended the Bakamjian-Thomas construction for two particles to a general group-theoretic setting for which Dirac's forms of the dynamics appear a special cases of a general construction. Lev (Le 83) has reviewed the three-body problem in the front form . Three-body calculations have been discussed in the instant form by Glöckle *et al.* (Gl 86), and in the front form by Kondratyuk and Terent'ev (Ko 80), by Bakker, *et al.* (Ba 79), and by Cao and Keister (Ca 90). Meson form factors in quark/parton models have been studied by Terent'ev (Te 76),

by Chung, *et al.* (Ch 86, Ch 88b), and by Dziembowski (Dz 88b). Meson-nucleon scattering with pion absorption has been examined by Berestetskiĭ (Be 81). Various aspects of electron-deuteron elastic scattering and breakup have been studied in the instant form by Coester and Ostebee (Co 75), and in the front form by Kondratyuk and collaborators (Ko 83, Ko 84, Gr 84), by Chung and collaborators (Ch 88a, Ch 89) and by Keister (Ke 88). Related front-form work based upon field theory can be found in the review of Frankfurt and Strikman (Fr 81). Nucleon form factors have been examined in the front form by Berestetskiĭ and Terent'ev (Be 76, Be 77), by Dziembowski (Dz 88a, Dz 88b), and by Chung and Coester (Ch 90). Nuclear structure functions have been reviewed by Berger and Coester (Be 85, Be 87).

Two examples of the Bakamjian-Thomas construction are given in the next two sections. One is a model of confined quarks in the "instant" form, and the other is a model of nucleon-nucleon scattering in the "front" form. In each case, the model is constructed first for spinless constituents, and then extended to spin-$\frac{1}{2}$ constituents. For both examples, the models are analytically solvable.

2.3. **Example: Confined Spinless Quarks**

We now consider a simple model of two spinless quarks of mass m with a confining interaction. The first step is to determine the mass spectrum. In quantum mechanics, this is done by computing the eigenvalues of a self-adjoint operator that represents the mass, or rest energy, of the bound system. The mass M plays the same role in relativistic models as the internal Hamiltonian h in nonrelativistic models.

The basis states of the Hilbert space for this model can be taken as the tensor products of single-particle states:

$$|\mathbf{p}_1\,\mathbf{p}_2\rangle := |\mathbf{p}_1\rangle \otimes |\mathbf{p}_2\rangle. \tag{2.14}$$

Equivalently, we can change variables and use state vectors labeled by the total momentum $\mathbf{P}$ and relative momentum $\mathbf{k}$ of two free quarks. The total momentum is defined as

$$\mathbf{P} := \mathbf{p}_1 + \mathbf{p}_2. \tag{2.15}$$

To construct the relative momentum vector $\mathbf{k}$, let $L_c^{-1}(Q)$ be the rotationless Lorentz transformation that transforms the momentum of two non-interacting quarks to zero. The relative momentum is defined as the three-vector components of

$$k^\mu := L_c^{-1}(Q_0)^\mu{}_\nu p_1^\nu. \tag{2.16}$$

The rotationless Lorentz transformation $L_c^{-1}(Q)$ is defined by its action on a four-vector A^μ:

$$\begin{pmatrix} A'^0 \\ \mathbf{A}' \end{pmatrix} = L_c^{-1}(Q) \begin{pmatrix} A^0 \\ \mathbf{A} \end{pmatrix} = \begin{pmatrix} A^0\sqrt{1+\mathbf{Q}^2} - \mathbf{Q}\cdot\mathbf{A} \\ \mathbf{A} - A^0\mathbf{Q} + \mathbf{Q}\,(\mathbf{Q}\cdot\mathbf{A})(1+\sqrt{1+\mathbf{Q}^2})^{-1} \end{pmatrix}. \tag{2.17}$$

The quantity $Q_0 = P/M_0$ is the four-velocity of the non-interacting system, where

$$M_0 := (H_0^2 - \mathbf{P}^2)^{\frac{1}{2}}; \qquad H_0 := \omega_m(\mathbf{p}_1) + \omega_m(\mathbf{p}_2), \tag{2.18}$$

and

$$\omega_m(\mathbf{p}) := \sqrt{m^2 + \mathbf{p}^2}. \tag{2.19}$$

The three-vector part of k is then given by the variable change

$$\mathbf{k} = \mathbf{k}(\mathbf{p}_1, \mathbf{p}_2) = \mathbf{p}_1 + \frac{\mathbf{P}}{M_0}\left[\frac{\mathbf{P}\cdot\mathbf{p}_1}{M_0 + H_0} - \omega_m(\mathbf{p}_1)\right]. \tag{2.20}$$

If the plane-wave states are given delta function normalizations, *i.e.,*

$$\langle \mathbf{p}_1'\, \mathbf{p}_2' | \mathbf{p}_1\, \mathbf{p}_2\rangle = \delta(\mathbf{p}_1' - \mathbf{p}_1)\delta(\mathbf{p}_2' - \mathbf{p}_2); \quad \langle \mathbf{P}'\, \mathbf{k}' | \mathbf{P}\, \mathbf{k}'\rangle = \delta(\mathbf{P}' - \mathbf{P})\delta(\mathbf{k}' - \mathbf{k}), \tag{2.21}$$

then the state vectors $|\mathbf{P}\,\mathbf{k}\rangle$ and $|\mathbf{p}_1\,\mathbf{p}_2\rangle$ are related to each other via

$$|\mathbf{P}\,\mathbf{k}\rangle := \left|\frac{\partial(\mathbf{p}_1\,\mathbf{p}_2)}{\partial(\mathbf{P}\,\mathbf{k})}\right|^{\frac{1}{2}} |\mathbf{p}_1\,\mathbf{p}_2\rangle \tag{2.22}$$

where the Jacobian in Eq. (2.22) is

$$\left|\frac{\partial(\mathbf{p}_1\,\mathbf{p}_2)}{\partial(\mathbf{P}\,\mathbf{k})}\right| = \frac{\omega_m(\mathbf{p}_1)\omega_m(\mathbf{p}_2)M_0}{\omega_m(\mathbf{k})\omega_m(\mathbf{k})\omega_{M_0}(\mathbf{P})}. \tag{2.23}$$

The angles $\hat{\mathbf{k}}$ can be eliminated in favor of discrete quantum numbers using spherical harmonics:

$$|kl; \mathbf{P}\mu\rangle := \int d\hat{\mathbf{k}}\, Y_\mu^l(\hat{\mathbf{k}})|\mathbf{k}; \mathbf{P}\rangle. \tag{2.24}$$

The Hilbert space then consists of the set of functions $\langle kl; \mathbf{P}\mu|\Psi\rangle$, with scalar product

$$\langle\Psi|\Phi\rangle := \sum_{l=0}^{\infty}\sum_{\mu=-l}^{l}\int d^3P \int_0^\infty k^2 dk\, \langle kl; \mathbf{P}\mu|\Psi\rangle^*\langle kl; \mathbf{P}\mu|\Phi\rangle, \tag{2.25}$$

where $\langle \mathbf{P}\,\mathbf{k}|\Psi\rangle$ satisfying

$$\langle\Psi|\Psi\rangle < \infty. \tag{2.26}$$

The mass operator M is assumed to be the sum of an operator M_0 that represents the

invariant mass of two non-interacting quarks, plus a phenomenological confining interaction:

$$M = M_0 + U. \tag{2.27}$$

For computational purposes, it is convenient to write equation (2.27) in the form

$$M^2 = M_0^2 + V, \tag{2.28}$$

where U and V are related by

$$V = \{M_0, U\}_+ + U^2. \tag{2.29}$$

Equations (2.27) and (2.28) are equivalent: the choice between them is a matter of convenience. To be consistent with experimental observation, the interaction must chosen so that M has only positive eigenvalues.

In the basis (2.24), the non-interacting Hamiltonian and mass operator are multiplication operators:

$$H_0(\mathbf{P}, \mathbf{k}\,) := \sqrt{M_0^2 + \mathbf{P}^2}; \qquad M_0 = M_0(\mathbf{k}\,) := 2\sqrt{m^2 + \mathbf{k}^2}. \tag{2.30}$$

The eigenvalue problem for the square of the mass is:

$$\langle kl; \mathbf{P}\mu | 4(m^2 + \mathbf{k}^2) + V | \mathbf{P}\,'\Psi \rangle = \lambda^2 \langle kl; \mathbf{P}\mu | \mathbf{P}'\Psi \rangle. \tag{2.31}$$

In applications, the interaction V, or equivalently U, is determined by the physics of the system being modeled. An analytically solvable choice is to take V so that in the representation (2.25), Eq. (2.31) is equivalent to the eigenvalue problem for a harmonic oscillator. The following V has this property:

$$\langle k'l'; \mathbf{P}'\mu' | V | kl; \mathbf{P}\mu \rangle := -\frac{1}{g^4} \delta_{l'l} \delta_{\mu'\mu} \delta(\mathbf{P}' - \mathbf{P}) \frac{1}{k^2} \delta(k' - k) \nabla_{kl}^2, \tag{2.32}$$

where g is a constant with dimensions of length, and ∇_{kl}^2 is the partial wave Laplacian:

$$\nabla_{kl}^2 = -\frac{1}{k^2} \frac{d}{dk} k^2 \frac{d}{dk} + \frac{l(l+1)}{k^2}. \tag{2.33}$$

The corresponding U is obtained from Eq. (2.29).

With this interaction, the eigenvalue problem for the mass operator is analytically solvable. Equation (2.31) can be put in the form

$$(-\nabla_{kl}^2 + 4g^4k^2)\langle kl;\mu|\Psi\rangle = g^4(\lambda^2 - 4m^2)\langle kl;\mu|\Psi\rangle, \tag{2.34}$$

where

$$\langle kl;\mathbf{P}'\mu'|\mathbf{P}\,\Psi\rangle = \delta(\mathbf{P}'-\mathbf{P})\langle kl;\mu|\Psi\rangle, \tag{2.35}$$

which is mathematically equivalent to the eigenvalue problem for a three dimensional harmonic oscillator.

The following mass spectrum results from diagonalizing M^2 and taking the square roots of the eigenvalues:

$$M_{nl} := \lambda_{nl} = 2\sqrt{m^2 + (2n + l + \tfrac{3}{2})/g^2}. \tag{2.36}$$

Note that M_{nl} is positive for all n and l. It represents the mass of a physical particle (meson) with intrinsic angular momentum l. The spectrum for the mass operator is the square root of a shifted oscillator spectrum, rather than the oscillator spectrum itself. It is interesting to compare this spectrum to that obtained from a mass operator of the form

$$M' = 2\sqrt{m^2 + k^2} + \frac{c_1}{r} + c_2 r, \tag{2.37}$$

which is motivated by quark phenomenology (Ca 83a). The parameters in (Ca 83a) are $m = 0.313$ GeV, $c_1 = 0.5$ and $c_2 = 0.197$ GeV^2. The spectrum for this operator is to be compared to Eq. (2.36) with the same quark mass, with the parameter g adjusted so the nominal *rms* ground state oscillator radius is 0.54 fm, namely, the weighted (3:1) average of the ρ and π meson Compton wavelengths. This corresponds to $g = 1.59$ GeV^{-1}. For the case $n = 0$ and $l = 0, 1, 2, 3$ the meson spectra predicted by these two different mass operators (Ca 83a, Po 87) are given in Table 1:

Meson Masses ($n = 0$, $l \leq 4$)

l	Eq. (2.37)	Eq. (2.36)
0	1.66 GeV	1.42 GeV
1	2.08 GeV	1.94 GeV
2	2.43 GeV	2.32 GeV
3	2.74 GeV	2.65 GeV
4	3.01 GeV	2.93 GeV

Table 1

The ground state energies of both calculations are unphysically high. It is possible to shift them down by adding a constant term to the interactions. The remaining dynamical prediction is the splitting between states of different l, which is shown for the two cases in Table 2:

Meson Mass Splitting ($n = 0$)

$$m(l+1) - m(l)$$

l	Eq. (2.37)	Eq. (2.36)
0	0.42 GeV	0.52 GeV
1	0.35 GeV	0.38 GeV
2	0.31 GeV	0.33 GeV
3	0.27 GeV	0.28 GeV

Table 2

Except for the lowest lying case, the splitting predicted by these two mass operators agree to within 0.03 GeV, and the agreement improves with increasing l. Better agreement could be obtained by varying the parameters. Neither of these models should be taken seriously without including spin-spin (Ca 83b) and spin-orbit contributions. However, the predictions of this oscillator model are not significantly different from those of a relativistic Coulomb-plus-linear model. The reason is that the ratio of the mass eigenvalue to the *rms* oscillator radius is constant in the limit that the oscillator quantum number becomes large. This leads asymptotically to a linear confinement.

The eigenvectors of the oscillator model can be labeled by a principal quantum number, an orbital quantum number, the total linear momentum of two free quarks, and a magnetic quantum number:

$$|nl;\mathbf{P}\mu\rangle. \tag{2.38}$$

With the internal wave functions $\langle\mathbf{k}|nl\mu\rangle$ normalized to unity, the normalization of the vectors (2.38) is determined by Eq. (2.35) to be:

$$\langle n'l';\mathbf{P}'\mu'|nl;\mathbf{P}\mu\rangle = \delta_{n'n}\delta_{l'l}\delta_{\mu'\mu}\delta(\mathbf{P}'-\mathbf{P}). \tag{2.39}$$

The wave functions of these state vectors in the plane wave basis are related to standard nonrelativistic harmonic oscillator wave functions $\phi_{nl}(k)Y_{l\mu}(\hat{\mathbf{k}})$ by

$$\langle kl';\mathbf{P}'\mu'|n\,l;\mathbf{P}\mu\rangle = \delta_{l'l}\delta_{\mu'\mu}\delta(\mathbf{P}'-\mathbf{P})\phi_{nl}(k). \tag{2.40}$$

The wave functions $\langle kl;\mathbf{P}'\mu|nl;\mathbf{P}\mu\rangle$ and the spectrum (2.36) represent the dynamical solution of this model. Note that the dynamics of this model are determined by the mass operator, or, equivalently, the square of the mass operator. The total momentum does not appear in the dynamical equations (2.34) or (2.37). The solutions $|nl;\mathbf{P}\mu\rangle$ are simultaneous eigenstates of the mass and linear momentum. However, the model is not yet relativistic.

To interpret the model relativistically, we must construct a unitary representation $U(\Lambda,a)$ of the Poincaré group, which defines the transformation properties of these states under changes of inertial coordinate systems. The representation must be consistent with the dynamics developed above. This is done by defining certain basic Poincaré transformations between pairs of eigenstates of the four-momentum, and then fixing the remaining transformations by group theory. This procedure is straightforward, but not unique. The construction in this section will be done in a manner that leads to an instant form of dynamics, namely, one in which the Euclidean subgroup, consisting of space translations and rotations, acts in a manner that is independent of the mass eigenvalue. In an instant-form dynamics, the Euclidean subgroup is called the "kinematic subgroup."

It is important to note that solving the eigenvalue problem for the mass operator is separate from the specific choices that lead to an instant-form dynamics. In particular, given the eigenstates $|nl;\mathbf{P}\mu\rangle$, one can construct different forms of the dynamics by making different choices concerning which eigenstates states are related by a given Poincaré transformation. There are many ways in which this can be done that are both consistent with the group theory and the dynamics. The examples with scattering which follow the confining models exhibit a different choice, which leads to a front-form dynamics.

The construction is based on the following observation: for a particle of mass M_{ln} and spin l, the transformation properties of states of that particle are fixed uniquely by the set of transformations that leave the particle at rest, along with the action of one specific momentum dependent transformation that associates states of the particle at rest to states where it has momentum $\mathbf{P}$. A general Poincaré transformation can be written as a Lorentz transformation, followed by a spacetime translation:

$$U(\Lambda, a) = T(a)U(\Lambda), \tag{2.41}$$

where the following shorthand notation has been introduced:

$$T(a) := U(I, a); \qquad U(\Lambda) := U(\Lambda, 0). \tag{2.42}$$

We define the transformation for spacetime translations of a mass eigenstate with mass eigenvalue M_{nl} and zero total momentum in a manner consistent with this dynamics:

$$T(a)|nl;0\mu\rangle := e^{-iM_{nl}a^0}|nl;0\mu\rangle. \tag{2.43}$$

The action of a pure rotation on a rest eigenstate with intrinsic angular momentum l is defined so that the rest state transforms as a spin-l irreducible representation under rotations:

$$U(R)|nl;0\mu\rangle := \sum_{\bar{\mu}} |nl;0\bar{\mu}\rangle D^l_{\bar{\mu}\mu}(R). \tag{2.44}$$

We now define the action of the *rotationless* Lorentz transformation $L_c(Q)$ which maps the rest

eigenstate to the corresponding eigenstate with momentum $\mathbf{P}$:

$$U\left[L_c(Q_{nl})\right]|nl;0\mu\rangle := \sqrt{\frac{\omega_{M_{nl}}(\mathbf{P})}{M_{nl}}}|nl;\mathbf{P}\mu\rangle. \tag{2.45}$$

The coefficient with the square root is needed to ensure unitarity of $U\left[L_c(Q_{nl})\right]$ if the states are normalized according to Eq. (2.39). Different choices of normalization will lead to different coefficients. The transformation depends on the four-velocity $Q^\mu = Q^\mu_{nl} = P^\mu_{nl}/M_{nl}$ of the *interacting* system.

Equations (2.43), (2.44) and (2.45) can be combined with the group representation property to define $U(\Lambda, a)$ uniquely on all states. This in turn defines a relativistic quantum mechanics. Note that Eqs. (2.43), (2.44), (2.45) and the group representation properties *define* a representation of $U(\Lambda, a)$. This is not the only possible representation consistent with this dynamics, as will be shown in later examples.

To show how Eqs. (2.43)-(2.45) define $U(\Lambda, a)$ on all states, consider an arbitrary Lorentz transformation Λ, and define the transformed four-momentum and four-velocity by

$$P'^\mu_{nl} := \Lambda^\mu{}_\nu P^\nu_{nl}; \qquad Q'^\mu_{nl} := \Lambda^\mu{}_\nu Q^\nu_{nl}. \tag{2.46}$$

To determine the effect of an arbitrary Lorentz transformation Λ on a state vector, consider the successive transformations

$$R(\Lambda, Q_{nl}) := L_c^{-1}(Q'_{nl})\,\Lambda\,L_c(Q_{nl}) \tag{2.47}$$

that map the three momentum as follows: $0 \to \mathbf{P} \to \mathbf{P}' \to 0$. Since the transformation $R(\Lambda, Q_{nl})$ maps the rest four-velocity $(1,0,0,0)$ onto itself, it is therefore a rotation, often called a Wigner rotation. Equation (2.47) implies that we can write

$$\Lambda = L_c(Q'_{nl})R(\Lambda, Q_{nl})L_c^{-1}(Q_{nl}), \tag{2.48}$$

and therefore, assuming the group representation property, the unitary transformation $U(\Lambda)$ can

be expressed as:

$$U(\Lambda) = U\left[L_c(Q'_{nl})\right] U\left[R(\Lambda, Q_{nl})\right] U\left[L_c^{-1}(Q_{nl})\right]. \tag{2.49}$$

This is used to compute the action of $U(\Lambda, a)$ on an arbitrary eigenvector $|nl; \mathbf{P}\mu\rangle$. Applying the transformations in Eq. (2.49) from right to left, we utilize Eq. (2.45), followed by Eq. (2.44), and finally Eq. (2.45) again to get

$$U(\Lambda)|nl; \mathbf{P}\mu\rangle = \sqrt{\frac{\omega_{M_{nl}}(\mathbf{P}')}{\omega_{M_{nl}}(\mathbf{P})}} \sum_{\bar{\mu}} |nl; \mathbf{P}'\bar{\mu}\rangle D^l_{\bar{\mu}\mu}\left[R(\Lambda, Q_{nl})\right]. \tag{2.50}$$

The action of $T(a)$ can be computed by observing that $T(a)U(\Lambda) = U(\Lambda)T(\Lambda^{-1}a)$, which, together with Eqs. (2.43) and (2.50), yields

$$U(\Lambda, a)|nl; \mathbf{P}\mu\rangle = e^{iP'_{nl}\cdot a} \sqrt{\frac{\omega_{M_{nl}}(\mathbf{P}')}{\omega_{M_{nl}}(\mathbf{P})}} \sum_{\bar{\mu}} |nl; \mathbf{P}'\bar{\mu}\rangle D^l_{\bar{\mu}\mu}\left[R(\Lambda, Q_{nl})\right]. \tag{2.51}$$

Equation (2.51) expresses the action of $U(\Lambda, a)$ on an eigenstate as a linear combination of eigenstates of the same physical mass M_{nl} and spin l, with uniquely determined algebraic coefficients. This defines the action of $U(\Lambda, a)$ on a basis of eigenstates of M, and thus completes a specific construction of $U(\Lambda, a)$.

The operator $U(\Lambda, a)$ defines a relativistic quantum mechanics which is consistent with Wigner's theorem. It specifies the desired correspondence between states in two different inertial coordinate systems in a manner that preserves all probabilities, and is consistent with the dynamics defined by Eq. (2.34).

The interaction dependence of an arbitrary Lorentz transformation that is demanded from the diagram in Fig. 1 appears in the Wigner rotation, the normalization factor $[\omega_{M_{nl}}(\mathbf{P}')/\omega_{M_{nl}}(\mathbf{P})]^{\frac{1}{2}}$, and the phase factor $e^{iP'_{nl}\cdot a}$ through the dependence on the mass eigenvalue M_{nl}. In this representation, for the special case of a pure spatial translation, there is no Wigner rotation, the normalization factor is unity, and the phase factor is $e^{i\mathbf{P}\cdot\mathbf{a}}$. This means that the coefficients of the transformation do not depend on the mass eigenvalue. For the case of a pure rotation, there is no

phase factor, the normalization factor is unity, and the Wigner rotation $R_c(R, Q_{nl})$ corresponding to a rotation $\Lambda = R$ can be calculated using Eqs. (2.17) and (2.47), with the result:

$$R_c(R, Q_{nl}) = R, \tag{2.52}$$

which is independent of the mass eigenvalue. It follows that in the representation (2.51), the subgroup generated by spatial translations and rotations has the same effect on all mass eigenstates, independent of their mass eigenvalue. This property is a consequence of our definition of $U(\Lambda, a)$. A relativistic model with this property is called an instant-form dynamics. The property (2.52) is characteristic of Wigner rotations associated with *rotationless* Lorentz transformations only. In other forms of dynamics, such as the *front form,* Eq. (2.17) is replaced with a different transformation, and the corresponding Wigner rotation of the rotation R generally depends on the mass eigenvalue.

Equation (2.51) can be used to compute matrix elements of $U(\Lambda, a)$ in the basis of eigenstates of the mass operator:

$$\begin{aligned}\langle n'l'; \mathbf{P}'\mu'|U(\Lambda, a)|nl; \mathbf{P}\mu\rangle = \delta_{n'n}\delta_{l'l}\delta(\mathbf{P}' - \mathbf{P}_{\Lambda nl})e^{iP'_{nl}\cdot a} \\ \times \sqrt{\frac{\omega_{M_{nl}}(\mathbf{P}')}{\omega_{M_{nl}}(\mathbf{P})}} D^l_{\mu'\mu}\left[R(\Lambda, Q_{nl})\right],\end{aligned} \tag{2.53}$$

where $P_{\Lambda nl} = \Lambda P_{nl}$ is the transformed four-momentum. In the basis of plane-wave states $|\mathbf{P}\,\mathbf{k}\rangle$, the transformation $U(\Lambda, a)$ can be obtained by inserting a complete set of mass eigenstates and using Eqs. (2.53) and standard properties of the rotation group:

$$\begin{aligned}\langle \mathbf{P}'\,\mathbf{k}'|U(\Lambda, a)|\mathbf{P}\mathbf{k}\rangle = \sum_{nl} \delta(\mathbf{P}' - \mathbf{P}_{\Lambda nl})\delta(\hat{\mathbf{k}}' - R(\Lambda, Q_{nl})\hat{\mathbf{k}}]) \\ \times e^{iP'_{nl}\cdot a}\phi_{nl}(k')\phi^*_{nl}(k),\end{aligned} \tag{2.54}$$

where $\phi_{nl}(k)$ is a radial momentum wave function, and

$$P_{nl} = P_{nl}(\mathbf{p}_1 + \mathbf{p}_2) := \big(\omega_{M_{nl}}(\mathbf{p}_1 + \mathbf{p}_2), \mathbf{p}_1 + \mathbf{p}_2\big). \tag{2.55}$$

In the tensor-product basis $|\mathbf{p}_1\,\mathbf{p}_2\rangle$, Eq. (2.54) becomes

$$\begin{aligned}\langle \mathbf{p}_1'\,\mathbf{p}_2'|U(\Lambda,a)|\mathbf{p}_1\,\mathbf{p}_2\rangle = \sum_{nl}\delta(\mathbf{p}_1'+\mathbf{p}_2'-\Lambda P_{nl})\delta(\mathbf{k}'-R(\Lambda,\mathbf{Q}_{nl})\mathbf{k})e^{iP_{nl}'\cdot a}\\ \times\sqrt{\frac{\omega_{M_{nl}}(\mathbf{P}_\Lambda)}{\omega_{M_{nl}}(\mathbf{P})}}\left|\frac{\partial(\mathbf{p}_1'\mathbf{p}_2')}{\partial(\mathbf{P}'\mathbf{k}')}\right|^{\frac{1}{2}}\left|\frac{\partial(\mathbf{p}_1\mathbf{p}_2)}{\partial(\mathbf{P}\mathbf{k})}\right|^{\frac{1}{2}}\phi_{nl}(k')\phi_{nl}^*(k).\end{aligned} \tag{2.56}$$

The interaction dependence enters Eqs. (2.54) and (2.56) in the l- and n-dependent quantities in the Jacobian factors, in the phase factor, and in the Wigner rotations that rotate the relative momentum vector. These expressions illustrate the complexity of $U(\Lambda,a)$ when it is expressed in a plane wave basis. These expressions are used when a relativistic two-body system is imbedded in a many-body system.

It is useful to examine the nonrelativistic limit of this model, and compare it to the familiar nonrelativistic harmonic oscillator. A nonrelativistic limit can be obtained formally by assuming that

$$m >> \mathbf{k}^2,\, V. \tag{2.57}$$

The mass operator is then expressed as a power series in

$$\zeta := \frac{h_{NR}}{m} := \frac{\mathbf{k}^2}{m^2}+\frac{V}{4m^2}, \tag{2.58}$$

where h_{NR} is the nonrelativistic Hamiltonian. However, since both $\mathbf{k}^2$ and V are unbounded operators and m is a c-number, it is not clear that the limit (2.57) or the expansion (2.58) is ever meaningful. This is a general problem which accompanies expansions about a nonrelativistic limit in relativistic quantum mechanical models. Nevertheless, the above assumption is the one usually made, and we will trace its implications for this particular model.

The expansion can be written as follows:

$$\begin{aligned}M &= \sqrt{4(m^2+\mathbf{k}^2)+V}\\ &= 2m\left[1+\sum_{n=1}^{\infty}\frac{(-)^{n-1}(2n-3)!!}{2^n n!}\zeta^n\right]\\ &= 2m+h_{NR}+2m\left[\sum_{n=2}^{\infty}\frac{(-)^{n-1}(2n-3)!!}{2^n n!}\zeta^n\right].\end{aligned} \tag{2.59}$$

In the nonrelativistic limit, only the first two terms are kept.

The exact eigenvalue equation can be expressed as follows:

$$M^2|\psi\rangle = [4(m^2 + \mathbf{k}^2) + V]|\psi\rangle = (4m^2 + 4mh_{NR})|\psi\rangle = \lambda^2|\psi\rangle. \tag{2.60}$$

That is, $|\psi\rangle$ is also an eigenstate of h_{NR}:

$$h_{NR}|\psi\rangle = (\frac{\lambda^2 - 4m^2}{4m})|\psi\rangle, \tag{2.61}$$

where λ is the exact eigenvalue of M. In the nonrelativistic limit, the eigenvalue equation is

$$h_{NR}|\psi\rangle = (\lambda_{NR} - 2m)|\psi\rangle. \tag{2.62}$$

The eigenstates $|\psi\rangle$ in Eqs. (2.61) and (2.62) are the same, with eigenvalue

$$\eta = \eta(n, l) = \frac{2n + l + \frac{3}{2}}{4mg^2}, \tag{2.63}$$

but the correspondence between $\eta(n, l)$ and the mass spectra λ and λ_{NR} is not. The difference is

$$|\lambda - \lambda_{NR}| = 2m\{\sqrt{1 + \eta/m} - 1 - \eta/2m\}. \tag{2.64}$$

For small η/m, the right-hand side of Eq. (2.64) is of order $(\eta/m)^2$. The relativistic correction to the spectrum is small for sufficiently low lying states, *i.e.,* for values of n and l such that $(2n + l + \frac{3}{2}) \ll 4m^2g^2$.

For models in which an interaction V is added to the *square* of the free mass operator, the eigenvalues change in the nonrelativistic limit, but the state vectors do not. However, the *interpretation* of the state vectors in terms of wave functions is also different in the nonrelativistic

limit. In our model, two-body wave functions have the form

$$\langle \mathbf{p}_1 \mathbf{p}_2 | nl; \mathbf{P}\mu \rangle = \left| \frac{\partial(\mathbf{P}\,\mathbf{k})}{\partial(\mathbf{p}_1\,\mathbf{p}_2)} \right|^{\frac{1}{2}} \delta(\mathbf{p}_1 + \mathbf{p}_2 - \mathbf{P}) \phi_{nl\mu}(\mathbf{k}), \tag{2.65}$$

where

$$\mathbf{k} = \mathbf{p}_1 + \frac{\mathbf{P}}{M_0} \left[\frac{\mathbf{P} \cdot \mathbf{p}_1}{M_0 + H_0} - \omega_m(\mathbf{p}_1) \right]. \tag{2.66}$$

In the nonrelativistic limit, we have

$${}_{NR}\langle \mathbf{p}_1 \mathbf{p}_2 | nl; \mathbf{P}\mu \rangle = \delta(\mathbf{p}_1 + \mathbf{p}_2 - \mathbf{P}) \phi_{nl\mu}(\mathbf{k}_{NR}), \tag{2.67}$$

where

$$\mathbf{k}_{NR} = \tfrac{1}{2}(\mathbf{p}_1 - \mathbf{p}_2). \tag{2.68}$$

These expressions of wave functions in terms of one-body degrees of freedom typically appear in calculations where the two-body system is embedded in a larger system, or an external probe is introduced. Thus, even though the nonrelativistic limit has the same state vector as the original model, calculations which employ the wave functions in Eqs. (2.65) and (2.67) will yield different results. Whether these differences can be interpreted in terms of a systematic expansion in powers of h_{NR}/m is not at all certain, since h_{NR} is an unbounded operator.

2.4. **Example: Confined Relativistic Quarks - With Spin**

We now consider an extension of the previous model in which spin-$\frac{1}{2}$ quarks are bound by a confining interaction.

In the example of spinless quarks just given, it was useful to work in a plane-wave basis $|kl;\mathbf{P}\mu\rangle$, characterized by the total momentum $\mathbf{P}$, the relative momentum k and the orbital angular momentum l of the non-interacting system. This suggests that we use basis states $|[ls]kj;\mathbf{P}\mu\rangle$ for models with spin. The state vectors labelled in this way can also be written in terms of tensor products of single-particle basis states in a manner similar to that of the previous section. We could proceed as before by showing the explicit connection between the tensor-product states and this partial-wave basis. Since this connection represents a change of basis only, and does not affect the discussion of dynamics which follows, it will be deferred to the end of this section. However, tensor products of single-particle states are frequently used in calculations which go beyond the two-body problem, either in terms of an external probe or a few-body system, and the connection to the partial-wave basis is not always the same as one finds in a nonrelativistic approach. The reader is therefore encouraged to examine that discussion, even though it does not bear directly on the development of a dynamical model which follows.

The labels l and s do not correspond to physical observables. Nevertheless, we anticipate in advance that for two spin-$\frac{1}{2}$ particles, s can take on the values 0 or 1, and the relative orbital angular momentum can take on any non-negative integer value. The model Hilbert space is taken to be the space of functions $\langle[ls]kj;\mathbf{P}\mu|\Psi\rangle$, satisfying

$$\langle\Psi|\Psi\rangle < \infty, \tag{2.69}$$

where the scalar product is given by

$$\langle\Psi|\Phi\rangle := \int_{\mathcal{R}^3} d^3P \int_0^\infty k^2 dk \sum_{j=0}^{\infty} \sum_{\mu=-j}^{j} \sum_{s=0}^{1} \sum_{l=|j-s|}^{|j+s|} \langle[l\,s]k\,j;\mathbf{P}\,\mu|\Psi\rangle^* \langle[l\,s]k\,j;\mathbf{P}\,\mu|\Phi\rangle. \tag{2.70}$$

In this representation, the square of the non-interacting mass operator is still the multipli-

cation operator:

$$M_0^2 := 4(m^2 + k^2). \tag{2.71}$$

The form of the eigenvalue problem for the square of the mass is unchanged from the spinless case:

$$\langle [ls]kj; \mathbf{P}\mu | \left[4(k^2 + m^2) + V \right] | \Psi \rangle = \lambda^2 \langle [ls]kj; \mathbf{P}\mu | \Psi \rangle, \tag{2.72}$$

except that the interaction can now have additional degrees of freedom. For the spin-$\frac{1}{2}$ case, an interaction V that leads to a solvable dynamical model is:

$$\langle [l's']k'j'; \mathbf{P}'\mu' | V | [ls]kj; \mathbf{P}\mu \rangle := -\frac{1}{g_{js}^4} \delta_{j'j} \delta_{\mu'\mu} \delta_{s's} \delta_{l'l} \delta(\mathbf{P}' - \mathbf{P}) \frac{1}{k^2} \delta(k' - k) \nabla_{kl}^2, \tag{2.73}$$

where g_{js} is a constant with dimensions of length. The interaction now depends on the spin s, the orbital angular momentum l, and the intrinsic angular momentum j. Since this interaction commutes with the total three-momentum, the intrinsic angular momentum, and the operators whose eigenvalues are l and s, the eigenstates of the mass operator can be taken to be simultaneous eigenstates of the operators whose eigenvalues are $j, \mathbf{P}, \mu, l$, and s:

$$\langle [l's']k'j'; \mathbf{P}'\mu' | [ls]nj; \mathbf{P}\mu \rangle = \delta_{j'j} \delta_{\mu'\mu} \delta_{s's} \delta_{l'l} \delta(\mathbf{P}' - \mathbf{P}) \phi_{l's'n'j'}(k), \tag{2.74}$$

where $\phi_{lsnj}(k)$ is the solution of

$$(-\nabla_{kl}^2 + 4g_{js}^4 k^2) \phi_{lsnj}(k) = g_{js}^4 (\lambda_n^2 - 4m^2) \phi_{lsnj}(k), \tag{2.75}$$

which is identical to the eigenvalue equation (2.34) in the spinless case. The oscillator functions $\phi_{lsnj}(k)$ are normalized to unity, which implies the following normalization for the eigenstates:

$$\langle [l's']n'j'; \mathbf{P}'\mu' | [ls]nj; \mathbf{P}\mu \rangle = \delta_{j'j} \delta_{\mu'\mu} \delta_{s's} \delta_{l'l} \delta_{n'n} \delta(\mathbf{P}' - \mathbf{P}). \tag{2.76}$$

The following mass spectrum results from diagonalizing M^2 and taking the square roots of the eigenvalues:

$$M_{lsnj} = \lambda_{lsnj} = 2\sqrt{m^2 + (2n + l + \tfrac{3}{2})/g_{js}^2}. \tag{2.77}$$

This is similar to the mass spectrum in the spinless case, except for the dependence upon l and s in addition to j and n. Note that M_{lsnj} is positive for all n, j, l and s, as required.

The eigenstates $|[ls]nj;\mathbf{P}\mu\rangle$ form a basis on the Hilbert space. In the representation (2.74), the only difference between this case and the spinless case is the appearance of the degeneracy parameters l and s, which appear in both the mass eigenstates and plane-wave basis elements.

In general, a two-body interaction need not be diagonal in l and s. When that is the case, the quantum numbers l and s no longer appear as labels of the eigenstates, although they are variables in the wave functions.

To develop a representation of the Poincaré group for our harmonic-oscillator model using instant-form dynamics, one simply takes over the entire development surrounding Eqs. (2.43)–(2.51), with the substitution $M_{nl} \to M_{lsnj}$. The actions of translations, rotations, and rotationless boosts on states with $\mathbf{P} = \mathbf{0}$ are *defined* as follows:

$$T(a)|[ls]nj;\mathbf{0}\mu\rangle := e^{-iM_{lsnj}\cdot a}|[ls]nj;\mathbf{0},\mu\rangle \tag{2.78}$$

$$U(R)|[ls]nj;\mathbf{0}\mu\rangle := \sum_{\bar{\mu}} |[ls]nj;\mathbf{0},\bar{\mu}\rangle D^{j}_{\bar{\mu}\mu}(R) \tag{2.79}$$

$$U[L_c(Q_{lsnj})]|[ls]nj;\mathbf{0}\mu\rangle := \sqrt{\frac{\omega_{M_{lsnj}}(\mathbf{P})}{M_{lsnj}}}|[ls]nj;\mathbf{P},\mu\rangle. \tag{2.80}$$

The group representation property can then be used again to give the general transformation law for the eigenstates of the four momentum and intrinsic angular momentum:

$$U(\Lambda,a)|[ls]nj;\mathbf{P}\mu\rangle = e^{i\Lambda P\cdot a}\sqrt{\frac{\omega_{M_{lsnj}}(\mathbf{P}_\Lambda)}{\omega_{M_{lsnj}}(\mathbf{P})}}|[ls]nj;\mathbf{P}_\Lambda,\bar{\mu}\rangle D^{j}_{\bar{\mu}\mu}[R(\Lambda,P)], \tag{2.81}$$

where

$$P_\Lambda := \Lambda P_{lsnj}, \tag{2.82}$$

corresponding to Eq. (2.51) in the spinless case. In this basis, matrix elements of $U(\Lambda,a)$ have the form

$$\begin{aligned}\langle[l's']n'j';\mathbf{P}'\mu'|U(\Lambda,a)|[ls]nj;\mathbf{P}\mu\rangle = \delta_{s's}\delta_{\mu'\mu}\delta_{n'n}\delta_{l'l}\delta(\mathbf{P}'-\mathbf{P}_\Lambda)e^{iP_\Lambda\cdot a}\\ \times\sqrt{\frac{\omega_{M_{nljs}}(\mathbf{P}')}{\omega_{M_{nljs}}(\mathbf{P})}}D^{(l)}_{\mu'\mu}\left[R_c(\Lambda,Q_{nljs})\right].\end{aligned} \tag{2.83}$$

Equation (2.81) defines a unitary representation of the Poincaré group consistent with the dynamics defined by Eq. (2.72), and thus represents a non-trivial relativistic quantum mechanical

model. Note that there are two steps to this general process. First, one finds a simultaneous set of eigenstates of the mass and spin. In the oscillator example, the oscillator coupling constant is adjusted to obtain the best fit to the measured mass/spin spectrum. After this is done, one defines a unitary representation of the Poincaré group that relates different eigenstates. Relativity does not affect the fitting of the mass/spin spectrum of the mesons, but the mass/spin spectrum constrains the transformation properties. Relativity becomes important in processes where two different inertial coordinate systems become relevant. Examples include the interaction of a two-body system with an external probe such as an electron, or a two-body subsystem within a many-body problem.

We end the discussion of this example by providing the connection between the basis states $|[ls]kj;\mathbf{P}\mu\rangle$ and the tensor-product states:

$$|\mathbf{p}_1\mu_1\,\mathbf{p}_2\mu_2\rangle := |\mathbf{p}_1\mu_1\rangle \otimes |\mathbf{p}_2\mu_2\rangle. \tag{2.84}$$

To construct the relative momentum basis, we need to introduce a "free-particle dynamics". The single-quark states are assumed to transform under a one-body representation of the Poincaré group like a bound state of mass M_{nl} and spin l in Eq. (2.51), with M_{nl} replaced by the quark mass m, and l replaced by $\frac{1}{2}$. We denote this transformation by $U_1(\Lambda, a)$ for quark 1 and $U_2(\Lambda, a)$ for quark 2. The tensor product of the single-particle transformations is

$$U_0(\Lambda, a) := U_1(\Lambda, a) \otimes U_2(\Lambda, a) \tag{2.85}$$

This is purely a kinematic definition that facilitates the construction of a variable change; it has nothing to do with the physical interacting representation of the Poincaré group.

We begin by defining a new vector labelled by relative and total momenta:

$$|\mathbf{k};\mathbf{P}\mu_1\mu_2\rangle := \left|\frac{\partial(\mathbf{p}_1\,\mathbf{p}_2)}{\partial(\mathbf{P}\,\mathbf{k})}\right|^{\frac{1}{2}} |\mathbf{p}_1\mu_1\,\mathbf{p}_2\mu_2\rangle, \tag{2.86}$$

where the relevant kinematic relations and Jacobians are given above. We now consider the set of state vectors $|\mathbf{k};\mathbf{0}\mu_1\mu_2\rangle$, corresponding to a non-interacting two-body system at rest. The angles

$\hat{\mathbf{k}}$ can be eliminated in favor of discrete quantum numbers using spherical harmonics:

$$|kl;\mathbf{0}\mu_l\mu_1\mu_2\rangle := \int d\hat{\mathbf{k}}\, Y^l_{\mu_l}(\hat{\mathbf{k}})|\mathbf{k};\mathbf{0}\mu_1\mu_2\rangle. \tag{2.87}$$

We now wish to couple the spins and the internal angular momentum together. Under rotations, a single-particle state $|\mathbf{p}_i\mu_i\rangle$ transforms as follows:

$$U_i(R)|\mathbf{p}_i\mu_i\rangle = \sum_{\bar{\mu}_i} |R\mathbf{p}_i\bar{\mu}_i\rangle D^{\frac{1}{2}}_{\bar{\mu}_i\mu_i}(R). \tag{2.88}$$

We have chosen the single-particle states to have the same properties as outlined in the previous example, namely, that the states $|\mathbf{0}_i\mu_i\rangle$ and $|\mathbf{p}_i\mu_i\rangle$ are related by kinematic rotationless boosts, which means that the Wigner rotation in Eq. (2.88) is in fact the rotation R. Combining the properties of the spherical harmonic, we find that the state vector $|\mathbf{k};\mathbf{0}\mu_1\mu_2\rangle$ transforms as follows under the rotation $U_0(R)$:

$$U_0(R)|kl;\mathbf{0}\mu_l\mu_1\mu_2\rangle = \sum |kl;\mathbf{0}\bar{\mu}_l\bar{\mu}_1\bar{\mu}_2\rangle D^l_{\bar{\mu}_l\mu_l}(R) D^{\frac{1}{2}}_{\bar{\mu}_1\mu_1}(R) D^{\frac{1}{2}}_{\bar{\mu}_2\mu_2}(R). \tag{2.89}$$

Since the rotation matrices all contain the same argument, the indices can be combined using standard rotational Clebsch-Gordan coefficients to define a new state vector:

$$|[ls]kj;\mathbf{0},\mu\rangle := \sum \langle \tfrac{1}{2}\mu_1\, \tfrac{1}{2}\mu_2|s\mu_s\rangle\langle l\mu_l s\mu_s|j\mu\rangle |kl;\mathbf{0}\mu_l\mu_1\mu_2\rangle. \tag{2.90}$$

Under rotations, this state transforms as follows:

$$U_0(R)|[ls]kj;\mathbf{0},\mu\rangle = \sum |[ls]kj;\mathbf{0}\bar{\mu}\rangle D^j_{\bar{\mu}\mu}(R). \tag{2.91}$$

The state vector $|[ls]kj;\mathbf{P}\mu\rangle$ describes two free particles with total three-momentum $\mathbf{P}$. The total four-momentum of the free-particle system is therefore

$$P_0 := ((\omega_m(\mathbf{p}_1) + \omega_m(\mathbf{p}_2), \mathbf{P}). \tag{2.92}$$

We now apply a noninteracting rotationless boost to the state $|[ls]kj;\mathbf{0},\mu\rangle$:

$$|[ls]kj;\mathbf{P}\mu\rangle := \sqrt{\frac{M_0}{\omega_{M_0}(\mathbf{P})}} U_0[L_c(P_0/M_0)]|[ls]kj;\mathbf{0}\mu\rangle, \tag{2.93}$$

where $M_0^2 = P_0^2$ is the square of the mass of the free-particle system. Note that the action of

$U_0[L_c(p_i/m_i)]$ induces Wigner rotations on the product states:

$$U_0[L_c(P_0/M_0)]|\mathbf{k}\mu_1\ -\mathbf{k}\mu_2\rangle = \sqrt{\frac{\omega_m(\mathbf{p}_1)}{\omega_m(\mathbf{k})}}\sqrt{\frac{\omega_m(\mathbf{p}_2)}{\omega_m(\mathbf{k})}}\sum|\mathbf{p}_1\bar{\mu}_1\,\mathbf{p}_2\bar{\mu}_2\rangle \times D^{\frac{1}{2}}_{\bar{\mu}_1\mu_1}[R(L_c(P/M_0),k_1/m)]D^{\frac{1}{2}}_{\bar{\mu}_2\mu_2}[R(L_c(P/M_0),k_2/m)], \tag{2.94}$$

where

$$k_i := L_c^{-1}(P_0/M_0)p_i. \tag{2.95}$$

For half-integer spins, the arguments of the Wigner rotations in Eq. (2.94) should be specified using $SU(2)$ in order to get the correct phase. This is discussed further in the following sections. Putting everything together, we get

$$\begin{aligned}|[ls]kj;\mathbf{P}\mu\rangle = \left|\frac{\omega_m(\mathbf{p}_1)\omega_m(\mathbf{p}_2)M_0}{\omega_m(\mathbf{k})\omega_m(\mathbf{k})\omega_{M_0}(\mathbf{P})}\right|^{\frac{1}{2}}\sum\int d\hat{\mathbf{k}}\,Y^l_{\mu_l}(\hat{\mathbf{k}}) \\ \times\langle\tfrac{1}{2}\mu_1\tfrac{1}{2}\mu_2|s\mu_s\rangle\langle l\mu_l s\mu_s|j\mu\rangle|\mathbf{p}_1\bar{\mu}_1\,\mathbf{p}_2\bar{\mu}_2\rangle \\ \times D^{\frac{1}{2}}_{\bar{\mu}_1\mu_1}[R(L_c(P/M_0),k_1/m)]D^{\frac{1}{2}}_{\bar{\mu}_2\mu_2}[R(L_c(P/M_0),k_2/m)].\end{aligned} \tag{2.96}$$

Given the above definitions, these vectors are normalized as follows:

$$\langle[l's']k'j';\mathbf{P}'\mu'|[ls]kj;\mathbf{P}\mu\rangle = \delta_{\mu'\mu}\delta_{l'l}\delta_{s's}\delta_{j'j}\delta(\mathbf{P}'-\mathbf{P})\frac{1}{k^2}\delta(k'-k). \tag{2.97}$$

The action of $U(\Lambda,a)$ in the tensor-product basis is obtained by combining Eqs. (2.96) and (2.83):

$$\begin{aligned}&\langle\mathbf{p}_1\mu_1\mathbf{p}_2\mu_2|U(\Lambda,a)|[ls]nj;\mathbf{P}\mu\rangle \\ &\quad= \delta(\mathbf{p}_1+\mathbf{p}_2-\mathbf{P}_\Lambda)\sqrt{\frac{\omega_{M_{nl}}(\mathbf{P}_\Lambda)}{\omega_{M_{nl}}(\mathbf{P})}}\left|\frac{\partial(\mathbf{p}_1'\mathbf{p}_2')}{\partial(\mathbf{P}'\mathbf{k}')}\right|^{\frac{1}{2}}\left|\frac{\partial(\mathbf{p}_1\mathbf{p}_2)}{\partial(\mathbf{P}\mathbf{k})}\right|^{\frac{1}{2}} \\ &\qquad\times e^{i\Lambda P_{lsnj}\cdot a}\phi_{lsnj}(k)\sum Y^l_{\mu'_l}(\hat{\mathbf{k}})\langle\tfrac{1}{2}\bar{\mu}_1\tfrac{1}{2}\bar{\mu}_2|s\mu_s\rangle\langle l\mu_l s\mu_s|j\bar{\mu}\rangle \\ &\qquad\times D^{\frac{1}{2}}_{\bar{\mu}_1\bar{\mu}_1}[R(L_c(\Lambda Q_{lsnj}),k_1/m)]D^{\frac{1}{2}}_{\bar{\mu}_2\bar{\mu}_2}[R(L_c(\Lambda Q_{lsnj}),k_2/m)] \\ &\qquad\times D^j_{\bar{\mu}\mu}[R_c(\Lambda,P_{lsnj})].\end{aligned} \tag{2.98}$$

As in the spinless case, these equations again illustrate the complexity of $U(\Lambda,a)$ when it is expressed in a plane-wave basis. Their use may be required when a two-body system is imbedded in a larger system.

2.5. **Example: Spinless Two-Particle Scattering**

In this section, a solvable relativistic model of two spinless particles of mass m that exhibits scattering is constructed. This model is formulated using "front-form" dynamics. The light front is the three-dimensional surface in spacetime characterized by the condition $x^+ = x^0 + x^3 = 0$. This surface has the property that it is invariant under a seven-parameter subgroup of the Poincaré group. The subgroup includes a three-parameter subgroup of spacetime translations, a three-parameter *subgroup* of "front-form" boosts, and a one-parameter subgroup of rotations. These transformations are discussed in detail in Sections 3, 4, and Appendix B. In a front-form dynamics, the unitary transformations $U(\Lambda, a)$ contain no interactions for (Λ, a) restricted to this seven-parameter subgroup, which replaces the Euclidean subgroup as the *kinematic* subgroup in a front-form dynamics. A front-form dynamics has the following properties:

i. It has the largest (7 parameters) possible kinematic subgroup in a dynamical model. It is the only kinematic subgroup containing both three independent boosts and three independent spacetime translations.

ii. Because the "front-form" boosts form a subgroup (unlike the set of rotationless boosts), there are no Wigner rotations associated with front-form boosts. There are, however, Wigner rotations associated with other Lorentz transformations.

Apart from illustrating a solvable two-body scattering problem, the purpose of the construction in this section is to illustrate the similarities and differences in relation to the instant-form constructions used to formulate the quark models above. As was mentioned earlier, the treatment of the mass eigenvalue problem is separate from the choice of dynamics in constructing a unitary representation of the Poincaré group.

In practice the construction of a front-form dynamics is almost identical to that of an instant form dynamics. It is a two-step process. The first step, which does not depend on the form of dynamics, is to construct simultaneous eigenstates of the four-momentum and the spin. The second is to define a unitary representation of the Poincaré group that relates eigenstates with the same mass and spin. The form of the dynamics is not fixed until the second step.

Before introducing the dynamics, we must select a set of plane-wave basis states. As with the

previous examples, we seek a partial-wave representation as the basis of choice for implementing the dynamics. We show here the explicit steps needed to obtain this basis, starting from tensor products of single-particle states:

$$|\tilde{\mathbf{p}}_1\,\tilde{\mathbf{p}}_2\rangle := |\tilde{\mathbf{p}}_1\rangle \otimes |\tilde{\mathbf{p}}_2\rangle, \tag{2.99}$$

where $\tilde{\mathbf{p}}_1$ and $\tilde{\mathbf{p}}_2$ are the light-front components of the four-momenta p_1 and p_2, and $\tilde{\mathbf{p}}$ is defined by:

$$\tilde{\mathbf{p}} := (\mathbf{p}_\perp, p^+); \quad \mathbf{p}_\perp := (p^1, p^2); \quad p^+ := p^3 + p^0. \tag{2.100}$$

These three components of the four-momentum generate translations tangent to the three-surface $x^+ = 0$ (the light front). The components of $\tilde{\mathbf{p}}$ make up what we call a *light-front vector.* The remaining component of the four momentum, $p^- := p^0 - p^3$, is related to the mass operator by the relation

$$p^- := \frac{m^2 + \mathbf{p}_\perp^2}{p^+}. \tag{2.101}$$

The operator p^- generates spacetime translations normal to the light front (*i.e.,* along the x^+ axis), and thus plays the role of the Hamiltonian in a front-form dynamics.

Transformations $L_f(Q)$ which leave the light front invariant and transform the rest four-velocity $(1,0,0,0)$ to the four-velocity Q are called *front-form* boosts. They have the property that for any four-vector A:

$$\begin{pmatrix} A^{+\prime} \\ \mathbf{A}'_\perp \\ A^{-\prime} \end{pmatrix} = L_f(Q) \begin{pmatrix} A^+ \\ \mathbf{A}_\perp \\ A^- \end{pmatrix} = \begin{pmatrix} Q^+ A^+ \\ \mathbf{A}_\perp + \mathbf{Q}_\perp A^+ \\ (Q^+)^{-1} \left(\mathbf{Q}_\perp^2 A^+ + 2\mathbf{Q}_\perp \cdot \mathbf{A}_\perp + A^- \right) \end{pmatrix}. \tag{2.102}$$

The composition of two front-form boosts yields a third front-form boost, which means that the set of transformations $L_f(Q)$ forms a subgroup. It also means that the three light-front components of the four-momentum transform among themselves.

As in the instant-form example above, we now switch to state vectors labeled by the total light-front momentum $\tilde{\mathbf{P}}$ and relative momentum $\mathbf{k}$ of two free quarks. The total momentum is defined as

$$\tilde{\mathbf{P}} := \tilde{\mathbf{p}}_1 + \tilde{\mathbf{p}}_2. \tag{2.103}$$

The relative momentum $\mathbf{k}$ is equal to the three-momentum of particle 1 when $\tilde{\mathbf{P}} = \tilde{\mathbf{0}}$, where

$$\tilde{\mathbf{0}} := (\mathbf{P}_\perp = 0, P^+ = M). \tag{2.104}$$

For arbitrary $\tilde{\mathbf{p}}_1$ and $\tilde{\mathbf{p}}_2$, k is defined by applying a front-form boost to p_1:

$$k := L_f^{-1}(Q_0)p_1, \tag{2.105}$$

where $Q_0 = P/M_0$ is the four-velocity of the non-interacting system, and

$$M_0 := (P^+P_0^- - \mathbf{P}_\perp^2)^{\frac{1}{2}}; \qquad P_0^- := \frac{m^2 + \mathbf{p}_{1\perp}^2}{p_1^+} + \frac{m^2 + \mathbf{p}_{2\perp}^2}{p_2^+}. \tag{2.106}$$

The three-vector part of k is then $\mathbf{k} = (\mathbf{k}_\perp, k^3)$, where

$$\mathbf{k}_\perp = \frac{p_2^+}{P^+}\mathbf{p}_{1\perp} - \frac{p_1^+}{P^+}\mathbf{p}_{2\perp}; \quad k^3 = \tfrac{1}{2}(p_1^+ - p_2^+)\sqrt{\frac{m^2 + \mathbf{k}_\perp^2}{p_1^+ p_2^+}}. \tag{2.107}$$

Note that the three-vector $\mathbf{k}$ as defined in Eq. (2.105) and given explicitly in Eq. (2.107) is not the same three-vector $\mathbf{k}$ as defined in Eq. (2.16) and given explicitly in Eq. (2.20). While they both give a relative three-momentum for a zero-momentum plane-wave state, the Lorentz transformations used to obtain each k are not the same. Since the two forms of k must have the same invariant length and are purely spatial, they must differ only by a rotation. If the plane-wave states are given delta-function normalizations, *i.e.,*

$$\langle \tilde{\mathbf{p}}_1' \, \tilde{\mathbf{p}}_2' | \tilde{\mathbf{p}}_1 \, \tilde{\mathbf{p}}_2 \rangle = \delta(\tilde{\mathbf{p}}_1' - \tilde{\mathbf{p}}_1)\delta(\tilde{\mathbf{p}}_2' - \tilde{\mathbf{p}}_2); \quad \langle \tilde{\mathbf{P}}' \, \mathbf{k}' | \tilde{\mathbf{P}} \, \mathbf{k} \rangle = \delta(\tilde{\mathbf{P}}' - \tilde{\mathbf{P}})\delta(\mathbf{k} - \mathbf{k}'), \tag{2.108}$$

then the state vectors $|\tilde{\mathbf{P}}\,\mathbf{k}\rangle$ and $|\tilde{\mathbf{p}}_1\,\tilde{\mathbf{p}}_2\rangle$ are related to each other via

$$|\tilde{\mathbf{P}}\,\mathbf{k}\rangle := \left|\frac{\partial(\tilde{\mathbf{p}}_1\,\tilde{\mathbf{p}}_2)}{\partial(\tilde{\mathbf{P}}\,\mathbf{k})}\right|^{\frac{1}{2}} |\tilde{\mathbf{p}}_1\,\tilde{\mathbf{p}}_2\rangle, \tag{2.109}$$

where the Jacobian in Eq. (2.109) is

$$\left|\frac{\partial(\tilde{\mathbf{p}}_1\,\tilde{\mathbf{p}}_2)}{\partial(\tilde{\mathbf{P}}\,\mathbf{k})}\right| = \frac{p_1^+ p_2^+ M_0}{\omega_m(\mathbf{k})\omega_m(\mathbf{k})P^+}. \tag{2.110}$$

The angles $\hat{\mathbf{k}}$ can be eliminated in favor of discrete quantum numbers using spherical harmonics:

$$|kl;\tilde{\mathbf{P}}\mu\rangle := \int d\hat{\mathbf{k}}\, Y_\mu^l(\hat{\mathbf{k}})|\mathbf{k};\tilde{\mathbf{P}}\rangle. \tag{2.111}$$

The Hilbert space for this model is taken to be the space of functions $\langle kl;\tilde{\mathbf{P}}\mu|\Psi\rangle$, with scalar product

$$\langle\Psi|\Phi\rangle := \sum_{l=0}^{\infty}\sum_{\mu=-l}^{l}\int_0^\infty dP^+ \int d^2P_\perp \int_0^\infty k^2 dk\, \langle kl;\tilde{\mathbf{P}}\mu|\Psi\rangle^*\langle kl;\tilde{\mathbf{P}}\mu|\Phi\rangle. \tag{2.112}$$

In this basis, $\tilde{\mathbf{P}}$ denotes light-front components of the four-momentum

$$\tilde{\mathbf{P}} = (\mathbf{P}_\perp, P^+) \qquad \mathbf{P}_\perp := (P^1, P^2) \qquad P^+ := P^0 + P^3. \tag{2.113}$$

The relative momentum k is related to the invariant mass of two non-interacting nucleons by

$$M_0^2 = 4(m^2 + k^2), \tag{2.114}$$

and l is the relative orbital angular momentum.

An interacting mass operator M can be constructed in the same manner as the previous examples by adding an interaction to M_0:

$$M = M_0 + U. \tag{2.115}$$

As in the case of the quark model, it is useful to define the interaction as the difference of the squares of the interacting and non-interacting mass operators:

$$M^2 = M_0^2 + V. \tag{2.116}$$

The interactions U and V are related by Eq. (2.29). As in the case of the quark model, the interactions must be such that M^2 has positive eigenvalues. This requirement is more restrictive in models that have scattering than in confining models because the mass of bound states is lower than the rest mass of the constituent non-interacting particles. The spectral condition, $M \geq 0$, is violated when the binding energy exceeds the rest mass of the non-interacting system.

The interaction V is taken to be a rank-one separable interaction in each partial wave:

$$\langle k'\, l'; \tilde{\mathbf{P}}'\, \mu' | V | k\, l; \tilde{\mathbf{P}}\, \mu \rangle := \delta_{l'l}\delta_{\mu'\mu}\delta(\tilde{\mathbf{P}}' - \tilde{\mathbf{P}})V^l(k', k), \tag{2.117}$$

where

$$\delta(\tilde{\mathbf{P}}' - \tilde{\mathbf{P}}) := \delta^2(\mathbf{P}'_\perp - \mathbf{P}_\perp)\delta(P'^+ - P^+), \tag{2.118}$$

and

$$V^l(k', k) := -g_l f_l(k') f_l(k)^*. \tag{2.119}$$

The form factors $f_l(k)$ and coupling constants g_l represent the phenomenological input to this model. The eigenvalue problem for M^2 is solvable up to a quadrature involving $f_l(k)$. The structure of this operator differs from the interaction appearing in the quark model in that it contains a delta function in the light-front vector components of the four-momentum, rather than the three-vector components of the four-momentum.

Eigenstates of the four-momentum are eigenstates of both M^2 and P^-, where

$$P^- = \frac{M^2 + \mathbf{P}_\perp^2}{P^+} \tag{2.120}$$

is the "front-form Hamiltonian" that generates translations normal to the light front. To solve for the eigenvectors, it is convenient to use the eigenvalue equation for M^2:

$$M^2|\Psi^{(-)}\rangle = \lambda^2|\Psi^{(-)}\rangle, \tag{2.121}$$

which can be expressed in integral form as:

$$|\Psi^{(-)}\rangle = |\Psi_0\rangle + \frac{1}{\lambda^2 - M_0^2 + i0^+} V|\Psi^{(-)}\rangle, \tag{2.122}$$

where $|\Psi_0\rangle$ must be a solution of

$$(M_0^2 - \lambda^2)|\Psi_0\rangle = 0. \tag{2.123}$$

Note that the $(-)$ superscript on the scattering state indicates an asymptotic condition as $t \to -\infty$, corresponding to the $+i0^+$ prescription in the denominator of the integral equation (2.122). The mass eigenvalue for the scattering state is the invariant mass of the incident plane wave state:

$$\lambda^2 = 4(k^2 + m^2). \tag{2.124}$$

For the interaction (2.117), the solutions $\langle k'l'; \tilde{\mathbf{P}}'\mu'|kl; \tilde{\mathbf{P}}\mu^{(-)}\rangle$ of Eq. (2.122) can be taken to be simultaneous eigenstates of $\tilde{\mathbf{P}}$ and μ:

$$\langle k'l'; \tilde{\mathbf{P}}'\mu'|kl; \tilde{\mathbf{P}}\mu^{(-)}\rangle = \delta_{l'l}\delta_{\mu'\mu}\delta(\tilde{\mathbf{P}}' - \tilde{\mathbf{P}})\phi_{kl}^{(-)}(k'), \tag{2.125}$$

where

$$\phi_{kl}^{(-)}(k') := \frac{1}{k^2}\delta(k' - k) - \frac{g_l f_l(k') f_l^*(k)}{(4k^2 - 4k'^2 + i0^+)\Delta_l(\lambda^2)}, \tag{2.126}$$

and

$$\Delta_l(\lambda^2) := 1 + g_l \int_0^\infty k^2 dk \, \frac{f_l^*(k) f_l(k)}{\lambda^2 - 4m^2 - 4k^2 + i0^+}. \tag{2.127}$$

The scattering eigenstates defined by Eq. (2.125) have the same normalization as the plane-

wave basis vectors:

$$\langle k'l';\tilde{\mathbf{P}}'\mu'^{(-)}|kl;\tilde{\mathbf{P}}\mu^{(-)}\rangle = \delta_{l'l}\delta_{\mu'\mu}\delta(\tilde{\mathbf{P}}'-\tilde{\mathbf{P}})\frac{1}{k^2}\delta(k'-k). \tag{2.128}$$

Bound state vectors satisfy the homogeneous form of Eq. (2.122). A bound state of angular momentum l will exist for each value of λ_b^2 satisfying

$$\Delta_l(\lambda_b^2) = 0. \tag{2.129}$$

All of the real λ_b^2's satisfying Eq. (2.129) must be positive or the interaction will violate the spectral condition. The mass of the bound system is $m_b = \lambda_b$. The bound state wave function is

$$\langle k'\,l';\tilde{\mathbf{P}}'\,\mu'|b\,l;\tilde{\mathbf{P}}\,\mu\rangle = \delta_{l'l}\delta_{\mu'\mu}\delta(\tilde{\mathbf{P}}'-\tilde{\mathbf{P}})\phi_{bl}(k'), \tag{2.130}$$

where

$$\phi_{bl}(k) := N_l\frac{f_l(k)}{\lambda_b^2-4m^2-4k^2}, \tag{2.131}$$

with the normalization constant

$$N_l := \left[\int_0^\infty k^2dk\,\frac{f_l(k)^*f_l(k)}{(\lambda_b^2-4m^2-4k^2)^2}\right]^{-\frac{1}{2}} \tag{2.132}$$

chosen so that

$$\langle b'\,l';\tilde{\mathbf{P}}'\,\mu'|b\,l;\tilde{\mathbf{P}}\,\mu\rangle = \delta_{b'b}\delta_{l'l}\delta_{\mu'\mu}\delta(\tilde{\mathbf{P}}'-\tilde{\mathbf{P}}). \tag{2.133}$$

In general, an interaction of the form (2.119) is determined phenomenologically. This is done by calculating the scattering cross section and binding energy as a function of the interaction and varying parameters of the interaction to obtain the best agreement with experiment.

The relation between the scattering wave functions and the invariant scattering cross section (Mo 45) is developed identically in the nonrelativistic and relativistic cases. For two spinless particles of equal mass, the invariant cross section is

$$d\sigma = (2\pi)^4 \frac{1}{sv} |\langle \mathbf{p}_1' \, \mathbf{p}_2' \| T(E + i0^+) \| \mathbf{p}_1 \, \mathbf{p}_2 \rangle|^2 d\Phi, \tag{2.134}$$

where v is the relative speed of the projectile and target, s is a statistical factor (which is 2 if the particles are identical and 1 otherwise), and $d\Phi$ is a phase space factor defined by

$$d^3p_1' d^3p_2' = d^4p' d\Phi. \tag{2.135}$$

The transition matrix elements $\langle \mathbf{p}_1' \, \mathbf{p}_2' \| T(E + i0^+) \| \mathbf{p}_1 \, \mathbf{p}_2 \rangle$ are related to the scattering state vectors and the interaction by

$$\langle \mathbf{p}_1' \, \mathbf{p}_2' \| T(E + i0^+) \| \mathbf{p}_1 \, \mathbf{p}_2 \rangle \delta(\mathbf{p}_1 + \mathbf{p}_2 - \mathbf{p}_1' - \mathbf{p}_2') = \langle \mathbf{p}_1' \, \mathbf{p}_2' | (H - H_0) | \mathbf{p}_1 \, \mathbf{p}_2^{(-)} \rangle \Big|_{p^- = p'^-}. \tag{2.136}$$

In the scattering model formulated in this section, the right-hand side of Eq. (2.136) has a delta function in the light-front components of the four-momentum. This can be transformed to a delta function in the ordinary three-momenta if the matrix element is evaluated on the P^- shell. With the conventions of Brenig and Haag (Br 59) used to derive Eq. (2.134), the plane-wave and scattering states have delta function normalizations. The use of the double bar $\|$ denotes a reduced matrix element in the sense that the momentum-conserving delta function is factored out. The transition matrix element itself is evaluated on shell.

The cross section (2.134) is a Poincaré invariant quantity. It can be written in a *manifestly* invariant form by rearranging the terms in the right-hand side as follows (Mo 45):

$$d\Phi \to \frac{d\Phi}{\omega_m(\mathbf{p}_1')\omega_m(\mathbf{p}_2')} = \frac{d^3p_1'}{\omega_m(\mathbf{p}_1')} \frac{d^3p_2'}{\omega_m(\mathbf{p}_2')} \delta^4(p - p_1' - p_2'); \tag{2.137}$$

$$\frac{1}{v} \to \frac{1}{v\omega_m(\mathbf{p}_1)\omega_m(\mathbf{p}_2)} = \frac{1}{\sqrt{(p_1 \cdot p_2)^2 - m^4}}; \tag{2.138}$$

and the transition matrix element by the invariant matrix elements:

$$\langle \mathbf{p}_1' \, \mathbf{p}_2' \| T(E + i0^+) \| \mathbf{p}_1 \, \mathbf{p}_2 \rangle \to \mathcal{M};$$

$$\mathcal{M} := 4(2\pi)^6 \sqrt{\omega_m(\mathbf{p}_1')\omega_m(\mathbf{p}_2')} \langle \mathbf{p}_1' \, \mathbf{p}_2' \| T(E + i0^+) \| \mathbf{p}_1 \, \mathbf{p}_2 \rangle \sqrt{\omega_m(\mathbf{p}_1)\omega_m(\mathbf{p}_2)}. \tag{2.139}$$

The invariant amplitude $\mathcal{M}$ has the same normalization as the amplitude $\mathcal{M}$ in the *Review of Particle Properties* (Yo 88).

As was the case in the instant-form quark models, it is convenient to use V rather than U in order to make contact with phenomenology. We must therefore replace $H - H_0$ by $V = M^2 - M_0^2$ in Eq. (2.136), making use of the following:

$$\begin{aligned} (H - H_0) &= \frac{1}{2}(P^- + P^+ - P_0^- - P^+) = \frac{1}{2}\left(\frac{\mathbf{P}_\perp^2 + M^2}{P^+} - \frac{\mathbf{P}_\perp^2 + M_0^2}{P^+} \right) \\ &= \frac{1}{2P^+}(M^2 - M_0^2) = \frac{V}{2P^+}. \end{aligned} \tag{2.140}$$

The last quotient in Eq. (2.140) is well defined because P^+ commutes with V in this model.

We must also replace the representations used to describe the initial and final states with those used to formulate our dynamical equations:

$$|\mathbf{p}_1 \, \mathbf{p}_2\rangle = \left| \frac{\partial(\tilde{\mathbf{P}} \, \mathbf{k})}{\partial(\mathbf{p}_1 \, \mathbf{p}_2)} \right|^{\frac{1}{2}} Y_\mu^{l*}(\hat{\mathbf{k}}) |k \, l; \tilde{\mathbf{P}} \, \mu\rangle. \tag{2.141}$$

Note that the single-particle states are labelled with ordinary three-momenta rather than light-front momenta. However, k is defined according to Eq. (2.105). The Jacobian is

$$\left| \frac{\partial(\tilde{\mathbf{P}} \, \mathbf{k})}{\partial(\mathbf{p}_1 \, \mathbf{p}_2)} \right| = \frac{\omega_m(\omega_m(\mathbf{k})\omega_m(\mathbf{k})P^+}{\mathbf{p}_1)\omega_m(\mathbf{p}_2)M}. \tag{2.142}$$

By means of the identification

$$\langle k'l'; \tilde{\mathbf{P}}'\mu' | V | kl; \tilde{\mathbf{P}}\mu^{(-)} \rangle = \delta_{l'l}\delta_{\mu'\mu}\delta(\tilde{\mathbf{P}}' - \tilde{\mathbf{P}}) \langle k'l' \| V \| kl^{(-)} \rangle, \tag{2.143}$$

we can write

$$\begin{aligned}\langle \mathbf{p}_1' \mathbf{p}_2' \| T(E+i0^+) \| \mathbf{p}_1 \mathbf{p}_2 \rangle = \frac{1}{2P^+} \left| \frac{\partial(\tilde{\mathbf{P}}'\mathbf{k}')}{\partial(\mathbf{p}_1'\mathbf{p}_2')} \right|^{\frac{1}{2}} \left| \frac{\partial(\tilde{\mathbf{P}}\mathbf{k})}{\partial(\mathbf{p}_1\,\mathbf{p}_2)} \right|^{\frac{1}{2}} \left| \frac{H}{P^+} \right| \\ \times \sum_l \frac{(2l+1)}{4\pi} P_l(\hat{\mathbf{k}}' \cdot \hat{\mathbf{k}}) \langle k' l' \| V \| k l^{(-)} \rangle.\end{aligned} \tag{2.144}$$

The matrix element is evaluated with the initial and final four-momenta having the same mass eigenvalue. It is simplest to evaluate the invariant cross section in the center-of-momentum frame. In that case, the Jacobians become unity, the velocity and phase space factors in Eq. (2.134) become

$$v \to \frac{2|\mathbf{k}|}{\omega_m(\mathbf{k})}; \quad d\Phi \to \frac{|\mathbf{k}|\omega_m(\mathbf{k})}{2} d\Omega; \quad P^+ \to 2\omega_m(k), \tag{2.145}$$

where $d\Omega$ is the differential solid angle. It follows that

$$\frac{d\sigma}{d\Omega} = \frac{(2\pi)^4}{64s} \left| \langle \mathbf{k}' \| V \| \mathbf{k}^{(-)} \rangle \right|^2, \tag{2.146}$$

where

$$\langle \mathbf{k}' \| V \| \mathbf{k}^{(-)} \rangle := -\sum_{l=0}^{\infty} (2l+1) P_l(\hat{\mathbf{k}}' \cdot \hat{\mathbf{k}}) \frac{g_l f_l^*(k) f_l(k)}{\Delta_l} \tag{2.147}$$

This is the desired expression that must be compared with scattering data to determine the phenomenological form factors $f_l(k)$ and the coupling constants g_l. Note that it is possible to write Eq. (2.146) in the form:

$$\frac{d\sigma}{d\Omega} = (2\pi)^4 \frac{m}{2ks} \left| \frac{1}{4m} \langle \mathbf{k}' \| V \| \mathbf{k}^{(-)} \rangle \right|^2 \frac{km}{2}, \tag{2.148}$$

which is the *nonrelativistic* expression for the cross section for a potential $V' = V/4m$. If we express Eq. (2.117) in terms of V' rather than V, then the eigenvalue problem can be written as follows:

$$\left(\frac{\mathbf{k}^2}{m} + V' \right) |\Psi\rangle = \frac{\lambda^2 - 4m^2}{4m} |\Psi\rangle, \tag{2.149}$$

which has the same form as the nonrelativistic Schrödinger equation. This means that one can take a solution to the nonrelativistic Schrödinger equation with an interacting V' determined by

a phase shift analysis, and the resulting scattering wave functions can be taken as eigenfunctions of our model, with $M^2 - M_0^2 = 4mV'$ fit to the same phase shifts.

This observation is very important, and is a consequence of the fact that phase shift analysis is a fitting procedure. In this example, the same equation fit to the same data can be interpreted either relativistically or nonrelativistically. There are no "relativistic corrections" in passing from one interpretation to the other. The relativistic and nonrelativistic interpretations lead to different predictions when these two-body interactions are used as input to describe dynamics of more than two particles.

In general, the binding energy ε_{nl} must be refit. It is related to the mass λ_{nl} by

$$\varepsilon_{nl} = \lambda_{nl} - 2m. \tag{2.150}$$

Substituting into the right-hand side of Eq. (2.149), we get

$$\frac{\lambda_{nl}^2 - 4m^2}{4m} = -\varepsilon_{nl}(1 - \frac{\varepsilon_{nl}}{4m}) \tag{2.151}$$

In the nonrelativistic Schrödinger equation, the multiplier on the right-hand side of Eq. (2.149) would be identified with the binding energy. The difference between the nonrelativistic and relativistic eigenvalues is small when

$$\left|\frac{\varepsilon_{nl}}{4m}\right| \ll 1. \tag{2.152}$$

This correction is extremely small for a weakly bound system such as the deuteron. This example was done in a front-form model, but the same remarks hold for any choice of continuous variables.

The eigenfunctions in Eqs. (2.125) and (2.130) form a complete set of functions on the model Hilbert space. A unitary representation $U(\Lambda, a)$ with a light-front kinematic subgroup consistent with the above dynamics can be defined by modifying the construction used in the case of the quark model. Note that the dynamics are always defined first. The *form* of the dynamics involves choices about the relations between physical interacting representations of the Poincaré group and unphysical non-interacting representations, and is defined second.

In the instant-form quark model, the representation $U(\Lambda, a)$ was defined so that the magnetic quantum number of a zero-momentum eigenstate state $|nl; (\mathbf{P} = 0)\mu\rangle$ does not change when Λ is a rotationless Lorentz transformation $L_c(Q)$. In a front-from dynamics, the representation $U(\Lambda, a)$ is defined so that the magnetic quantum number of a zero-momentum eigenstate $|nl; \tilde{\mathbf{0}}\mu\rangle$ does not change when Λ is a *kinematic front-form boost* $L_f(Q)$, where $Q := P/M$ is the four-velocity, and M is the mass eigenvalue of the *interacting* state. In this case, the rest eigenstate corresponds to

$$\tilde{\mathbf{P}} = \tilde{\mathbf{0}} := (\mathbf{P}_\perp = 0, P^+ = M). \tag{2.153}$$

We are now in a position to *define* a unitary representation of the Poincaré group that (1) is consistent with the dynamics (2.122) and (2) has the kinematic subgroup of the light front. As in the instant-form examples discussed earlier, this is done by defining the action of spacetime translations and rotations on the rest eigenstates of the four-momentum and the action of one boost for each value of the four-velocity that relates states with any fixed momentum to states with zero momentum. Thus, we define translations and rotations on the rest *scattering eigenstates* by

$$T(a)|k\, l; \tilde{\mathbf{0}}\, \mu^{(-)}\rangle := e^{-iMa^0}|k\, l; \tilde{\mathbf{0}}\, \mu^{(-)}\rangle, \tag{2.154}$$

and

$$U(R)|k\, l; \tilde{\mathbf{0}}\, \mu^{(-)}\rangle := \sum_{\bar{\mu}=-l}^{l} |k\, l; \tilde{\mathbf{0}}\, \bar{\mu}^{(-)}\rangle D^l_{\bar{\mu}\mu}(R), \tag{2.155}$$

respectively. Front-form boosts on the rest eigenstates are defined by

$$U[L_f(P)]|k\, l; \tilde{\mathbf{0}}\, \mu^{(-)}\rangle := \sqrt{\frac{P^+}{M}}|k\, l; \tilde{\mathbf{P}}\, \mu^{(-)}\rangle, \tag{2.156}$$

where the multiplicative factor in Eq. (2.156) is needed for unitarity with the normalization (2.128). We must also define the action of $U(\Lambda, a)$ on the rest *eigenstates* of the bound system:

$$T(a)|b\, l; \tilde{\mathbf{0}}\, \mu\rangle := e^{-iM_b a^0}|b\, l; \tilde{\mathbf{0}}\, \mu\rangle; \tag{2.157}$$

$$U(R)|b\,l;\tilde{\mathbf{0}}\,\mu\rangle := \sum_{\bar{\mu}=-l}^{l} |b\,l;\tilde{\mathbf{0}}\,\bar{\mu}\rangle D^{l}_{\bar{\mu}\mu}(R); \tag{2.158}$$

$$U[L_f(P)]|b\,l;\tilde{\mathbf{0}}\,\mu\rangle := \sqrt{\frac{P^+}{M_b}}|b\,l;\tilde{\mathbf{P}}\,\mu\rangle. \tag{2.159}$$

These relations, together with the group representation property, fix the form of $U(\Lambda, a)$ on all states. This construction is similar to that in Eqs. (2.44)–(2.51): it makes use of the fact that any Poincaré transformation can be expressed as a composition of four basic transformations:

$$U(\Lambda, a) = U[L_f(\Lambda Q)]\,T[L_f^{-1}(\Lambda Q)a]\,U[R_f(\Lambda, Q)]\,U[L_f^{-1}(Q)], \tag{2.160}$$

The transformation $R_f(\Lambda, Q) := L_f^{-1}(\Lambda Q)\Lambda L_f(Q)$ is a *front-form* Wigner rotation: it maps the rest four-velocity $(1, 0, 0, 0)$ onto itself. However, since $R_f(\Lambda, Q)$ employs front-form boosts rather than rotationless boosts, it has different properties from those of $R_c(\Lambda, Q)$ defined in the instant-form examples in the previous sections. First, because the set of front-form boosts form a closed subgroup, there is no Wigner rotation associated with a front-form boost. However, unlike the case in the instant form, the Wigner rotation corresponding to an arbitrary rotation is not necessarily the rotation itself. Only for pure rotations about the z axis (*i.e.,* an element of the kinematic subgroup) are the rotations the same.

Applying these transformations to a state with four-momentum $P = MQ$, and using Eqs. (2.154)–(2.159), we obtain, with $P' := \Lambda P$:

$$U(\Lambda, a)|k\,l;\tilde{\mathbf{P}}\,\mu^{(-)}\rangle = e^{iP'\cdot a}\sqrt{\frac{P'^+}{P^+}}\sum_{\bar{\mu}=-l}^{l} |k\,l;\tilde{\mathbf{P}}'\,\bar{\mu}^{(-)}\rangle D^{l}_{\bar{\mu}\mu}[R_f(\Lambda, Q)] \tag{2.161}$$

for scattering states, and

$$U(\Lambda, a)|b\,l;\tilde{\mathbf{P}}\,\mu\rangle = e^{iP'\cdot a}\sqrt{\frac{P'^+}{P^+}}\sum_{\bar{\mu}=-l}^{l} |b\,l;\tilde{\mathbf{P}}'\,\bar{\mu}\rangle D^{l}_{\bar{\mu}\mu}[R_f(\Lambda, Q)] \tag{2.162}$$

for bound states. The transformation properties of the scattering wave functions can be determined by combining Eqs. (2.125) and (2.161):

$$\langle k'l';\tilde{\mathbf{P}}'\mu'|U(\Lambda, a)|kl;\tilde{\mathbf{P}}\mu^{(-)}\rangle = \delta_{l'l}\delta(\tilde{\mathbf{P}}' - \tilde{\mathbf{P}}_\Lambda)e^{iP'\cdot a}\sqrt{\frac{P'^+}{P^+}}\phi^{(-)}_{kl}(k')D^{l}_{\mu'\mu}[R_f(\Lambda, Q)], \tag{2.163}$$

where $\tilde{\mathbf{P}}_\Lambda$ is the light-front vector component of the transformed four-momentum $P_\Lambda := \Lambda P$, and

P is the four-momentum of the interacting system. For bound-state wave functions, the result is

$$\langle k'\, l'; \tilde{\mathbf{P}}'\, \mu' | U(\Lambda, a) | b\, l; \tilde{\mathbf{P}}\, \mu \rangle = \delta_{l'l} \delta(\tilde{\mathbf{P}}' - \tilde{\mathbf{P}}_\Lambda) e^{iP'\cdot a} \sqrt{\frac{P'^+}{P^+}} \phi_{bl}(k') D^l_{\mu'\mu}[R_f(\Lambda, Q)]. \tag{2.164}$$

Equations (2.163) and (2.164) define a unitary representation of the Poincaré group consistent with the model dynamics. To show that this is a front-form dynamics we must demonstrate that the coefficients of the transformations $U(\Lambda, a)$ are independent of the *mass eigenvalue* when (Λ, a) is restricted to the subgroup of the Poincaré group that leaves the light-front invariant. There are three type of transformations that must be considered. First, there are spacetime translations tangent to the light front which have the form

$$T(a^-, \mathbf{a}_\perp) | k\, l\, \tilde{\mathbf{P}}\, \mu^{(-)} \rangle = e^{i(-\frac{1}{2}P^+ a^- + \mathbf{P}_\perp \cdot \mathbf{a}_\perp)} | k\, l\, \tilde{\mathbf{P}}\, \mu^{(-)} \rangle. \tag{2.165}$$

These are clearly independent of the mass eigenvalue. Next, there are the three independent light-front boosts:

$$U(L_f(Q)) | k\, l\, \tilde{\mathbf{P}}\, \mu^{(-)} \rangle = \sqrt{Q^+} | k\, l; (Q^+ P^+, \mathbf{P}_\perp + P^+ \mathbf{Q}_\perp)\, \mu^{(-)} \rangle, \tag{2.166}$$

where again the right-hand side is independent of the mass eigenvalue. Note that in this expression, the four-velocity Q is an argument of a Lorentz transformation, and is not related to the eigenvalues of any operator. The last type of kinematic transformation is a rotation about the axis that defines the light-front. As discussed above the Wigner rotation associated with this particular transformation is the rotation itself. Thus, we have

$$U[R_z(\phi)] | kl\, \tilde{\mathbf{P}} \mu^{(-)} \rangle = e^{-i\mu\phi} | kl\, (P^+, R_z(\phi) \tilde{\mathbf{P}}_\perp) \mu^{(-)} \rangle, \tag{2.167}$$

where ϕ is the angle of rotation. Once again, the mass eigenvalue does not appear anywhere on the right hand-side of this equation. Analogous remarks apply to the bound state vectors. It follows that the dynamics that we have constructed has the kinematic subgroup of the light front. For translations in the z direction or rotations about the x or y axis, the transformed quantities will depend on the mass eigenvalue, in contrast to an instant-form dynamics.

We now return to a comment made earlier, namely, that solving the eigenvalue problem for the mass operator is separate from the specific choices that lead to a particular *form* of dynamics. The key to this observation is that the zero-momentum state vectors $|k\,l;\mathbf{0}\,\mu\rangle$ in the instant form and $|k\,l;\tilde{\mathbf{0}}\,\mu\rangle$ in the front form are in fact identical. This is because the wave functions satisfy identical equations in both cases, and the interactions are determined by fitting to the same binding energies and partial wave phase shifts.

The difference between the instant-form and front-form constructions lies in the choice of kinematic components of the momentum, and in the way in which the spins in the zero-momentum states are related to the spins in states with non-zero momentum. These representations are unitarily equivalent. The transformation between instant-form and front-form state vectors is obtained by utilizing their identity at zero momentum:

$$\begin{aligned}
|k\,l;\mathbf{P}\,\mu\rangle_c &= U[L_f(P)]U^\dagger[L_f(P)]U[L_c(P)]U^\dagger[L_c(P)]|kl;\mathbf{P}\mu\rangle \\
&= \sqrt{\frac{M}{P^0}}U[L_f(P)]U[R_{fc}(P)]|kl;\mathbf{0}\mu\rangle \\
&= \sqrt{\frac{M}{P^0}}\sum U[L_f(P)]|kl;\tilde{\mathbf{0}}\bar{\mu}\rangle D^l_{\bar{\mu}\mu}[R_{fc}(P)] \\
&= \sqrt{\frac{P^+}{P^0}}\sum |kl;\tilde{\mathbf{P}}\bar{\mu}\rangle_f D^l_{\bar{\mu}\mu}[R_{fc}(P)].
\end{aligned} \tag{2.168}$$

Apart from normalization factors, the two state vectors differ by a rotation:

$$R_{fc}(P) = L_f^{-1}(P)L_c(P), \tag{2.169}$$

commonly called a Melosh rotation (Me 74).

As in the case of the quark model, we can consider the nonrelativistic limit of this model with scattering. Since interactions have again been added to the *square* of the non-interacting mass operator, the state vectors are identical in the nonrelativistic limit to the exact state vectors. For the scattering problem, the exact eigenvalue equation is

$$M^2|\psi\rangle = [4(m^2+\mathbf{k}^2)+V]|\psi\rangle = (4m^2+4mh_{NR})|\psi\rangle = 4(m^2+k_0^2)|\psi\rangle, \tag{2.170}$$

where k_0 is the relative momentum of the asymptotic state. This can be expressed as follows:

$$h_{NR}|\psi\rangle = \frac{k_0^2}{m}|\psi\rangle. \tag{2.171}$$

If M is expanded as in Eq. (2.59) and one computes the nonrelativistic limit, the resulting eigenvalue equation has the same form as Eq. (2.171). As in the instant form, the exact wave functions and the and the corresponding nonrelativistic wave functions are identical (in this representation).

The cross section in the nonrelativistic limit is given by the replacements

$$U \to \frac{V}{4m} \qquad \omega_m(\mathbf{k}) \to m \tag{2.172}$$

$$\frac{d\sigma}{d\Omega} = (2\pi)^4 \frac{\omega_m(\mathbf{k})}{2ks} |\langle \mathbf{k}'|U|\mathbf{k}^{(-)}\rangle|^2 \frac{k'\omega_m(\mathbf{k}')}{2} \to (2\pi)^4 \frac{m}{2ks} |\langle \mathbf{k}'|\frac{V}{4m}|\mathbf{k}^{(-)}\rangle|^2 \frac{k'm}{2}. \tag{2.173}$$

The nonrelativistic expression for the cross section is identical to the exact expression. As in the confining case, there are differences between the preditions of the relativistic model and the corresponding nonrelativistic limit for the spectra of bound states. In addition, when the two-body system is embedded in a larger system or an external probe is introduced, there are differences associated with the interpretation of the eigenstates in terms of one-body degrees of freedom. We note that these considerations are independent of the form of the dynamics.

2.6. **Example: Nucleon-Nucleon Scattering - With Spin**

We now extend the previous discussion to describe a model of interacting nucleons of spin $\frac{1}{2}$ using front-form dynamics.

The approach is analogous to the way in which we added spin to the confined-quark model in the instant form. The plane-wave basis $|kl;\tilde{\mathbf{P}}\mu\rangle$ used for spinless nucleons is replaced by the set of states $|[ls]kj;\tilde{\mathbf{P}}\mu\rangle$ for nucleons with spin. The state vectors labelled in this way can again be written in terms of tensor products of single-particle basis states. This connection will be deferred to the end of this section. However, the reader is again encouraged to examine that discussion. The connection to the partial-wave basis is not the same as one finds in a nonrelativistic approach, and it is not the same as the connection for particles with spin in the instant form. Nevertheless, the change of basis does not bear directly on the development of a dynamical model which we now address. The labels l and s do not correspond to physical observables. We again anticipate that for two spin-$\frac{1}{2}$ particles, s can take on the values 0 or 1, and the relative orbital angular momentum can take on any non-negative integer value. The model Hilbert space is taken to be the space of square integrable functions with respect to the scalar product:

$$\begin{aligned}\langle\Psi|\Phi\rangle := &\sum_{j=0}^{\infty}\sum_{\mu=-j}^{j}\sum_{s=0}^{1}\sum_{l=|j-s|}^{|j+s|} \\ &\times\int d^2P_\perp\int_0^\infty dP^+\int_0^\infty k^2dk\langle[ls]kj;\tilde{\mathbf{P}}\mu|\Psi\rangle^*\langle[ls]kj;\tilde{\mathbf{P}}\mu|\Phi\rangle.\end{aligned} \tag{2.174}$$

The invariant mass of two free nucleons is identical to the corresponding operator in the spinless case:

$$M_0^2 := 4(k^2+m^2). \tag{2.175}$$

In order to obtain an analytically solvable model, consider a separable interaction V of the form:

$$\langle[l's']k'j';\tilde{\mathbf{P}}'\mu'|V|[ls]kj;\tilde{\mathbf{P}}\mu\rangle := -\delta_{j'j}\delta_{\mu'\mu}\delta(\tilde{\mathbf{P}}'-\tilde{\mathbf{P}})\lambda_j f_j(k'l's')f_j^*(kls). \tag{2.176}$$

The form factors $f_j(kls)$ and coupling constants λ_j form the phenomenological input to this model. This interaction is a rank-one separable potential in each angular momentum channel, which makes the eigenvalue problem for M^2 solvable up to a quadrature involving $f_j(kls)$.

The solution to the dynamical problem is determined by diagonalizing

$$M^2 = M_0^2 + V, \tag{2.177}$$

with bound states being associated with discrete mass eigenvalues and scattering states with continuous mass eigenvalues. The bound states are solutions of

$$M^2|\Psi\rangle = \mu^2|\Psi\rangle, \tag{2.178}$$

and the scattering solutions with incoming wave asymptotic conditions are solutions of

$$|\Psi^{(-)}\rangle = |\Psi_0\rangle + \frac{1}{\mu^2 - M_0^2 + i0^+} V|\Psi^{(-)}\rangle, \tag{2.179}$$

where $|\Psi_0\rangle$ is a solution of

$$(M_0^2 - \mu^2)|\Psi_0\rangle = 0. \tag{2.180}$$

As operator equations, these equations are identical to the spinless case. For the interaction (2.176), the solutions $\langle [l's']k'j'; \tilde{\mathbf{P}}'\mu'|[ls]kj; \tilde{\mathbf{P}}\mu^{(-)}\rangle$ of Eq. (2.179) have the form

$$\langle [l's']k'j'; \tilde{\mathbf{P}}'\mu'|[ls]kj; \tilde{\mathbf{P}}\mu^{(-)}\rangle = \delta_{j'j}\delta_{\mu'\mu}\delta(\tilde{\mathbf{P}}' - \tilde{\mathbf{P}})\phi^{(-)}_{lskj}(l'\ s'\ k'), \tag{2.181}$$

where

$$\phi^{(-)}_{lskj}(l'\ s'\ k') := \delta_{l'l}\delta_{s's}\frac{1}{k^2}\delta(k-k') - \frac{\lambda_j f_j(k'l's')f_j^*(kls)}{(4k^2 - 4k'^2 + i0^+)\Delta_j(\mu^2(k))}, \tag{2.182}$$

and

$$\Delta_j(\mu^2) := 1 + \lambda_j \sum_{s=0}^{1} \sum_{l=|j-s|}^{|j+s|} \int_0^\infty k^2 dk\, \frac{f_j^*(kls)f_j(kls)}{\mu^2 - 4m^2 - 4k^2 + i0^+}. \tag{2.183}$$

The scattering eigenfunctions defined by Eq. (2.181) have the same normalization as the plane-wave basis vectors:

$$\langle [l's']k'j'; \tilde{\mathbf{P}}'\mu'^{(-)}|[ls]kj; \tilde{\mathbf{P}}\mu^{(-)}\rangle = \delta_{l'l}\delta_{s's}\delta_{j'j}\delta_{\mu'\mu}\delta(\tilde{\mathbf{P}}' - \tilde{\mathbf{P}})\frac{1}{k^2}\delta(k'-k). \tag{2.184}$$

Bound state vectors satisfy the homogeneous form of equation (2.179). A bound state of

angular momentum j will exist for each value of μ_b^2 satisfying:

$$\Delta_j(\mu_b^2) = 0. \tag{2.185}$$

All of the real μ_b^2's satisfying Eq. (2.185) must be positive or the interaction will violate the spectral condition. The bound state wave function is

$$\langle [l's']k'j'; \tilde{\mathbf{P}}'\mu'|bj; \tilde{\mathbf{P}}\mu\rangle = \delta_{j'j}\delta_{\mu'\mu}\delta(\tilde{\mathbf{P}}' - \tilde{\mathbf{P}})\phi_{blsj}(l'\,s'\,k'), \tag{2.186}$$

where

$$\phi_{lsbj}(l'\,s'\,k') := N_j \frac{f_j(l'\,s'\,k')}{\mu_b^2 - 4m^2 - 4k'^2}, \tag{2.187}$$

and the normalization constant

$$N_j := \left[\sum_{s=0}^{1}\sum_{l=|j-s|}^{|j+s|}\int_0^\infty \frac{f_j(l\,s\,k)^* k^2 dk f_j(l\,s\,k)}{(\mu_b^2 - 4m^2 - 4k^2)^2}\right]^{-\frac{1}{2}} \tag{2.188}$$

is chosen so that

$$\langle b'\,j'; \tilde{\mathbf{P}}'\,\mu'|b\,j; \tilde{\mathbf{P}}\,\mu\rangle = \delta_{b'b}\delta_{j'j}\delta_{\mu'\mu}\delta(\tilde{\mathbf{P}}' - \tilde{\mathbf{P}}). \tag{2.189}$$

The eigenfunctions (2.181) and (2.186) form a complete set of functions on the model Hilbert space. A unitary representation $U(\Lambda, a)$ with a light-front kinematic subgroup can be constructed exactly as in the spinless case. One starts by defining the action of the Euclidean group (rotations and translations) on rest eigenstates, and operators that transform each rest eigenstate to a four-momentum eigenstate:

$$T(a)|[ls]kj; \tilde{\mathbf{0}}\mu^{(-)}\rangle := e^{-2i\omega_m(k)a^0}|[ls]kj; \tilde{\mathbf{0}}\mu^{(-)}\rangle; \tag{2.190}$$

$$U(R)|[ls]kj; \tilde{\mathbf{0}}\mu^{(-)}\rangle := \sum_{\bar{\mu}=-j}^{j}|[ls]kj; \tilde{\mathbf{0}}\bar{\mu}^{(-)}\rangle D^j_{\bar{\mu}\mu}(R), \tag{2.191}$$

$$U[L_f(q)]|[ls]kj;\tilde{\mathbf{0}}\mu^{(-)}\rangle := \sqrt{\frac{P^+}{2\omega_m(k)}}|[ls]kj;\tilde{\mathbf{P}}\mu^{(-)}\rangle; \tag{2.192}$$

where $\tilde{\mathbf{0}} := (2\omega_m(k), 0, 0)$ for scattering states, and

$$T(a)|bj;\tilde{\mathbf{0}}\mu\rangle := e^{-iM_b a^0}|bj;\tilde{\mathbf{0}}\mu\rangle, \tag{2.193}$$

$$U(R)|bj;\tilde{\mathbf{0}}\mu^{(-)}\rangle = \sum_{\bar{\mu}=-j}^{j}|b,j;\tilde{\mathbf{0}},\bar{\mu}\rangle D^j_{\bar{\mu}\mu}(R), \tag{2.194}$$

$$U[L_f(P)]|bj;\tilde{\mathbf{0}}\mu\rangle := \sqrt{\frac{P^+}{\mu_b}}|bj;\tilde{\mathbf{P}}\mu\rangle \tag{2.195}$$

where $\tilde{\mathbf{0}} := (\mu_b, 0, 0)$ for bound states. The remaining transformations are fixed in terms of these transformations by the group representation properties exactly as in the spinless case. The only difference with the spinless case is the appearance of degeneracy parameters. The resulting unitary representation of the Poincaré group is given by

$$U(\Lambda,a)|[ls]kj;\tilde{\mathbf{P}}\mu^{(-)}\rangle = e^{i\Lambda P\cdot a}\sqrt{\frac{P^+_\Lambda}{P^+}}\sum_{\bar{\mu}=-j}^{j}|[ls]kj;\tilde{\mathbf{P}}_\Lambda\bar{\mu}^{(-)}\rangle D^j_{\bar{\mu}\mu}[R_f(\Lambda,p_s)], \tag{2.196}$$

for scattering states, and

$$U(\Lambda,a)|bj;\tilde{\mathbf{P}}\mu^{(-)}\rangle = e^{i\Lambda P\cdot a}\sqrt{\frac{P^+_\Lambda}{P^+}}\sum_{\bar{\mu}=-j}^{j}|bj;\tilde{\mathbf{P}}_\Lambda\bar{\mu}\rangle D^j_{\bar{\mu}\mu}[R_f(\Lambda,p_b)], \tag{2.197}$$

for bound states where $P_\Lambda := \Lambda P$.

The transformation properties of the scattering wave functions are therefore

$$\begin{aligned}\langle[l's']k'j';\tilde{\mathbf{P}}'\mu'|U(\Lambda,a)|[ls]kj;\tilde{\mathbf{P}}\mu^{(-)}\rangle = \delta_{j'j}\delta(\tilde{\mathbf{P}}'-\tilde{\mathbf{P}}_\Lambda)e^{iP'\cdot a}\sqrt{\frac{P'^+}{P^+}}\\ \times\phi_{klsj}(l'\,s'\,k')D^{(l)}_{\mu'\mu}[R_f(\Lambda,Q)],\end{aligned} \tag{2.198}$$

where $\tilde{\mathbf{P}}_\Lambda$ is the light-front vector component of the transformed four-momentum $P_\Lambda := \Lambda P$. For

bound states, the result is

$$\langle [l's']k'j';\tilde{\mathbf{P}}'\mu'|U(\Lambda,a)|[ls]bj;\tilde{\mathbf{P}}\mu\rangle = \delta_{j'j}\delta(\tilde{\mathbf{P}}'-\tilde{\mathbf{P}}_\Lambda)e^{iP'\cdot a}\sqrt{\frac{P'^+}{P^+}} \times \phi_{blsj}(l'\,s'\,k')D^{(l)}_{\mu'\mu}[R_f(\Lambda,Q)]. \tag{2.199}$$

Equations (2.196) and (2.197) define a unitary representation of the Poincaré group, consistent with the underlying dynamics, by specifying its action on a basis of interacting eigenstates. It can be shown that this is a front-form dynamics in the same manner that this was shown in the spinless case. Note that the mass and spin eigenvalues determine the transformation properties under Poincaré transformations. If additional quantum numbers are included, they will have nontrivial dynamical consequences under Poincaré transformations if the mass eigenvalues depend on these quantum numbers.

We now provide the connection between the basis states $|[ls]kj;\tilde{\mathbf{P}}\mu\rangle$ and the tensor-product states

$$|\tilde{\mathbf{p}}_1\mu_1\,\tilde{\mathbf{p}}_2\mu_2\rangle := |\tilde{\mathbf{p}}_1\mu_1\rangle\otimes|\tilde{\mathbf{p}}_2\mu_2\rangle. \tag{2.200}$$

In this case, as in the case of spin $\frac{1}{2}$ quarks, we introduce a purely kinematic one-body representation of the Poincaré group for each particle, where in a front-form dynamics, the single particle states transform like bound states with mass M_{nl} and spin l in Eq. (2.162), with M_{nl} replaced by the nucleon mass, and the angular momentum l replaced by $\frac{1}{2}$. As in the instant form case we let $U_0(\Lambda,a)$ denote the tensor product of the one-body representation associated with each particle. A new vector is defined which is labelled by relative and total momenta:

$$|\mathbf{k};\tilde{\mathbf{P}}\mu_1\mu_2\rangle := \left|\frac{\partial(\tilde{\mathbf{p}}_1\,\tilde{\mathbf{p}}_2)}{\partial(\tilde{\mathbf{P}}\,\mathbf{k})}\right|^{\frac{1}{2}}|\tilde{\mathbf{p}}_1\mu_1\,\tilde{\mathbf{p}}_2\mu_2\rangle, \tag{2.201}$$

where

$$\tilde{\mathbf{P}} := \tilde{\mathbf{p}}_1+\tilde{\mathbf{p}}_2; \qquad k := L_f^{-1}(Q_0)p_1; \qquad \left|\frac{\partial(\tilde{\mathbf{P}}\,\mathbf{k})}{\partial(\tilde{\mathbf{p}}_1\,\tilde{\mathbf{p}}_2)}\right| = \frac{\omega_m(\mathbf{k})\omega_m(\mathbf{k})P^+}{p_1^+p_2^+M_0}, \tag{2.202}$$

as in the spinless case. We now consider the set of state vectors $|\mathbf{k};\tilde{\mathbf{0}}\mu_1\mu_2\rangle$, corresponding to a non-interacting two-body system at rest. The angles in $\hat{\mathbf{k}}$ can be eliminated in favor of discrete

quantum numbers using spherical harmonics:

$$|kl;\tilde{\mathbf{0}}\mu_l\mu_1\mu_2\rangle := \int d\hat{\mathbf{k}}\, Y^l_{\mu_l}(\hat{\mathbf{k}})|\mathbf{k};\tilde{\mathbf{0}}\mu_1\mu_2\rangle. \tag{2.203}$$

Note that k refers to a four-vector in Eq. (2.202) and the magnitude of a three-vector in Eq. (2.203). As in the case of particles with spin in the instant form, we wish to couple the spins and the internal angular momentum together. In the instant form, we made use of the fact that the indices of the spherical harmonic and of the particle spins all transform in the same way under rotations, thus allowing us to combine them in the standard fashion. In the front form, because we use state vectors which are related to zero-momentum states via front-form boosts, the argument of the rotation matrix for the particle spins is not the rotation itself, but rather its *front-form Wigner rotation,* which depends upon the particle momenta. However, in the instant-form examples given earlier, we made use of state vectors related to zero-momentum states by rotationless boosts, for which an arbitrary rotation is the same as its Wigner rotation. As discussed in the previous example, these two kinds of state vectors are related by a Melosh rotation. This suggests the following definition:

$$|kl;\tilde{\mathbf{0}}\mu_l\mu_1\mu_2\rangle_c := \sum |kl;\tilde{\mathbf{0}}\mu_l\bar{\mu}_1\bar{\mu}_2\rangle D^{\frac{1}{2}}_{\bar{\mu}_1\mu_1}[R_{fc}(k_1)]D^{\frac{1}{2}}_{\bar{\mu}_2\mu_2}[R_{fc}(k_2)], \tag{2.204}$$

where

$$k_i := L_f^{-1}(P_0)p_i. \tag{2.205}$$

This state vector describes two free particles with zero total momentum. It transforms as follows under a tensor product $U_0(R) = U_1(R)U_2(R)$ of single-particle rotations:

$$U_0(R)|kl;\tilde{\mathbf{0}}\mu_l\mu_1\mu_2\rangle_c = \sum |kl;\tilde{\mathbf{0}}\bar{\mu}_l\bar{\mu}_1\bar{\mu}_2\rangle_c D^l_{\bar{\mu}_l\mu_l}(R)D^{\frac{1}{2}}_{\bar{\mu}_1\mu_1}(R)D^{\frac{1}{2}}_{\bar{\mu}_2\mu_2}(R). \tag{2.206}$$

The indices can therefore be combined using standard rotational Clebsch-Gordan coefficients to

define a new state vector:

$$|[ls]kj;\tilde{\mathbf{0}},\mu\rangle := \sum \langle \tfrac{1}{2}\mu_1 \tfrac{1}{2}\mu_2|s\mu_s\rangle\langle l\mu_l s\mu_s|j\mu\rangle|kl;\tilde{\mathbf{0}}\mu_l\mu_1\mu_2\rangle_c. \tag{2.207}$$

Under a tensor product of free-particle rotations, this state transforms as follows:

$$U_0(R)|[ls]kj;\tilde{\mathbf{0}},\mu\rangle = \sum |[ls]kj;\tilde{\mathbf{0}}\bar{\mu}\rangle D^j_{\bar{\mu}\mu}(R). \tag{2.208}$$

We now apply a product of kinematic front-form boosts to the state $|[ls]kj;\tilde{\mathbf{0}},\mu\rangle$:

$$|[ls]kj;\tilde{\mathbf{P}}\mu\rangle := \sqrt{\frac{M_0}{P^+}}U_0[L_f(P_0/M_0)]|[ls]kj;\tilde{\mathbf{0}}\mu\rangle. \tag{2.209}$$

Note that, since the front-form boosts form a subgroup, the action of $U_0[L_f(p_i)]$ does *not* induce Wigner rotations on the product states:

$$U_1[L_f(P_0/M_0)]U_2[L_f(P_0/M_0)]|\mathbf{k}\mu_1\ -\mathbf{k}\mu_2\rangle = \sqrt{\frac{p_1^+}{k_1^+}}\sqrt{\frac{p_2^+}{k_2^+}}|\tilde{\mathbf{p}}_1\mu_1\,\tilde{\mathbf{p}}_2\mu_2\rangle. \tag{2.210}$$

Putting everything together, we get an expression for the two-body plane-wave states in terms of the tensor product of one-body states:

$$\begin{aligned}|[ls]kj;\tilde{\mathbf{P}}\mu\rangle = \sum \int d\hat{\mathbf{k}} \left|\frac{p_1^+ p_2^+ M_0}{k_1^+ k_2^+ P^+}\right|^{\frac{1}{2}} Y^l_{\mu_l}(\hat{\mathbf{k}}) \\ \times \langle \tfrac{1}{2}\mu_1 \tfrac{1}{2}\mu_2|s\mu_s\rangle\langle l\mu_l s\mu_s|j\mu\rangle|\tilde{\mathbf{p}}_1\bar{\mu}_1\,\tilde{\mathbf{p}}_2\bar{\mu}_2\rangle \\ \times D^{\frac{1}{2}}_{\bar{\mu}_1\mu_1}[R_{fc}(k_1/m)]D^{\frac{1}{2}}_{\bar{\mu}_2\mu_2}[R_{fc}(k_2/m)].\end{aligned} \tag{2.211}$$

Given the above definitions, these vectors are normalized as follows:

$$\langle[l's']k'j';\tilde{\mathbf{P}}'\mu'|[ls]kj;\tilde{\mathbf{P}}\mu\rangle = \delta_{\mu'\mu}\delta_{l'l}\delta_{s's}\delta_{j'j}\delta(\tilde{\mathbf{P}}'-\tilde{\mathbf{P}})\frac{1}{k^2}\delta(k'-k). \tag{2.212}$$

The action of $U(\Lambda,a)$ in the tensor-product basis for scattering states is obtained by com-

bining Eqs. (2.211) and (2.198):

$$\begin{aligned}
&\langle \tilde{\mathbf{p}}_1\mu_1\tilde{\mathbf{p}}_2\mu_2|U(\Lambda,a)|[ls]kj;\tilde{\mathbf{P}}\mu^{(-)}\rangle \\
&\qquad = \delta(\tilde{\mathbf{p}}_1+\tilde{\mathbf{p}}_2-\tilde{\mathbf{P}}_\Lambda)e^{i\Lambda P_{lskj}\cdot a}\phi^{(-)}_{lskj}(k'l's')\sum Y^l_{\mu'_l}(\hat{\mathbf{k}}) \\
&\qquad\quad \times\sqrt{\frac{P^+_\Lambda}{P^+}}\left|\frac{\partial(\tilde{\mathbf{P}}\mathbf{k})}{\partial(\tilde{\mathbf{p}}_1\tilde{\mathbf{p}}_2)}\right|^{\frac{1}{2}}\langle\tfrac{1}{2}\bar{\mu}_1\tfrac{1}{2}\bar{\mu}_2|s\mu_s\rangle\langle l\mu_l s\mu_s|j\bar{\mu}\rangle \\
&\qquad\quad \times D^{\frac{1}{2}}_{\mu_1\bar{\mu}_1}[R_{fc}(k_1/m))]D^{\frac{1}{2}}_{\mu_2\bar{\mu}_2}[R_{fc}(k_2/m))]D^j_{\bar{\mu}\mu}[R_c(\Lambda,P)].
\end{aligned} \tag{2.213}$$

For bound states, the result is

$$\begin{aligned}
&\langle \tilde{\mathbf{p}}_1\mu_1\tilde{\mathbf{p}}_2\mu_2|U(\Lambda,a)|[ls]bj;\tilde{\mathbf{P}}\mu\rangle \\
&\qquad = \delta(\tilde{\mathbf{p}}_1+\tilde{\mathbf{p}}_2-\tilde{\mathbf{P}}_\Lambda)e^{i\Lambda P_{kjls}\cdot a}\phi_{blsj}(k'l's')\sum Y^l_{\mu'_l}(\hat{\mathbf{k}}) \\
&\qquad\quad \times\sqrt{\frac{P^+_\Lambda}{P^+}}\left|\frac{\partial(\tilde{\mathbf{P}}\mathbf{k})}{\partial(\tilde{\mathbf{p}}_1\tilde{\mathbf{p}}_2)}\right|^{\frac{1}{2}}\langle\tfrac{1}{2}\bar{\mu}_1\tfrac{1}{2}\bar{\mu}_2|s\mu_s\rangle\langle l\mu_l s\mu_s|j\bar{\mu}\rangle \\
&\qquad\quad \times D^{\frac{1}{2}}_{\mu_1\bar{\mu}_1}[R_{fc}(k_1/m))]D^{\frac{1}{2}}_{\mu_2\bar{\mu}_2}[R_{fc}(k_2/m))]D^j_{\bar{\mu}\mu}[R_c(\Lambda,P)].
\end{aligned} \tag{2.214}$$

As in the previous examples, one needs expressions such as (2.213) and (2.214) to compute matrix elements of a one-body operator involving a boosted state vector.

The invariant differential cross section can be conveniently evaluated in the center-of-momentum frame with the result:

$$\frac{d\sigma}{d\Omega} = (2\pi)^4\frac{m}{2ks}\left|\langle \mathbf{k}\,\mu'_1\,\mu'_2\|\frac{V}{4m}\|\mathbf{k}^-\,\mu_1\,\mu_2\rangle\right|^2\frac{km}{2}, \tag{2.215}$$

where

$$\begin{aligned}
&\langle \mathbf{k}'\,\mu'_1\,\mu'_2\|V\|\mathbf{k}\mu_1\,\mu_2^{(-)}\rangle \\
&\qquad := -\sum Y^{l'}_{\mu'_l}(\hat{\mathbf{k}}')Y^{l*}_{\mu_l}(\hat{\mathbf{k}})\frac{\lambda_j f^*_j(kl's')f_j(kls)}{\Delta_l[2\omega_m(k)]} \\
&\qquad\quad \times\langle\tfrac{1}{2}\bar{\mu}'_1\tfrac{1}{2}\bar{\mu}'_2|s'\mu'_s\rangle\langle l'\mu'_l s'\mu'_s|j\mu\rangle\langle\tfrac{1}{2}\bar{\mu}_1\tfrac{1}{2}\bar{\mu}_2|s\mu_s\rangle\langle l\mu_l s\mu_s|j\mu\rangle \\
&\qquad\quad \times D^{\frac{1}{2}}_{\mu'_1\bar{\mu}'_1}[R_{fc}(k'_1/m))]D^{\frac{1}{2}}_{\mu'_2\bar{\mu}'_2}[R_{fc}(k'_2/m))] \\
&\qquad\quad \times D^{\frac{1}{2}\dagger}_{\bar{\mu}_1\mu_1}[R_{fc}(k_1/m))]D^{\frac{1}{2}\dagger}_{\bar{\mu}_2\mu_2}[R_{fc}(k_2/m))].
\end{aligned} \tag{2.216}$$

Note that the Melosh rotations which appear for the case of particles with spin in the front form. For experiments where the magnetic quantum numbers are summed, these rotations do

not contribute to the cross section. For experiments where target and/or beam polarizations are measured, the above formula can be used directly if one considers the spin rotations as a change of representation. An important point is that the partial-wave representation of the internal interaction is invariant. If V is given in a partial wave representation, then it can be used to define a perturbation of M_0^2 which will provide the same fit to experimental data as a nonrelativistic interaction $V' = V/4m$ which has been fit to the same data.

2.7. **Summary of Examples**

We conclude this section with some general observations about the examples presented above. We wish to stress again the difference between the dynamics contained in a model for the mass operator M or its square, and the *form* of dynamics expressed in a particular unitary representation of the Poincaré group. The eigenvalue equation which must be solved is identical in structure to the Schrödinger equation, enabling us to make use of interactions fitted within a nonrelativistic framework, *but without sacrificing relativistic invariance.* The choice of *form* then determines the manner in which a system with four-momentum P is connected to the same system at rest. It is at this point that relativity plays a role, and it is also clear that there are choices involved.

Different choices of forms of dynamics also affect the manner in which the spin labels of state vectors are related under Lorentz transformations. In general, spin labels undergo Wigner rotations under arbitrary Lorentz transformations. Canonical spins have the special property that the Wigner rotation associated with a pure rotation is the rotation itself. This makes it possible to combine spins of more than one particle with ordinary rotation Clebsch-Gordan coefficients. Front-form spins can still be combined, but only after transforming to canonical spins with a Melosh rotation. On the other hand, because the front-form boosts form a subgroup, front-form spins do not undergo Wigner rotations under front-form boosts.

The interaction dependence of Lorentz transformations can be seen in the presence of the mass eigenvalue in at least some of the transformations. The choice of form is also a choice as to which subgroup of transformations does *not* depend upon the mass eigenvalue, but only on free-particle kinematic variables. For instant-form dynamics, this subgroup includes rotations and translations, but not boosts. In the front form, the subgroup involves certain combinations of boosts and rotations. Transformations between two different forms of dynamics is relatively straightforward when working with mass eigenstates. As stated above, the zero-momentum eigenstates are independent of the form of dynamics. For non-zero momentum, they are related by kinematic normalization factors and a Melosh rotation. Transformations between two different forms of dynamics are considerably more complicated in a basis of single-particle product states, because the mass operator is not diagonal in this basis.

3. **Symmetries in Quantum Mechanics**

In Sections 3–7, we present a more formal study of relativistic quantum mechanical models. We begin in this section by examining the general requirements of Poincaré invariance for a quantum mechanical system. We then introduce infinitesimal generators and their commutation relations, along with commuting Hermitian operators which are functions of the generators. The corresponding treatment for systems satisfying Galilean invariance is also presented so that the reader may better understand nonrelativistic systems within this context. In the relativistic case, we pay particular attention to the spin operator, which has noticeably different features from the nonrelativistic spin operator.

In quantum mechanics, symmetries imply constraints on the structure of a dynamical model. In general, a symmetry transformation has the property that its action on a quantum mechanical system leaves the physics unchanged. The predictions of quantum mechanics are probabilities. A quantum mechanical symmetry transformation therefore has the property that its action on states leaves probabilities unchanged. In a quantum theory, probabilities are expressed in terms of a scalar product on the Hilbert space $\mathcal{H}$. The probability of measuring a system prepared in a state represented by $|\Psi\rangle$ to be found in a state represented by $|\Phi\rangle$ is

$$P_{\Psi\Phi} := \frac{|\langle\Psi|\Phi\rangle|^2}{\langle\Psi|\Psi\rangle\,\langle\Phi|\Phi\rangle}. \tag{3.1}$$

A symmetry transformation is a correspondence of the form

$$|\Psi\rangle \to |\Psi'\rangle; \qquad |\Phi\rangle \to |\Phi'\rangle, \tag{3.2}$$

which satisfies

$$P_{\Psi\Phi} = P_{\Psi'\Phi'} \tag{3.3}$$

for all normalizable vectors $|\Psi\rangle$, $|\Phi\rangle \in \mathcal{H}$. Wigner's theorem (Go 66) shows that the most general correspondence that preserves condition (3.3) is either a unitary or an antiunitary transformation

T:

$$|\Psi\rangle \to |\Psi'\rangle = T(|\Psi\rangle) \qquad |\Phi\rangle \to |\Phi'\rangle = T(|\Phi\rangle). \tag{3.4}$$

In the cases of Galilean invariance and Poincaré invariance, there is an infinite number of symmetry transformations that can be parameterized by elements of a continuous group $\mathcal{G}$:

$$T \to T_g \qquad g \in \mathcal{G}. \tag{3.5}$$

The group transformation properties require that

$$T_{g_2 \circ g_1}(|\Psi\rangle) \qquad \text{and} \qquad T_{g_2}(T_{g_1}(|\Psi\rangle)) \tag{3.6}$$

correspond to the same physical state. Because two normalized vectors that differ by an overall phase correspond to the same physical state, the physical equivalence of the vectors in (3.6) implies

$$T_{g_2}(T_{g_1}(|\Psi\rangle)) = e^{i\phi(g_2,g_1)} T_{g_2 \circ g_1}(|\Psi\rangle), \tag{3.7}$$

where the phase factor $\phi(g_2, g_1)$ depends on the group elements. This means that T_g is a "ray representation" of $\mathcal{G}$.

In the neighborhood of the identity, each group element is the square of another group element:

$$T_g = T_{g_1} \circ T_{g_1} \tag{3.8}$$

(up to a possible phase factor), for g in a neighborhood of the identity. The composition of two identical unitary or antiunitary transformations is unitary. Thus, T_g is unitary in the neighborhood of the identity. If the group is pathwise connected, then all elements can be represented as finite products of group elements in the neighborhood of the identity (A group is pathwise connected if any two group elements can be continuously transformed from one to the other using a group-valued curve $\gamma(t)$, where $0 \leq t \leq 1$). Since a finite product of unitary operators is unitary, it follows that T_g is unitary for any g in the component of the group pathwise connected to the identity.

Thus, for the component T_g of the Poincaré group or Galilean group pathwise connected to the identity, we write $T_g \to U_g$, where U_g is unitary, and satisfies

$$U_{g_2} U_{g_1} = e^{i\phi(g_1, g_2)} U_{g_2 \circ g_1}. \tag{3.9}$$

A representation U_g satisfying Eq. (3.9) is called a *unitary ray representation* of the group $\mathcal{G}$. Because the assumption of invariant probabilities leads to a unitary symmetry, and unitary operators preserve the eigenvalues of self-adjoint operators, it also follows that expectation values and ensemble averages, which are composed of probabilities and eigenvalues, are also invariant.

This last statement requires a brief explanation with regard to tensor operators. For example, consider the four-momentum, which transforms as a four-vector operator under the action of a Lorentz transformation:

$$U(\Lambda, a) P^\mu U(\Lambda, a)^\dagger = (\Lambda^{-1})^\mu{}_\nu P^\nu. \tag{3.10}$$

This equation appears to violate the above statement concerning invariance, since the components of the momentum are not individually invariant.. That this is not the case can be seen by considering the following:

$$P^\mu |p^\mu\rangle = p^\mu |p^\mu\rangle \to U(\Lambda, a) P^\mu U(\Lambda, a)^\dagger U(\lambda, a) |p^\mu\rangle = p^\mu U(\Lambda, a) |p^\mu\rangle. \tag{3.11}$$

From Eq. (3.11), we see that $P'^\mu := U(\Lambda, a) P^\mu U(\Lambda, a)^\dagger$ and P^μ have identical eigenvalues corresponding to *different* eigenvectors. In particular, the eigenvector of P'^μ with eigenvalue p^μ is the eigenvector of P^μ with eigenvalue $\Lambda^\mu{}_\nu P^\nu$. In making coordinate changes, both the states and observables need to be transformed, and this results in identical expectation values. The same result holds for ensemble averages.

3.1. **Galilean Relativity**

The Galilean principle of relativity (Ba 54, Le 63, Wa 70) states that the *laws* of quantum mechanics do not distinguish between different inertial coordinate systems. This requires the existence of a unitary ray representation of the group of transformations that relate different inertial coordinate systems. What separates Galilean relativity from special relativity is the assumed relation between different inertial coordinate systems. A system is consistent with the principle of Galilean relativity if any two inertial coordinate systems are related by a coordinate transform that preserves the form of Newton's Second Law for a free particle.

The continuous transformations that preserve Newton's Second Law for a free particle are well known. They are generated by the following ten coordinate transformations:

$$(\mathbf{x}, t) \rightarrow (\mathbf{x}', t') = (\mathbf{x}, t + t_0); \tag{3.12}$$

$$(\mathbf{x}, t) \rightarrow (\mathbf{x}', t') = (\mathbf{x} + \mathbf{a}, t); \tag{3.13}$$

$$(\mathbf{x}, t) \rightarrow (\mathbf{x}', t') = (R\mathbf{x}, t), \qquad R \in SO(3); \tag{3.14}$$

$$(\mathbf{x}, t) \rightarrow (\mathbf{x}', t') = (\mathbf{x} + \mathbf{v}t, t), \tag{3.15}$$

which represent time translations, space translations, rotations and Galilean boosts (uniform rectilinear motion), respectively. These transformations generate the Galilean group of transformations, which has a linear representation by 5×5 matrices of the form (Wa 70):

$$g = \begin{pmatrix} R & \mathbf{v} & \mathbf{a} \\ 0 & 1 & t_0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{3.16}$$

where R is a 3×3 rotation matrix, and $\mathbf{v}$ and $\mathbf{a}$ are real three-vectors. The matrix g acts on column vectors that represent spacetime coordinates by

$$x = \begin{pmatrix} \mathbf{x} \\ t \\ 1 \end{pmatrix} \rightarrow x' := gx = \begin{pmatrix} R\mathbf{x} + \mathbf{v}t + \mathbf{a} \\ t + t^0 \\ 1 \end{pmatrix}. \tag{3.17}$$

In this matrix representation, the 1 in the last row does not change when multiplied by the matrix

g. A Galilean invariant quantum theory requires that vectors corresponding to states in different inertial coordinate systems are related by a unitary ray representation of the Galilean group:

$$|\Psi'\rangle = U_g|\Psi\rangle, \tag{3.18}$$

with U_g satisfying

$$U_{g_2}U_{g_1} = e^{i\phi(g_2,g_1)}U_{g_2\cdot g_1}. \tag{3.19}$$

One of the complications of Galilean relativity is that the true representations (*i.e.,* those where the phase is not needed) do not have a reasonable physical interpretation (In 52). The problem is that if one deduces the transformation properties of the infinitesimal generators of time translations (the Hamiltonian) and spatial translations (linear momentum) from the group structure, they do not transform as would be expected in classical mechanics. To see this, consider a simultaneous eigenstate of the linear momentum operator and Hamiltonian (energy) $|\mathbf{p}, E\rangle$. If $U(\mathbf{v})$ is the unitary representation of a Galilean boost $U(\mathbf{x}, t)$ is the unitary representation of a spacetime translation, and U_g is a single valued unitary representation of the Galilean group, then the group representation property (3.9) and the group multiplication law (Eqs. (3.12)–(3.15)) imply

$$U(\mathbf{x}, t)U(\mathbf{v}) = U(\mathbf{v})U(\mathbf{x} - \mathbf{v}t, t). \tag{3.20}$$

We now apply $U(\mathbf{x}, t)U(\mathbf{v})$ to a simultaneous eigenstate $|\mathbf{p}; E\rangle$ of energy and momentum, observing that

$$U(\mathbf{x}, t)|\mathbf{p}; E\rangle = e^{i(\mathbf{p}\cdot\mathbf{x} - Et)}|\mathbf{p}; E\rangle. \tag{3.21}$$

The result is

$$U(\mathbf{x}, t)U(\mathbf{v})|\mathbf{p}; E\rangle = U(\mathbf{v})U(\mathbf{x} - \mathbf{v}t, t)|\mathbf{p}; E\rangle = e^{i(\mathbf{p}\cdot\mathbf{x} - (E + \mathbf{p}\cdot\mathbf{v})t)}U(\mathbf{v})|\mathbf{p}, E\rangle. \tag{3.22}$$

We can read off the energy and momentum of the transformed state $U(\mathbf{v})|\mathbf{p}; E\rangle$ as the coefficients

of t and $\mathbf{x}$ in the exponent:

$$E \to E' = E + \mathbf{p} \cdot \mathbf{v} \qquad \mathbf{p} \to \mathbf{p}' = \mathbf{p}. \tag{3.23}$$

This differs from the expected transformation:

$$E \to E' = E + \mathbf{p} \cdot \mathbf{v} + \frac{1}{2} m \mathbf{v}^2; \qquad \mathbf{p} \to \mathbf{p}' = \mathbf{p} + m\mathbf{v}. \tag{3.24}$$

The difference between Eqs. (3.23) and (3.24) is contained in the terms proportional to the mass m. By assuming a single valued representation, the terms involving the mass have been lost.

Bargmann (Ba 54) showed that the correct relations between energy and momentum are recovered if one introduces a ray representation with the following phase factor:

$$\phi(g_1, g_2) = \frac{m}{2}(\mathbf{a}_2^T R_2 \mathbf{v}_1 - \mathbf{v}_2^T R_1 \mathbf{a}_1 + t_1 \mathbf{v}_2^T R_2 \mathbf{v}_1), \tag{3.25}$$

where the T superscript indicates the transpose of a vector. In this expression, m is the mass of the particle. It is possible to treat Eq. (3.19) with the phase factor $\phi(g_1, g_2)$ using standard group representation theory. This is done by extending the group to include a phase: $g \to (g, \phi)$, with the product:

$$(g_1, \phi_1) \circ (g_2, \phi_2) = (g_1 \circ g_2, \phi_1 + \phi_2 + \phi(g_1, g_2)). \tag{3.26}$$

The physical representation is not yet a true representation because of the usual phase ambiguity associated with half-integral spin particles. This can be eliminated if the $SO(3)$ group labels are replaced by $SU(2)$ labels. Given these two changes it is possible to realize the changes of inertial coordinate systems as single valued unitary representations of the form

$$U(\tilde{g}_2, \phi_2) U(\tilde{g}_1, \phi_1) = U(\tilde{g}_2 \circ \tilde{g}_1, \phi_{21}), \tag{3.27}$$

where $g \to \tilde{g}$ denotes the replacement of the $O(3)$ rotation R by an $SU(2)$ rotation U and ϕ_{12} denotes the phase on the right hand side of (3.26). Technically, a unitary ray representation of the Galilean group has been replaced by a single valued unitary representation of the central

extension (*i.e.,* extending the group to include the phase factor $\phi(g_1, g_2)$) of the universal covering group (*i.e.,* replacing $SO(3)$ with $SU(2)$) of the Galilean group. Neither of these changes alters the probabilities in Eq. (3.27). Their only effect is to allow the application of standard group representation theory to the evaluation of such probabilities.

The central extension of the Galilean group is an eleven-parameter group. A unitary representation has eleven infinitesimal Hermitian generators. These are the Hamiltonian H, which is the infinitesimal generator of time translations; the linear momentum operators $\mathbf{P}$, which are the generators of spatial translations, the angular momentum operators $\mathbf{J}$, which are the infinitesimal generators of rotations, the operators $\mathbf{K}$, which are the infinitesimal generators of uniform rectilinear motion (Galilean boosts), and the mass M, which is the generator of phase transformations in the central extension. The group transformation properties require that theses generators satisfy certain commutation relations. The non-zero commutators are listed below:

$$[J^i, J^k]_- = i\epsilon^{ijk}J^k; \qquad [J^i, K^k]_- = i\epsilon^{ijk}K^k; \qquad [J^i, P^k]_- = i\epsilon^{ijk}P^k \tag{3.28}$$

$$[K^j, H]_- = -iP^j; \qquad [K^j, P^k]_- = -i\delta_{jk}M. \tag{3.29}$$

There are two observations that can be made about these commutation relations that distinguish them from those of the Poincaré group. The first is that in a Galilean invariant theory, the mass operator is a generator. It commutes with all of the other generators. The second is that the Hamiltonian never appears on the right-hand side of the non-vanishing commutators. As will be seen shortly, this differs from a Poincaré invariant theory, where the mass operator is not a generator and the Hamiltonian appears on the right-hand side of some of the commutators.

The generator $\mathbf{K}$ is related to the position operator by

$$\mathbf{X} := -\mathbf{K}/M. \tag{3.30}$$

An internal angular momentum or spin observable can also be defined in terms of these generators

by

$$\mathbf{j} := \mathbf{J} - \mathbf{X} \times \mathbf{P}. \tag{3.31}$$

The above analysis exhibits the operators normally used in nonrelativistic quantum mechanics as functions of the infinitesimal generators associated with the underlying Galilean symmetry.

Before discussing Poincaré invariance in quantum mechanics, it is instructive to consider the constraints that Galilean invariance places on two-body interactions in nonrelativistic quantum mechanics. These will be compared to similar constraints in the Poincaré invariant case.

For a free particle, one can construct the generators of the Galilean group from the operators M, $\mathbf{P}$, $\mathbf{j}$ and $\mathbf{X}$. For two free particles, the generators are sums of the one-body generators. The position operator is then given by

$$\mathbf{X} = -\frac{\mathbf{K}_1 + \mathbf{K}_2}{M_1 + M_2} = \frac{M_1\mathbf{X}_1 + M_2\mathbf{X}_2}{M_1 + M_2}, \tag{3.32}$$

and the spin operator is

$$\mathbf{j} = \mathbf{J} - \mathbf{X} \times \mathbf{P} = (\mathbf{j}_1 + \mathbf{X}_1 \times \mathbf{P}_1 + \mathbf{j}_2 + \mathbf{X}_2 \times \mathbf{P}_2) - \mathbf{X} \times (\mathbf{P}_1 + \mathbf{P}_2). \tag{3.33}$$

It is possible to include an interaction V in the Hamiltonian in a manner that preserves the commutation relations of the Galilean group without modifying any other generators. Since H never appears on the right-hand side of any commutator, the commutation relations will be satisfied if V commutes with the remaining generators. Equivalently, we can write

$$[\mathbf{P}, V]_- = [\mathbf{j}, V]_- = [\mathbf{X}, V]_- = [M, V]_- = 0, \tag{3.34}$$

since it is possible to construct the remaining generators as functions of these operators. These conditions require that the interaction commutes with the total momentum, the total mass, is independent of the total momentum, and is rotationally invariant. More general realizations are possible if the other ten generators are allowed to have interactions, but these do not appear to be of any practical interest.

3.2. **Special Relativity - The Poincaré group**

In the case of the Poincaré group, Bargmann showed that Wigner's theorem could be modified without loss of generality by replacing "ray representation of the component of the Poincaré group connected to the identity" by "single valued unitary representation of the covering group of the Poincaré group." For completeness, the theorem is stated below (Ba 54):

> *A quantum mechanical model formulated on a Hilbert space preserves probabilities in all inertial coordinate systems if and only if the correspondence between states in different inertial coordinate systems can be realized by single valued unitary representation of the covering group of the Poincaré group.*

This theorem will not be proved here, but the source of the difficulty is related phase ambiguities that occur when one rotates particles with half-integer spin through an angle of 2π. The covering group does nothing more that put two distinct labels on each Lorentz transformation that keeps track of these phases. The covering group of the Poincaré group is inhomogeneous $SL(2,C)$, or $ISL(2,C)$. Elements of the group are ordered pairs of 2×2 matrices $(\underline{\Lambda}, \underline{a})$, where $\underline{\Lambda}$ has determinant = 1 and $\underline{a}$ is Hermitian. To understand the relation of $ISL(2,C)$ to the Poincaré group, we note that any space-time coordinate x^μ can be represented by a 2×2 Hermitian matrix:

$$\underline{X} := x^\mu \sigma_\mu = \begin{pmatrix} x^0 + x^3 & x^1 - ix^2 \\ x^1 + ix^2 & x^0 - x^3 \end{pmatrix}; \qquad x^\mu = \tfrac{1}{2}\,\mathrm{Tr}(\sigma_\mu \underline{X}), \tag{3.35}$$

where σ_0 is the 2×2 identity matrix and $\boldsymbol{\sigma}$ are the three traceless Hermitian Pauli matrices,

$$\sigma_0 := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}; \quad \sigma_1 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}; \quad \sigma_2 := \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}; \quad \sigma_3 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{3.36}$$

In all that follows, an underscore will indicate a 2×2 matrix, except for the case of the Pauli matrices, where we use the notation above. In this representation, the most general transformations that preserve the proper time between events, the Hermiticity of X, the handedness of space and the direction of time can be put in the form

$$\underline{X} \to \underline{X}' = \underline{\Lambda} \underline{X} \underline{\Lambda}^\dagger + \underline{a}, \tag{3.37}$$

with $\det|\underline{\Lambda}|=1$ and $\underline{a} = \underline{a}^\dagger$. These transformations preserve the proper time between events

because

$$\tau^2_{AB} = \det|\underline{X}_A - \underline{X}_B| = \det|\underline{X}'_A - \underline{X}'_B| = \tau^2_{A'B'}. \tag{3.38}$$

Each of the pairs $(\underline{\Lambda}, \underline{a})$ and $(-\underline{\Lambda}, \underline{a})$ correspond to the same Poincaré transformation satisfying $\det|\Lambda| = 1$ and ${\Lambda^0}_0 \geq 1$. From Eqs.(3.35)–(3.38), this correspondence is given by

$${\Lambda^\mu}_\nu := \tfrac{1}{2}\,\mathrm{Tr}(\sigma_\mu \underline{\Lambda} \sigma_\nu \underline{\Lambda}^\dagger); \qquad a^\mu = \tfrac{1}{2}\,\mathrm{Tr}(\sigma_\mu \underline{a}). \tag{3.39}$$

The inverse expressions can be found in (Wi 60). In general, the notation ${\Lambda^\mu}_\nu$, Λ, and $\underline{\Lambda}$ will be used interchangeably. When there is any ambiguity, such as in the argument of a D function, the 2×2 matrix is the correct form to use. Similarly a^μ, a and $\underline{a}$ will be used interchangeably. The ordered pairs of matrices $(\underline{\Lambda}, \underline{a})$ form a group under composition

$$(\underline{\Lambda}_2, \underline{a}_2) \circ (\underline{\Lambda}_1, \underline{a}_1) := (\underline{\Lambda}_2\underline{\Lambda}_1, \underline{\Lambda}_2\underline{a}_1\underline{\Lambda}_2^\dagger + \underline{a}_2), \tag{3.40}$$

with inverse

$$(\underline{\Lambda}, \underline{a})^{-1} = (\underline{\Lambda}^{-1}, -\underline{\Lambda}^{-1}\underline{a}(\underline{\Lambda}^\dagger)^{-1}). \tag{3.41}$$

and identity $(\underline{I}, \underline{0})$. The component of the Poincaré group connected to the identity is isomorphic to the group obtained by identifying $(\underline{\Lambda}, \underline{a})$ and $(-\underline{\Lambda}, \underline{a})$ in $ISL(2, C)$. In the remainder of this article, we use the notation $\mathcal{P}$ to denote $ISL(2, C)$.

Inertial coordinate systems can be labeled by elements (Λ, a) of the Poincaré group, or by elements $(\underline{\Lambda}, \underline{a})$ of $\mathcal{P}$. The difference is that $\mathcal{P}$ provides two distinct labels $(\underline{\Lambda}, \underline{a})$ and $(-\underline{\Lambda}, \underline{a})$ for each inertial coordinate system, while elements of the component of the Poincaré group connected to the identity provide a unique label for each inertial coordinate system.

A single valued unitary representation $U(\underline{\Lambda}, \underline{a})$ of $\mathcal{P}$ is a function from the group $\mathcal{P}$ to the space of linear operators on the model Hilbert space $\mathcal{H}$ satisfying:

$$U(\underline{\Lambda}_2, \underline{a}_2)U(\underline{\Lambda}_1, \underline{a}_1) = U[(\underline{\Lambda}_2, \underline{a}_2) \circ (\underline{\Lambda}_1, \underline{a}_1)] = U(\underline{\Lambda}_2\underline{\Lambda}_1, \underline{\Lambda}_2\underline{a}_1(\underline{\Lambda}_2)^\dagger + \underline{a}_2), \tag{3.42}$$

and

$$U(\underline{\Lambda}, \underline{a})^{\dagger} = U(\underline{\Lambda}, \underline{a})^{-1} = U[(\underline{\Lambda}, \underline{a})^{-1}]. \tag{3.43}$$

The discussion to this point has identified a mathematical characterization of the concept of relativistic invariance. The construction of a relativistic quantum mechanical model is equivalent to construction of a unitary representation of the group $\mathcal{P}$ on the model Hilbert space. Below, we will work out the consequences of the assumption that probabilities have values that are independent of coordinate systems which are related by Poincaré transformations.

3.3. **Parameterization of Poincaré Transformations**

The group $\mathcal{P}$ is parameterized by an ordered pair $(\underline{\Lambda}, \underline{a})$ of 2×2 matrices, where the matrix $\underline{\Lambda}$ is restricted to have $\det|\underline{\Lambda}| = 1$, and the matrix $\underline{a}$ is restricted to be Hermitian.

The most general 2×2 matrix $\underline{\Lambda}$ with $\det|\underline{\Lambda}| = 1$ can be represented by

$$\underline{\Lambda} = \underline{\Lambda}(\boldsymbol{\theta}, \boldsymbol{\rho}) = \exp\big(-\frac{i}{2}(\boldsymbol{\theta} + i\boldsymbol{\rho}) \cdot \boldsymbol{\sigma}\big). \tag{3.44}$$

This follows because $\det|\exp(\mathbf{C})| = \exp(\mathrm{Tr}(\mathbf{C}))$, and $\mathrm{Tr}(\sigma) = 0$.

The parameters $\boldsymbol{\theta}$ represent the angle and axis of a rotation, while $\boldsymbol{\rho}$ represents the direction and rapidity of a rotationless Lorentz transformation. In general, unitary $\underline{\Lambda}$'s represent rotations, while positive Hermitian $\underline{\Lambda}$'s represent rotationless Lorentz transformations. The polar decomposition theorem ensures that any non-singular 2×2 matrix can be always be represented as the product of a unitary matrix and a positive Hermitian matrix.

The most general 2×2 Hermitian matrix $\underline{a}$ has the form:

$$\underline{a} = a^{\mu}\sigma_{\mu}, \tag{3.45}$$

where the coefficients a^{μ} are real. The parameters $\boldsymbol{\rho}$, $\boldsymbol{\theta}$ and a^{μ} have the special property that if any nine of them are set to zero, the subgroup generated by the remaining one is a one-parameter Abelian subgroup. These quantities are taken as the ten parameters of $\mathcal{P}$.

3.4. **Definition of Infinitesimal Generators**

Differentiation of the abstract representation $U(\underline{\Lambda}, \underline{a})$ with respect to angle, rapidity and spacetime coordinate in the neighborhood of the identity defines abstract generators. Because the representation $U(\underline{\Lambda}, \underline{a})$ is a one-parameter unitary Abelian group with respect to any of the parameters $\boldsymbol{\rho}$, $\boldsymbol{\theta}$ or b^μ with the other parameters set to zero, we can express $U(\underline{\Lambda}(\boldsymbol{\theta}, \boldsymbol{\rho}), \underline{a})$ in the form

$$U(\underline{\Lambda}(\boldsymbol{\theta}, \boldsymbol{\rho}), \underline{a}) := e^{iP\cdot a} e^{-i(\mathbf{J}\cdot\boldsymbol{\theta} + \mathbf{K}\cdot\boldsymbol{\rho})} \tag{3.46}$$

Note that the generators defined here sometimes have sign differences with generators defined elsewhere. The source of this difference is in the sign of the generator of rotationless Lorentz transformations. Our convention is fixed by Eq. (3.46) and the definition of rapidity in Eq. (3.44). Equation (3.46) implies the following definitions for the infinitesimal generators:

$$P^\mu := -ig^{\mu\nu} \left.\frac{\partial}{\partial a^\nu} U(\underline{\Lambda}, \underline{a})\right|_{\boldsymbol{\rho}=\boldsymbol{\theta}=a^\mu=0}; \tag{3.47}$$

$$K^j := i \left.\frac{\partial}{\partial \rho^j} U(\underline{\Lambda}, \underline{a})\right|_{\boldsymbol{\rho}=\boldsymbol{\theta}=a^\mu=0}; \tag{3.48}$$

$$J^j := i \left.\frac{\partial}{\partial \theta^j} U(\underline{\Lambda}, \underline{a})\right|_{\boldsymbol{\rho}=\boldsymbol{\theta}=a^\mu=0}. \tag{3.49}$$

It follows from Stone's theorem that these generators are self-adjoint operators (Yo 80). The Hamiltonian H is the operator for the total energy of the system, which is the infinitesimal generator of time translations; $\mathbf{P}$ is the total linear momentum and $\mathbf{J}$ is the total angular momentum. The generator $\mathbf{K}$ has no physical interpretation.

3.5. **Commutation Relations - Canonical Form**

The commutation relations of the generators defined in the previous subsection are computed from the group representation property (3.42). These are given in three equivalent forms that are useful in different applications. The commutation relations are needed to identify complete sets of commuting self-adjoint operators, and to determine the spectrum of these operators. Given

these operators and a knowledge of their spectrum, it possible to construct representations of the Hilbert space as a space of functions. This will be done in the next section.

The commutators of the generators can be computed from the definitions (3.47)–(3.49) and the group representation property (3.42), using the formula:

$$\begin{aligned}\frac{\partial^2}{\partial c_1 \partial c_2}\left[U(g_1)U(g_2)U(g_1^{-1})U(g_2^{-1})\right]_{|\boldsymbol{\rho}=\boldsymbol{\theta}=a^\mu=0} \\ = \frac{\partial^2}{\partial c_1 \partial c_2} U(g_1 \cdot g_2 \cdot g_1^{-1} \cdot g_2^{-1})_{|\boldsymbol{\rho}=\boldsymbol{\theta}=a^\mu=0},\end{aligned} \tag{3.50}$$

where $g_i = (\underline{\Lambda}_i, \underline{a}_i)$, and $g_i^{-1} = (\underline{\Lambda}_i^{-1}, -\underline{\Lambda}_i^{-1}\underline{a}_i(\underline{\Lambda}_i^{-1})^\dagger)$, and c_i are parameters chosen among ρ^i, θ^i, and a^μ.

These are most efficiently computed by using the infinitesimal forms and expanding in a neighborhood of the identity to leading non-trivial order in the parameters ρ^i, θ^i and a^μ. For example, to compute the commutator of K^1 with J^2, we use Eq. (3.46) to expand $g_1 = \exp(-\frac{i}{2} i\rho_1\sigma_1)$ and $g_2 = \exp(-\frac{i}{2}\theta_2\sigma_2)$ to first order in rapidity ρ_1 and angle θ_2 of the corresponding finite transformations:

$$\underline{\Lambda}_1 = 1 - \frac{i}{2} i\rho_1\sigma_1 + o(\rho_1^2); \qquad \underline{\Lambda}_2 = 1 - \frac{i}{2}\theta_2\sigma_2 + o(\theta_2^2). \tag{3.51}$$

Keeping only the coefficients of $\rho_1\theta_2$, we get

$$\begin{aligned}&\underline{\Lambda}_1\underline{\Lambda}_2\underline{\Lambda}_1^{-1}\underline{\Lambda}_2^{-1} \\ &\quad = I - \tfrac{1}{4}\rho_1\theta_2\left[(i\sigma_1)(\sigma_2) + (i\sigma_1)(-\sigma_2) + (\sigma_2)(-i\sigma_1) + (-i\sigma_1)(-\sigma_2)\right] + \cdots \\ &\quad = I - \frac{i}{2}\rho_1\theta_2(i\sigma_3) + \cdots.\end{aligned} \tag{3.52}$$

On comparison with Eq. (3.46), we obtain

$$U(\underline{\Lambda}_1\underline{\Lambda}_2\underline{\Lambda}_1^{-1}\underline{\Lambda}_2^{-1}) = U(I - \frac{i}{2}\rho_1\theta_2(i\sigma_3) + \cdots) = I - i\rho_1\theta_2 K^3 + \cdots. \tag{3.53}$$

We also use Eq. (3.46) to expand $U(g_1)$ and $U(g_2)$ to first order in ρ_1 and θ_2:

$$U(\underline{\Lambda}_1, 0) = I - i\rho_1 K^1 + o(\rho_1^2); \qquad U(\underline{\Lambda}_2, 0) = I - i\theta_2 J^2 + o(\theta_2^2), \tag{3.54}$$

with the result:

$$\begin{aligned} &U(\underline{\Lambda}_1)U(\underline{\Lambda}_2)U(\underline{\Lambda}_1^{-1})U(\underline{\Lambda}_2^{-1}) \\ &\quad = I + \rho_1\theta_2\left[(-iK^1)(-iJ^2) + (-iK^1)(iJ^2) + (-iJ^2)(iK^1) + (iK^1)(iJ^2)\right] + \cdots. \end{aligned} \tag{3.55}$$

Comparing the coefficients of $\rho_1\theta_2$ in Eqs. (3.53) and Eq. (3.55) (or, equivalently, differentiating and setting the parameters to zero), we get

$$[J^2, K^1]_- = J^2K^1 - K^1J^2 = -iK^3. \tag{3.56}$$

If this procedure is repeated for all pairs of generators, one obtains the following forty-five commutators:

$$[J^j, J^k]_- = i\epsilon^{jkl}J^l; \qquad [K^j, K^k]_- = -i\epsilon^{jkl}J^l; \tag{3.57}$$

$$[J^j, K^k]_- = i\epsilon^{jkl}K^l; \tag{3.58}$$

$$[P^\mu, P^\nu]_- = 0; \tag{3.59}$$

$$[K^j, P^0]_- = -iP^j; \qquad [J^j, P^0]_- = 0; \tag{3.60}$$

$$[K^k, P^j]_- = -i\delta^{jk}P^0; \qquad [J^j, P^k]_- = i\epsilon^{jkl}P^l. \tag{3.61}$$

Equations (3.57)–(3.58) are the commutation relations for the Lorentz group, Eqs. (3.59) give the commutation relations for the group of spacetime translations, and Eqs. (3.60)-(3.61) constrain P^μ to transform as a four-vector under Lorentz transformations.

Commutation Relations - Covariant Form The commutation relations can be put in a

manifestly covariant form if they are expressed in terms of the angular momentum tensor $J^{\alpha\beta}$:

$$J^{0j} := K^j; \tag{3.62}$$

$$J^{jk} := \epsilon^{jkl} J^l; \tag{3.63}$$

$$J^{\alpha\beta} := -J^{\beta\alpha}. \tag{3.64}$$

In terms of these operators, the commutation relations (3.57)-(3.61) become

$$[J^{\alpha\beta}, J^{\rho\sigma}]_- = i(g^{\beta\sigma} J^{\alpha\rho} - g^{\nu\rho} J^{\alpha\sigma} + g^{\alpha\rho} J^{\nu\sigma} - g^{\alpha\sigma} J^{\nu\rho}); \tag{3.65}$$

$$[P^\mu, P^\nu]_- = 0; \tag{3.66}$$

$$[J^{\rho\sigma}, P^\mu]_- = i(g^{\mu\rho} P^\sigma - g^{\mu\sigma} P^\rho). \tag{3.67}$$

Commutation Relations - Front Form Any linear combinations of Poincaré generators are also generators. Another set of generators that is useful in applications is the set of "front-form" generators. Since these generators are important in applications, we provide them explicitly. They have the property that seven of them form a closed Lie subalgebra that does not involve the generator that plays the role of the Hamiltonian. These are called "front form" generators because the seven operators generate those Poincaré transformations which leave the "light front", $x^+ := x^0 + x^3 = 0$, invariant. If x^+ is set to any *non-zero* constant, this surface is left invariant by only a six-parameter subgroup. The seven generators are P^1 ,P^2, J^3 and K^3, which have been defined previously, and

$$P^+ := P^0 + P^3 \tag{3.68}$$

$$\mathbf{E}_\perp := \mathbf{K}_\perp - \hat{\mathbf{z}} \times \mathbf{J}_\perp. \tag{3.69}$$

In addition to these, there are three other generators that take points on the light front to points

away from the light front:

$$P^- := P^0 - P^3 \tag{3.70}$$

$$\mathbf{F}_\perp := \mathbf{K}_\perp + \hat{\mathbf{z}} \times \mathbf{J}_\perp. \tag{3.71}$$

Sometimes $\mathbf{F}_\perp$ is replaced by $\mathbf{J}_\perp$. The operator P^- plays the role of the Hamiltonian in the front form, generating translations in x^+. The finite Lorentz transformations generated by E^1, E^2, F^1 and F^2 correspond to one-parameter groups $\underline{\Lambda}(\boldsymbol{\rho}_\lambda, \boldsymbol{\theta}_\lambda)$, with $(\boldsymbol{\theta}_\lambda, \boldsymbol{\rho}_\lambda)$ given by $\lambda(\hat{\mathbf{x}}, \hat{\mathbf{y}})$, $\lambda(-\hat{\mathbf{y}}, \hat{\mathbf{x}})$, $\lambda(-\hat{\mathbf{x}}, \hat{\mathbf{y}})$ and $\lambda(\hat{\mathbf{y}}, \hat{\mathbf{x}})$, respectively. The commutation relations among the front-form generators are fixed by the definitions (3.68)–(3.71). The commutation relations are given in Appendix B.

3.6. **Commuting Self-Adjoint Operators**

The generators $\{H, \mathbf{P}, \mathbf{J}, \mathbf{K}\}$ and the corresponding front-from generators can now be used to construct a set of commuting self-adjoint operators. The eigenvalues of these operators are used to label different irreducible representations of the Poincaré group, and to label different vectors in an irreducible subspace.

The Mass Operator The square of the mass operator is a second degree polynomial function of the generators defined by

$$M^2 := -P^\mu P_\mu = H^2 - \mathbf{P}^2 = P^+P^- - \mathbf{P}_\perp^2. \tag{3.72}$$

The commutation relations (3.57)–(3.61) imply that M^2 commutes with all of the operators $\{H, \mathbf{P}, \mathbf{J}, \mathbf{K}\}$ and their front-form equivalents. If M^2 represents the mass of a real system, it is a physical requirement that the eigenvalues of this operator are non-negative. Any self-adjoint operator with strictly non-negative eigenvalues has a unique non-negative square root (Re 72). Formally, this is the operator obtained by diagonalizing M^2 and replacing all of the eigenvalues by their non-negative square roots. The mass operator associated with M^2 is defined by:

$$M := \sqrt{H^2 - \mathbf{P}^2} = \sqrt{P^+P^- - \mathbf{P}_\perp^2}. \tag{3.73}$$

The mass operator is defined by this expression in any Poincaré invariant quantum theory that satisfies the *spectral condition*, $M^2 \geq 0$. This is also true in local relativistic quantum field

theories, although the mass operator is seldom used in practice in field theories. The mass operator commutes with all of the Poincaré generators. For a single free particle, M has one eigenvalue, which is the mass m. Relation (3.73) can also be used to express the Hamiltonian in terms of the mass and the momentum operators:

$$H = \sqrt{M^2 + \mathbf{P}^2}. \tag{3.74}$$

The corresponding expression for P^- is:

$$P^- = \frac{M^2 + \mathbf{P}_\perp^2}{P^+}. \tag{3.75}$$

The Pauli-Lubanski Operator In relativistic systems, the spin (total intrinsic angular momentum) is related to the *Pauli-Lubanski vector* (Lu 42), which is defined as follows:

$$W^\mu := \tfrac{1}{2}\epsilon^{\mu\alpha\beta\gamma} P_\alpha J_{\beta\gamma}, \tag{3.76}$$

where $\epsilon^{\mu\alpha\beta\gamma}$ is the completely antisymmetric tensor in four dimensional spacetime with $\epsilon^{0123} = 1$. W^μ is a pseudo-four-vector, with components

$$W^0 = \mathbf{P}\cdot\mathbf{J}; \qquad \mathbf{W} = H\mathbf{J} - \mathbf{P}\times\mathbf{K}. \tag{3.77}$$

It follows from the commutation relations and the self-adjointness of the generators that

$$W^0 = (W^0)^\dagger; \qquad \mathbf{W} = (\mathbf{W})^\dagger. \tag{3.78}$$

From the commutation relations for the generators and Eqs. (3.77)–(3.78), it can also be shown that W^μ has the following commutation relations with the generators

$$[P^\mu, W^\nu]_- = 0; \tag{3.79}$$

$$[\mathbf{J}, W^0]_- = 0; \qquad [J^j, W^k]_- = i\epsilon^{jkl} W^l; \tag{3.80}$$

$$[\mathbf{K}, W^0]_- = -i\mathbf{W}; \qquad [K^j, W^k]_- = -i\delta^{jk} W^0. \tag{3.81}$$

The relations (3.80) and (3.81) have the same form as Eqs. (3.60)–(3.61), with the generators P^μ replaced by W^μ. These relations ensure that the components of W^μ transform like a four-

vector under Lorentz transformations. The components of W^μ have the following commutation relations:

$$[W^\mu, W^\nu]_- = i\epsilon^{\mu\nu\rho\sigma} W_\rho P_\sigma. \tag{3.82}$$

The quantity

$$W^2 := W^\mu W_\mu := M^2 j^2 \tag{3.83}$$

is a polynomial in the generators that is independent of M^2 and commutes with all of the generators. The operators W^2 and M^2 are the only independent polynomial functions of the generators that commute with all generators. The operator j^2 is the total intrinsic spin operator of the system.

Boosts Although the spin operator for the particle is unambiguously defined in terms of the generators, there is an infinite number of spin vector valued functions of the generators that satisfy angular momentum commutation relations and whose square is the total spin. The different types of spin vectors are distinguished by their behavior under Lorentz boosts. To see this, note that Eq. (3.76) implies the relation

$$W^\mu P_\mu = 0. \tag{3.84}$$

For timelike P^μ, Eq. (3.84) implies that the Pauli-Lubanski vector is spacelike. If P^μ and W^μ were c-numbers, then W^μ could be transformed to a three-vector by a suitable Lorentz transformation. Equation (3.83) implies that the resulting three-vector should be divided by M to obtain a spin vector. The actual construction of a spin vector *operator* is more complicated, because relations (3.79) and (3.84) are operator relations rather than c-number relations. In addition, there is more than one way to transform a spacelike four-vector to a three-vector, since any rotation will not change the three-vector nature of an operator.

We now construct a set of boost operators $L_g(Q)^\mu{}_\nu$, where $Q := P/M$ is the four-velocity, with the properties

$$L_g(Q)^\mu{}_\nu (1,0,0,0)^\nu = M^{-1} P^\mu, \tag{3.85}$$

and

$$L_g(1,0,0,0)^\mu{}_\nu = g^\mu{}_\nu. \tag{3.86}$$

Equation (3.85) is to be interpreted as defining sixteen operator-valued functions of the operators $\{P^\mu\}$ which, when applied to a simultaneous eigenstate of the operators $\{P^\mu\}$, take on the values of the sixteen components of a Lorentz transformation that maps $(m,0,0,0)$ to P^μ. This type of Lorentz transformation will be called a "boost" which generalizes the notion of a rotationless Lorentz transformation which will be referred to as a "canonical boost." The index g differentiates between different possible choices of Lorentz transformations that satisfy Eqs. (3.85) and (3.86). Three different Lorentz transformations with this property are given below. These are known as *canonical* (or rotationless) boosts, *front-form* boosts, and *helicity* boosts. These transformations are used to construct the spin vectors which appear in most applications. The rotationless Lorentz transformation $L_c^{-1}(Q)$ was introduced in Section 2, and is defined by its action on a four-vector A^μ:

$$\begin{pmatrix} A'^0 \\ \mathbf{A}' \end{pmatrix} = L_c^{-1}(Q) \begin{pmatrix} A^0 \\ \mathbf{A} \end{pmatrix} = \begin{pmatrix} A^0\sqrt{1+\mathbf{Q}^2} - \mathbf{Q}\cdot\mathbf{A} \\ \mathbf{A} - A^0\mathbf{Q} + \mathbf{Q}\,(\mathbf{Q}\cdot\mathbf{A})(1+\sqrt{1+\mathbf{Q}^2})^{-1} \end{pmatrix}. \tag{3.87}$$

The 2×2 matrix representation of this transformation is

$$\underline{L}_c(Q) = \exp(\tfrac{1}{2}\boldsymbol{\omega}\cdot\boldsymbol{\sigma}), \tag{3.88}$$

where

$$\boldsymbol{\omega} := \widehat{\mathbf{Q}}\,\sinh^{-1} Q, \tag{3.89}$$

and $\mathbf{Q} := \mathbf{P}/M$ and $Q := |\mathbf{Q}|$. The inverse transformation of (3.87) is obtained by reversing the sign of $\mathbf{Q}$.

The front-form Lorentz boost $L_f(Q)$, which was also introduced in Section 2, is defined by the following action on any four-vector A:

$$\begin{pmatrix} A^{+\prime} \\ \mathbf{A}'_\perp \\ A^{-\prime} \end{pmatrix} = L_f(Q) \begin{pmatrix} A^+ \\ \mathbf{A}_\perp \\ A^- \end{pmatrix} = \begin{pmatrix} Q^+A^+ \\ \mathbf{A}_\perp + \mathbf{Q}_\perp A^+ \\ (Q^+)^{-1}\left(\mathbf{Q}_\perp^2 A^+ + 2\mathbf{Q}_\perp\cdot\mathbf{A}_\perp + A^-\right) \end{pmatrix}, \tag{3.90}$$

where $A^\pm := A^0 \pm A^3$. This is called a front-form boost because it is a type of Lorentz transformation that leaves the light front ($x^+ = 0$) invariant. The 2×2 matrix representation of this

transformation is:

$$\begin{aligned}\underline{L}_f(Q) &= \exp\left[\frac{1}{2Q^+}(Q^1+iQ^2)(\sigma_1-i\sigma_2)\right]\exp(\tfrac{1}{2}\sigma_3\ln Q^+)\\ &= \exp(\tfrac{1}{2}\sigma_3\ln Q^+)\exp\left[\tfrac{1}{2}(Q^1+iQ^2)(\sigma_1-i\sigma_2)\right]\\ &= \frac{1}{\sqrt{Q^+}}\begin{pmatrix} Q^+ & 0\\ (Q^1+iQ^2) & 1\end{pmatrix}.\end{aligned} \tag{3.91}$$

The inverse transformation is obtained by the substitution $\mathbf{Q}_\perp \to -\mathbf{Q}_\perp/Q^+$, $Q^+ \to 1/Q^+$:

$$\underline{L}_f^{-1}(Q) = \frac{1}{\sqrt{Q^+}}\begin{pmatrix} 1 & 0\\ -(Q^1+iQ^2) & Q^+\end{pmatrix}. \tag{3.92}$$

The third type of Lorentz transformation is associated with the helicity. The Lorentz transformation $L_h(Q)$ is a canonical boost in the z direction to a momentum of the desired magnitude, followed by a rotation from the z axis to the axis that defines the direction of momentum (We 64):

$$\underline{L}_h(Q) = \underline{R}(\hat{\mathbf{z}} \to \hat{\mathbf{p}})\underline{L}_c(|\mathbf{p}|\hat{\mathbf{z}}), \tag{3.93}$$

where the axis of rotation is

$$\hat{\boldsymbol{\theta}} = \frac{\hat{\mathbf{z}}\times\mathbf{P}}{|\hat{\mathbf{z}}\times\mathbf{P}|}, \tag{3.94}$$

and the angle of rotation is

$$\theta = \cos^{-1}(\hat{\mathbf{z}}\cdot\hat{\mathbf{P}}). \tag{3.95}$$

Spin Vectors Each set of operators in the previous subsection has the property that when their inverses are applied to a simultaneous eigenstate of $\mathbf{P}$ and M, they take on the value of a Lorentz transformation that maps P^μ to $(M,0,0,0)$. Since each of these operators is a function of operators which in turn commute with all components of the Pauli-Lubanski vector, (see Eq. (3.79)), it follows that

$$(0,\mathbf{j}_g) := \frac{1}{M}L_g^{-1}(Q)^\mu{}_\nu W^\nu \tag{3.96}$$

has three non-vanishing components when it is applied to any simultaneous eigenstate of P^μ. Note that although the right-hand side of (3.96) has the formal appearance of a four-vector,

and the left-hand side has the formal appearance of a three-vector, the quantity above does *not* transform like a four-vector under Lorentz transformations, and does *not* generally transform like a three-vector under rotations. This is because the argument of the Lorentz transformation is an operator rather that a parameter. Only in the special case of the canonical spin does $\mathbf{j}_c$ transform like a three-vector under rotations. A subscript g has been included on $\mathbf{j}_g$ to emphasize the fact that *different operators are obtained for different choices of* $L_g(Q)^\mu{}_\nu$. The canonical spin $\mathbf{j}_c$, is obtained from the canonical Lorentz boost, the front-form spin $\mathbf{j}_f$ is obtained from the front-form Lorentz transformation, and the helicity spin is obtained from the "helicity boost" $L_h(Q)$. In general, an infinite number of different types of spins is possible.

It is an immediate consequence of Eq. (3.96) and the transformation properties of four-vectors under Lorentz transformations that

$$\mathbf{j}_g \cdot \mathbf{j}_g = \frac{1}{M^2} W^2 = j^2 \tag{3.97}$$

independent of g. Making use of the commutation relations among components of the Pauli-Lubanski vector W^μ, we obtain

$$M^2[j_g^k, j_g^l] = i\epsilon^{kln} M^2 j_g^n \tag{3.98}$$

for any choice of boost g. When M has no vanishing eigenvalues, the M^2 factor can be canceled from both sides of this equation. The result is that any of the spin operators defined by Eq. (3.96) satisfies $SU(2)$ commutation relations:

$$[j_g^k, j_g^l] = i\epsilon^{kln} j_g^n. \tag{3.99}$$

Thus when $M \neq 0$, the spectrum of $\mathbf{j}_g^2$ is necessarily of the form $s(s+1)$, where s is an integer or half integer. Similarly, the spectrum of any component of $\mathbf{j}_g$ is $(-s, -s+1, \cdots, s-1, s)$.

It is possible to extract the spin directly from the angular momentum tensor without explicitly constructing the Pauli-Lubanski vector. The components are given by

$$j_g^k = \tfrac{1}{2}\epsilon^{kln} j_g^{ln}; \qquad j_g^{ln} := L_g^{-1}(Q)^l{}_\alpha L_g^{-1}(Q)^n{}_\beta J^{\alpha\beta}. \tag{3.100}$$

Note that in the same way that $\mathbf{j}_g$ is *not* a four-vector, the quantities j_g^{jk} are *not* the spatial components of a tensor operator.

The components of the canonical spin vector, the front-form spin vector, and the helicity spin vector can be computed explicitly in terms of the generators using the definition (3.96), and the inverse of the Lorentz transformations $L_g(Q)$ given in Eqs. (3.87), (3.91) and (3.93), respectively. The canonical spin is

$$\mathbf{j}_c = \frac{1}{M}\left(\mathbf{W} - \frac{\mathbf{P}W^0}{M+H}\right) = \frac{1}{M}\left[(H\mathbf{J} - \mathbf{P}\times\mathbf{K}) - \frac{\mathbf{P}(\mathbf{P}\cdot\mathbf{J})}{M+H}\right]. \tag{3.101}$$

The front-form spin is

$$j_f^3 = \frac{W^+}{P^+} = \frac{1}{P^+}\left[P^+J^3 + \hat{\mathbf{z}}\cdot(\mathbf{E}_\perp\times\mathbf{P}_\perp)\right]; \tag{3.102}$$

$$\begin{aligned}\mathbf{j}_{f\perp} &= \frac{1}{M}\left(\mathbf{W}_\perp - \frac{W^+}{P^+}\mathbf{P}_\perp\right)\\ &= \frac{1}{M}\left\{\hat{\mathbf{z}}\times\left[\tfrac{1}{2}(P^-\mathbf{E}_\perp - P^+\mathbf{F}_\perp) + \mathbf{P}_\perp K^3\right] - \frac{\mathbf{P}_\perp}{P^+}\left[P^+J^3 + \hat{\mathbf{z}}\cdot(\mathbf{E}_\perp\times\mathbf{P}_\perp)\right]\right\},\end{aligned} \tag{3.103}$$

where $W^\pm := W^0 \pm W^3$. For helicity spin, we have

$$j_h^3 = \frac{\mathbf{P}\cdot\mathbf{J}}{|\mathbf{P}|}; \tag{3.104}$$

$$\mathbf{j}_h\times\hat{\mathbf{z}} = \frac{1}{M|\mathbf{P}|}\left[\hat{\mathbf{z}}\times(\mathbf{W}\times\mathbf{P}) + (\hat{\mathbf{z}}\times\mathbf{P})\frac{\mathbf{W}\cdot(\hat{\mathbf{z}}\times\mathbf{P})}{|\mathbf{P}| + \mathbf{P}\cdot\hat{\mathbf{z}}}\right]\times\hat{\mathbf{z}}. \tag{3.105}$$

The third component j_h^3 is recognized as the usual expression for the helicity. The transverse components $\mathbf{j}_h\times\hat{\mathbf{z}}$ complete the $SU(2)$ Lie algebra in this case. Note that only the helicity has a well defined limit as the mass vanishes. The other two components of the helicity spin have mass terms in the denominator.

Since spin vectors are constructed by applying Lorentz transformation valued functions of the four-momentum whose value on eigenstates of the four-momentum is a Lorentz transformation that maps the four-momentum eigenvalue to its rest value, the net effect of the inverse of one of these transformations, followed by one of the other transformations is a rotation-valued function of the four-velocity operators. This implies the following relation between two different spin

vectors:

$$j_a^j = L_a^{-1}(Q)^j{}_\nu L_b(Q)^\nu{}_k j_b^k = R_{ab}(Q)^j{}_k j_b^k, \tag{3.106}$$

or, equivalently:

$$j_a^{kl} = R_{ab}(Q)^k{}_m R_{ab}(Q)^l{}_n j_b^{mn}. \tag{3.107}$$

It is important to realize that this is an *operator* relation. The angles appearing in the rotation matrices only take on c-number values when the operator is applied to a simultaneous eigenstate of P^μ. The rotations $R_{cf}(Q)$ which transform the front-form spin to the canonical spin are called Melosh rotations (Me 74). These Melosh rotations play an important role in combining front-form spins, as will be seen in Section 5. The term *generalized Melosh rotation* will be used to describe the general transformation $R_{ab}(Q)$ which relates two different types of spin vectors. Although only the three most common Lorentz transformations satisfying Eqs. (3.85) and (3.86) were given explicitly, there is an infinite number of possible choices. A different spin operator is obtained for each $SU(2)$-valued function $\underline{R}(\mathbf{Q})$. From the polar decomposition theorem (Re 72), any 2×2 matrix can be expressed as the product of a unitary matrix multiplied by a positive matrix, which implies that the most general Lorentz transformation satisfying Eqs. (3.85) and (3.86) has the 2×2 matrix form

$$\underline{L}(Q) := \underline{L}_c(Q)\underline{R}(Q), \tag{3.108}$$

with $\underline{R}[Q = (1, \mathbf{0})] = I$. The observable quantity associated with a given spin vector $\mathbf{j}_g$ is equal to the value of the canonical spin that would be measured if the state of the particle were transformed to its rest frame using the Lorentz transformation L_g. The operator $L_g(Q)$ is an operator, and is not tied to any reference frame.

3.7. **Other Considerations**

In this section, we have concentrated on Poincaré invariance as a necessary property of any sensible relativistic theory. Most formulations of local relativistic quantum field theories are Poincaré invariant, though perturbative calculations, truncations, or cutoffs may destroy such invariance. Field theories are a proper subset of the general class of Poincaré invariant

quantum models which are distinguished by the additional requirement of microscopic locality. Microscopic locality means that there are observables associated with each bounded open subset of space time, and that observables associated with any two causally disconnected subsets of spacetime necessarily commute.

The principle of microscopic locality applies to observables associated with arbitrarily small sets of spacetime, though any experimental test of this condition has only a finite resolution. In an ideal scattering experiment, what is actually measured is the set of probability distributions of momenta and spins of the initial and final particles at points which are asymptotically separated from the spacetime volume in which the reaction occurs. We must therefore assume that the dynamics permits states which behave asymptotically like systems of free particles. In quantum field theory, this property is ensured by microscopic locality. However, in general, this condition can be realized directly, by demanding that the dynamical model satisfy cluster separability properties or macroscopic locality. This is discussed in detail in Sections 6 and 7. Poincaré invariant models which replace microscopic locality with cluster properties cannot be distinguished from local systems by any finite set of experiments.

4. The One-Body Problem: Irreducible Representations

The quantum mechanical description of a single particle of mass m and spin j is equivalent to the construction of an irreducible representation of $\mathcal{P}$ with mass m and spin j. Irreducible representations of $\mathcal{P}$ play a central role in all that follows. In addition to providing a mathematical description of a single particle, they are used to formulate asymptotic conditions in the mathematical description of scattering theory and cluster properties.

Consider a particle of mass $m > 0$ and spin j, with the property that transition probabilities between different states of this particle are independent of the choice of inertial coordinate system. From the discussion in Section 3, this is only possible if there exists an abstract representation of $\mathcal{P}$ on the abstract one-particle Hilbert space $\mathcal{H}$. Given this starting point, we develop a mathematical description of this particle as follows:

1. Construct a complete set of commuting self-adjoint operators from the generators. Determine the eigenvalue spectrum of the complete set of commuting self-adjoint operators in terms of the mass and spin of the particle. Construct a representation of the model Hilbert space as the space of square summable (integrable) functions of these eigenvalues.

2. Construct an explicit unitary representation of $\mathcal{P}$ on this representation of the Hilbert space using transformation properties of the commuting operators.

3. Construct an explicit representation of the generators of $\mathcal{P}$ on this space.

4.1. The Hilbert Space

From the previous section, we know that the operators $\{\mathbf{P}, j_g^3, M, \mathbf{j}^2\}$ form a set of commuting self-adjoint operators that are constructed from the generators for any type (g) of spin. The existence of these quantities as observable attributes of a particle is a consequence of the assumption that one-body transition probabilities have values independent of inertial coordinate system. The assumption of invariant transition probabilities ensures the existence of $U(\underline{\Lambda}, \underline{a})$ by the Wigner-Bargmann theorem, that the infinitesimal generators are well defined functions

of $U(\underline{\Lambda}, \underline{a})$, and that the operators $\{\mathbf{P}, j_g^3, M, \mathbf{j}^2\}$ are well defined function of the infinitesimal generators.

If the operators $\{\mathbf{P}, j_g^3, M, \mathbf{j}^2\}$ form a *complete* set of commuting self-adjoint operators, then the particle will be called structureless. Since additional quantum numbers such as flavor, color, charge, *etc.,* play no special role in the description of the transformation properties of a free particle under the action of the Poincaré group, it is assumed in this section that the particle is structureless.

The Hilbert space of any quantum mechanical system can be represented as the space of square integrable functions of the eigenvalues of a complete set of self-adjoint operators. Thus, the determination of the spectrum of the operators $\{\mathbf{P}, j_g^3, M, \mathbf{j}^2\}$ fixes a representation of the model Hilbert space

The spectrum of these operators for a single particle of mass m and spin j is completely determined by the specification of the mass and spin of the particle. The operator M has one discrete eigenvalue which is m. The operator $\mathbf{j}^2$ also has only one discrete eigenvalue which is $j(j+1)$. For particles with $m > 0$ the components of $\mathbf{j}_g$ satisfy $SU(2)$ commutation relations, *i.e.,* Eq. (3.99). This requires that j can take on only integral or half-integral values, and that the spectrum of j_g^3 can take the $2j+1$ values $\{-j, -j+1, \ldots, j-1, j\}$. The spectrum of the components of $\mathbf{P}$ range over the possible physical values of the momentum of the particle, which is $(-\infty, \infty)$. In a point-from or front-form description, the three components of the three-momentum are replaced by three independent functions of the three-momentum. The spectrum of these operators is determined by the range of these three independent functions as $\mathbf{p}$ varies over its spectrum. These considerations completely determine the spectrum of the complete set of commuting self-adjoint operators.

Of special interest are cases where the type of spin is canonical, front-form, or helicity, and the continuous variables are the three components of the linear momentum, the three components of the four-velocity, or the three components of the four-momentum that generate translations that leave the light front invariant. Dirac's three forms (Di 49) of the dynamics are associated with specific combinations of spin and continuous variables. The natural commuting operators for the instant form are the mass M, the linear momentum $\mathbf{P}$, the spin $\mathbf{j}^2$ and the third component of

the canonical spin operator. For the point form, is natural to replace $\mathbf{P}$ by the three independent components of the four-velocity $\mathbf{Q} := \mathbf{P}/M$. For the front form, the momentum operators are replaced by $\tilde{\mathbf{P}} = (P^+, P^1, P^2)$, which are the three components of a light-front vector and the canonical spin is replaced by the front-form spin. Each of these three cases will be considered explicitly.

A representation of the quantum mechanical Hilbert space $\mathcal{H}^s_m$ for this particle is the space of square summable functions of these eigenvalues, *i.e.,* the space of complex valued functions ${}_g\langle m\, s; \mathbf{p}\, \mu|\Psi\rangle$ satisfying:

$$\|\Psi\|^2 := \langle\Psi|\Psi\rangle < \infty, \tag{4.1}$$

where

$$\langle\Psi|\Phi\rangle := \sum_{\mu=-j}^{j} \int_{\mathcal{R}^3} d^3p\, {}_g\langle m\, j; \mathbf{p}\, \mu|\Psi\rangle^*\, {}_g\langle m\, j; \mathbf{p}\, \mu|\Phi\rangle. \tag{4.2}$$

For a single particle, there is only one mass and spin eigenvalue, so there is no sum over m and j. Because these eigenvalues do not appear in sums, these states will equivalently be denoted ${}_g\langle\, \mathbf{p}\, \mu|\Psi\rangle$.

For the three special combinations of spin and continuous variables associated with Dirac's forms of dynamics, the representations of the Hilbert space are:

$$\langle\Psi|\Phi\rangle := \sum_{\mu=-j}^{j} \int_{\mathcal{R}^3} d^3p\, {}_c\langle\mathbf{p}\, \mu|\Psi\rangle^*\, {}_c\langle\mathbf{p}\, \mu|\Phi\rangle; \tag{4.3}$$

$$\langle\Psi|\Phi\rangle := \sum_{\mu=-j}^{j} \int_{\mathcal{R}^2} d^2p_\perp \int_0^\infty dp^+\, {}_f\langle\tilde{\mathbf{p}}\, \mu|\Psi\rangle^*\, {}_f\langle\tilde{\mathbf{p}}\, \mu|\Phi\rangle; \tag{4.4}$$

$$\langle\Psi|\Phi\rangle := \sum_{\mu=-j}^{j} \int_{\mathcal{R}^3} d^3q\, {}_c\langle\mathbf{q}\, \mu|\Psi\rangle^*\, {}_c\langle\mathbf{q}\, \mu|\Phi\rangle, \tag{4.5}$$

for the instant, front and point form, respectively. The subscripts on the bras and kets indicate the type of spin vector associated with each of these representations. In all cases, the mass and

spin quantum numbers have been suppressed. The vector $\mathbf{q}$ in Eq. (4.5) represents the eigenvalue of the four-velocity operator $\mathbf{Q} := \mathbf{P}/M$. The vector $\tilde{\mathbf{p}}$ in Eq. (4.4) is the light-front three-vector (p^+, p^1, p^2). In general the continuous variables $\mathbf{p}$ can be replaced by any other equivalent set of quantities, independent of the choice of spin. The pairings of continuous variables with spin vectors associated with Dirac's forms of the dynamics have the special property that under the action of certain subgroups of the Poincaré group they transform in a manner such that the transformed quantities have values that do not depend on the mass of the particle. This property is not important for the description of a single particle, but is an important simplification for systems of interacting particles. The case of arbitrary spin and continuous variables is treated in (Po 89).

4.2. **Unitary Representations**

We now use the transformation properties of the observables that define the Hilbert space to construct a realization of $U(\underline{\Lambda}, \underline{a})$ on that space. The transformation properties of the operators $\{\mathbf{P}, j_g^3, M, \mathbf{j}^2\}$, which can be deduced from the definitions in Section 3, are

$$U(\underline{\Lambda}, \underline{a})^\dagger P^\mu U(\underline{\Lambda}, \underline{a}) = \Lambda^\mu{}_\nu P^\nu; \tag{4.6}$$

$$U(\underline{\Lambda}, \underline{a})^\dagger \mathbf{j}_g U(\underline{\Lambda}, \underline{a}) = R_g(\underline{\Lambda}, Q)\mathbf{j}_g. \tag{4.7}$$

Here, R_g is a 3×3 matrix of operators constructed from the matrix expression for the g-spin Wigner rotation:

$$\underline{R}_g(\underline{\Lambda}, q) := \underline{L}_g^{-1}(\Lambda q)\underline{\Lambda}\underline{L}_g(q), \tag{4.8}$$

and replacing the four-velocity variable q by the four-velocity operator. Note that the form of this Wigner rotation depends on the type of boost used to define the spin. When this matrix is applied to a simultaneous eigenstate of the four-momentum, the matrix elements are evaluated by replacing the operators by the eigenvalues. Equations (4.6) and (4.7) determine the transformation properties of the eigenvalues of the operators $\{\mathbf{P}, j_g^3, M, \mathbf{j}^2\}$. The four-momentum transforms like a four-vector under Poincaré transformations. The transformation properties of the spin can be determined using two properties of the Wigner rotations in Eq. (4.8):

i. For $p = p_0 := (m, \mathbf{0})$, $q_0 = p_0/m$ and $\underline{\Lambda} = \underline{R}$ (a pure rotation), Eqs. (3.86) and (4.8) imply that

$$\underline{R}_g(\underline{R}, q_0) = \underline{L}_g^{-1}(Rq_0)\underline{R}\underline{L}_g(q_0) = \underline{R}, \tag{4.9}$$

which is the $SU(2)$ representative of R.

ii. For $p = p_0$ and $\underline{\Lambda} = \underline{L}_g(q)$, the Wigner rotation is the identity:

$$\underline{R}_g[\underline{L}_g(q), q_0] := \underline{L}_g^{-1}(q)\underline{L}_g(q)\underline{L}_g(q_0) = \underline{L}_g(q_0) = I. \tag{4.10}$$

Equations (4.6), (4.7), (4.9) and (4.10) will be used to construct the action of $U(\underline{\Lambda}, \underline{a})$ on simultaneous eigenstates of the operators $\{\mathbf{P}, j_g^3, M, \mathbf{j}^2\}$. The following normalization convention is used:

$${}_g\langle\, \mathbf{p}\,\mu|\mathbf{p}'\,\mu'\rangle_g := \delta_{\mu'\mu}\delta(\mathbf{p}' - \mathbf{p}) \tag{4.11}$$

and

$$I = \sum_{\mu=-j}^{j} \int_{\mathcal{R}^3} d^3p\, |\mathbf{p}, \mu\rangle_g\, {}_g\langle\, \mathbf{p}\,\mu|. \tag{4.12}$$

The notation $T(\underline{a}) := U(\underline{I}, \underline{a})$ and $U(\underline{\Lambda}) := U(\underline{\Lambda}, \underline{0})$ will be used to denote the unitary representative of spacetime translations and a Lorentz transformation, respectively. From Eq. (4.9), we see that $U(\underline{R})$ behaves like an ordinary rotation on eigenstates of the four-momentum with $p = p_0$:

$$U(\underline{R})|\mathbf{0}\mu\rangle_g = \sum_{\bar{\mu}=-j}^{j} |\mathbf{0}\,\bar{\mu}\rangle_g D^j_{\bar{\mu}\mu}(\underline{R}). \tag{4.13}$$

The function $D^j_{\bar{\mu}\mu}(\underline{R})$ is a $(2j+1)$-dimensional unitary irreducible representation of $SU(2)$ (We 64):

$$\begin{aligned} D^j_{\bar{\mu}\mu}(\underline{R}) = \sum_{\nu=0}^{2j} & \frac{[(j+\bar{\mu})!(j-\bar{\mu})!(j+\mu)!(j-\mu)!]^{\frac{1}{2}}}{(j+\bar{\mu}-\nu)!\nu!(\nu-\bar{\mu}+\mu)!(j-\mu-\nu)!} \\ & \times R_{11}^{j+\bar{\mu}-\nu} R_{12}^{\nu} R_{21}^{\nu-\bar{\mu}+\mu} R_{22}^{j-\mu-\nu}. \end{aligned} \tag{4.14}$$

This is a finite degree polynomial in the matrix elements of $\underline{R}$ with real coefficients. Note that $D(\underline{R})$ is a function on $SU(2)$ rather than $O(3)$: for particles with half-integral spins rotations

through an angle 2π about any axis are equivalent to multiplication by -1. For this reason, a Wigner rotation is properly labeled by an element of $SL(2,C)$ rather than a Lorentz transformation.

If $\underline{L}$ is the boost used to define the spin, then from item *ii.* above,

$$U[\underline{L}_g(q)]|\mathbf{0}\,\mu\rangle_g = \text{constant} \times |\mathbf{p}\,\mu\rangle_g. \tag{4.15}$$

The constant is needed to ensure that $U[\underline{L}_g(q)]$ acts unitarily. The specific form of this constant depends on the choice of normalization of the basis vectors. For the normalization (4.11), it is determined up to a phase to be

$$\text{constant} = \sqrt{\frac{\omega_m(\mathbf{p})}{m}}; \qquad \omega_m(\mathbf{p}) := \sqrt{m^2 + \mathbf{p}^2}. \tag{4.16}$$

Any phase factors can be absorbed into the definition of the simultaneous eigenstates. The result is

$$U[\underline{L}_g(p)]|\mathbf{0}\,\mu\rangle_g = \sqrt{\frac{\omega_m(\mathbf{p})}{m}}|\mathbf{p}\,\mu\rangle_g. \tag{4.17}$$

Because the four-momentum commutes with the Pauli-Lubanski vector (and therefore the spin), the spacetime translation is given by a multiplication in this representation:

$$T(\underline{a})|\mathbf{p}\,\mu\rangle_g := e^{ip\cdot a}|\mathbf{p}\,\mu\rangle_g. \tag{4.18}$$

The action of $U(\underline{\Lambda}, \underline{a})$ on an arbitrary vector is uniquely fixed by Eqs. (4.13), (4.17), (4.18) and the group representation properties. The key relation is that for any $\mathbf{p}$, $\underline{\Lambda}$ and $\underline{a}$, it follows from the group representation properties that $U(\underline{\Lambda}, \underline{a})$ can be uniquely represented in the form:

$$U(\underline{\Lambda}, \underline{a}) = T(\underline{a})U[\underline{L}_g(\Lambda q)]U[R_g(\underline{\Lambda}, q)]U[\underline{L}_g^{-1}(q)]. \tag{4.19}$$

If $U(\underline{\Lambda}, \underline{a})$ is applied to the state vector $|m\,j;\mathbf{p}\,\mu\rangle_g$, the result is a sequence of four transformations of the type (4.13), (4.17) and (4.18). The inverse of Eq. (4.17) eliminates the first transformation

in Eq. (4.19):

$$T(\underline{a})U[\underline{L}_g(\Lambda q)]U[\underline{R}_g(\underline{\Lambda}, q)]\sqrt{\frac{m}{\omega_m(\mathbf{p})}}|\mathbf{0}\,\mu\rangle_g.$$

The new state vector is a rest state. From Eq. (4.13), the effect of the rotation is

$$\sqrt{\frac{m}{\omega_m(\mathbf{p})}}\sum_{\bar{\mu}=-j}^{j} T(\underline{a})U[\underline{L}_g(\Lambda q)]|m\,j;\mathbf{0}\,\bar{\mu}\rangle_g D^j_{\bar{\mu}\mu}[\underline{R}_g(\underline{\Lambda}, q),]$$

The rotated rest state can now be boosted using Eq. (4.17):

$$\sqrt{\frac{\omega_m(\mathbf{p}_\Lambda)}{m}}\sqrt{\frac{m}{\omega_m(\mathbf{p})}}\sum_{\bar{\mu}=-j}^{j} T(\underline{a})|\mathbf{p}_\Lambda\,\bar{\mu}\rangle_g D^j_{\bar{\mu}\mu}[\underline{R}_g(\underline{\Lambda}, q)].$$

Finally, Eq. (4.18) can be used to compute the action of the translation. The result is

$$U(\underline{\Lambda}, \underline{a})|\mathbf{p}\,\mu\rangle_g = e^{ip_\Lambda\cdot a}\sqrt{\frac{\omega_m(\mathbf{p}_\Lambda)}{\omega_m(\mathbf{p})}}\sum_{\bar{\mu}=-j}^{j}|\mathbf{p}_\Lambda, \bar{\mu}\rangle_g D^j_{\bar{\mu}\mu}[\underline{R}_g(\underline{\Lambda}, q)], \tag{4.20}$$

where

$$p_\Lambda := \Lambda p; \qquad p = (\omega_m(\mathbf{p}), \mathbf{p}). \tag{4.21}$$

Equation (4.20) is the key result of this section.

The operator $U(\underline{\Lambda}, \underline{a})$ is a single-valued unitary representation of $\mathcal{P}$. Unitarity follows because it can be expressed using Eq. (4.19) as the composition of four elementary unitary transformations. From Eq. (4.20), a wave function has the following transformation property:

$${}_g\langle\mathbf{p}\,\mu|U(\underline{\Lambda}, \underline{a})|\Psi\rangle = e^{ip\cdot a}\sqrt{\frac{\omega_m(\mathbf{p}_{\Lambda^{-1}})}{\omega_m(\mathbf{p})}}\sum_{\bar{\mu}=-j}^{j} D^j_{\mu\bar{\mu}}[\underline{R}_g(\underline{\Lambda}, q_{\Lambda^{-1}})]_g\,\langle\mathbf{p}_{\Lambda^{-1}}\,\bar{\mu}|\Psi\rangle. \tag{4.22}$$

The matrix elements of $U(\underline{\Lambda}, \underline{a})$ are

$${}_g\langle\mathbf{p}'\,\mu'|U(\underline{\Lambda}, \underline{a})|\mathbf{p}\,\mu\rangle_g = \delta(\mathbf{p}' - \mathbf{p}_\Lambda)e^{ip\cdot a}\sqrt{\frac{\omega_m(\mathbf{p}')}{\omega_m(\mathbf{p})}}D^j_{\mu'\mu}[\underline{R}_g(\underline{\Lambda}, p)]. \tag{4.23}$$

The above expressions give an explicit representation for $U(\underline{\Lambda}, \underline{a})$ on a basis of simultaneous eigenstates of the three-momentum, the mass, and an arbitrary type of spin. These equations

are appropriate for Dirac's instant form of dynamics if the spin is chosen to be the canonical spin ($g = c$). In the point and front form of dynamics, the three components of the three-momentum are replaced by three components of the four-velocity, or the front-form components of the four-momentum, respectively. These choices have different natural normalizations for the continuum variables, which require a different constant to maintain unitarity.

For the front form, the independent components of the momentum are replaced by

$$\mathbf{p} \to \tilde{\mathbf{p}} := (p^+, p_1, p_2), \tag{4.24}$$

where $p^+ = \omega_m(\mathbf{p}) + p_3$. The spin is taken to be the front-form spin and the normalization condition is

$${}_f\langle \tilde{\mathbf{p}'}\, \mu' | \tilde{\mathbf{p}}\, \mu \rangle_f = \delta_{\mu'\mu} \delta(p'^+ - p^+) \delta^2(\mathbf{p}'_\perp - \mathbf{p}_\perp). \tag{4.25}$$

For the front form, Eqs. (4.20), (4.22) and (4.23) are replaced as follows:

$$U(\underline{\Lambda}, \underline{a}) |\tilde{\mathbf{p}}\, \mu\rangle_f = e^{ip_\Lambda \cdot a} \sqrt{\frac{p^+_\Lambda}{p^+}} \sum_{\bar{\mu}=-j}^{j} |\tilde{\mathbf{p}}_\Lambda\, \bar{\mu}\rangle_f D^j_{\bar{\mu}\mu}[\underline{R}_f(\underline{\Lambda}, p)]; \tag{4.26}$$

$${}_f\langle \tilde{\mathbf{p}}\, \mu | U(\underline{\Lambda}, \underline{a}) | \Psi \rangle = e^{ip\cdot a} \sqrt{\frac{p^+_{\Lambda^{-1}}}{p^+}} \sum_{\bar{\mu}=-j}^{j} D^j_{\mu\bar{\mu}}[\underline{R}_f(\underline{\Lambda}, p_{\Lambda^{-1}})]_f \, \langle \tilde{\mathbf{p}}_{\Lambda^{-1}}\, \bar{\mu} | \Psi \rangle; \tag{4.27}$$

$${}_f\langle \tilde{\mathbf{p}}'\, \mu' | U(\underline{\Lambda}, \underline{a}) | \tilde{\mathbf{p}}\, \mu \rangle_f = \delta(\tilde{\mathbf{p}}' - \tilde{\mathbf{p}}_\Lambda) e^{ip\cdot a} \sqrt{\frac{p'^+}{p^+}} D^j_{\mu'\mu}[\underline{R}_f(\underline{\Lambda}, p)]. \tag{4.28}$$

For the point form, let $\mathbf{q} := \mathbf{p}/m$, and use the basis

$$|\mathbf{q}\, \mu\rangle_c = |m\, s; \mathbf{q}\, \mu\rangle_c, \tag{4.29}$$

with normalization

$${}_c\langle \mathbf{q}'\, \mu' | \mathbf{q}\, \mu \rangle_c = \delta_{\mu'\mu} \delta(\mathbf{q}' - \mathbf{q}). \tag{4.30}$$

Let $q_\Lambda := p_\Lambda/m$. With these conventions, Eqs. (4.20), (4.22) and (4.23) are replaced as follows:

$$U(\underline{\Lambda},\underline{a})|\mathbf{q}\,\mu\rangle_c = e^{ip_\Lambda\cdot a}\sqrt{\frac{\omega_1(\mathbf{q}_\Lambda)}{\omega_1(\mathbf{q})}}\sum_{\bar\mu=-j}^{j}|\mathbf{q}_\Lambda,\bar\mu\rangle_c D^j_{\bar\mu\mu}[\underline{R}_c(\underline{\Lambda},p)]; \tag{4.31}$$

$${}_c\langle\mathbf{q}\,\mu|U(\underline{\Lambda},\underline{a})|\Psi\rangle = e^{ip\cdot a}\sqrt{\frac{\omega_1(\mathbf{q}_{\Lambda^{-1}})}{\omega_1(\mathbf{q})}}\sum_{\bar\mu=-j}^{j}D^j_{\mu\bar\mu}[\underline{R}_c(\underline{\Lambda},p_{\Lambda^{-1}})]_c\,\langle\mathbf{q}_{\Lambda^{-1}},\bar\mu|\Psi\rangle; \tag{4.32}$$

$${}_g\langle\mathbf{q}'\,\mu'|U(\underline{\Lambda},\underline{a})|\mathbf{q}\,\mu\rangle_c = \delta(\mathbf{q}'-\mathbf{q}_\Lambda)e^{ip\cdot a}\sqrt{\frac{\omega_1(\mathbf{q}')}{\omega_1(\mathbf{q})}}D^j_{\mu'\mu}[\underline{R}_c(\underline{\Lambda},p')]. \tag{4.33}$$

The above equations define several concrete realizations of single-valued unitary representations of $\mathcal{P}$, corresponding to a single particle of mass m and spin j. This construction is the analog of a plane wave basis for a single nonrelativistic particle. Although we have constructed a variety of different unitary representations, they are all unitarily equivalent. The relations between them can be deduced from the definitions. A more general basis of state vectors $|\mathbf{g}\mu\rangle_g$ can be constructed, with $\mathbf{g}(\mathbf{p},m)$ representing any three independent functions of the three-momentum (*i.e.,* the front-form components, the three components of the four-velocity, or $\mathbf{p}$ itself). If there are normalized in the usual way:

$${}_g\langle\mathbf{g}'\,\mu'|\mathbf{g}\,\mu\rangle_g = \delta_{\mu'\mu}\delta(\mathbf{g}'-\mathbf{g}), \tag{4.34}$$

then these states are related to the set $|\mathbf{p}\,\mu\rangle_t$ by a unitary transformation with coefficients

$$\begin{aligned}{}_g\langle\mathbf{g}\,\mu'|\mathbf{p}\,\mu\rangle_t &= \delta(\mathbf{p}-\mathbf{p}(\mathbf{g};m))\left|\frac{\partial\mathbf{p}(\mathbf{g};m)}{\partial\mathbf{g}}\right|^{\frac{1}{2}}D^j_{\mu'\mu}[\underline{R}_{gt}(p)]\\ &= \delta(\mathbf{g}-\mathbf{g}(\mathbf{p};m))\left|\frac{\partial\mathbf{g}(\mathbf{p};m)}{\partial\mathbf{p}}\right|^{\frac{1}{2}}D^j_{\mu'\mu}[\underline{R}_{gt}(p)].\end{aligned} \tag{4.35}$$

The transformations

$$\underline{R}_{gt}(p) := \underline{L}_g^{-1}(p)\underline{L}_t(p) \tag{4.36}$$

are the generalized Melosh rotations (Me 74) defined in Eqs. (3.106) and (3.107), which transform between g and t type of spin. The square roots of the Jacobians maintain unitarity. The relevant

Jacobians relating $\tilde{\mathbf{p}}$ and $\mathbf{q}$ to $\mathbf{p}$ are

$$\left|\frac{\partial\tilde{\mathbf{p}}(\mathbf{p};m)}{\partial\mathbf{p}}\right| = \frac{p^+}{\omega_m(\mathbf{p})}; \qquad \left|\frac{\partial\mathbf{q}(\mathbf{p};m)}{\partial\mathbf{p}}\right| = \frac{1}{m^3}. \tag{4.37}$$

4.3. **Lie Algebra**

Given representations of finite Poincaré transformations in a given basis, it is possible to construct representations of the Lie algebra in the same basis by differentiation using Eqs. (3.47)–(3.49), along with the general formula (4.22) for the action of a unitary transformation on a wave function. Representations of the Lie algebra are needed to derive constraints on model interactions in the next section.

The generators of spacetime translations are multiplication operators:

$$\mathbf{P}|m\,j;\mathbf{p}\,\mu\rangle_g = \mathbf{p}|m\,j;\mathbf{p}\,\mu\rangle_g; \tag{4.38}$$

$$H|m\,j;\mathbf{p}\,\mu\rangle_g = \omega_m(\mathbf{p})|m\,j;\mathbf{p}\,\mu\rangle_g. \tag{4.39}$$

The formulas for the representations of the generators of Lorentz transformations depend on the choice of spin. They are constructed from the definitions

$$\begin{aligned} &{}_g\langle\mathbf{p}\,\mu|K^j|\Psi\rangle \\ &\quad := i\frac{\partial}{\partial\rho_j}\left\{\sum_{\bar\mu=-j}^{j}\int_{\mathcal{R}^3} d^3p'\,{}_g\langle\mathbf{p}\,\mu|U[\underline{\Lambda}(\boldsymbol{\theta},\boldsymbol{\rho})]|\mathbf{p}'\,\bar\mu\rangle_g\,{}_g\langle\mathbf{p}'\,\bar\mu|\Psi\rangle\right\}\Bigg|_{\boldsymbol{\theta}=\boldsymbol{\rho}=0}; \end{aligned} \tag{4.40}$$

$$\begin{aligned} &{}_g\langle\mathbf{p},\mu|J^j|\Psi\rangle \\ &\quad := i\frac{\partial}{\partial\theta_j}\left\{\sum_{\bar\mu=-j}^{j}\int_{\mathcal{R}^3} d^3p'\,{}_g\langle\mathbf{p}\,\mu|U[\underline{\Lambda}(\boldsymbol{\theta},\boldsymbol{\rho})]|\mathbf{p}'\,\bar\mu\rangle_g\,{}_g\langle\mathbf{p}'\,\bar\mu|\Psi\rangle\right\}\Bigg|_{\boldsymbol{\theta}=\boldsymbol{\rho}=0}, \end{aligned} \tag{4.41}$$

where $\underline{\Lambda}(\boldsymbol{\theta},\boldsymbol{\rho})$ is given by Eq. (3.44). If Eq. (4.22) is used in these relations and the result differentiated, then the chain rule gives some factors and derivatives of the continuous variables.

The following result is obtained after performing the required differentiations:

$$K^j = -\frac{1}{2}\{\omega(\mathbf{p}), X_g^j\}_+ + iC_K^{jk}(q)j_g^k; \tag{4.42}$$

$$J^j = (\mathbf{X}_g \times \mathbf{p})^j + C_J^{jk}(q)j_g^k, \tag{4.43}$$

where $X_g^j := i(\partial/\partial p^j)$ in the representation (4.22), and the matrices $C_J^{jk}(\mathbf{q})$ and $C_K^{jk}(\mathbf{q})$ are given by

$$C_K^{jk}(q) := C_{1g}^{jk}(\mathbf{q}) - q^0 C_{2g}^{jk}(\mathbf{q}); \tag{4.44}$$

$$C_J^{jk}(q) := C_{1g}^{jk}(\mathbf{q}) + i\epsilon_{jlm}C_{2g}^{lk}(\mathbf{q})q^m, \tag{4.45}$$

where the coefficients $C_{1g}^{jk}(\mathbf{q})$ and $C_{2g}^{jk}(\mathbf{q})$ are defined as follows:

$$C_{1g}^{jk}(\mathbf{q}) := \tfrac{1}{2}\mathrm{Tr}\left[\underline{L}_g^{-1}(\mathbf{q})\sigma_j\underline{L}_g(\mathbf{q})\sigma_k\right] \tag{4.46}$$

$$C_{2g}^{jk}(\mathbf{q}) := \mathrm{Tr}\left[\underline{L}_g^{-1}(\mathbf{q})\frac{\partial}{\partial q^j}\underline{L}_g(\mathbf{q})\sigma_k\right]. \tag{4.47}$$

For the case of canonical spin ($\underline{L}_g = \underline{L}_c$), a direct computation yields

$$\mathbf{K} = -\frac{1}{2}\{\omega_m(\mathbf{p}), \mathbf{X}_c\}_+ - \frac{1}{\omega_m(\mathbf{p}) + m}(\mathbf{p} \times \mathbf{j}_c); \tag{4.48}$$

$$\mathbf{J} = \mathbf{X}_c \times \mathbf{p} + \mathbf{j}_c. \tag{4.49}$$

For massive particles, $(m + \omega)$ is positive. Equation (4.49) has the form of the nonrelativistic relation (3.33). It is clear from the general relation (4.43) for $\mathbf{J}$ that this connection is special to canonical spin. Other types of spins are multiplied by a momentum dependent matrix that is the identity only for $\mathbf{p} = 0$ (by assumption (3.86)) . Thus, although all of the spins defined in this paper coincide with the total angular momentum on states with $\mathbf{p} = 0$, only the canonical spin can be added to the orbital angular momentum with the usual rules of combining angular momenta to obtain the total angular momenta. The canonical spin thus plays a special role in relativistic dynamics.

The identification $\mathbf{X}_g = i\nabla_{\mathbf{p}}$ is also representation dependent. The operator $\mathbf{X}_g$ is represented by the derivative when the representation is irreducible and the spin is the g-spin. In these cases, the partial derivative must be computed holding the degeneracy labels of the irreducible representation constant. In representations where $(m, \mathbf{p})$ is replaced by $(m, \mathbf{g})$, such as the point form or the front form, the operator $\mathbf{X}_g$ is represented by

$$X_g^j = \frac{\partial g^l}{\partial p^j} Y_g^l, \tag{4.50}$$

where

$$\mathbf{Y}_g := i\nabla_{\mathbf{g}}. \tag{4.51}$$

In the front form, the generators of front-form boosts play the same role as $\mathbf{X}_g$. The general case is treated in (Po 89). The operator $\mathbf{X}_g$ can be expressed directly in terms of the Poincaré generators by inverting Eq. (4.42) and replacing all eigenvalues by the corresponding operators:

$$X_g^j = -\frac{1}{2}\{\frac{1}{H}, K^j - iC_K^{jk}(q) j_g^k\}_+, \tag{4.52}$$

which is representation independent. For the case of canonical spin, this operator becomes

$$\mathbf{X}_c = -\frac{1}{2}\{\frac{1}{H}, \mathbf{K}\}_+ - \frac{\mathbf{P} \times (H\mathbf{J} - \mathbf{P} \times \mathbf{K})}{MH(H+M)}. \tag{4.53}$$

For massive particles, all of the inverted operators are bounded. The operator $\mathbf{X}_c$ is called the Newton-Wigner position operator (Ne 49). Note that although all of the operators $\mathbf{X}_g$ are derivatives with respect to $\mathbf{p}$, up to normalization, this is true only in the representation in which they are defined.

Equations (4.38), (4.39), (4.42) and (4.43) yield a representation for the Poincaré Lie algebra on the Hilbert space with normalization defined in Eq. (4.11). In order to construct the transverse components of the spin operator, $\mathbf{j}_g$ must be replaced by the appropriate linear combinations of the raising and lowering operator for that spin.

In the instant form, the generators $\mathbf{K}$ and $\mathbf{J}$ are given explicitly by Eqs. (4.48) and (4.49), respectively. In the point form, where the Hilbert space is represented by (4.30), the instant form formulas can be used with the point form representation for $\mathbf{X}$. In the front form, where the Hilbert space is represented by states normalized according to Eq. (4.25), it is more convenient to start from the finite transformations associated with E^1, E^2, and K^3. These finite transformations leave the three-component of the front-form spin invariant. In the front-form representation, these three operators are:

$$E^i = -ip^+ \frac{\partial}{\partial p^i}; \tag{4.54}$$

$$K^3 = -ip^+ \frac{\partial}{\partial p^+}. \tag{4.55}$$

From these representations, it is possible to construct a representation of the Lie algebra. The spacetime translation generators are simply multiplication operators:

$$P^+ = p^+; \qquad P^i = p^i \, (i = 1, 2); \qquad P^- = \frac{m^2 + \mathbf{p}_\perp^2}{p^+} := p^-, \tag{4.56}$$

while the remaining Lorentz generators can be expressed in terms of these operators and the front-form spin operators by inverting Eqs. (3.102)–(3.103):

$$J^3 = j_f^3 - \frac{\hat{\mathbf{z}}}{P^+} \cdot (\mathbf{P} \times \mathbf{E}); \tag{4.57}$$

$$\mathbf{F}_\perp = 2\mathbf{P}_\perp \frac{\hat{\mathbf{z}} \cdot \mathbf{K}}{P^+} + \frac{P^-}{P^+}\mathbf{E}_\perp + \frac{2}{P^+}\hat{\mathbf{z}} \times [\mathbf{P}_\perp (\hat{\mathbf{z}} \cdot \mathbf{j}_f) + M\mathbf{j}_{f\perp}]. \tag{4.58}$$

It is sometimes customary to use the transverse components of the total angular momentum to replace $\mathbf{F}_\perp$. These operators have the form

$$\mathbf{J}_\perp = \frac{1}{P^+} \left[\tfrac{1}{2}(P^+ - P^-)(\hat{\mathbf{z}} \times \mathbf{E}_\perp) - (\hat{\mathbf{z}} \times \mathbf{P}_\perp) K^3 + \mathbf{P}_\perp \, j_f^3 + M\mathbf{j}_{f\perp} \right]. \tag{4.59}$$

4.4. **Position in Relativity**

Our development of relativistic quantum mechanics has been done in a momentum representation. This is because the four-momentum operators arise naturally in a relativistic model as the infinitesimal generators of spacetime translations. It is no accident that we have not made a parallel development in configuration space. The problem is that there is no suitable position observable in relativistic quantum mechanics. The most conventional approach is a field theoretic argument that follows from the uncertainty principle: if one attempts to localize the position of a particle of mass m in a region of dimension Δx, then one must use a probe with momentum transfer greater than $\Delta p = \hbar/\Delta x$. When Δp is of order m, it is possible to add enough energy to the system to create a particle identical to the one whose position is being localized. Since the newly created particle cannot be distinguished from the original, we find the following limitation: $\Delta x \geq \hbar/m$, which is the Compton wavelength of the particle.

This argument is based on the assumption that the model allows particle production. It is possible to show that difficulties occur in defining a position operator, whether the model admits particle production or not. The argument below is due to Haag (Ha 00, Sc 61).

We begin by considering the form of the wave function of a spinless particle at the origin at time $t = 0$. Let us denote the wave function of this particle by $\langle \mathbf{p}|\mathbf{x} = 0; t = 0\rangle$. If such a state is invariant under homogeneous Lorentz transformations, then

$$\langle \mathbf{p}|\mathbf{x} = 0; t = 0\rangle = \langle \mathbf{p}|U(\Lambda)|\mathbf{x} = 0; t = 0\rangle = \sqrt{\frac{\omega_m(\mathbf{p}_{\Lambda^{-1}})}{\omega_m(\mathbf{p})}}\langle \mathbf{p}_{\Lambda^{-1}}|\mathbf{x} = 0; t = 0\rangle. \tag{4.60}$$

The wave function $\langle \mathbf{p}|\mathbf{x} = 0; t = 0\rangle$ must therefore have the form

$$\langle \mathbf{p}|\mathbf{x} = 0; t = 0\rangle = \frac{1}{\sqrt{\omega_m(\mathbf{p})}} f(p^2) = \frac{1}{\sqrt{\omega_m(\mathbf{p})}} f(m^2) = \frac{C}{\sqrt{\omega_m(\mathbf{p})}}, \tag{4.61}$$

where C is constant. We can now translate this eigenstate to construct an eigenstate corresponding to a particle localized at $\mathbf{x}$:

$$\langle \mathbf{p}|\mathbf{x}; t = 0\rangle = \langle \mathbf{p}|T(\mathbf{x})|\mathbf{x} = 0; t = 0\rangle = e^{-i\mathbf{p}\cdot\mathbf{x}} \frac{C}{\sqrt{\omega_m(\mathbf{p})}}. \tag{4.62}$$

If we take the overlap between the state at $(\mathbf{x} = 0; t = 0)$ with a state at $(\mathbf{x} \neq 0; t = 0)$, the result

is (Bo 59):

$$\begin{aligned}\langle \mathbf{0}|\mathbf{x}\rangle &= |C|^2 \int \frac{d^3p}{\omega_m(\mathbf{p})} e^{-i\mathbf{p}\cdot\mathbf{x}} \\ &= (2\pi)^3 |C|^2 \frac{i}{2} D_+(0,\mathbf{x}) \qquad , \\ &= (2\pi)^3 |C|^2 \frac{i}{2}\left[\lim_{t\to 0}\frac{1}{4\pi}\epsilon(t)\delta(\mathbf{x}^2) + \frac{mi}{4\pi^2|\mathbf{x}|}K_1(m|\mathbf{x}|)\right]\end{aligned} \tag{4.63}$$

where $D_+(x)$ is the positive frequency part of the Pauli-Jordan commutator fucntion. For $\mathbf{x} \neq 0$, this expression is non-zero, but falls off like $K_1(m|\mathbf{x}|)$, vanishing as $|m\mathbf{x}|^{-1/2}e^{-|m\mathbf{x}|}$ as $|\mathbf{x}| \to \infty$. Thus, these two states have an overlap which falls off when the coordinates are separated by more than a Compton wavelength. The assumption that a particle localized at the origin can be described in an invariant way implies that it is not orthogonal to a state at a different point at the same time. The Compton wavelength of the particle again sets the scale for the violation of orthogonality. The same argument also works in free field theory.

An alternative is to use one of the generalized Newton-Wigner position operators. These have the advantage that they are defined to be canonically conjugate to the momenta in a given representation. They will satisfy $\langle \mathbf{x}|\mathbf{x}'\rangle = \delta(\mathbf{x}-\mathbf{x}')$ at equal times. However, (1) the state corresponding to a particle at $(\mathbf{x} = 0; t = 0)$ is *not* invariant under Lorentz transformations (otherwise we would be reduced to the previous case), (2) they are not unique, and (3) the lack of uniqueness is manifest at the operator level, due to the dependence of the explicit expressions (4.52) for these operators on an arbitrary but fixed type of Lorentz boost. This lack of uniqueness is easily exhibited in the case of particles with spin. The Newton-Wigner operator for a given type of spin, defined by Eq. (4.52) is canonically conjugate to the momentum, *and* it commutes with the given spin operator. Since different spins are related by *momentum* dependent Melosh rotations, the partial derivative with respect to the momentum holding one spin constant is not the same operator as the partial derivative with respect to the momentum holding the other spin constant. This is a well known phenomena in classical thermodynamics, and is relevant in the interpretation of momentum distributions in exclusive processes. To understand this, consider two spin-$\frac{1}{2}$ wave functions in a canonical spin and helicity spin basis, respectively:

$${}_c\langle m\, j; \mathbf{p}\, \mu|\phi\rangle = f_\mu(\mathbf{p}); \tag{4.64}$$

$$ {}_h\langle m\, j; \mathbf{p}\, \mu|\psi\rangle = f_\mu(\mathbf{p}). \tag{4.65} $$

The wave function $f_\mu(\mathbf{p})$ is chosen to be the same in each case, although the representations are different. In both expressions, $\mathbf{p}$ is the three-momentum. The Fourier transforms of each of these wave functions are clearly the same. On the other hand, if we take the wave function ${}_c\langle m\, s; \mathbf{p}\, \mu|\phi\rangle$, and perform the unitary transformation that puts it into the same representation as the wave function ${}_h\langle m\, s; \mathbf{p}\, \mu|\psi\rangle$, then the new wave function is

$$ f_\mu(\mathbf{p}) \to f'_\mu(\mathbf{p}) = \sum_{\bar{\mu}} D^{\frac{1}{2}}_{\mu\bar{\mu}}[\underline{R}_{hc}(\mathbf{p}/m)] f_{\bar{\mu}}(\mathbf{p}). \tag{4.66} $$

If we Fourier transform $f'_\mu(\mathbf{p})$, which is $f_\mu(\mathbf{p})$ in the representation (4.65), it will have different $\mathbf{x}$ dependence than the Fourier transform of $f_\mu(\mathbf{p})$ in the representation (4.64). Thus, the $\mathbf{x}$ dependence for a given momentum distribution is different depending on the choice of spin observable. Because the generalized Melosh rotation is irrelevant when we sum over magnetic quantum numbers, this difficulty will be most apparent if one attempts to make spacetime interpretations of processes where spin degrees of freedom are measured.

The conclusion is that although configuration space wave functions can be used as well as momentum space wave functions, one should never attempt to interpret the coordinates as observable quantities, especially on distance scales on the order of a Compton wavelength of a particle. We note that the concept of position gets even more complicated in models with interactions (Fo 64).

4.5. **Summary**

This completes the discussion of the description of a single relativistic particle of mass m and spin j. The starting point was the assumption that there exists an abstract unitary representation of $\mathcal{P}^0$ on an abstract Hilbert space. Abstract expressions for the infinitesimal generators were then developed, and their commutation relations were determined. The generators were used to construct a complete set of commuting Hermitian operators, including the mass and spin. The spectrum of the complete set of commuting self-adjoint operators was determined, and representations of the model Hilbert space were constructed as spaces of square-integrable

functions of the eigenvalues of these operators. The condition of unitarity and a knowledge of how the operators behave under finite Poincaré transformations led to unitary representations of $\mathcal{P}^0$ on the these Hilbert spaces. Finally, formal differentiation of these unitary operators with respect to parameters of the Poincaré group led to representations of the infinitesimal generators.

The only point where the assumption was made that this space describes a single particle was in the specification of the spectrum of the mass and spin operators. A similar analysis applies to systems of noninteracting particles. The construction of a representation of the Hilbert space is the same, except the spectrum of the mass and spin operators is richer.

The transformations and spaces constructed in this section define irredcuible representations of the Poincaré group. All of the results of this section extend immediately to systems of particles, provided the representaions are first reduced to superpositions of irreducible represenations. From a mathematical point of view, there are additional classes of irreducible representations of $\mathcal{P}$ (Mo 65). The representations discussed in this section are appropriate to particles with positive real mass and positive energy.

5. **The Two-Body Problem**

As shown in the previous section, irreducible representations of the Poincaré group are characterized by a mass m and spin j. As far as the representation is concerned, it does not matter whether the physical object described is in any sense 'elementary' or composite, nor does such an irreducible representation depend upon any dynamical theory behind it, once the mass and spin are specified. It is only at the level of the two-body problem that dynamics can be specified in distinguishable ways. In this section, the Bakamjian-Thomas construction for two interacting particles is developed in detail, using results from the previous section on the one-body problem. The construction proceeds as follows:

1. The two-particle Hilbert space is defined as the tensor product of two one-particle Hilbert spaces.

2. A two-body unitary representation of $\mathcal{P}$ on the two-particle Hilbert space is defined as the tensor product of two one-body representations. In general, this representation is reducible.

3. Clebsch-Gordan coefficients for the Poincaré group are constructed and used to reduce the unitary representation of $\mathcal{P}$ on the two-particle Hilbert space to a linear superposition (direct integral) of irreducible representations of $\mathcal{P}$.

4. Poincaré generators for irreducible representations of the non-interacting two-particle system are constructed, along with operators for the mass and spin, and the generalized Newton-Wigner position operators.

5. Following Bakamjian and Thomas, interactions are added to the mass operator in the irreducible free-particle representation, which, together with the non-interacting spin and generalized Newton-Wigner position operators, are used to construct Poincaré generators for the *interacting* system. These generators are used to construct a new unitary representation of $\mathcal{P}$ with interactions.

6. Special choices of the Clebsch-Gordan coefficients in Step 3 yield two-body models associated with Dirac's different forms of dynamics. These are discussed and related.

The methods presented are sufficiently flexible to permit the formulation of models consistent with existing two-body scattering data and spectral properties.

5.1. **The Two-Body Hilbert Space**

The Hilbert space $\mathcal{H}(2)$ for a system of two particles of mass m_i and spin s_i $(i = 1, 2)$ is the tensor product, $\mathcal{H}^{s_1}_{m_1} \otimes \mathcal{H}^{s_2}_{m_2}$, of the single-particle Hilbert spaces associated with each particle. A basis on this space can be constructed from single-particle bases:

$$|\mathbf{p}_1\, \mu_1\, \mathbf{p}_2\, \mu_2\rangle_g := |m_1\, s_1; \mathbf{p}_1\, \mu_1\rangle_g \otimes |m_2\, s_2; \mathbf{p}_2\, \mu_2\rangle_g, \tag{5.1}$$

with normalization

$${}_g\langle \mathbf{p}'_1\, \mu'_1\, \mathbf{p}'_2\, \mu'_2|\mathbf{p}_1\, \mu_1\, \mathbf{p}_2\, \mu_2\rangle_g = \delta_{\mu'_1\mu_1}\delta_{\mu'_2\mu_2}\delta(\mathbf{p}'_1 - \mathbf{p}_1\,)\delta(\mathbf{p}'_2 - \mathbf{p}_2\,). \tag{5.2}$$

The subscript g denotes the type of boost used to define the spin.

5.2. **Relativistic Dynamics of Two Free Particles**

The unitary representation $U_0(\underline{\Lambda}, \underline{a})$ of $\mathcal{P}$ for two free particles is the tensor product of two single-particle representations:

$$U_0(\underline{\Lambda}, \underline{a}) := U_1(\underline{\Lambda}, \underline{a}) \otimes U_2(\underline{\Lambda}, \underline{a}). \tag{5.3}$$

In this representation, the Poincaré generators are sums of the generators for each particle:

$$P^\mu_0 := P^\mu_1 \otimes I_2 + I_1 \otimes P^\mu_2; \tag{5.4}$$

$$\mathbf{K}_0 := \mathbf{K}_1 \otimes I_2 + I_1 \otimes \mathbf{K}_2; \tag{5.5}$$

$$\mathbf{J}_0 := \mathbf{J}_1 \otimes I_2 + I_1 \otimes \mathbf{J}_2. \tag{5.6}$$

Equations (5.5) and (5.6) can be replaced by the covariant relation:

$$J^{\alpha\beta}_0 := J^{\alpha\beta}_1 \otimes I_2 + I_1 \otimes J^{\alpha\beta}_2. \tag{5.7}$$

The single-particle generators are given in Section 4, and the operator I_i is the identity operator on

$\mathcal{H}^{s_i}_{m_i}$. The zero subscript on the generators indicates that they correspond to the non-interacting system.

The front-form generators for two free particles, P_0^+, P_0^-, $\mathbf{P}_{0\perp}$, $\mathbf{E}_{0\perp}$, K_0^3, J_0^3 and $\mathbf{F}_{0\perp}$ are obtained by taking the linear combinations of the generators defined in Eqs. (3.68)–(3.71).

Operators corresponding to the total mass and spin for the non-interacting system can be constructed from these generators using the definitions in Section 3. Since the mass and spins are nonlinear functions of the two-body generators, they *cannot* be represented as sums of single-particle operators.

5.3. **Clebsch-Gordan Coefficients**

In nonrelativistic quantum mechanics, both translational invariance and rotational invariance of the system lead to simplifications of the dynamical equations if the mass and spin of the combined system are used as variables in the dynamical equations. A similar simplification occurs in relativistic quantum mechanics. In the latter case, this involves a change of representation, in which the single-particle momenta and spins are replaced by an overall system momentum and internal angular momentum. Mathematically, the two-particle basis (5.1) is a tensor product of irreducible representation spaces of $\mathcal{P}$. The problem of changing to a basis in which the variables are the total four-momentum and the spin is mathematically equivalent to the problem of constructing Clebsch-Gordan coefficients for the Poincaré group. These are coefficients of the unitary transformation that reduces a tensor product of two irreducible representations of $\mathcal{P}$ to a linear superposition (direct integral) of irreducible representations of $\mathcal{P}$. The resulting basis still describes a system of free particles. Working with irreducible representation is central to the Bakamjian-Thomas method for adding interactions used in this paper. Although expressions for the Clebsch-Gordan coefficients can be found in the literature (Mo 65), their explicit form depends on normalization conventions and spin conventions. They can be constructed in a straightforward manner. We consider several different cases corresponding to different choices of spin and continuous variables.

The problem of constructing the Clebsch-Gordan coefficients of the Poincaré group is equivalent to the problem of expressing products of single-particle eigenstates as linear combinations

of two-body states that have the same transformation properties as a free particle. A particle of mass m and spin j is characterized by a timelike four-momentum with rest energy m, and with rest eigenstates which transform as a spin-j irreducible representation under rotations, *i.e.,*

$$U(\underline{R})|mj;\mathbf{0}\,\mu\rangle = \sum_{\bar{\mu}} |mj;\mathbf{0}\,\bar{\mu}\rangle D^{j}_{\bar{\mu},\mu}(\underline{R}). \tag{5.8}$$

The construction presented below corresponds to the special case that both representations correspond to massive physical particles. Treatments of general representations can be found in the original papers of Wigner (Wi 39) and Bargmann (Ba 47), and also from a more modern point of view in Mackey's theory of induced representations (Ma 66).

The goal of the construction of the Clebsch-Gordan coefficient is to express the tensor-product states (5.1) as linear combinations of eigenstates of the total four-momentum, with the property that the *rest* eigenstates transform as spin-j irreducible representation under rotations.

The first step in the construction of the Clebsch-Gordan coefficients is to identify eigenstates of the four-momentum. From Eq. (5.4), it follows that the four-momentum of the non-interacting two-body system is the sum of the four-momenta of each constituent particle. Since for massive particles the individual four-momenta are timelike, it follows that the total four-momentum is a timelike four-vector. It is always possible to find a Lorentz boost that transforms a state with a timelike four-momentum to a rest state.

The fundamental observation is that the rest four-momentum is invariant under rotations and spacetime translations. For each fixed mass, the spacetime translations of the rest eigenstates are given by multiplication by a phase which is independent of any internal degrees of freedom of the rest eigenstate. In general, the internal degrees of freedom transform among themselves non-trivially under the action of rotations. The next step is to express the rest eigenstate as a linear combination of terms, each of which transforms irreducibly under rotations. Rest eigenstates with different spins can be boosted to yield a direct sum of irreducible representation of the Poincaré group for a given mass. The desired linear superposition or direct integral is obtained by including the contribution of these direct sums for each mass. The Clebsch-Gordan coefficients can be read off once the normalization of these combined states are fixed.

We begin with the problem of coupling two-particle states with canonical spin to a superposition of states with canonical spin. The four-momentum of a tensor product state,

$$P^\mu|\mathbf{p}_1\,\mu_1\,\mathbf{p}_2\,\mu_2\rangle_c = (p_1^\mu + p_2^\mu)|\mathbf{p}_1\,\mu_1\,\mathbf{p}_2\,\mu_2\rangle_c, \tag{5.9}$$

is the sum of the four-momenta of each particle. This is related to a rest eigenstate by the inverse canonical boost with the four-velocity associated with this four-momentum. In what follows, it is more convenient to label the arguments of boost and Wigner rotation by four-momenta rather than four-velocities. Thus, $L_c(P)$ will be used interchangeably with $L_c(Q) := L_c(P/\sqrt{-P^2})$, and similarly for Wigner rotations. This rest eigenstate is defined as follows:

$$\begin{aligned}&\sqrt{\frac{\omega_{m_1}(\mathbf{p}_1)\omega_{m_2}(\mathbf{p}_2)}{\omega_{m_1}(\mathbf{k})\omega_{m_2}(\mathbf{k})}}U\big(\underline{L}_c^{-1}(P)\big)\sum|\mathbf{p}_1\,\bar\mu_1\,\mathbf{p}_2\,\bar\mu_2\rangle_c\\ &\times D^{j_1}_{\bar\mu_1\mu_1}[\underline{R}_c^{-1}(\underline{L}_c^{-1}(P),p_1)]D^{j_2}_{\bar\mu_2\mu_2}[\underline{R}_c^{-1}(\underline{L}_c^{-1}(P),p_2)].\end{aligned} \tag{5.10}$$

This is an eigenstate of the four-momentum with eigenvalue

$$P^\mu = (M_0,0,0,0); \qquad M_0 := \omega_{m_1}(\mathbf{k}) + \omega_{m_2}(\mathbf{k}). \tag{5.11}$$

This rest eigenstate of the four-momentum has the following transformation property under rotations:

$$U(\underline{R})|\mathbf{k}\,\mu_1\ -\mathbf{k}\,\mu_2\rangle_c = \sum|R\mathbf{k}\,\bar\mu_1\ -R\mathbf{k}\,\bar\mu_2\rangle_c D^{s_1}_{\bar\mu_1\mu_1}[\underline{R}_c(\underline{R},k_1)]D^{s_2}_{\bar\mu_2\mu_2}[\underline{R}_c(\underline{R},k_2)] \tag{5.12}$$

where $k_1 = (\omega_{m_1}(\mathbf{k}),\mathbf{k})$, and $k_2 = (\omega_{m_2}(\mathbf{k}),-\mathbf{k})$. The problem is to express the rest eigenstate as a linear combinations of terms which transform like Eq. (5.8). For the special case of canonical spin, this can be done by appealing to what is done in the case of nonrelativistic quantum mechanics.

For state vectors which are related to rest eigenstates by canonical boosts, the Wigner rotation associated with a rotation $\underline{R}$ is the rotation $\underline{R}$ itself:

$$\underline{L}_c^{-1}(R\mathbf{k})\underline{R}\,\underline{L}_c(\mathbf{k}) = \underline{R}. \tag{5.13}$$

This property is not shared by Wigner rotations associated with other type of boosts such as

front-form or helicity boosts. Equation (5.12) then becomes

$$U(\underline{R})|\mathbf{k}\,\mu_1\ -\mathbf{k}\,\mu_2\rangle_c = \sum |R\mathbf{k}\,\bar{\mu}_1\ - R\mathbf{k}\,\bar{\mu}_2\rangle_c D^{s_1}_{\bar{\mu}_1\mu_1}(\underline{R}) D^{s_2}_{\bar{\mu}_2\mu_2}(\underline{R}), \tag{5.14}$$

which is identical to the transformation properties of a nonrelativistic two-particle rest eigenstate under rotations. In this form, it is possible to use all of the well known properties of angular momentum coupling to decompose this into irreducible representations. The angles in $\hat{\mathbf{k}}$ are eliminated in favor of discrete quantum numbers using spherical harmonics. These can be coupled with the single-particle spins to obtain the irreducible representations (under rotations). The linear combinations of the rest eigenstates constructed in this way are

$$|[l\,s]k\,j;\mathbf{0}\mu\rangle := \sum \int d\hat{\mathbf{k}}\, Y^l_{\mu_l}(\hat{\mathbf{k}})|\mathbf{k}\,\mu_1\ -\mathbf{k}\,\mu_2\rangle_c \langle s_1\,\mu_1\,s_2\mu_2|s\,\mu_s\rangle\langle l\,\mu_l\,s\,\mu_s|j\,\mu\rangle. \tag{5.15}$$

It follows from Eqs. (5.14), (5.15), and the properties of spherical harmonics and the $SU(2)$ Clebsch-Gordan coefficients that the state vector just defined is a spin-j irreducible representation under rotations:

$$U(\underline{R})|[l\,s]kj;\mathbf{0}\mu\rangle = \sum |[ls]k\,j;\mathbf{0}\bar{\mu}\rangle D^j_{\bar{\mu}\mu}(\underline{R}). \tag{5.16}$$

This is identical to the transformation properties (5.8). Since the state (5.15) is a superposition of eigenstates of the four-momentum, each with eigenvalue P^μ_0, it follows that it is also a rest eigenstate of the four-momentum with mass M_0. The quantum numbers l and s are degeneracy parameters that distinguish different linear combinations of rest eigenstates that transform with the same value of j.

To construct an eigenstate of the four-momentum with canonical spin, it is sufficient to boost the state (5.15) with a canonical boost:

$$|[l\,s]k,j;\mathbf{P}\mu\rangle_c := \sqrt{\frac{M_0}{\omega_{M_0}(\mathbf{P})}} U\big[\underline{L}_c(P)\big]|[l\,s]k,j;\mathbf{0}\,\mu\rangle_c. \tag{5.17}$$

With this definition, the quantum numbers that label these states acquire a meaning as eigenvalues of commuting self-adjoint operators. Expressions for these operators will be given explicitly

in the next section. The factors under the square roots fix the normalization. This choice is consistent with the normalization used in the single-particle states:

$$ {}_c\langle [l'\ s']k'\ j';\mathbf{P}'\ \mu'|[l\ s]k\ j;\mathbf{P}\ \mu\rangle_c = \delta_{\mu'\mu}\delta_{l'l}\delta_{s's}\delta_{j'j}\delta(\mathbf{P}'-\mathbf{P})\frac{1}{k^2}\delta(k'-k), \tag{5.18} $$

except that the Kronecker delta in the single-particle masses is replaced with $\delta(k'-k)/k^2$, and there are two additional quantum numbers, l and s, that label degeneracies. It follows from these definitions and the group representation property that the state vector $|[l\ s]k\ j;\mathbf{P}\ \mu\rangle_c$ transforms irreducibly under the action of $\mathcal{P}$:

$$ U(\underline{\Lambda},\underline{a})|[l\ s]k\ j;\mathbf{P}\ \mu\rangle_c = e^{i\Lambda P\cdot a}\sqrt{\frac{\omega_{M_0}(\mathbf{P}_\Lambda)}{\omega_{M_0}(\mathbf{P})}}\sum|[l\ s]k\ j;\mathbf{P}_\Lambda\ \bar{\mu}\rangle_c D^j_{\bar{\mu}\mu}[\underline{R}_c(\underline{\Lambda},Q)]. \tag{5.19} $$

The steps in the proof of Eq. (5.19) are identical to those used to reach Eq. (4.20) in the single-particle case.

The Clebsch-Gordan coefficients can now be computed by means of the following steps:

1. Expand the rest eigenstate appearing in Eq. (5.17) in terms of the tensor-product states as it is defined in Eq. (5.15);
2. Apply the boost operator in Eq. (5.17) to this rest eigenstate in terms of a product of single-particle boosts;
3. Take the inner product of the result with the tensor-product state $|\mathbf{p}_1\mu_1\mathbf{p}_2\mu_2\rangle$.

The result is

$$ \begin{aligned} &{}_c\langle \mathbf{p}'_1\ \mu'_1\ \mathbf{p}'_2\ \mu'_2|[l\ s]k\ j;\mathbf{P}\ \mu\rangle_c \\ &\qquad = \delta(\mathbf{P}-\mathbf{p}'_1-\mathbf{p}'_2)\frac{1}{|\mathbf{k}|^2}\delta(|\mathbf{k}(\mathbf{p}'_1,\mathbf{p}'_2)|-|\mathbf{k}|)\left|\frac{\partial(\mathbf{P}\ \mathbf{k})}{\partial(\mathbf{p}_1\ \mathbf{p}_2)}\right|^{\frac{1}{2}} \\ &\qquad\quad \times\sum D^{s_1}_{\mu'_1\mu_1}[\underline{R}_c(\underline{L}_c(P),k_1)]D^{s_2}_{\mu'_2\mu_2}[\underline{R}_c(\underline{L}_c(P),k_2)]Y^l_{\mu_l}(\hat{\mathbf{k}}) \\ &\qquad\quad \times\langle s_1\ \mu_1\ s_2\ \mu_2|s\ \mu_s\rangle\langle l\ \mu_l\ s\ \mu_s|j\ \mu\rangle, \end{aligned} \tag{5.20} $$

where

$$ k_i = L_c^{-1}(P)p_i, \tag{5.21} $$

and $P = (\omega_M(\mathbf{P}), \mathbf{P})$, $k_1 = (\omega_{m_1}(\mathbf{k}), \mathbf{k})$ and $k_2 = (\omega_{m_2}(\mathbf{k}), -\mathbf{k})$. The Jacobian is

$$\left| \frac{\partial(\mathbf{P}\,\mathbf{k})}{\partial(\mathbf{p}_1\,\mathbf{p}_2)} \right| = \frac{\omega_{m_1}(\mathbf{k})\omega_{m_2}(\mathbf{k})\omega_{M_0}(\mathbf{P})}{\omega_{m_1}(\mathbf{p}_1)\omega_{m_2}(\mathbf{p}_2)M_0}. \tag{5.22}$$

Equation (5.20) defines a specific Clebsch-Gordan coefficient, corresponding to the coupling of two representations with canonical spin to a superposition of representations with canonical spin. The construction presented above is unique to canonical spin because the single-particle states transform via a Wigner rotation $D(\underline{R})$ for a rotation R. This permits us to combine spins in the manner that one sees nonrelativistically. In general, particle spins transform with a Wigner rotation which is not the same as the rotation itself. However, an irreducible representation with a given type of spin and continuous variable is related to one with canonical spin and ordinary three-momentum by a unitary transformation of the form (4.35):

$${}_g\langle \mathbf{g}\,\mu|\mathbf{p}'\,\mu'\rangle_c = \delta(\mathbf{p}' - \mathbf{p}(\mathbf{g};m)) \left| \frac{\partial \mathbf{p}(\mathbf{g};m)}{\partial \mathbf{p}} \right|^{\frac{1}{2}} D^s_{\mu\mu'}[\underline{R}_{gc}(p)]. \tag{5.23}$$

By exploiting this relation, we can obtain an expression for a general Clebsch-Gordan coefficient by inserting a complete set of canonical states:

$$\begin{aligned} {}_g\langle \mathbf{g}_1\,\mu_1\,\mathbf{g}_2\,\mu_2|[l\,s]k\,j;\mathbf{G}\,\mu\rangle_g = \sum \int d^3p_1 \int d^3p_2 \int d^3P \\ \times {}_g\langle \mathbf{g}_1\,\mu_1|\mathbf{p}_1\,\bar{\mu}_1\rangle_{cg}\langle \mathbf{g}_2\,\mu_2|\mathbf{p}_2\,\bar{\mu}_2\rangle_c \\ \times {}_c\langle \mathbf{p}_1\,\bar{\mu}_1\,\mathbf{p}_2\,\bar{\mu}_2|[l\,s]k\,j;\mathbf{P}\mu'\rangle_{cc}\langle \mathbf{P}\,\mu'|\mathbf{G}\,\mu\rangle_g. \end{aligned} \tag{5.24}$$

The result is

$$\begin{aligned} &{}_g\langle \mathbf{g}'_1\,\mu'_1\,\mathbf{g}'_2\,\mu'_2|[l\,s]k\,j;\mathbf{G}\,\mu\rangle_g \\ &\quad = \delta\big((\mathbf{G} - \mathbf{G}(\mathbf{g}'_1,\mathbf{g}'_2)\big)\frac{1}{|\mathbf{k}|^2}\delta(|\mathbf{k}(\mathbf{g}'_1,\mathbf{g}'_2)| - |\mathbf{k}|) \\ &\quad\quad \times \left|\frac{\partial \mathbf{p}_1}{\partial \mathbf{g}_1}\right|^{\frac{1}{2}} \left|\frac{\partial \mathbf{p}_2}{\partial \mathbf{g}_2}\right|^{\frac{1}{2}} \left|\frac{\partial \mathbf{G}}{\partial \mathbf{P}}\right|^{\frac{1}{2}} \left[\frac{\omega_{m_1}(\mathbf{k})\omega_{m_2}(\mathbf{k})\omega_{M_0}(\mathbf{P})}{\omega_{m_1}(\mathbf{p}_1)\omega_{m_2}(\mathbf{p}_2)M_0}\right]^{\frac{1}{2}} \\ &\quad\quad \times \sum D^{s_1}_{\mu'_1\mu_1}[\underline{R}_g(\underline{L}_g(P),k_1)\underline{R}_{gc}(k_1)]D^{s_2}_{\mu'_2\mu_2}[\underline{R}_g(\underline{L}_g(P),k_2)\underline{R}_{gc}(k_2)]Y^l_{\mu_l}(\hat{\mathbf{k}}) \\ &\quad\quad \times \langle s_1\,\mu_1\,s_2\,\mu_2|s\,\mu_s\rangle\langle l\,\mu_l\,s\,\mu_s|j\,\mu\rangle, \end{aligned} \tag{5.25}$$

where k_i is defined by the boost associated with the given spin:

$$k_i \to k_i := L_g^{-1}(P)p_i. \tag{5.26}$$

In general, both Wigner and Melosh rotation are required in the Clebsch-Gordan coefficients.

They are different type of rotations: one depends on the system four-velocity and the relative momentum, while the other depends only on the relative momentum.

For front-form state vectors $|[l\,s]k\,j;\tilde{\mathbf{P}}\,\mu\rangle_f$ normalized as follows:

$$ {}_f\langle[l'\,s']k'\,j';\tilde{\mathbf{P}}'\,\mu'|[l\,s]k\,j;\tilde{\mathbf{P}}\,\mu\rangle_f = \delta_{\mu'\mu}\delta_{j'j}\delta_{l'l}\delta_{ss'}\delta(P'^+ - P^+)\delta^2(\mathbf{P}'_\perp - \mathbf{P}_\perp)\frac{1}{k^2}\delta(k-k'), \quad (5.27)$$

the Clebsch-Gordan coefficient is

$$\begin{aligned} &{}_f\langle\tilde{\mathbf{p}}'_1\,\mu'_1\,\tilde{\mathbf{p}}'_2\,\mu'_2|[l\,s]k\,j;\tilde{\mathbf{P}}\,\mu\rangle_f \\ &\quad= \delta(\tilde{\mathbf{P}} - \tilde{\mathbf{p}}'_1 - \tilde{\mathbf{p}}'_2)\frac{1}{|\mathbf{k}|^2}\delta(|\mathbf{k}(\tilde{\mathbf{p}}'_1,\tilde{\mathbf{p}}'_2)| - |\mathbf{k}|)\sqrt{\frac{\omega_{m_1}(\mathbf{k})\omega_{m_2}(\mathbf{k})P^+}{p_1^+p_2^+M_0}} \\ &\qquad\times\sum D^{s_1}_{\mu'_1\mu_1}[\underline{R}_{fc}(k_1)]D^{s_2}_{\mu'_2\mu_2}[\underline{R}_{fc}(k_2)]Y^l_{\mu_l}(\hat{\mathbf{k}})\langle s_1\,\mu_1\,s_2\,\mu_2|s\,\mu_s\rangle\langle l\,\mu_l\,s\,\mu_s|j\,\mu\rangle, \end{aligned} \quad (5.28)$$

where

$$k_i := L_f^{-1}(P)p_i. \quad (5.29)$$

Equation (5.28) contains Melosh rotations, but no Wigner rotations. Because the front-form boosts form a subgroup, the front-form Wigner rotation associated with a front-form boost is the identity. This is special to Clebsch-Gordan coefficients associated with front-form spin.

State vectors in the point form have the normalization

$$ {}_c\langle[l'\,s']k'\,j';\mathbf{Q}'\,\mu'|[l\,s]k\,j;\mathbf{Q}\,\mu\rangle_c = \delta_{\mu'\mu}\delta_{j'j}\delta_{l'l}\delta_{s's}\delta(\mathbf{Q}'-\mathbf{Q})\frac{1}{k^2}\delta(k'-k). \quad (5.30)$$

The only change from Eq. (5.20) for instant-form state vectors is the replacement of the three-momenta by the three components of the four-velocity, with an associated change in normalization:

$$\begin{aligned} &{}_c\langle\mathbf{q}'_1\,\mu'_1\,\mathbf{q}'_2\,\mu'_2|[l\,s]k\,j;\mathbf{Q}\,\mu\rangle_c \\ &\quad= \delta\big(\mathbf{Q} - \mathbf{Q}(\mathbf{q}'_1,\mathbf{q}'_2)\big)\frac{1}{|\mathbf{k}|^2}\delta(|\mathbf{k}(\mathbf{q}'_1,\mathbf{q}'_2)| - |\mathbf{k}|)\left|\frac{m_1m_2}{M_0}\right|^{\frac{3}{2}}\sqrt{\frac{\omega_{m_1}(\mathbf{k})\omega_{m_2}(\mathbf{k})\omega_{M_0}(\mathbf{P})}{\omega_{m_1}(\mathbf{p}_1)\omega_{m_2}(\mathbf{p}_2)M_0}} \\ &\qquad\times\sum D^{s_1}_{\mu'_1\mu_1}[\underline{R}_c(\underline{L}_c(P),k_1)]D^{s_2}_{\mu'_2\mu_2}[\underline{R}_c(\underline{L}_c(P),k_2)]Y^l_{\mu_l}(\hat{\mathbf{k}}) \\ &\qquad\times\langle s_1\,\mu_1\,s_2\,\mu_2|s\,\mu_s\rangle\langle l\,\mu_l\,s\,\mu_s|j\,\mu\rangle. \end{aligned} \quad (5.31)$$

The Clebsch-Gordan coefficients developed above will be used extensively in the material which follows. The fundamental property of these coefficients is that they define a basis on which $U_1(\underline{\Lambda},\underline{a})\otimes U_2(\underline{\Lambda},\underline{a})$ acts irreducibly.

5.4. **Free-Particle Generators and Other Operators**

The quantum numbers that appear in the states $|[l\,s]k\,j;\mathbf{P}\,\mu\rangle_c$ are eigenvalues of commuting self-adjoint operators that can be expressed in terms of the one-body generators. The relation between these operators and the one-body generators can be determined by considering properties of the Clebsch-Gordan coefficients. We consider first the case of canonical spin.

The content of Eq. (5.19) is that $U_0(\underline{\Lambda},\underline{a}) = U_1(\underline{\Lambda},\underline{a})\otimes U_2(\underline{\Lambda},\underline{a})$ acts irreducibly on the linear combination of tensor product states defined by $|[l\,s]k\,j;\mathbf{P}\,\mu\rangle$. The transformation properties of $|[l\,s]k\,j;\mathbf{P}\,\mu\rangle$ under the action of $U_1(\underline{\Lambda},\underline{a})\otimes U_2(\underline{\Lambda},\underline{a})$ are identical to those a particle of mass M_0 and canonical spin j, where

$$M_0 = \omega_{m_1}(\mathbf{k}) + \omega_{m_2}(\mathbf{k}). \tag{5.32}$$

Because $U_0(\underline{\Lambda},\underline{a})$ is a tensor product, the generators are sums of the one-body generators:

$$P_0^\mu := p_1^\mu + p_2^\mu; \tag{5.33}$$

$$J_0^{\alpha\beta} := J_1^{\alpha\beta} + J_2^{\alpha\beta}. \tag{5.34}$$

Using these relations, any function of the infinitesimal generators for the two-body system can be expressed in terms of the one-body generators. Of interest are the operators M_0, W_0^μ, $\mathbf{j}_{c0}$ and $\mathbf{X}_{c0}$:

$$M_0 := \sqrt{P_0^\mu P_{0\mu}}\,; \tag{5.35}$$

$$W_0^\mu := \frac{1}{2}\epsilon^{\mu\alpha\beta\gamma}(P_0)_\alpha(J_0)_{\beta\gamma}; \qquad (0,\mathbf{j}_{c0}) := \frac{1}{M_0}L_c^{-1}(P_0)^{\mu}{}_{\bar\mu}W_0^{\bar\mu}; \tag{5.36}$$

$$\mathbf{X}_{c0} = \frac{1}{2}\{\frac{1}{H_0},\mathbf{K}_0\}_+ - \frac{\mathbf{P}_0\times(H_0\mathbf{J}_0 - \mathbf{P}_0\times\mathbf{K}_0)}{M_0H_0(M_0+H_0)}. \tag{5.37}$$

In addition, there are operators which are functions of the one-body generators, but which are not explicit functions of the *sum* of the one-body generators. For example, from Eq. (5.21), the

relative momentum is

$$k_i := L_c^{-1}(P_0)p_i, \tag{5.38}$$

where $L_c^{-1}(P_0)$ is a function of the total four-momentum *operator*. This can be evaluated explicitly, using the expression for the canonical boost in Section 3 (Eq. (3.87)):

$$\mathbf{k} = \mathbf{k}_1 = \mathbf{p}_1 + \frac{\mathbf{P}_0}{M_0}\left[\frac{\mathbf{P}_0 \cdot \mathbf{p}_1}{M_0 + H_0} + \omega_{m_1}(\mathbf{p}_1)\right]. \tag{5.39}$$

Note that k_i is *not* a four-vector operator. Instead it transforms with a Wigner rotation:

$$U(\underline{\Lambda})^\dagger k_i U(\underline{\Lambda}) = R_c(\underline{\Lambda}, Q_0)k_i, \tag{5.40}$$

where $Q_0 = P_0/M_0$ is the four-velocity operator. The magnitude of $\mathbf{k}_i$ is related to the mass operator by

$$\mathbf{k}_i^2 = \frac{1}{4M_0^2}[M_0^4 - 2M_0^2(m_1^2 + m_2^2) + (m_1^2 - m_2^2)^2]. \tag{5.41}$$

In order to obtain angular momentum operators $\mathbf{l}$ and $\mathbf{s}$, whose eigenvalues label the degeneracies l and s, respectively, we examine the structure of the Clebsch-Gordan coefficients. The last rotational Clebsch-Gordan coefficient in Eq. (5.20) implies the following relation:

$$\mathbf{j}_{c0} = \mathbf{l} + \mathbf{s}. \tag{5.42}$$

The next rotational Clebsch-Gordan coefficient in Eq. (5.20) implies a sum of the single-particle spins, each with a different Wigner rotation:

$$\mathbf{s} = R_c[\underline{L}_c^{-1}(P_0), p_1]\mathbf{j}_1 + R_c[\underline{L}_c^{-1}(P_0), p_2]\mathbf{j}_2. \tag{5.43}$$

In this expression, the rotations are to be interpreted as matrices of operators. The operator $\mathbf{s}$ is therefore a function of the one-body generators. The operator $\mathbf{l}$ does not have a simple expression, but it can be constructed in terms of one-body generators from the relation

$$\mathbf{l} = \mathbf{j}_{c0} - \mathbf{s}, \tag{5.44}$$

where $\mathbf{j}_{c0}$ and $\mathbf{s}$ are given in terms of the one-body generators via Eqs. (5.36) and (5.43), respectively.

The states $|[l\,s]k\,j;\mathbf{P}\,\mu\rangle$ are simultaneous eigenstates of the operators $\mathbf{k}^2$, $\mathbf{j}_{c0}^2$, $\mathbf{P}_0$, j_{c0}^3, $\mathbf{l}^2$ and $\mathbf{s}^2$, which are all functions of the single-particle generators. That they all mutually commute follows from their construction. The representation of the generators in this basis is completely analogous to that of a free particle.

When the spin is not canonical the relations must be modified. The required modifications follow from the replacements:

$$k_i \to k_i = R_{gc}(Q_0)L_c^{-1}(Q_0)p_i = L_g^{-1}(Q_0)p_i; \tag{5.45}$$

$$\mathbf{j}_{c0} \to \mathbf{j}_{g0} = R_{gc}(Q_0)\mathbf{j}_{c0}; \tag{5.46}$$

$$\mathbf{s} = R_{cg}(k_1)R_g[\underline{L}_g^{-1}(Q_0), p_1]\mathbf{j}_{1g} + R_{cg}(k_2)R_g[\underline{L}_c^{-1}(Q_0), p_2]\mathbf{j}_{2g}, \tag{5.47}$$

where the relation

$$\mathbf{l} = \mathbf{j}_{g0} - \mathbf{s} \tag{5.48}$$

is unchanged, although $\mathbf{l}$ is not the same operator as before. In addition, the operator $\mathbf{X}_{c0}$ in Eq. (5.37) is replaced by $\mathbf{X}_{g0}$, which is obtained by using the sums of the one-body generators in Eq. (4.52). This operator has canonical commutation relations with the momentum and commutes with $\mathbf{j}_{g0}$. The commuting Hermitian operators are defined in terms of these operators.

5.5. **The Bakamjian-Thomas Construction**

The previous two sections illustrate techniques for constructing different non-interacting representations of the Poincaré group and the Lie algebra for two particles as superpositions (direct integrals) of one-body (irreducible) representations of the Poincaré group. This construction is the relativistic analog of constructing a two-particle plane-wave basis using total and relative momenta in nonrelativistic quantum mechanics.

In this section, we consider how to add interactions to the non-interacting representations. This is more difficult than the corresponding nonrelativistic construction, because it must be

done in a manner that preserves the group structure, or, equivalently, the commutation relations. The difficulty can be seen most easily by considering the commutation relation

$$[P^j, K^k]_- = P^0 \delta^{jk} = H\delta^{jk}. \tag{5.49}$$

Clearly, an interaction dependence on the right-hand side requires that at least one of the generators on the left-hand side must contain interactions.

The Bakamjian-Thomas (Ba 53) construction provides one means for the invariant addition of interactions.

We assume at this point that an irreducible basis for two free particles is given. These may be of the type constructed using any choice of the Clebsch-Gordan coefficients in Eqs. (5.20), (5.28), (5.31), or the general form (5.25). We denote generalized state vectors in this basis by

$$|[l\, s]k\, j; \mathbf{P}\, \mu\rangle_g. \tag{5.50}$$

In this representation finite Poincaré transformations associated with the non-interacting system are given by

$$U_0(\underline{\Lambda}, \underline{a})|[l\, s]k\, j; \mathbf{P}\, \mu\rangle_g = e^{i\Lambda P_0 \cdot a}\sqrt{\frac{\omega_{M_0}(\mathbf{P}_{0\Lambda})}{\omega_{M_0}(\mathbf{P}_0)}}\sum |[l\, s]k\, j; \mathbf{P}_\Lambda\, \bar{\mu}\rangle_g D^j_{\bar{\mu}\mu}[\underline{R}_g(\underline{\Lambda}, P_0)]. \tag{5.51}$$

As in the one-body case, the infinitesimal generators can be expressed in terms of the operators $\{M_0, \mathbf{P}_0, \mathbf{X}_{g0}\, , \mathbf{j}_{g0}\}$. In this basis, M_0 and $\mathbf{P}_0$ are multiplication operators, $\mathbf{X}_{g0} = i\nabla_{\mathbf{P}}$, and $\mathbf{j}_{g0}$ can be expressed in terms of raising and lowering operators. Conversely, given the set $\{M_0, \mathbf{P}_0, \mathbf{X}_{g0}\, , \mathbf{j}_{g0}\}$, it is possible to reconstruct the generators and $U_0(\underline{\Lambda}, \underline{a})$. The infinitesimal generators will satisfy the commutation relations of the Poincaré group if and only if the operators in the latter set satisfy

$$[X^j_{g0}, P^k_0]_- = i\delta^{jk}; \qquad [j^j_{g0}, j^k_{g0}]_- = i\epsilon^{jkl} j^k_{g0} \tag{5.52}$$

with all other commutators in this set vanishing.

Since M_0 commutes with all of the generators of $U_0(\underline{\Lambda}, \underline{a})$, one possible approach is to add an interaction operator to M_0 that commutes with the set $\{\mathbf{P}_0, \mathbf{j}_{g0}, \mathbf{X}_{g0}\}$. The new mass operator is defined by

$$M := M_0 + V. \tag{5.53}$$

If V is any operator that satisfies the following conditions:

$$M = M^\dagger; \qquad M > 0; \tag{5.54}$$

$$[\mathbf{P}_0, V]_- = [\mathbf{X}_{g0}, V]_- = [\mathbf{j}_{g0}, V]_- = 0. \tag{5.55}$$

then the set $\{M, \mathbf{P}_0, \mathbf{X}_{g0}, \mathbf{j}_{g0}\}$ will satisfy the same commutation relations as the set $\{M_0, \mathbf{P}_0, \mathbf{X}_{g0}, \mathbf{j}_{g0}\}$. Note the similarity between these relations and the corresponding relations (3.34) in the Galilean invariant case. If the generators are then constructed from the set $\{M, \mathbf{P}_0, \mathbf{X}_{g0}, \mathbf{j}_{g0}\}$ using the inverse of Eqs. (5.35)–(5.37), then the resulting operators will satisfy the commutation relations of the Poincaré group.

Given such an interaction V it is useful to define two other related interactions:

$$U := M^2 - M_0^2 = V^2 + \{M_0, V\}_+ \tag{5.56}$$

and

$$W := \frac{1}{4}M^2 + \frac{(m_1^2 - m_2^2)^2}{4M^2} - \frac{1}{4}M_0^2 - \frac{(m_1^2 - m_2^2)^2}{4M_0^2}. \tag{5.57}$$

Note that W is obtained by solving first for $\mathbf{k}^2$ as a function of the non-interacting mass, and then computing the difference between that expression with the interacting mass less the corresponding quantity with the non-interacting mass. These operators, defined in terms of V, also commute with the set $\{\mathbf{P}_0, \mathbf{j}_{g0}, \mathbf{X}_{g0}\}$. Any of V, U, W can be expressed in terms of any one of the others. The interactions U and W are introduced for practical reasons, because the eigenvalue problem for the mass can be put in a form that resembles the nonrelativistic Schrödinger equation for the case of particles with equal mass in Eq. (5.56), and unequal masses in equation Eq. (5.57).

The conditions (5.55) are satisfied in the representation $|[l\,s]k\,j;\mathbf{P}\mu\rangle_{g0}$ if and only if the matrix elements of V (or U or W) have the form

$$_{g0}\langle[l'\,s']k'\,j';\mathbf{P}'\mu'|V|[l\,s]k\,j;\mathbf{P}\mu\rangle_{g0} = \delta_{j'j}\delta_{\mu'\mu}\delta(\mathbf{P}'-\mathbf{P})\langle k'l's'\|V^j\|kls\rangle, \tag{5.58}$$

where g_0 denotes a general spin and the reduced matrix element does not depend on μ or $\mathbf{P}$. At this point, it is possible to construct formal expressions for the Poincaré generators by replacing M_0 with M in the equations that define the generators in terms of $\mathbf{P}_0$, $\mathbf{j}_{g0}$, $\mathbf{X}_{g0}$ and M_0. This leads to expressions in which every Lorentz generator becomes interaction dependent. Although this will be discussed subsequently, it is not a practical procedure for solving the dynamics. In order to solve the dynamics using this representation, we need to solve the eigenvalue problem for the mass or some function of the mass operator. This leads to eigenstates of mass and spin that transform irreducibly. The eigenvalue problem takes on the three equivalent forms:

$$[\sqrt{m_1^2+k^2}+\sqrt{m_2^2+k^2}+V]|\Psi\rangle = \lambda|\Psi\rangle; \tag{5.59}$$

$$[m_1^2+m_2^2+2k^2+2\sqrt{(m_1^2+k^2)(m_2^2+k^2)}+U]|\Psi\rangle = \lambda^2|\Psi\rangle; \tag{5.60}$$

$$(k^2+W)|\Psi\rangle = \eta|\Psi\rangle, \tag{5.61}$$

where

$$\lambda^2 = 2\eta+(m_1^2+m_2^2)+2\sqrt{\eta(\eta+m_1^2+m_2^2)+m_1^2m_2^2}. \tag{5.62}$$

Each of these is a well defined eigenvalue problem. The eigenvector $|\Psi\rangle$ and the mass eigenvalue λ are the same in all three cases. Since the spin is already diagonal in the representation $|[l\,s]k,j;\mathbf{P}\,\mu\rangle_{g0}$, the solution of any of the above eigenvalue problems will lead to eigenfunctions

of the form

$$ {}_{g0}\langle [l'\,s']k'\,j';\mathbf{P}'\,\mu'|\lambda\,j;\mathbf{P}\,\mu\rangle_g = \delta_{\mu'\mu}\delta_{j'j}\delta(\mathbf{P}'-\mathbf{P})\Psi^j_\lambda(k'\,l'\,s'), \tag{5.63} $$

where $\Psi^j_\lambda(kls)$ satisfies:

$$ \begin{aligned} (\sqrt{m_1^2+k^2}+\sqrt{m_2^2+k^2})\Psi^j_\lambda(kls) \\ +\sum_{l's'}\int_0^\infty k'^2dk'\langle kls\|V^j\|k'l's'\rangle\Psi^j_\lambda(k'l's') = \lambda\Psi^j_\lambda(kls); \end{aligned} \tag{5.64} $$

$$ \begin{aligned} [m_1^2+m_2^2+2k^2+2\sqrt{(m_1^2+k^2)(m_2^2+k^2)}]\Psi^j_\lambda(kls) \\ +\sum_{l',s'}\int_0^\infty k'^2dk'\langle kls\|U^j\|k'l's'\rangle\Psi^j_\lambda(k'l's') = \lambda^2\Psi^j_\lambda(kls), \end{aligned} \tag{5.65} $$

or

$$ k^2\Psi^j_\lambda(kls)+\sum_{l',s'}\int_0^\infty k'^2dk'\langle kls\|W^j\|k'l's'\rangle\Psi^j_\lambda(k'l's') = \eta\Psi^j_\lambda(kls). \tag{5.66} $$

The solution of any one of these eigenvalue problems provides the complete relativistic two-body dynamics. The reason is that for each fixed value of λ, the eigenstate $|\lambda j;\mathbf{P}\mu\rangle_g$ must have the same transformation properties as the corresponding one-body state:

$$ U(\underline{\Lambda},\underline{a})|\lambda\,j;\mathbf{P}\,\mu\rangle_g = e^{i\Lambda P\cdot a}\sqrt{\frac{\omega_{M_0}(\mathbf{P}_\Lambda)}{\omega_{M_0}(\mathbf{P})}}|\lambda\,j;\mathbf{P}_\Lambda\,\bar{\mu}\rangle_g D^j_{\bar{\mu}\mu}[\underline{R}_g(\underline{\Lambda},P/\lambda)]. \tag{5.67} $$

If the mass eigenstates are normalized as follows:

$$ {}_g\langle(\lambda',j')\mathbf{P}',\mu'|(\lambda,j)\mathbf{P},\mu\rangle_g = \delta[\lambda';\lambda]\delta_{\mu'\mu}\delta_{j'j}\delta(\mathbf{P}'-\mathbf{P}), \tag{5.68} $$

then the matrix elements of $U(\underline{\Lambda},\underline{a})$ will have the form

$$ \begin{aligned} {}_g\langle\lambda'\,j';\mathbf{P}'\,\mu'|U(\underline{\Lambda},\underline{a})|\lambda\,j;\mathbf{P}\,\mu\rangle_g = \delta_{j'j}\delta(\mathbf{P}'-\mathbf{P}_\Lambda)\delta[\lambda';\lambda]e^{iP'\cdot a}\sqrt{\frac{\omega_\lambda(\mathbf{P}')}{\omega_\lambda(\mathbf{P})}} \\ \times D^j_{\mu'\mu}[\underline{R}_g(\underline{\Lambda},P/\lambda)], \end{aligned} \tag{5.69} $$

where $\delta[\lambda';\lambda] := \delta(\lambda'-\lambda)$ when λ is in the continuous spectrum of M, and $\delta[\lambda';\lambda] := \delta_{\lambda'\lambda}$ when λ is in the point spectrum of M. Equation (5.69) can be used to compute matrix elements of

$U(\underline{\Lambda}, \underline{a})$ in either of the free-particle bases by inserting a complete set of intermediate eigenstates. In the irreducible basis, the matrix elements are

$$\begin{aligned} {}_{g0}\langle [l'\, s']k\, j'; \mathbf{P}'\, \mu' | U(\underline{\Lambda}, \underline{a}) | [l\, s]k\, j; \mathbf{P}\, \mu \rangle_{g0} \\ = \int d^3\bar{P}' \int d^3\bar{P} \int d[\lambda'] \int d[\lambda] \sum {}_{g0}\langle [l'\, s']k'\, j'; \mathbf{P}'\, \mu' | \lambda'\, j'; \bar{\mathbf{P}}' \mu' \rangle_g \\ \times {}_g\langle \lambda'\, j'; \bar{\mathbf{P}}'\, \mu' | U(\underline{\Lambda}, \underline{a}) | \lambda\, j; \bar{\mathbf{P}}\, \mu \rangle_g\, {}_g\langle \lambda\, j; \bar{\mathbf{P}}\, \mu | [l\, s]k\, j; \mathbf{P}\, \mu \rangle_{g0}, \end{aligned} \tag{5.70}$$

where $\int d[\lambda]$ indicates an integral over the complete set of states associated with the continuous spectrum of M, and a sum over the complete set of states associated with the discrete spectrum of M. The input to this expression consists of the matrix elements of $U(\underline{\Lambda}, \underline{a})$ in the basis of mass eigenstates, together with the eigenfunctions. In the tensor-product representation, the matrix elements can be expressed in terms of the irreducible-basis matrix elements as follows:

$$\begin{aligned} {}_g\langle \mathbf{p}'_1\, \mu'_1\, \mathbf{p}'_2\, \mu'_2 | U(\underline{\Lambda}, \underline{a}) | \mathbf{p}_1\, \mu_1\, \mathbf{p}_2\, \mu_2 \rangle_g \\ = \int d^3P' \int k'^2 dk' \int d^3P \int k^2 dk \sum {}_g\langle \mathbf{p}'_1\, \mu'_1\, \mathbf{p}'_2\, \mu'_2 | [l'\, s']k'\, j'; \mathbf{P}'\, \mu' \rangle_{g0} \\ \times {}_{g0}\langle [l'\, s']k'\, j'; \mathbf{P}'\, \mu' | U(\underline{\Lambda}, \underline{a}) | [l, s]k\, j; \mathbf{P}\, \mu \rangle_{g0} \\ \times {}_{g0}\langle [l\, s]k\, j; \mathbf{P}\, \mu | \mathbf{p}_1\, \mu_1\, \mathbf{p}_2\, \mu_2 \rangle_g. \end{aligned} \tag{5.71}$$

To evaluate Eq. (5.71), one also needs one of the sets of Clebsch-Gordan coefficients defined above. All of these quantities are known explicitly once the eigenfunctions and eigenvalues of the mass operator are known. The transformation properties are thus reduced to algebra. Representations for the generators in this basis can be computed by differentiating Eq. (5.70) with respect to the parameters that label the one-parameter subgroups of the Poincaré group. The results are analogous to Eqs. (5.33) and (5.34) in the one-body case:

$$\mathbf{P} := \mathbf{P}; \tag{5.72}$$

$$H := \sqrt{\lambda^2 + \mathbf{P}^2}; \tag{5.73}$$

$$K^j := -\frac{1}{2}\{H, X^j_{\lambda g}\}_+ + iC^{jk}_{Kg}(\mathbf{P}/\lambda) j^k_{g0}; \tag{5.74}$$

$$J^j := (\mathbf{X}_{\lambda g} \times \mathbf{P})^j + C^{jk}_{Jg}(\mathbf{P}/\lambda) j^k_{g0}, \tag{5.75}$$

where $X^j_{\lambda g} := i(\partial/\partial P^j)$, holding λ, μ and j constant. Each of these operators can be expressed

in the free-particle representations using the eigenfunctions (5.63) and the Clebsch-Gordan coefficients (5.24).

When the two-body system allows scattering, there is also the problem of finding model interactions consistent with existing two-body scattering data. For the case of the nucleon-nucleon system, there has been a great deal of work invested in phase shift analysis, together with the development of nucleon-nucleon interaction potentials with parameters adjusted to fit these phase shifts. The phase shifts, when parameterized in terms of the relative momentum of the interacting pair, are relativistic invariants. Because the equations for the wave functions can be manipulated in a variety of ways without changing the solutions, *it is possible to use a previously fitted nonrelativistic nucleon-nucleon potential V_{NN} to construct model interactions V, U, or W (as in Eqs.* (5.53)*,* (5.56) *and* (5.57)*, respectively) which yield the same wave functions and invariant cross sections as the nonrelativistic potential V_{NN}.* This result does not depend on the form of the dynamics and is discussed in terms of transition operators in Appendix A. In applications, the binding energy may have to be refit, although the corrections for a weakly bound system like the deuteron are small (on the order of 0.1 KeV for the deuteron).

This completes the discussion of general constructions of relativistic two-body models. The Schrödinger equation has been replaced by one of Eqs. (5.59), (5.60) or (5.61). Once the eigenvalues λ_m are determined, the appropriate unitary representation of the Poincaré group can be constructed. The necessary tools consist of the Clebsch-Gordan coefficients that decompose the product of two one-body irreducible representation spaces into the direct integral of irreducible representations, together with a solution of the mass eigenvalue problem in the direct integral representation.

5.6. **Special Cases**

We now discuss three specific methods for adding interactions. The three cases of interest correspond to the forms of the dynamics introduced by Dirac (Di 49). They are distinguished by the property that there is a non-trivial subgroup of the Poincaré group whose unitary representation is the same for both the interacting and the non-interacting system. This subgroup is called the *kinematic* subgroup for that particular form of dynamics. Note that for the general

construction discussed above, the only kinematic transformations are those associated with spatial translations. These three methods differ in the choice of Clebsch-Gordan coefficients that relate the tensor product of two irreducible representations of $\mathcal{P}$ to a superposition of irreducible representations. The three Clebsch-Gordan coefficients have the special property that a minimal number of generators become interaction dependent. In the discussion which follows, we concentrate on the algebraic properties of the interacting and non-interacting generators for each form of dynamics.

The Instant Form We recall that for a single particle, the following relations were derived:

$$M = \sqrt{H^2 - \mathbf{P}^2}; \tag{5.76}$$

$$\mathbf{X}_c = -\frac{1}{2}\{\frac{1}{H}, \mathbf{K}\}_+ - \frac{\mathbf{P} \times (H\mathbf{J} - \mathbf{P} \times \mathbf{K})}{MH(M+H)}; \tag{5.77}$$

$$\mathbf{j}_c = \frac{1}{M}(H\mathbf{J} - \mathbf{P} \times \mathbf{K}) - \frac{\mathbf{P}(\mathbf{P} \cdot \mathbf{J})}{M(M+H)}, \tag{5.78}$$

along with the inverse relations

$$H = \sqrt{M^2 + \mathbf{P}^2}; \tag{5.79}$$

$$\mathbf{K} = -\frac{1}{2}\{H, \mathbf{X}_c\}_+ - \frac{\mathbf{P} \times \mathbf{j}_c}{H+M}; \tag{5.80}$$

$$\mathbf{J} = \mathbf{X}_c \times \mathbf{P} + \mathbf{j}_c. \tag{5.81}$$

These relations are consequences of using canonical spin. They show that a knowledge of the set $\{H, \mathbf{P}, \mathbf{J}, \mathbf{K}\}$ is equivalent to that of the set $\{M, \mathbf{P}, \mathbf{j}_c, \mathbf{X}_c\}$. The commutation relations between operators in one set implies the commutators between operators in the other set.

The realization of the dynamics is a four-step construction. The first step is to construct free-particle generators on $\mathcal{H}(2)$ by adding single-particle generators following Eqs. (5.4), (5.5) and (5.6).

The second step is to construct the set $\{M, \mathbf{P}, \mathbf{j}_c, \mathbf{X}_c\}$ as a functions of these *non-interacting* two-body generators by means of Eqs. (5.76), (5.77) and (5.78). The resulting operators are denoted by $\{M_0, \mathbf{P}_0, \mathbf{j}_{c0}, \mathbf{X}_{c0}\}$. The free mass operator M_0 commutes with any function of the generators, since it is the square root of a Casimir operator for the Poincaré group. In particular, it commutes with the set $\{\mathbf{P}_0, \mathbf{j}_{c0}, \mathbf{X}_{c0}\}$.

The third step is to add a two-body interaction to M_0 (or any function of the mass operator and other invariants as in (5.56) and (5.57)). This leads to an interacting mass operator:

$$M := M_0 + V. \tag{5.82}$$

The set $\{M,\, \mathbf{P}_0,\, \mathbf{j}_{c0},\, \mathbf{X}_{c0}\}$ will have the same commutation relations as the set $\{M_0,\, \mathbf{P}_0, \mathbf{j}_{c0},\, \mathbf{X}_{c0}\}$ provided that V (U or W) commute with the set $\{\mathbf{P}_0,\, \mathbf{j}_{c0},\, \mathbf{X}_{c0}\}$. The interaction V (or U or W) is assumed to satisfy these conditions and the spectral condition. This is similar to the construction outlined earlier, except that it is formulated without using the direct integral representation of the Hilbert space.

The fourth step is to use Eqs. (5.79), (5.80) and (5.81) to construct generators as functions of M, $\mathbf{P}_0$, $\mathbf{j}_{c0}$, and $\mathbf{X}_{c0}$. These generators automatically satisfy the commutation relations for the Poincaré Lie algebra because of the conditions on the interaction.

Note that in this procedure, H and $\mathbf{K}$ depend on M and become interaction dependent. The angular momentum operator does not depend on M, and thus $\mathbf{J} = \mathbf{J}_0$. The generators $\mathbf{P}_0$ and $\mathbf{J}_0$ form a closed Lie subalgebra of the Poincaré group corresponding to space translations and rotations. This is the subgroup of the Poincaré group that leaves the instant $t = t_0$ invariant, hence the name *instant form.*

As a practical matter, this representation is not constructed by integrating the Lie algebra to construct the group. Instead, one diagonalizes the mass operator in the irreducible representation associated with a system of two free particles given the Clebsch-Gordan coefficients in Eq. (5.20), and then proceeds as in the previous subsection. In this representation, the interaction matrix elements must have the form (5.58), the spin is canonical, and the continuous variable is the three-momentum.

The important property of the instant form is that the generators of translations and rotations are sums of the single-particle generators. If the corresponding unitary representation (5.67) is restricted to the Euclidean subgroup (rotations and translations), the coefficients on the right-hand side of that equation are independent of the mass eigenvalue λ.

The Front Form The front form has the largest kinematic subgroup of the various forms of the dynamics. The construction of a front-form dynamics is a four-step process, as in the instant form. The first step is the same as before, namely, to construct generators for two free particles by adding single-particle generators using Eqs. (5.4), (5.5) and (5.6). It is convenient to utilize the appropriate linear combinations of the generators associated with the front form. The second step is to use the relations:

$$M^2 = P^+P^- - \mathbf{P}_\perp^2; \tag{5.83}$$

$$j_f^3 = \frac{1}{P^+}[P^+J^3 - \hat{\mathbf{z}}\cdot(\mathbf{P}_\perp \times \mathbf{E}_\perp)]; \tag{5.84}$$

$$\begin{aligned}\mathbf{j}_{f\perp} = \frac{1}{M}\Big[&-\tfrac{1}{2}(P^+ - P^-)(\hat{\mathbf{z}}\times\mathbf{E}_\perp) + \hat{\mathbf{z}}\times\mathbf{P}_\perp K^3 + P^+\mathbf{J}_\perp \\ &- \frac{\mathbf{P}_\perp}{P^+}[P^+J^3 - \hat{\mathbf{z}}\cdot(\mathbf{P}_\perp\times\mathbf{E}_\perp)]\Big].\end{aligned} \tag{5.85}$$

to eliminate the generators P^- and $\mathbf{J}$. A knowledge of the set

$$\{M, P^+, \mathbf{P}_\perp, \mathbf{E}_\perp, K^3, \mathbf{j}_f\}$$

is equivalent to a knowledge of the set

$$\{P^-, P^+, \mathbf{P}_\perp, \mathbf{E}_\perp, K^3, \mathbf{J}_\perp, J^3\}.$$

For *non-interacting* two-body generators, the set of operators is denoted by

$$\{M_0, P_0^+, \mathbf{P}_{\perp 0}, \mathbf{E}_{\perp 0}, K_0^3, \mathbf{j}_{f0}\}.$$

The third step is to add an interaction term V to the free mass operator M_0:

$$M := M_0 + V \qquad \text{or} \qquad M^2 := M_0^2 + U \tag{5.86}$$

For the front form, the interaction V must satisfy:

$$[\mathbf{E}_\perp, V]_- = [K^3, V]_- = [\mathbf{j}_{f0}, V]_- = [\mathbf{P}_\perp, V]_- = [P^+, V]_- = 0. \tag{5.87}$$

These conditions mean that the set of operators

$$\{M, P^+, \mathbf{P}_\perp, \mathbf{E}_\perp, K^3, \mathbf{j}_f\}$$

satisfy the same commutation relations among themselves as the set

$$\{M_0, P_0^+, \mathbf{P}_{\perp 0}, \mathbf{E}_{\perp 0}, K_0^3, \mathbf{j}_{f0}\}.$$

The fourth step is to invert the relations (5.83)–(5.85), with M replacing M_0. The resulting operators necessarily satisfy the Poincaré commutation relations, and are formally given by

$$P^- = \frac{M^2 + \mathbf{P}_{\perp 0}^2}{P_0^+}; \tag{5.88}$$

$$J^3 = j_{f0}^3 + \frac{1}{P_0^+}\hat{\mathbf{z}} \cdot (\mathbf{P}\perp\, 0 \times \mathbf{E}_{\perp 0}); \tag{5.89}$$

$$\mathbf{J}_\perp = \frac{1}{P^+}\left[\tfrac{1}{2}(P_0^+ - P^-)(\hat{\mathbf{z}} \times \mathbf{E}_{\perp 0}) - (\hat{\mathbf{z}} \times \mathbf{P}_{\perp 0})K^3 + \mathbf{P}_{\perp 0} j_{f0}^3 + M\mathbf{j}_{f\perp 0}\right]. \tag{5.90}$$

The six remaining generators are sums of the one-body generators. Note that the right-hand side of Eq. (5.89) does not involve M, so that J^3 is also the sum of one-body operators. This shows that seven generators do not involve interactions. Inspection of the commutation relations in Appendix B shows that these seven generators form a closed Lie subalgebra. The kinematic subgroup is the set of transformations that leave the light front $x^+ = 0$ invariant.

The actual construction of a front-form dynamics is similar to that of the instant form, except it uses the Clebsch-Gordan coefficients given in Eq. (5.28). In this representation, the

generators E_0^i and K_0^3 have same the form as Eqs. (4.54) and (4.55) in the one-body case:

$$E_0^i = -iP^+ \frac{\partial}{\partial P^i}; \tag{5.91}$$

$$K_0^3 = -iP^+ \frac{\partial}{\partial P^+}. \tag{5.92}$$

It follows from these equation that the most general interaction V satisfying the commutation relations (5.87) in this representation has the form

$${}_{f0}\langle [l'\, s']k'\, j'; \tilde{\mathbf{P}}'\, \mu'|V|[l\, s]k\, j; \tilde{\mathbf{P}}\, \mu\rangle_{f0} = \delta_{j'j}\delta_{\mu'\mu}\delta(\tilde{\mathbf{P}}' - \tilde{\mathbf{P}})\langle k'l's'\|V^j\|kls\rangle. \tag{5.93}$$

Matrix elements of the unitary representation of the Poincaré group are constructed by diagonalizing M or M^2 in the irreducible representation basis, and then using the Clebsch-Gordan coefficients to transform to free-particle bases if necessary. In this case, irreducible eigenfunctions of the four-momentum and spin observables have the form

$${}_{f0}\langle [l\, s]k\, j'; \tilde{\mathbf{P}}\, \mu'|\lambda\, j; \tilde{\mathbf{P}}'\, \mu\rangle_f = \delta_{j'j}\delta_{\mu'\mu}\delta(\tilde{\mathbf{P}}' - \tilde{\mathbf{P}})\Psi^j_\lambda(kls), \tag{5.94}$$

where $\Psi^j_\lambda(kls)$ satisfies one of the eigenvalue equations (5.64)–(5.66), as in the instant form. These states transform irreducibly:

$$U(\underline{\Lambda}, \underline{a})|\lambda\, j; \tilde{\mathbf{P}}\, \mu\rangle_f = e^{i\Lambda P\cdot a}\sqrt{\frac{P^+_\Lambda}{P^+}}\sum |\lambda\, j; \tilde{\mathbf{P}}_\Lambda\, \bar{\mu}\rangle_f D^j_{\bar{\mu}\mu}[(\underline{R}_f(\Lambda, P)] \tag{5.95}$$

The coefficients of the transformations that leave the light front $x^+ = 0$ invariant do not depend on λ.

The Point Form In the point form of dynamics, the full Lorentz group is kinematic. These are the transformations that leave the point $x^\mu = 0$ invariant. The generators of spacetime translations are interaction dependent in the point form.

The point-form representations utilize canonical spin, but replace the three-momentum variables by the space components of the four-velocity: $\mathbf{q} = \mathbf{p}/m$. For single-particle states, this modification changes only the normalization coefficient. These state vectors are given explicitly for the case of single-particle states in the previous section. For systems of non-interacting particles, this corresponds to using the Clebsch-Gordan coefficients given in Eq. (5.31).

The point form construction is also a four-step construction. The first step is identical to the first step in the instant form, which is to construct a set of generators for two free particles as sums of single-particle generators using Eqs. (5.4), (5.5) and (5.6).

These generators for two-non-interacting particles are used to construct operators $\{M_0,\, \mathbf{X}_{q0},\, \mathbf{Q}_0,\, \mathbf{j}_{c0}\}$ using the general relations:

$$M := \sqrt{H^2 - \mathbf{P}^2}; \tag{5.96}$$

$$\mathbf{Q} := \mathbf{P}/M; \tag{5.97}$$

$$X_q^j := -\frac{1}{2}\{\frac{M}{H}, K^j\}_+ - \frac{M\mathbf{Q}\times(H\mathbf{J} - M\mathbf{Q}\times\mathbf{K})}{H(M+H)}; \tag{5.98}$$

$$\mathbf{j}_c = \frac{1}{M}(H\mathbf{J} - M\mathbf{Q}\times\mathbf{K}) - \frac{M^2\mathbf{Q}(\mathbf{Q}\cdot\mathbf{J})}{M+H}. \tag{5.99}$$

The third step is to add an interaction V to the free mass operator M_0 that satisfies:

$$[\mathbf{X}_{q0}, V]_- = [\mathbf{Q}_0, V]_- = [\mathbf{j}_{c0},\, V]_- = 0. \tag{5.100}$$

This ensures that

$$M := M_0 + V \tag{5.101}$$

commutes with $\mathbf{X}_{q0}$, $\mathbf{Q}_0$ and $\mathbf{j}_{c0}$.

The fourth step is to use M, $\mathbf{X}_{q0}$, $\mathbf{Q}_0$ and $\mathbf{j}_{c0}$ in the inverse relations:

$$\mathbf{P} = M\mathbf{Q} \tag{5.102}$$

$$H = M\sqrt{1+\mathbf{Q}^2}; \tag{5.103}$$

$$\mathbf{K} = -\frac{1}{2}\{\sqrt{1+\mathbf{Q}^2}, \mathbf{X}_q\}_+ - \frac{\mathbf{Q}\times\mathbf{j}_c}{(1+\sqrt{1+\mathbf{Q}^2})}; \tag{5.104}$$

$$\mathbf{J} = \mathbf{X}_q \times \mathbf{Q} + \mathbf{j}_c \tag{5.105}$$

to construct generators that include interactions. With this construction, the expressions for the Lorentz generators are independent of M, which means that they are identical to the corresponding non-interacting generators.

In practice, the interaction V is given in the basis defined by the Clebsch-Gordan coefficients of Eq. (5.31). In this basis the commutation relations (5.100) imply that V has the form:

$${}_{p0}\langle [l'\,s']k'\,j';\mathbf{Q}'\,\mu'|V|[l\,s]k\,j;\mathbf{Q}\,\mu\rangle_{p0} = \delta_{j'j}\delta_{\mu'\mu}\delta(\mathbf{Q}'-\mathbf{Q})\langle k'l's'\|V^j\|kls\rangle. \tag{5.106}$$

Irreducible eigenstates of the four-momentum and spin observables have the form

$${}_{p0}\langle [l\,s]k\,j';\mathbf{Q}'\,\mu|\lambda\,j;\mathbf{Q}\,\mu'\rangle_p = \delta_{j'j}\delta_{\mu'\mu}\delta(\mathbf{Q}'-\mathbf{Q})\Psi^j_\lambda(kls), \tag{5.107}$$

where $\Psi^j_\lambda(kls)$ satisfies one of the eigenvalue equations (5.64)–(5.66), as in the instant and front forms. These states transform irreducibly:

$$U(\underline{\Lambda},\underline{a})|\lambda\,j;\mathbf{Q}\,\mu\rangle_p = e^{i\lambda\Lambda Q\cdot a}\sqrt{\frac{\omega_1(\mathbf{Q}_\Lambda)}{\omega_1(\mathbf{Q})}}\sum|\lambda\,j;\mathbf{Q}_\Lambda\,\bar{\mu}\rangle_p D^j_{\bar{\mu}\mu}[\underline{R}_c(\underline{\Lambda},Q)]. \tag{5.108}$$

For the case of pure Lorentz transformations, the coefficient on the right-hand side of Eq. (5.108) are independent of the mass eigenvalue λ.

6. The 2+1 Body Problem

In principle, we could extend the Bakamjian-Thomas construction discussed in the previous section to cover arbitrary numbers of particles. However, for three or more particles, a new consideration enters, namely, that of *cluster separability,* or *macroscopic locality.* Cluster separability is a physical requirement that goes beyond that of relativistic invariance, although it is considerably more difficult to satisfy in relativistic systems than in their nonrelativistic counterparts. It was first studied for Bakamjian-Thomas constructions by Foldy (Fo 61), and the three-body problem in particular by Coester (Co 65).

A system of two interacting particles and a non-interacting spectator is the simplest for which the formulation of macroscopic locality becomes non-trivial. The solution of the 2+1 body problem is used to formulate the three-body problem in the next section. In this section, we define macroscopic locality, construct useful bases for systems of three particles, and construct two scattering equivalent (unitarily equivalent with the same scattering matrix) formulations of a model with two interacting particles and a non-interacting spectator. We show that one of the 2+1 body models trivially satisfies macroscopic locality, while the second violates it. The nature of this violation is discussed, and a unitary transformation that relates these two models is constructed. Both formulations of the 2+1 body problem, as well as this unitary transformation, play an important role in the formulation of a three-body model satisfying macroscopic locality.

6.1. Macroscopic Locality and the 2+1 Body Problem

The problem of two interacting particles with a third particle as spectator is the simplest system where macroscopic locality becomes non-trivial. This is strictly a feature of the relativistic problem that does not occur nonrelativistically. An example is a model of a deuteron with a third particle as spectator. If this problem is not formulated properly, the interaction between the constituents in the deuteron can vanish when the spectator and the deuteron are separated by a large spacelike distance. We will show that the correct formulation uses the relative momentum operator between the interacting two-body system and the spectator, while the incorrect formulation uses the relative momentum operator between the *non-interacting* two-body system and the

spectator. Problems occur because the relative momentum operator between a non-interacting pair and a spectator does not commute with the two-body interaction. The problem is subtle because the correct and incorrect (Mu 78) formulation of the problem are unitarily equivalent *and* have the same scattering matrix (Co 82). Both formulations are used in the construction of dynamical three-body models.

Macroscopic locality is the mathematical formulation of the physical picture that when a system is separated into disjoint subsystems by a sufficiently large spacelike separation, then the subsystems behave as independent systems. This differs from microscopic locality, which is an idealization of macroscopic locality to *arbitrarily* small spacelike separations. It is only macroscopic locality that is directly testable by experiment.

The importance of macroscopic locality can be understood without appealing either to relativity or to quantum mechanics. It is essential because our understanding of physical laws is a consequence of experiments performed on systems in varying degrees of isolation. Experiments done on different isolated systems in different regions of spacetime yield results which depend only on properties of the isolated system. Physical laws are formulated by abstracting properties of isolated subsystems to larger systems. An example of this is the law of conservation of momentum. We expect that the momentum is conserved in a translationally invariant universe. This is never tested by measuring the momentum of the universe at two different times. Instead, it is tested by checking it for different isolated systems and then abstracting these results to the universe. Macroscopic locality is the property that physical principles that apply to a system apply equally to isolated subsystems. It is the physical principle that justifies the abstraction from measured properties of isolated systems to physical laws that cannot always be directly tested by experiment.

For relativistic models, macroscopic locality means that an isolated system, such as a deuteron, must satisfy all of the physical properties that are demanded of a relativistic theory. For a model that permits a deuteron to exist asymptotically, there should be a unitary representation of $\mathcal{P}$ that relates free deuteron states in any inertial coordinate systems. The mass operator for this representation must have one eigenvalue, which is the deuteron mass. It is mathematically possible to have the mass operator for the universe to have a positive spectrum

and with the physical deuteron having a negative mass spectrum. This would satisfy the spectral condition, but would not cluster into a subsystem that satisfies its own spectral condition.

Macroscopic locality is also essential for building interacting many-body models from input determined from few-body subsystems. It ensures that the two-body interactions used in the many-body problem are the same as the two-body interactions used in the two-body problem.

The interesting feature of relativistic models is that it is easy to violate macroscopic locality. In general, macroscopic locality requires that *all* relevant physical properties which hold for the system must also hold for each possible isolated subsystem. For the purpose of this review, our considerations involve the role of macroscopic locality in the formulation of relativistic dynamics. In this case, the discussion of macroscopic locality will be limited to relativistic transformation laws.

To find a mathematical characterization of macroscopic locality, we need to define a means for dividing a system into isolated subsystems. In principle, this should be done by translating each subsystem until they have a sufficiently large spacelike separation. Since finite spacelike distances are not invariant under Lorentz transformations, a meaningful formulation of this condition can be made by considering the limit of an infinite spacelike separation. It can be shown that if a model satisfies all of the axioms of local quantum field theory with the exception of microscopic locality, which in turn is replaced by the assumption that observables commute only when they are separated beyond a fixed finite distance, then the model must satisfy microscopic locality. In particular, the distance must be zero (St 64).

To construct the operators that provide the desired translation of the two subsystems, we first consider two isolated subsystems, denoted by x and y. In principle, we can make independent measurements on each subsystem and find a maximal set of independent observables for each system. If each system behaves like a separate relativistic system, then we should be able to choose mutually commuting observables that include four-momentum operators and independent spin observables for each subsystem (*i.e.,* it should be possible to make independent measurements of the four-momentum and internal spin of each subsystem). This implies that the Hilbert space for this system can be represented as a tensor product of two Hilbert spaces, each one being represented as a space of square integrable functions of the values of a complete set of

measurements that can be performed on each isolated subsystem, with each space carrying a unitary representation of $\mathcal{P}$ associated with each subsystem:

$$\mathcal{H} = \mathcal{H}_x \otimes \mathcal{H}_y; \tag{6.1}$$

$$U_{x,y}(\underline{\Lambda}, \underline{a}) = U_x(\underline{\Lambda}, \underline{a}) \otimes U_y(\underline{\Lambda}, \underline{a}). \tag{6.2}$$

Both of these representations are assumed to have all of the physical properties expected of the physical system, such as time translation generators (subsystem Hamiltonians), with the desired symmetries and spectrum bounded from below.

In addition, it is assumed that there exists an interacting representation $U(\underline{\Lambda}, \underline{a})$ of $\mathcal{P}$ associated with the interacting system composed of the particles in both subsystems. Spacetime translation operators for particles or clusters of particles in each subsystem are defined by

$$T_x(b) = U_x(\underline{I}, \underline{b}) \otimes I_y \qquad T_y(c) = I_x \otimes U_y(\underline{I}, \underline{c}). \tag{6.3}$$

For a three-body system, x might represent the pair (12) and y might represent particle 3. In this case, $T_x(b)$ is the operator that translates the interacting (12) pair, and $T_y(c)$ is the operator that translates particle 3.

The physical representation $U(\underline{\Lambda}, \underline{a})$ of $\mathcal{P}$ satisfies *macroscopic locality* if it satisfies the following condition:

$$\lim_{(b-c)^2 \to \infty} \langle \Psi | T_x^\dagger(\Lambda b) T_y^\dagger(\Lambda c) \big[U(\underline{\Lambda}, \underline{a}) - U_{x,y}(\underline{\Lambda}, \underline{a}) \big] T_x(b) T_y(c) | \Psi \rangle = 0 \tag{6.4}$$

for all normalizable vectors $|\Psi\rangle$. Note that the argument of the translation operator on the left hand side of Eq. (6.4) is transformed by Λ. This is a consequence of the group representation properties in the absence of any interaction. This equation implies that for large spacelike separations of the subsystems, $U(\underline{\Lambda}, \underline{a})$ can be replaced by a tensor product of representations associated with each subsystem. This feature, coupled with the implicit assumption that the representations in each factor of the tensor product have all of the properties of the full system,

is the physically relevant form of macroscopic locality. Macroscopic locality is a property that must hold for matrix elements of operators. However, the rate at which Eq. (6.4) approaches zero may depend on the initial and final state vector. This is a weak limit. It is sufficient (but not necessary) replace Eq. (6.4) with a similar expression involving a strong limit:

$$\lim_{(b-c)^2\to\infty} \|\big[U(\underline{\Lambda},\underline{a}) - U_{xy}(\underline{\Lambda},\underline{a})\big]T_x(b)T_y(c)|\Psi\rangle\| = 0 \tag{6.5}$$

for all normalizable vectors $|\Psi\rangle$. This condition is called *strong macroscopic locality*. It implies that matrix elements vanish with a rate that depends only on the initial state vector. Strong macroscopic locality is a sufficient condition for weak macroscopic locality, and is of interest because it is easier to handle mathematically. In either case, the condition must hold for all values of $\underline{\Lambda}$ and $\underline{a}$.

The application of these conditions to infinitesimal Poincaré transformations leads to conditions on the infinitesimal generators of models satisfying macroscopic locality:

$$\lim_{(b-c)^2\to\infty} \langle\Psi|T_x^\dagger(b)T_y^\dagger(c)G^{x,y}T_x(b)T_y(c)|\Psi\rangle = 0, \tag{6.6}$$

and

$$\lim_{(b-c)^2\to\infty} \|G^{x,y}T_x(b)T_y(c)|\Psi\rangle\| = 0, \tag{6.7}$$

corresponding to the weak and strong formulations of macroscopic locality, respectively. In both of these expressions,

$$G^{x,y} := G - G_x \otimes I_y - I_x \otimes G_y \tag{6.8}$$

is the residual interaction between the actual generators and the sum of the subsystem generators. For instance, if $G = H$ is the Hamiltonian, then

$$H^{x,y} = H - H_x \otimes I_y - I_x \otimes H_y, \tag{6.9}$$

which includes the residual interactions of particles in cluster x with those in cluster y. Formally, the conditions (6.6) and (6.7) are obtained from Eqs. (6.4) and (6.5) by differentiating $U(\underline{\Lambda},\underline{a})$

with respect to coordinates, angles, and rapidities about the origin. Since the generators are typically unbounded operators, the initial and final states must be restricted to a suitable domain for Eqs. (6.6) and (6.7) to hold. For this reason, the conditions (6.4) and (6.5) are easier to use mathematically. However, the formulation of macroscopic locality has a simple physical interpretation when expressed in terms of the generators. For the Hamiltonian, it implies that the energy of two asymptotically separated subsystems is the sum of the individual subsystem energies. For the linear and angular momenta, it implies that the linear and angular momentum of two asymptotically separated subsystems is the sum of the individual subsystem linear and angular momenta . These characterizations make it difficult to tolerate models that do not respect macroscopic locality. Macroscopic locality applied to the boost generator does not have any simple physical interpretation.

Macroscopic Locality applies only to subsystems that can exist as stable asymptotically separated subsystems, which normally means subsystems that interact by short range interactions. It does not apply to quark-antiquark pairs coupled to the same $SU(3)$ color singlet, which interact with long range confining interactions. Also, there are generalizations of this condition that must hold when a system is broken up into more than two subsystems, and there are consistency conditions which relate different cluster limits. The general case is discussed in (Co 82, So 77).

6.2. **The Three-Body Hilbert Space**

The Hilbert space $\mathcal{H}(3)$ for a system of three particles of mass m_i and spin s_i $(i = 1, 2, 3)$ is the tensor product of the three single-particle Hilbert spaces:

$$\mathcal{H}(3) := \bigotimes_{i=1}^{3} \mathcal{H}_{m_i}^{j_i}. \tag{6.10}$$

The basis vectors associated with the j^{th} particle are defined in Eq. (4.20):

$$|i\rangle := |m_i\, j_i; \mathbf{p}_i\, \mu_i\rangle_g \tag{6.11}$$

where j_i^2 has eigenvalue $s_i(s_i + 1)$. In general the one-body representations are simultaneous eigenstates of three-momentum and canonical spin, light-front momentum vectors and front-form spin, four-velocity and canonical spin, or another combination of continuous and discrete

quantum numbers needed to fix a pure state of a particle. In this section, the continuous observables are taken to be the three-momentum and the three-component of the canonical spin. Other combinations of continuous and spin observables can be treated similarly; the primary modification involves the use of different Clebsch-Gordan coefficients.

The single-particle vectors $|i\rangle$ transform irreducibly under $\mathcal{P}$, and have the normalization

$$\langle i'|i\rangle = \delta_{\mu'_i\mu_i}\delta(\mathbf{p}'_i - \mathbf{p}_i). \tag{6.12}$$

The tensor product of single-particle bases:

$$|\mathbf{p}_1\,\mu_1\,\mathbf{p}_2\,\mu_2\,\mathbf{p}_3\,\mu_3\rangle_{c0} := |1\rangle \otimes |2\rangle \otimes |3\rangle \tag{6.13}$$

is a basis on $\mathcal{H}(3)$.

In order to describe a system in which particles 1 and 2 interact, it is convenient to represent the three-particle Hilbert space as a tensor product of the two-particle Hilbert space, associated with the interacting pair, and the one-particle Hilbert space, associated with the spectator:

$$\mathcal{H}(3) = \mathcal{H}(2) \otimes \mathcal{H}(1). \tag{6.14}$$

If we use the Clebsch-Gordan coefficients for $\mathcal{P}$ constructed in the previous section, the basis vectors on $\mathcal{H}(2)$ can be replaced by state vectors $|\mathbf{p}_1\,\mu_1\,\mathbf{p}_2\,\mu_2\rangle \;\rightarrow\; |[l\,s]k\,j\,\mathbf{p}\,\mu\rangle$, as defined in Eq. (5.20), which transform irreducibly under the action of $\mathcal{P}$ We abbreviate these as follows:

$$|[12]k\,j\,\mathbf{p}\,\mu\rangle := |[l\,s]k\,j\,\mathbf{p}\,\mu\rangle. \tag{6.15}$$

The tensor product of the basis (6.15) with the one-body basis is the first of two bases used to construct a model of two interacting particles with a non-interacting spectator:

$$|[12,3]\mathbf{p}_{12}\,\mu_{12};\mathbf{p}_3\,\mu_3\rangle_{TP} := |[12]k_{12}\,j_{12};\,\mathbf{p}_{12}\,\mu_{12}\rangle \otimes |m_3\,j_3;\,\mathbf{p}_3\,\mu_3\rangle. \tag{6.16}$$

We use the subscript TP to denote the tensor-product basis.

The second basis involves the following change of variables

$$\{\mathbf{p}_{12}, \mathbf{p}_3\} \to \{\mathbf{P}, \mathbf{q}\}, \tag{6.17}$$

where

$$\mathbf{P} := \mathbf{p}_{12} + \mathbf{p}_3; \tag{6.18}$$

$$\mathbf{q} := (\omega_{m_3}(\mathbf{q}), \mathbf{q}) = L_c^{-1}(P/M_0)(\omega_{m_3}(\mathbf{p}_3), \mathbf{p}_3) = \mathbf{p}_3 - \mathbf{P}\Phi_c(\mathbf{P}, \mathbf{p}_3, k_{12}), \tag{6.19}$$

where

$$\Phi_c(\mathbf{P}, \mathbf{p}_3, k_{12}) = \frac{1}{M_0}\left[\frac{\mathbf{P}\cdot\mathbf{p}_3}{M_0 + H_0} + \omega_{m_3}(\mathbf{p}_3)\right], \tag{6.20}$$

M_0 is the invariant mass of the non-interacting three-body system, and $H_0 := (\mathbf{P}^2 + M_0)^{\frac{1}{2}}$. Both of these operators are functions of k_{12} in this representation.

Because the operator $\mathbf{q}$ is defined in Eq. (6.19) by applying a Lorentz transformation function of the four-velocity of the *non-interacting* system to the four-momentum of the spectator particle, this operator has no physical interpretation in an *interacting* system. In the non-interacting system it can be interpreted as a relative momentum vector. As we will see, difficulties with macroscopic locality occur because there is no physical interpretation for this operator, which is nevertheless defined mathematically. The relevant property is that the coefficient matrix in this variable change $\{\mathbf{p}_{12}, \mathbf{p}_3\} \to \{\mathbf{P}, \mathbf{q}\}$ involves the invariant mass, M_0, of a system of three non-interacting particles.

If the transformed states are given a delta function normalization in the continuous variables, the new basis vectors are given by

$$|[12]\mathbf{P}\,\mathbf{q}\,k_{12}\,j_{12}\,\mu_{12}; j_3\,\mu_3\rangle_{BT} = \left|\frac{\partial(\mathbf{p}_{12}, \mathbf{p}_3)}{\partial(\mathbf{P}, \mathbf{q})}\right|^{\frac{1}{2}} |[12,3]\mathbf{p}_{12}\,\mu_{12}, \mathbf{p}_3\,\mu_3\rangle \tag{6.21}$$

where

$$\mathbf{p}_{12} = \mathbf{p}_{12}(\mathbf{P}, \mathbf{q}, k_{12}); \qquad \mathbf{p}_3 = \mathbf{p}_3(\mathbf{P}, \mathbf{q}, k_{12}) \tag{6.22}$$

are obtained by inverting Eqs. (6.18) and (6.19). The mass and spin quantum numbers of the spectator, which are fixed, are suppressed in this notation. We use the subscript BT (which

stands for Bakamjian-Thomas, although its relevance will not become completely clear until later) to distinguish this basis.

The TP basis and the BT basis, both of which are plane-wave bases, are used in the formulation of models of two interacting particles and a non-interacting spectator. The BT basis can be expressed as a linear combination of states that transform irreducibly under $\mathcal{P}$ by completing the Clebsch-Gordan coefficient:

$$\begin{aligned}|[12,3]q\,j;\mathbf{p}\,\mu\rangle_{BT} = \sum\int d\hat{\mathbf{q}}\,|[12]\mathbf{p}\,\mathbf{q}\,k_{12}\,j_{12}\,\mu_{12};j_3\,\mu_3\rangle_{BT}Y_{L\mu_L}(\hat{\mathbf{q}})\\ \times D^{j_{12}}_{\mu_{12}\mu_{12'}}[\underline{R}(\underline{L}_c(p),q_{12})]D^{j_3}_{\mu_3\mu_{3'}}[\underline{R}(\underline{L}_c(p),q)]\\ \times\langle j_{12}\,\mu'_{12}\,j_3\,\mu'_3|S\,\mu_S\rangle\langle L\,\mu_L\,S\,\mu_S|j\,\mu\rangle.\end{aligned} \tag{6.23}$$

In this expression, $q_{12} := (\omega_{m_{12}}(\mathbf{q}),-\mathbf{q})$ where m_{12} is the invariant mass of two *non-interacting* particles.

6.3. **Two 2+1 Body Models**

We now wish to add two-body interactions using the two bases constructed above. Interactions in both cases are described in terms of a two-body mass operator:

$$M_{12}^2 := m_{12}^2 + \tilde{v}, \tag{6.24}$$

where $\tilde{v}$ has the form

$$\langle[12]'k'\,j';\mathbf{p}'\,\mu'|\tilde{v}|[12]k\,j;\mathbf{p}\,\mu\rangle = \delta_{j'j}\delta_{\mu'\mu}\delta(\mathbf{p}'-\mathbf{p})\langle[12]'k'\|v^j\|[12]k\rangle. \tag{6.25}$$

The first step toward adding a spectator particle is to imbed this interaction in the three-particle Hilbert space. To do this, consider the following two interactions, which are related to $\tilde{v}$ by including delta functions associated with the spectator particle:

$$\begin{aligned}{}_{TP}\langle[12,3]'\mathbf{p}'_{12}\,\mu'_{12};\mathbf{p}'_3\,\mu'_3|v_{TP}|[12,3]\mathbf{p}_{12}\,\mu_{12};\mathbf{p}_3\,\mu_3\rangle_{TP}\\ = \delta_{j'_{12}j_{12}}\delta_{\mu'_{12}\mu_{12}}\delta_{\mu'_3\mu_3}\delta(\mathbf{p}'_{12}-\mathbf{p}_{12})\delta(\mathbf{p}'_3-\mathbf{p}_3)\langle[12]k|v^{j_{12}}|[12]'k'\rangle,\end{aligned} \tag{6.26}$$

and

$$\begin{aligned}{}_{BT}\langle[12]'\mathbf{P}'\,\mathbf{q}'\,k'_{12}\,j'_{12}\,\mu'_{12};\mu'_3|v_{BT}|[12]\mathbf{P}\,\mathbf{q}\,k_{12}\,j_{12}\,\mu_{12};\mu_3\rangle_{BT}\\ = \delta_{j'_{12}j_{12}}\delta_{\mu'_{12}\mu_{12}}\delta_{\mu'_3\mu_3}\delta(\mathbf{P}'-\mathbf{P})\delta(\mathbf{q}'-\mathbf{q})\langle[12]'k'|v^{j_{12}}|[12]k\rangle.\end{aligned} \tag{6.27}$$

The interaction v_{BT} commutes with the operators $\mathbf{P}_0$, $\mathbf{j}_{c0}$ and $\mathbf{X}_{c0}$ of the non-interacting system.

This means that it satisfies the condition for a Bakamjian-Thomas construction.

Note that both interactions lead to the same scattering matrix and two-body binding energies. This can be seen by realizing that the conserved kinematic variables that appear in the delta functions do not appear in either the scattering amplitudes or the phase shifts, since the scattering amplitudes and phase shifts only involve quantities that change in the interacting system relative to the non-interacting one. Because the delta functions account for the entire difference in these interactions, the scattering is identical for both interactions. All of these statements follow trivially by direct evaluation of the scattering matrix. Specifically, it follows from the nature of these interactions that

$$\begin{aligned} {}_{TP}\langle [12,3]' \mathbf{p}'_{12}\, \mu'_{12}; \mathbf{p}'_3\, \mu'_3 | S_{TP} | [12,3] \mathbf{p}_{12}\, \mu_{12}; \mathbf{p}_3\, \mu_3 \rangle_{TP} \\ = \delta_{j'_{12} j_{12}} \delta_{\mu'_{12}\mu_{12}} \delta_{\mu'_3 \mu_3} \delta(\mathbf{p}'_{12} - \mathbf{p}_{12}) \delta(\mathbf{p}'_3 - \mathbf{p}_3) \langle [12]' k | s^{j_{12}} | [12] k \rangle, \end{aligned} \tag{6.28}$$

and

$$\begin{aligned} {}_{BT}\langle [12]' \mathbf{P}'\, \mathbf{q}'\, k'_{12}\, j'_{12}\, \mu'_{12}; \mu'_3 | S_{BT} | [12] \mathbf{P}\, \mathbf{q}\, k_{12}\, j_{12}\, \mu_{12}; \mu_3 \rangle_{BT} \\ = \delta_{j'_{12} j_{12}} \delta_{\mu'_{12}\mu_{12}} \delta_{\mu'_3 \mu_3} \delta(\mathbf{P}' - \mathbf{P}) \delta(\mathbf{q}' - \mathbf{q}) \langle [12]' k' | s^{j_{12}} | [12] k \rangle, \end{aligned} \tag{6.29}$$

where $\langle [12] k' | s^j | [12] k \rangle$ is the two-particle S matrix. Because these expressions must be evaluated on shell, $(k' = k)$, the delta functions $\delta(\mathbf{p}'_{12} - \mathbf{p}_{12})\delta(\mathbf{p}'_3 - \mathbf{p}_3)$ and $\delta(\mathbf{P}' - \mathbf{P})\delta(\mathbf{q}' - \mathbf{q})$ become equivalent. This implies equality of the $2+1$ body S matrix for these two interactions.

Similar remarks apply to the two-body binding energy. Although the reduced matrix elements and associated scattering matrices are identical in both of these expressions, the operators are not identical. The interaction v_{TP} satisfies

$$[v_{TP}, \mathbf{p}_{12}]_- = [v_{TP}, \mathbf{p}_3]_- = 0, \tag{6.30}$$

while the interaction v_{BT} satisfies

$$[v_{BT}, \mathbf{P}]_- = [v_{BT}, \mathbf{q}]_- = 0, \tag{6.31}$$

where $\mathbf{p}_{12}$ is the total momentum associated with the (12) pair. The interactions v_{TP} and v_{BT} are different because the reduced interaction is not diagonal in k_{12}, which is involved in the

relation between $\mathbf{q}$ and $\mathbf{p}_3$:

$$\mathbf{q} = \mathbf{p}_3 - \mathbf{P}\Phi_c(\mathbf{P}, \mathbf{p}_3, k_{12}). \tag{6.32}$$

These interactions are used to construct mass operators for the system of two particles plus one spectator:

$$M_{TP}^2 := \left[\sqrt{\mathbf{p}_{12}^2 + {M_{12}}_{TP}^2} + \sqrt{\mathbf{p}_3^2 + m_3^2}\right]^2 - \mathbf{P}^2; \tag{6.33}$$

$$M_{BT}^2 := \left[\sqrt{\mathbf{q}^2 + {M_{12}}_{BT}^2} + \sqrt{\mathbf{q}^2 + m_3^2}\right]^2, \tag{6.34}$$

where

$${M_{12}}_{TP}^2 := m_{12}^2 + v_{TP}; \qquad {M_{12}}_{BT}^2 = m_{12}^2 + v_{BT}. \tag{6.35}$$

Given these mass operators and the corresponding Hamiltonians, it is possible to define the two-body interactions in the three-body Hilbert space:

$$V_{TP} := M_{TP} - M_0; \qquad U_{TP} := H - H_0; \tag{6.36}$$

$$V_{BT} := M_{BT} - M_0; \qquad U_{BT} := \bar{H} - H_0, \tag{6.37}$$

where M_0 is the invariant mass of three non-interacting particles. The interactions U_{TP}, V_{TP}, V_{BT} and U_{BT} satisfy the same properties as v_{TP} and v_{BT}, respectively:

$$[U_{TP}, \mathbf{p}_{12}]_- = [U_{TP}, \mathbf{p}_3]_- = [V_{TP}, \mathbf{p}_{12}]_- = [V_{TP}, \mathbf{p}_3]_- = 0, \tag{6.38}$$

and

$$[U_{BT}, \mathbf{P}]_- = [U_{BT}, \mathbf{q}]_- = [V_{BT}, \mathbf{P}]_- = [V_{BT}, \mathbf{q}]_- = 0. \tag{6.39}$$

Both of these operators lead to a relativistic dynamics. These models can be specified by giving representatives of the infinitesimal generators associated with either interaction in the appropriate basis.

The mass operator M_{TP} is associated with the tensor-product representation. The generators are sums of the generators for the interacting pair and the generators for the spectator particle:

$$P^{\mu} = p_{12}^{\mu} + p_{3}^{\mu}; \tag{6.40}$$

$$J^{\alpha\beta} = J_{12}^{\alpha\beta} + J_{3}^{\alpha\beta}. \tag{6.41}$$

The infinitesimal generators associated with the mass operator M_{BT} are

$$\mathbf{P} = \mathbf{P}; \tag{6.42}$$

$$H_{BT} = \sqrt{M_{BT}^2 + \mathbf{P}^2}; \tag{6.43}$$

$$\mathbf{J} = \mathbf{j}_c + i\nabla_{\mathbf{P}} \times \mathbf{P}; \tag{6.44}$$

$$\mathbf{K}_{BT} = -\frac{1}{2}\{H_{BT}, i\nabla_{\mathbf{P}}\}_{+} - \frac{\mathbf{P} \times \mathbf{j}_c}{H_{BT} + M_{BT}}, \tag{6.45}$$

where $\mathbf{j}_c$ can be expressed directly in terms of raising and lowering operators in the irreducible representation (6.23).

Either of the operators U_{TP} or U_{BT} (V_{TP} or V_{BT}), represent interactions between particles 1 and 2, with particle 3 as a spectator. The difficulty with macroscopic locality occurs when particle 3 is moved relative to the (12) pair in the BT model. This translation must not affect the interaction between particles 1 and 2. The interaction U_{TP} has the structure

$$U_{TP} = U_{12TP} \otimes I_3 \tag{6.46}$$

which implies that the Hamiltonian H_{TP} has the form

$$H_{TP} = H_{12TP} \otimes I_3 + I_{12} \otimes H_3. \tag{6.47}$$

The residual interaction $H^{12,3}$ is identically zero. This Hamiltonian trivially satisfies the macroscopic locality condition (6.6). It will now be shown by contradiction that the Hamiltonian

H_{BT} violates Eq. (6.6). To keep the analysis as simple as possible, we consider a pure spatial translation. If H_{BT} were to satisfy macroscopic locality, then we would have

$$\begin{aligned}
&\lim_{|\mathbf{b}-\mathbf{c}|\to\infty} \langle\Psi| e^{-i(\mathbf{p}_{12}\cdot\mathbf{b}+\mathbf{p}_3\cdot\mathbf{c})} \left(H_{BT} - H_{TP}\right) e^{i(\mathbf{p}_{12}\cdot\mathbf{b}+\mathbf{p}_3\cdot\mathbf{c})} |\Psi\rangle \\
&\quad = \lim_{|\mathbf{b}-\mathbf{c}|\to\infty} \langle\Psi| e^{-i(\mathbf{p}_{12}\cdot\mathbf{b}+\mathbf{p}_3\cdot\mathbf{c})} \left(U_{BT} - U_{TP}\right) e^{i(\mathbf{p}_{12}\cdot\mathbf{b}+\mathbf{p}_3\cdot\mathbf{c})} |\Psi\rangle \\
&\quad = 0,
\end{aligned} \tag{6.48}$$

in the limit that particle 3 is separated from the (12) pair. Since U_{TP} has delta functions in $\mathbf{p}_{12}$ and $\mathbf{p}_3$ it commutes with the translation operators resulting in:

$$\lim_{|\mathbf{b}-\mathbf{c}|\to\infty} \langle\Psi| e^{-i(\mathbf{p}_{12}\cdot\mathbf{b}+\mathbf{p}_3\cdot\mathbf{b})} U_{BT} e^{i(\mathbf{p}_{12}\cdot\mathbf{b}+\mathbf{p}_3\cdot\mathbf{c})} |\Psi\rangle = \langle\Psi| U_{TP} |\Psi\rangle, \tag{6.49}$$

Using $\mathbf{p}_{12} = \mathbf{P} - \mathbf{p}_3$, and the property that $\mathbf{P}$ commutes with U_{BT} in Eq. (6.48), we get

$$\lim_{|\mathbf{r}|\to\infty} \langle\Psi| e^{i\mathbf{p}_3\cdot\mathbf{r}} U_{BT} e^{-i\mathbf{p}_3\cdot\mathbf{r}} |\Psi\rangle = \langle\Psi| U_{TP} |\Psi\rangle, \tag{6.50}$$

where $\mathbf{r} = \mathbf{b} - \mathbf{c}$. Next, we make use of the relation (6.19) between $\mathbf{q}$ and $\mathbf{p}_3$ to obtain

$$\lim_{|\mathbf{r}|\to\infty} \langle\Psi| e^{i\mathbf{P}\cdot\mathbf{r}\Phi_c(\mathbf{P},\mathbf{q},k_{12})} U_{BT} e^{-i\mathbf{P}\cdot\mathbf{r}\Phi_c(\mathbf{P},\mathbf{q},k_{12})} |\Psi\rangle = \langle\Psi| U_{TP} |\Psi\rangle. \tag{6.51}$$

The important observation is that U_{BT} commutes with all of the arguments of Φ_c except k_{12}. Decomposing the state vectors with respect to $k = k_{12}$ leads to an expression of the form

$$\begin{aligned}
\lim_{|\mathbf{r}|\to\infty} \int d^3P \int d^3q \int k^2 dk \int k'^2 dk' e^{i\mathbf{P}\cdot\mathbf{r}\left(\Phi_c(\mathbf{P},\mathbf{q},k) - \Phi_c(\mathbf{P},\mathbf{q},k')\right)} \\
\times \Psi^*(k,\cdots) U_{BT}(k,k';\cdots) \Psi(k',\cdots) = \langle\Psi| U_{TP} |\Psi\rangle.
\end{aligned} \tag{6.52}$$

The ellipses in Eq. (6.52) mean that the remaining variables beyond k and k' are summed and/or integrated.

In order to make any precise mathematical statements, some assumptions need to be made about the regularity of the interaction and the wave function. These assumptions obscure the mechanism that leads to a violation of macroscopic locality. To understand this mechanism, note

that the phase factor vanishes in Eq. (6.52) only when $\mathbf{P} = 0$ or $k' = k$. These are both sets of measure zero and do not contribute to the integral (assuming the integrand is finite for each fixed value of $\mathbf{r}$). Equation (6.52) can be expressed in the form

$$\lim_{|\mathbf{r}|\to\infty} \int d^3P \int d^3q \int k^2 dk \int k'^2 dk'\, e^{i\mathbf{P}\cdot\mathbf{r}\,\Delta(\mathbf{P},\mathbf{q},k,k')} F[\mathbf{P},\mathbf{q},k,k']. \tag{6.53}$$

On the domain of integration where $k' \neq k$, we can make the change of variable $\mathbf{P} \to \mathbf{P'} = \mathbf{P}\Delta(\mathbf{P},\mathbf{q},k,k')$, which gives

$$\lim_{|\mathbf{r}|\to\infty} \int d^3P'\, e^{i\mathbf{P}'\cdot\mathbf{r}} \left\{ \left| \frac{\partial \mathbf{P}}{\partial \mathbf{P}'} \right| \int d^3q \int k^2 dk \int k'^2 dk'\, F[\mathbf{P}'/\Delta(\mathbf{P},\mathbf{q},k,k'),\mathbf{q},k,k'] \right\}. \tag{6.54}$$

This limit will vanish by the Riemann-Lebesgue lemma if the function

$$G(\mathbf{P}') := \left\{ \left| \frac{\partial \mathbf{P}}{\partial \mathbf{P}'} \right| \int d^3q \int k^2 dk \int k'^2 dk'\, F[\mathbf{P}'/\Delta(\mathbf{P},\mathbf{q},k,k'),\mathbf{q},k,k'] \right\} \tag{6.55}$$

is an L^1 function of $\mathbf{P}'$. Under this assumption, the left-hand side of this expression will vanish as $|\mathbf{r}| \to \infty$. Since the right-hand side does not generally vanish, this leads to the desired contradiction, and thus to a violation of macroscopic locality.

The violation of macroscopic locality is related to the asymptotic vanishing of the interaction between two particles that remain *close together* in the cluster limit. Note also that it is the *off-shell* behavior of the two-body subsystem that is inconsistent; the on-shell behavior associated with the two particles being highly separated is not affected by this discussion.

The failure of macroscopic locality occurs because translations of the separated subsystems are physically generated by $\mathbf{p}_3$ (up to overall momentum conservation) rather than by $\mathbf{q}$. It would appear that macroscopic locality would not be violated if the operator $\mathbf{q}$ were used to define the cluster limit rather than $\mathbf{p}_3$. Unfortunately, this operator has the wrong physical interpretation. It is the operator that generates translations of the spectator particle relative to the *non-interacting* (12) pair. The corresponding operator that generates translations of the

spectator particle relative to the interacting pair is defined by replacing M_0 with M in the relation

$$\mathbf{q} = L_c^{-1}(P/M_0)(\omega_{m_3}(\mathbf{p}_3), \mathbf{p}_3). \tag{6.56}$$

This concludes the demonstration of violations of macroscopic locality. These problems are not special to an instant-form formulation of the dynamics. In a front-form formulation, the problem appears in the generators of transverse rotation rather than in the "front-form" Hamiltonian P^-. In this case, a Bakamjian-Thomas construction leads to a system with a well defined angular momentum that does not cluster into subsystems for which the system angular momentum is a sum of the total subsystem angular momenta. The general rule is that TP models satisfy macroscopic locality trivially, while the generalized BT construction, which was used in Sections 2 and 4 to construct dynamical models, satisfies macroscopic locality only in the case of two-body models, or in N-body models with *no* $2, 3, \cdots, N-1$ body interactions (Mu 78). Nevertheless, the BT construction plays an important role in the formulation of the full three-body problem in the next section.

6.4. **Packing Operators**

The TP and BT formulations of the $2+1$ body dynamics are scattering equivalent. The unitary transformations which relate them are an important ingredient in the three-body formulation which follows in the next section. To understand the relationship between the mass operators M_{TP} and M_{BT} we note first that the eigenfunctions of M_{TP} have the general form

$$\begin{aligned} {}_{TP}\langle [12,3]' \mathbf{p}'_{12}\, \mu'_{12}; \mathbf{p}'_3\, \mu'_3 | \lambda_{12}\, j_{12}\, \mathbf{p}_{12}\, \mu_{12}; \mathbf{p}_3\, \mu_3 \rangle_{TP} \\ = \delta_{j'_{12} j_{12}} \delta_{\mu'_{12}\mu_{12}} \delta_{\mu'_3 \mu_3} \delta(\mathbf{p}'_{12} - \mathbf{p}_{12}) \delta(\mathbf{p}'_3 - \mathbf{p}_3) \langle [12] k_{12} | \lambda_{12}, j_{12} \rangle, \end{aligned} \tag{6.57}$$

while those of M_{BT} have the form (in the irreducible basis (6.23)) :

$$\begin{aligned} {}_{BT}\langle [12,3]' q'\, j'; \mathbf{P}'\, \mu' | [L\, S\, \lambda_{12}\, j_{12}] q\, j; \mathbf{P}\, \mu \rangle_{BT} \\ = \delta_{j'j} \delta_{\mu'\mu} \delta_{L'L} \delta_{S'S} \delta_{j'_{12} j_{12}} \delta(\mathbf{P}' - \mathbf{P}) \frac{1}{q^2} \delta(q' - q) \langle [12] k_{12} | \lambda_{12}, j_{12} \rangle, \end{aligned} \tag{6.58}$$

where the reduced two-body wave function $\langle [12] k_{12} | \lambda_{12}, j_{12} \rangle$ is the same in both cases. This wave function is the solution of any of the two-body equations (5.59)–(5.61).

The eigenstates in Eq. (6.57) of M_{TP} are in a tensor-product representation. By using the Clebsch-Gordan coefficients of the Poincaré group it is possible to find linear combinations of vectors with the same mass eigenvalue that transform irreducibly under the action of the $\mathcal{P}$. The irreducible eigenstates of M_{TP} are

$$\begin{aligned}|[L\,S\,\lambda_{12}\,j_{12}]q\,j;\mathbf{P}\,\mu\rangle_{TP} = \sum\int d^3p_{12}\int d^3p_3\int d\lambda'_{12}|\lambda'_{12}\,j'_{12}\,\mathbf{p}'_{12}\,\mu'_{12};\mathbf{p}'_3\,\mu'_3\rangle_{TP}\\ \times {}_{TP}\langle\lambda'_{12}\,j_{12}\,\mathbf{p}_{12}\,\mu_{12}\,\mathbf{p}_3\,\mu_3|[L\,S\,\lambda_{12}\,j_{12}]q\,j\,\mathbf{P}\,\mu\rangle,\end{aligned} \tag{6.59}$$

where it is important to note that the Clebsch-Gordan coefficients depend on the two-body mass eigenvalues λ_{12}. By comparison, the eigenstates

$$|[L\,S\,\lambda_{12}\,j_{12}]q\,j;\mathbf{p}\,\mu\rangle_{BT} \tag{6.60}$$

of M_{BT} already have the same form as the states (6.59). Although the arguments of the two state vectors are eigenvalues of different operators, the spectra of the corresponding operators in each vector are the same. For instance, the q in (6.59) is the relative momentum of the spectator and the *interacting* two-body system, while q in (6.60) is the relative momentum of the spectator and the *non-interacting* two-body system. Nevertheless, the spectrum of each of these operators is $(0,+\infty)$.

Because both sets of eigenstates are complete (for a given choice of two-body scattering asymptotic condition) with the same eigenvalues, the eigenstates (6.60) of M_{BT} can be mapped into the corresponding eigenstates (6.59) of M_{TP} by means of a unitary operator:

$$\begin{aligned}A_\pm := \int d^3P\int q^2dq\int k_{12}^2dk_{12}\sum_{L,S,l_{12},s_{12},j_{12},j,\mu}\\ \times|[L\,S\,k_{12}\,j_{12}\,l_{12}\,s_{12}]q\,j;\mathbf{P}\,\mu^{(\pm)}\rangle_{TP\;BT}\langle[L\,S\,k_{12}\,j_{12}\,l_{12}\,s_{12}]\,q\,j;\mathbf{P}\,\mu^{(\pm)}|\\ +\int d^3P\int q^2dq\sum_{L,S,j_b,j,\mu}|[L\,S\,\lambda_b\,j_b]q\,j;\mathbf{P}\,\mu\rangle_{TP\,BT}\langle[L\,S\,\lambda_b\,j_b]\,q\,j;\mathbf{P}\,\mu|.\end{aligned} \tag{6.61}$$

The operators A_+ and A_- are equal, provided the TP and BT representations are scattering equivalent (Co 82, Po 90). This was shown in Eqs. (6.28) and (6.29). We can therefore define

$$A := A_+ = A_-. \tag{6.62}$$

In addition, because the dynamical representation of the Poincaré group acts irreducibly on each

of these states, it follows that

$$U_{BT}(\underline{\Lambda}, \underline{a}) = A^{\dagger} U_{TP}(\underline{\Lambda}, \underline{a}) A \tag{6.63}$$

for all $\underline{\Lambda}$ and $\underline{a}$. It also follows from the definitions that

$$A|[L\,S\,k_{12}\,j_{12}\,l_{12}\,s_{12}]\,q\,j;\mathbf{P}\,\mu^{(\pm)}\rangle_{BT} = |[L\,S\,k_{12}\,j_{12}\,l_{12}\,s_{12}]q\,j;\mathbf{P}\,\mu^{(\pm)}\rangle_{TP}, \tag{6.64}$$

with a similar expression for states containing a two-body bound pair. The operator A is called a packing operator, or Sokolov operator.

We have shown that tensor products trivially satisfy macroscopic locality, and that violations in macroscopic locality typically involve the behavior of particles that remain close together when a spectator is removed, rather than the behavior of the spectator itself. The above discussion also demonstrates that unitary transformations that preserve the scattering operator and two-body binding energies do not necessarily preserve macroscopic locality. However, this is precisely the behavior we want, in order to be able to transform a representation such as the BT construction into one which does satisfy macroscopic locality.

The discussion in the section was done using an instant form of the dynamics. This was particularly convenient for the purpose of illustrating how macroscopic locality is violated. These violations occur whenever one adds interactions directly in a non-interacting irreducible representation of the Poincaré group using the Bakamjian-Thomas method outlined in the previous section. The generators in which the violation occurs and the nature of the violation may differ in different cases, but they always appear. What is relevant is that a packing operator A can always be found for asymptotically complete two-body models. In all cases of interest, it has the form (6.61), where the spins can be of any type, and the continuous variables $\mathbf{p}$ can be replaced by light-front components of the four-momentum or by three-components of the four-velocity. What is needed to construct matrix elements of A are the two-body solutions and the appropriate Clebsch-Gordan coefficients of $\mathcal{P}$.

The operator $A = A_{ij,k}$ can be expressed in a single basis using the wave functions and the Clebsch-Gordan coefficients. The result for a general spin, with the three-momentum as a

continuous variable is

$$
\begin{aligned}
&{}_{BTg0}\langle[12,3]'\mathbf{p}'_{12}\,\mu'_{12}\,\mathbf{p}'_3\,\mu'_3|A^{\dagger}_{12,3}|[12,3]\mathbf{p}_{12}\,\mu_{12}\,\mathbf{p}_3\,\mu_3\rangle_{g0TP} \\
&\quad=\sum\delta(\mathbf{P}'-\mathbf{P})\frac{1}{q^2}\delta(q'-q)\left|\frac{\partial(\mathbf{P}\mathbf{q}_g)}{\partial(\mathbf{p}_{12}\mathbf{p}_3)}\right|^{\frac{1}{2}}\left|\frac{\partial(\mathbf{P}\mathbf{q}_{ng})}{\partial(\mathbf{p}_{n12}\mathbf{p}_3)}\right|^{\frac{1}{2}} \\
&\qquad\times D^{j_{12}}_{\mu'_{12}\bar{\mu}'_{12}}[\underline{R}_g(\underline{L}_g(P/M'_0),-(\mathbf{q}'_g/m_{12}))\underline{R}_{gc}(-\mathbf{q}'_g/m_{12})] \\
&\qquad\times D^{j_3}_{\mu'_3\bar{\mu}'_3}[(\underline{R}_g(\underline{L}_g(P/M'_0),\mathbf{q}'_g/m_3)\underline{R}_{gc}(\mathbf{q}'_g/m_3)]Y^{l'}_{\mu'_l}(\hat{\mathbf{q}}'_g) \\
&\qquad\times\langle j_{12}\bar{\mu}'_{12}j_3\bar{\mu}'_3|\bar{s}'\bar{\mu}'_s\rangle\langle l'\mu'_l\bar{s}'\bar{\mu}'_s|j\mu\rangle\langle[12]'k'_{12}|nj_{12}\rangle\langle nj_{12}|[12]k_{12}\rangle \\
&\qquad\times\langle j\mu|l\mu_l\bar{s}\bar{\mu}_s\rangle\langle\bar{s}\bar{\mu}_s|j_{12}\bar{\mu}_{12}j_3\bar{\mu}_3\rangle Y^{l}_{\mu_l}(\hat{\mathbf{q}}_{ng}{}') \\
&\qquad\times D^{j_{12}}_{\bar{\mu}_{12}\mu_{12}}[(\underline{R}_{cg}(-\mathbf{q}_{ng}/\lambda_n)\underline{R}_g(\underline{L}_g(P_n/M_n),-\mathbf{q}_{ng}/\lambda_n)^{-1}] \\
&\qquad\times D^{j_3}_{\bar{\mu}_3\mu_3}[\underline{R}_{cg}(\mathbf{q}_{ng}/m_3)\underline{R}_g(\underline{L}_g(P_n/M_n),\mathbf{q}_{ng}/m_3)^{-1}],
\end{aligned}
\tag{6.65}
$$

and cyclic permutations. A sum over two-body internal eigenstates (*i.e.*, $|n\,j_{12}\rangle$ for both bound and scattering states) is implied. The masses M_n and M'_0 are

$$
M'_n=\sqrt{\lambda_n^2+\mathbf{q}'^2_{ng}}+\sqrt{m_3^2+\mathbf{q}'^2_{ng}};\quad M_0=\sqrt{m_{12}^2+\mathbf{q}_g^2}+\sqrt{m_3^2+\mathbf{q}_g^2},
\tag{6.66}
$$

where $\mathbf{q}_g$ and $\mathbf{q}_{ng}$ are given by the vector components of

$$
q_g=L_g^{-1}(p/M_0)p_3;\qquad q'_{ng}=L_g^{-1}(p'/M'_n)p'_3,
\tag{6.67}
$$

and m_{12} is the non-interacting (12) mass:

$$
m_{12}=\sqrt{m_1^2+\mathbf{k}^2}+\sqrt{m_2^2+\mathbf{k}^2}.
\tag{6.68}
$$

To obtain the packing operators for the instant form, replace $g\to c$ in the boosts used to define $\mathbf{q}_c$ and $\mathbf{q}_{nc}$. Note also that the Melosh rotations become the identity. To obtain the point-form packing operators from those of the instant form, the total momenta are replaced by the corresponding spatial components of the four-velocity. The Jacobians are replaced by

$$
\left|\frac{\partial(\mathbf{Q}\mathbf{q}_c)}{\partial(\mathbf{Q}_{12}\mathbf{Q}_3)}\right|^{\frac{1}{2}}\left|\frac{\partial(\mathbf{Q}\mathbf{q}_{nc})}{\partial(\mathbf{Q}_{n12}\mathbf{Q}_3)}\right|^{\frac{1}{2}}.
\tag{6.69}
$$

We have used upper case $\mathbf{Q}$'s to indicate three-components of the four-velocities to avoid confusion with the relative momentum operators $\mathbf{q}_c$ and $\mathbf{q}_{ng}$.

In the front form, the Wigner rotation are replaced by the identity, with a corresponding change in the Jacobian. Since this case is frequently used in applications, we give it explicitly:

$$
\begin{aligned}
&{}_{BT\,f0}\langle [12,3]'\tilde{\mathbf{p}}'_{12}\,\mu'_{12}\,\tilde{\mathbf{p}}'_3\,\mu'_3|A^{\dagger}_{12,3}|[12,3]\tilde{\mathbf{p}}_{12}\,\mu_{12}\,\tilde{\mathbf{p}}_3\,\mu_3\rangle_{f0\,TP} \\
&\qquad = \sum \delta(\tilde{\mathbf{P}}'-\tilde{\mathbf{P}})\frac{1}{q^2}\delta(q'-q)\left|\frac{\partial(\tilde{\mathbf{P}}\mathbf{q}_f)}{\partial(\tilde{\mathbf{p}}_{12}\tilde{\mathbf{p}}_3)}\right|^{\frac{1}{2}}\left|\frac{\partial(\tilde{\mathbf{P}}\mathbf{q}_{nf})}{\partial(\tilde{\mathbf{p}}_{n12}\tilde{\mathbf{p}}_3)}\right|^{\frac{1}{2}} \\
&\qquad\quad \times D^{j_{12}}_{\mu'_{12}\bar{\mu}'_{12}}[\underline{R}_{fc}(-\mathbf{q}'_f/m_{12})]D^{j_3}_{\mu'_3\bar{\mu}'_3}[\underline{R}_{fc}(\mathbf{q}'_f/m_3)]Y^{l'}_{\mu'_l}(\hat{\mathbf{q}}'_f) \\
&\qquad\quad \times \langle j_{12}\bar{\mu}'_{12}j_3\bar{\mu}'_3|\bar{s}'\bar{\mu}'_s\rangle\langle l'\mu'_l\bar{s}'\bar{\mu}'_s|j\mu\rangle\langle[12]'k'_{12}|nj_{12}\rangle\langle nj_{12}|[12]''k_{12}\rangle \\
&\qquad\quad \times \langle j\mu|l\mu_l\bar{s}\bar{\mu}_s\rangle\langle\bar{s}\bar{\mu}_s|j_{12}\bar{\mu}_{12}j_3\bar{\mu}_3\rangle Y^{l}_{\mu_l}(\hat{\mathbf{q}}_{nf}\,) \\
&\qquad\quad \times D^{j_{12}}_{\bar{\mu}_{12}\mu_{12}}[\underline{R}_{cf}(-\mathbf{q}_{nf}/\lambda_n)]D^{j_3}_{\bar{\mu}_3\mu_3}[\underline{R}_{cf}(\mathbf{q}_{nf}/m_3)],
\end{aligned}
\tag{6.70}
$$

plus cyclic permutations. Note that the packing operators for different choices of spin and continuous degrees of freedom are different operators, not the same operator in different representations. These operators were first constructed by Sokolov (So 77); specific cases have been examined in (Co 82, Le 83, Co 87). For the packing operators given in the general expression (6.65), when $\mathbf{P}=0$, the Wigner rotations all become the identity, $\mathbf{q}_n \to \mathbf{q}$, and the generalized Melosh rotations cancel. The result in this case is that the packing operator is diagonal in $\mathbf{P}$, and becomes the identity for $\mathbf{P}=0$. For the special case of the front form, matrix elements are independent of $\tilde{\mathbf{P}}$, except for a trivial factor of P^+ in the Jacobians which arises because of the normalization of the states used to define this representation. The condition for the front-form packing operator to become the identity is $m_{12}=\lambda_n$. The operators associated with these eigenvalues do not commute, and are involved in defining $\mathbf{p}\cdot\hat{\mathbf{z}}$. Thus, for the front form, this condition is not satisfied in any frame, which means that there is no frame for which $A=I$.

When the interactions U_{TP} and U_{BT} are short-range operators, the associated packing operators become the identity in the limit that the *interacting* particles are asymptotically separated. In the mixed representation, the packing operators have the form

$$
A = \sum_n \int |[12,3]_b\rangle_{BT\,TP}\langle[12,3]_b| + \sum_n \int |[12,3]^{(\pm)}\rangle_{BT\,TP}\langle[12,3]^{(\pm)}| \tag{6.71}
$$

where the sum runs over contributions from the two-body bound and scattering states. In the limit that the *interacting* particles are separated, the bound state contributions vanish and the

scattering solutions approach the plane-wave solutions. Since the plane-wave basis elements are the same in the TP and BT representations, the result is the resolution of the identity in the plane-wave basis. We might have expected this result, since the problem of macroscopic locality occurs for an interacting pair whose particles are close together. For this to be valid, there can only be a finite number of bound states, otherwise the order of the limit and the sum cannot be interchanged. To prove this, the Kato-Birman (Bi 62, Ka 65, Ch 76, Ob 78) invariance principle can be used to reduce the proof to the usual nonrelativistic proof of strong cluster properties of wave operators (Re 79) . Thus, for suitable short-range interactions, the packing operators will satisfy the following cluster property:

$$\lim_{(x-y)^2\to\infty} \|\big(A_{ij,k} - I\big)T_i(x)T_j(y)|\Psi\rangle\| = 0, \tag{6.72}$$

where in this case the interacting particles are being separated. This may not have a limit when the "non-interacting" particles are separated. In all that follows, it will be assumed that this condition holds.

7. The Three-Body Problem

In this section, we consider the problem of three particles with pairwise interactions. While the development builds on that of the previous section, it is not a trivial extension of the problem of a single interacting pair in the presence of a spectator. As will be seen shortly, the tensor-product representation no longer satisfies the commutation relations of the Poincaré group when three pairwise interactions are combined. Instead, the preferred route toward a three-body problem satisfying both macroscopic locality and the Poincaré Lie algebra turns out to be a combination of the Bakamjian-Thomas construction with appropriate packing operators. The presence of three-body interactions is a necessary outcome of this approach.

This section is divided into three parts. In the first part, we illustrate the general problem of satisfying the Poincaré Lie algebra for a general three-body system, and then consider a modification of the Bakamjian-Thomas constructions of the generators which satisfies both macroscopic locality and the Poincaré commutation relations. In the second part, we discuss the formulation of three-body integral equations. In the third part, we give an overview of an approach to three-body problems using the idea of symmetric coupling.

7.1. Three-Body Constructions

Poincaré Invariance in the Three-Body Problem In the previous section, we showed that generators for two interacting particles and a non-interacting spectator satisfying macroscopic locality are the generators of the appropriate tensor-product representation:

$$P^{\mu TP}_{ij,k} := P^{\mu}_{ij} \otimes I_k + I_{ij} \otimes P^{\mu}_{k}; \tag{7.1}$$

$$J^{\alpha\beta TP}_{ij,k} := J^{\alpha\beta}_{ij} \otimes I_k + I_{ij} \otimes J^{\alpha\beta}_{k}, \tag{7.2}$$

where P^{μ}_{ij} and $J^{\alpha\beta}_{ij}$ are the infinitesimal generators for the interacting pair consisting of particle i and particle j. Generators for three non-interacting particles are the generators of the tensor

product of three one-body representations:

$$P_0^\mu := P_1^\mu \otimes I_2 \otimes I_3 + I_1 \otimes P_2^\mu \otimes I_3 + I_1 \otimes I_2 \otimes P_3^\mu \tag{7.3}$$

$$J_0^{\alpha\beta} := J_1^{\alpha\beta} \otimes I_2 \otimes I_3 + I_1 \otimes J_2^{\alpha\beta} \otimes I_3 + I_1 \otimes I_2 \otimes J_3^{\alpha\beta}. \tag{7.4}$$

In what follows, we use the notation P_{ij}^μ for $P_{ij}^\mu \otimes I_k$, P_i^μ for $P_i^\mu \otimes I_k \otimes I_k$, *etc.* We also use P_0^μ for $P_1^\mu + P_2^\mu + P_3^\mu$ *etc.*

These generators are the limiting forms of the three-body generators under different asymptotic conditions, provided the three-body generators satisfy macroscopic locality: for three-body generators satisfying macroscopic locality, the difference between the generators of the tensor-product representation and the physical three-body generators vanishes on states where the particles in different clusters of a partition are asymptotically separated.

We call the operators, such as those defined in Eqs. (7.1)–(7.4) which generate tensor-product representations of $\mathcal{P}$, *partition generators.* The partition refers to the grouping of a set of particles into asymptotically separated subsets or clusters. In the three-body problem, there are four partitions of the particles into at least two non-empty clusters, which we denote by $(12,3)$, $(23,1)$, $(31,2)$ and $0 := (1,2,3)$. In the cluster limit, the particles in the same cluster of a partition remain close together as the different clusters are asymptotically separated.

If a three-body generator with short-range interactions satisfies macroscopic locality, then it can be reconstructed uniquely up to a short-range three-body operator in terms of the partition generators (Po 80). For example, a three-body Hamiltonian satisfying macroscopic locality necessarily has the form

$$H^{TP} = H_{12,3}^{TP} + H_{23,1}^{TP} + H_{31,2}^{TP} - 2H_0 + V_{123}, \tag{7.5}$$

where V_{123} is an operator that vanishes in all asymptotic configurations. This is easily seen if one represents the first three terms on the right-hand side of Eq. (7.5) as the sum of a three-body kinetic energy operator H_0 and three two-body interactions. The term $-2H_0$ on the right compensates for overcounting the three-body kinetic energy, and the remaining term V_{123} on the

right must vanish in all of the limiting cases. The Hamiltonian $H_{ij,k}$ and H_0 can be expressed as sums of tensor products of proper subsystem Hamiltonians with different identity operators, *i.e.,*

$$H^{TP}_{12,3} = H_{12} + H_3. \tag{7.6}$$

Similar expressions hold for each generator. In general macroscopic locality fixes N-body generators, up to an overall N-body interaction, in terms of the proper subsystem generators (Po 80, Co 82). In this case, macroscopic locality fixes the form of the three-body generators in terms of the various one- and two-body generators, up to an overall short-range three-body interaction for each generator. These interactions correspond to V_{123} for each generator; they may be zero for some generators.

Another constraint on the three-body generators comes from the requirement that these generators satisfy the commutation relations of the Poincaré Lie algebra. Although each of the partition generators satisfies the commutations relations, it does not follow that the *sums* of these generators will satisfy them, because interactions involving different pairs of interacting particles do not in general commute.

As an example, consider the case of rotations in a front-form dynamics. As seen in Eq. (5.90), the generators of transverse angular momentum have interactions. Let

$$\mathbf{J}^{TP}_{\perp ij,k} = \mathbf{J}_{\perp i} + \mathbf{J}_{\perp j} + \mathbf{J}_{\perp k} + \mathbf{Z}_{\perp ij}, \tag{7.7}$$

and

$$\mathbf{J}_{\perp 0} = \mathbf{J}_{\perp 1} + \mathbf{J}_{\perp 2} + \mathbf{J}_{\perp 3}, \tag{7.8}$$

where $\mathbf{Z}_{\perp ij}$ is an interaction term in the (ij) cluster, and the generators associated with each partition satisfy the Poincaré commutation relations. The unique linear combination of these operators satisfying macroscopic locality is

$$\mathbf{J}^{TP}_{\perp} = \mathbf{J}^{TP}_{\perp 12,3} + \mathbf{J}^{TP}_{\perp 23,1} + \mathbf{J}^{TP}_{\perp 31,2} - 2\mathbf{J}_{\perp 0}, \tag{7.9}$$

where we have for the moment assumed that the three-body interactions contributing to the angular momentum all vanish. For the components of $\mathbf{J}$ to satisfy $SU(2)$ commutation relations

the following condition must be satisfied:

$$\begin{aligned} 0 &= [J^1, J^2]_- - iJ^3 \\ &= [J^{1TP}_{12,3} + J^{1TP}_{23,1} + J^{1TP}_{31,2} - 2J^1_0, J^{2TP}_{12,3} + J^{2TP}_{23,1} + J^{2TP}_{31,2} - 2J^2_0]_- \\ &\quad - i(J^{3TP}_{12,3} + J^{3TP}_{2,31} + J^{3TP}_{31,2} - 2J^3_0) \\ &= [Z^{1TP}_{23,1}, Z^{2TP}_{31,2} + Z^{2TP}_{12,3}]_- + [Z^{1TP}_{31,2}, Z^{2TP}_{12,3} + Z^{2TP}_{23,1}]_- + [Z^{1TP}_{12,3}, Z^{2TP}_{23,1} + Z^{2TP}_{31,2}]_- . \end{aligned} \tag{7.10}$$

The interaction terms associated with different partitions do not generally commute; consequently, the right-hand side will not vanish. This implies a violation of the commutation relations. Similar remarks apply to all generators with interactions. However, the right-hand side of Eq. (7.10) is an operator that should vanish in all *asymptotic* regions since it is a sum of products of short-range two-body interactions that involve all three particles. This means that the following linear combinations of generators:

$$G^{TP} = G^{TP}_{12,3} + G^{TP}_{23,1} + G^{TP}_{31,2} - 2G_{1,2,3} \tag{7.11}$$

fail to satisfy the commutation relations by a discrepancy which is an operator that vanishes in all asymptotic regions, *i.e.,* a three-body operator. Alternative linear combinations of the partition generators would also have additional two-body interactions on the right-hand side of Eq. (7.10). This suggests that we try to find a three-body interaction $\mathbf{Z}_{\perp 123}$ to be added to $\mathbf{J}^{TP}_{\perp}$:

$$\mathbf{J}^{TP}_{\perp} \rightarrow \mathbf{J}^{TP}_{\perp} + \mathbf{Z}_{\perp 123}, \tag{7.12}$$

such that the *modified* generators $\mathbf{J}$ satisfy $SU(2)$ commutation relations. In general, all ten generators must be considered, rather than $\mathbf{J}$ alone, and the commutation relations must be those of $\mathcal{P}$ rather than $SU(2)$.

We will demonstrate shortly that a Bakamjian-Thomas construction with pairwise interactions *does* satisfy the Poincaré Lie algebra, although, as shown in the previous section, this construction does not satisfy the conditions of macroscopic locality. It was also shown in the previous section that macroscopic locality can be restored for the BT construction by means of packing operators.

We are now faced with two approaches for formulating a dynamical model. One is to use the BT construction to ensure the commutation relations, and then fix macroscopic locality, while the alternative is to use the TP representation to ensure macroscopic locality, and then fix the commutation relations. Since the commutation relations are non-linear relations among the generators, it is easier to get the commutation relations correct first, and then fix macroscopic locality, subject to the constraint that the commutation relations are preserved. Thus, we choose the BT construction as the starting point.

Bakamjian-Thomas Construction The development proceeds in two phases. First, a BT construction which satisfies the Poincaré Lie algebra is formulated following the results of the previous section. After that, the appropriate packing operators are derived which restore the property of macroscopic locality.

Our construction uses a front-form dynamics, although the procedure works whenever we can formulate a Bakamjian-Thomas type of construction. In the case of a front-form dynamics, the steps are outlined as follows:

1. Construct the front-form generators for three *non-interacting* particles.

2. Construct the front-form spin operator for the system of three *non-interacting* particles in terms of the generators using Eqs. (3.102) and (3.103).

3. Use the Clebsch-Gordan coefficients of $\mathcal{P}$ to construct irreducible bases for three non-interacting particles of the form

$$\begin{aligned}
&|[ij,k]q_k\, j;\tilde{\mathbf{p}}\,\mu\rangle_{f0} \\
&\quad := \sum \int d\tilde{\mathbf{p}}_1 \int d\tilde{\mathbf{p}}_2 \int d\tilde{\mathbf{p}}_3 \int d\tilde{\mathbf{p}}_{ij} \int k_{ij}^2 dk_{ij} |\tilde{\mathbf{p}}_i\,\mu_i;\tilde{\mathbf{p}}_j\,\mu_j;\tilde{\mathbf{p}}_k\,\mu_k\rangle \\
&\qquad \times \langle \tilde{\mathbf{p}}_i\,\mu_i;\tilde{\mathbf{p}}_j\,\mu_j|[l_{ij}\,s_{ij}]k_{ij}\,j_{ij};\tilde{\mathbf{p}}_{ij}\mu_{ij}\rangle\langle\tilde{\mathbf{p}}_{ij}\,\mu_{ij};\tilde{\mathbf{p}}_k\,\mu_k|[L_k\,S_k]q_k\,j;\tilde{\mathbf{p}}\mu\rangle.
\end{aligned} \tag{7.13}$$

4. For any (ij) pair of particles, define a two-body interaction v_{ij}^{BT}, whose matrix elements in the BT basis $|[ij,k]q_k\,j;\tilde{\mathbf{p}}\,\mu\rangle_{BT}$ have the form

$$\begin{aligned}
&{}_{f0}\langle[ij,k]'q_k'\,j';\tilde{\mathbf{p}}'\,\mu'|v_{ij}^{j_{ij}\,BT}|[ij,k]q_k\,j;\tilde{\mathbf{p}}\,\mu\rangle_{f0} \\
&\quad = \delta_{L_k'L_k}\delta_{S_k'S_k}\delta_{\mu'\mu}\delta_{j'j}\delta_{j'_{ij}j_{ij}}\delta(\tilde{\mathbf{p}}'-\tilde{\mathbf{p}})\frac{1}{q_k^2}\delta(q_k'-q_k)\langle[l'_{ij}\,s'_{ij}]k'_{ij}\|v\|[l_{ij}\,s_{ij}]k_{ij}\rangle,
\end{aligned} \tag{7.14}$$

where v is the two-body interaction.

5. Construct the "two-body" interactions in the three-particle space by

$$V^{BT}_{ij,k} := \sqrt{m^2_{ij} + v^{BT}_{ij} + q^2_k} - \sqrt{m^2_{ij} + q^2_k} \tag{7.15}$$

where $m_{ij} = \sqrt{m_i^2 + k_k^2} + \sqrt{m_j^2 + k_k^2}$ is the invariant mass of two free particles imbedded in the three-particle Hilbert space. Note that by construction each of these interactions commutes with the kinematic subgroup of the light front as well as the front-form spin of three non-interacting particles.

6. Define the three-body mass operator M_{BT}:

$$M_{BT} := M_0 + V^{BT}_{12,3} + V^{BT}_{23,1} + V^{BT}_{31,2} + V^{BT}_{123}\,, \tag{7.16}$$

where V^{BT}_{123} is any short-range three-body operator which commutes with the kinematic subgroup of the light front and the front-form spin of three non-interacting particles. A necessary and sufficient condition for this is that its matrix elements in the basis $|[ij,k]q_k\, j; \tilde{\mathbf{p}}\,\mu\rangle_{BT}$ have the form

$$\begin{aligned}&\langle [ij,k]' q'_k\, j'; \tilde{\mathbf{p}}'\,\mu' | V^{BT}_{123} | [ij,k] q_k\, j; \tilde{\mathbf{p}}\,\mu\rangle \\ &\quad = \delta_{\mu'\mu}\delta_{j'j}\delta(\tilde{\mathbf{p}}' - \tilde{\mathbf{p}})\langle [L'_k\, S'_k\, k'_{ij}\, j'_{ij}\, l'_{ij}\, s'_{ij}] q'_k \| V^{jBT}_{123} \| [L_k\, S_k\, k_{ij}\, j_{ij}\, l_{ij}\, s_{ij}] q_k\rangle.\end{aligned} \tag{7.17}$$

Since M_{BT} is a sum of terms, each of which commutes with the kinematic subgroup of the light front and the front-form spin of three non-interacting particles, it follows that M_{BT} also commutes with these operators.

7. Define a representation of $\mathcal{P}$ consistent with this dynamics using the front-form Bakamjian-Thomas construction. This is possible because M_{BT} commutes with the kinematic subgroup of the light front and the front-form spin of three non-interacting particles.

Formal expressions for the infinitesimal interacting generators of $\mathcal{P}$ can be constructed as functions of M_{BT}, $\mathbf{j}_{0f}$ and the kinematic generators using Eqs. (5.88)–(5.90), with M_{BT} replacing M in those expressions. This ensures that the commutation relations are satisfied. To construct representations for finite Poincaré transformations, it is sufficient to find simultaneous eigenstates

of M_{BT}, $\tilde{\mathbf{P}}$, j^2_{f0} and j^3_{0f} in any of the bases $|[ij,k]q_k\, j;\tilde{\mathbf{p}}\,\mu\rangle_{BT}$ which diagonalizes all of these operators except M_{BT}. These eigenstates have the form

$$_{f0}\langle[ij,k]q_k\, j';\tilde{\mathbf{p}}'\,\mu'|m_{BT}\, j;\tilde{\mathbf{p}}\,\mu\rangle = \delta_{j'j}\delta_{\mu'\mu}\delta(\tilde{\mathbf{p}}'-\tilde{\mathbf{p}})\langle[ij,k]q_k|m_{BT}\, j\rangle, \tag{7.18}$$

where $|m_{BT}\, j\rangle$ an eigenstate of the reduced mass operator M^j_{BT} which is defined by

$$_{f0}\langle[ij,k]'q'_k\, j';\tilde{\mathbf{p}}'\,\mu'|M_{BT}|[ij,k]q_k\, j;\tilde{\mathbf{p}}\,\mu\rangle = \delta_{j'j}\delta_{\mu'\mu}\delta(\tilde{\mathbf{p}}'-\tilde{\mathbf{p}})\langle[ij,k]'q'_k\|M^{jBT}\|[ij,k]q_k\rangle. \tag{7.19}$$

The representation $U_{BT}(\underline{\Lambda},\underline{a})$ is defined by its action on the states $|m_{BT}\, j;\tilde{\mathbf{p}}\,\mu\rangle$, which is an irreducible representation of mass m_{BT} with front-form spin:

$$U_{BT}(\underline{\Lambda},\underline{a})|m_{BT}\, j;\tilde{\mathbf{p}}\,\mu\rangle = e^{i\Lambda p\cdot a}\sqrt{\frac{p^+_\Lambda}{p^+}}|m_{BT}\, j;\tilde{\mathbf{p}}_\Lambda\,\bar{\mu}\rangle D^j_{\bar{\mu}\mu}[\underline{R}_f(\underline{\Lambda},q)], \tag{7.20}$$

where $\tilde{\mathbf{P}}_\Lambda$ is the light-front component of ΛP for $P^2 = m^2_{BT}$. This defines a unitary representation $U_{BT}(\underline{\Lambda},\underline{a})$ of $\mathcal{P}$ on $\mathcal{H}$, with the kinematic subgroup of the light front and a kinematic front-form spin.

Packing Operators Before we discuss the diagonalization of the mass operator, we must address the lack of macroscopic locality of the representation $U_{BT}(\underline{\Lambda},\underline{a})$. In order to understand how $U_{BT}(\underline{\Lambda},\underline{a})$ behaves when the particles are asymptotically separated, we note first that $U_{BT}(\underline{\Lambda},\underline{a})$ is a bounded operator valued function of nine kinematic operators (the kinematic generators and the transverse components of the front-form spin) and M_{BT}. By definition, M_{BT} has the structure

$$\begin{aligned} M_{BT} &= M_0 + V^{BT}_{12,3} + V^{BT}_{23,1} + V^{BT}_{31,2} + V^{BT}_{123} \\ &= M^{BT}_{12,3} + M^{BT}_{23,1} + M^{BT}_{31,2} - 2M_0 + V^{BT}_{123}\,, \end{aligned} \tag{7.21}$$

where $M^{BT}_{ij,k} := M_0 + V^{BT}_{ij,k}$. The operators $M^{BT}_{ij,k}$ are the front-form versions of the operators used in the BT formulation of the $2+1$ body problem. In that section, we showed that these were related to mass operators associated with a tensor-product representation using the unitary packing operators A introduced by Sokolov (So 77, Co 82, Le 83, Co 87). The development in the previous sections used an instant-form dynamics; to treat a front-form dynamics one only has to use the front-form packing operators given in Eq. (6.70) in place of the instant-form packing operators.

In a three-body problem, there is a different packing operator $A_{ij,k}$ associated with each interacting pair, and the different packing operators do not commute. The property of these operators that we need are

1. $A_{ij,k} M^{BT}_{ij,k} A^{\dagger}_{ij,k} = M^{TP}_{ij,k}$, where $M^{TP}_{ij,k}$ is the $2+1$ body mass operator associated with the tensor-product representation.

2. $A_{ij,k}$ commutes with the kinematic subgroup of the light front.

3. $A_{ij,k}$ does *not* commute with the free front-form spin.

4. $A_{ij,k}$ is unitary, and satisfies the asymptotic condition

$$\lim_{a^2 \to \infty} \|(A_{ij,k} - I) T_j(a) |\Psi\rangle\| = \lim_{a^2 \to \infty} \|(A_{ij,k} - I) T_{jk}(a) |\Psi\rangle\| = 0. \tag{7.22}$$

Note that property (1) follows from the construction of $A_{ij,k}$. Property (2) follows because the kinematic generators of the two-body subsystem are kinematic in a front-form dynamics, and generators are additive in the tensor-product representation. The front-form spin fails to commute with $A_{ij,k}$ (property (3)) because the spin of the tensor product is not the sum of the subsystem spins but instead it is a complicated function, via Eqs. (5.84) and (5.85), of these operators which involves the two-body mass operator. The asymptotic condition (7.22) is the same as condition (6.72), except that in the front form, additional care is needed, because $T_{jk}(a)$ is not kinematic for all choices of a. In what follows, we assume that this condition holds.

It follows from property (1) and the asymptotic condition (7.16) that we can express M_{BT} in terms of the mass operators in the tensor-product representations:

$$M_{BT} = A^{\dagger}_{12,3} M_{12,3} A_{12,3} + A^{\dagger}_{23,1} M_{23,1} A_{23,1} + A^{\dagger}_{31,2} M_{31,2} A_{31,2} - 2M_0 + V^{BT}_{123}. \tag{7.23}$$

In order to understand both the failure of macroscopic locality and the methods for restoring it, it is convenient to separate the analytic from the algebraic aspects of this problem. To treat the algebraic aspect of this problem, we define the notion of *algebraic clustering* (Co 82).

Algebraic clustering is the operation of physically setting to zero the part of an operator that should vanish asymptotically in a given region. If O is a function of M_{BT} and the kinematic operators, then O_a is defined as the result of setting to zero all of the interactions between particles in different clusters of a given partition a of three particles. We can then write

$$O = O_a + O^a, \tag{7.24}$$

where O^a is the residual interaction. For short-range interactions we expect that O^a will vanish in the cluster limit. The operation of algebraic clustering takes care of the contributions that should obviously vanish in a given limit, assuming short-range interactions. There are two problems which remain. The first is the need to show that the residual interaction vanishes. This is not generally expected to be a problem for "reasonable" interactions, and normally can be treated using methods that are similar to those used in time dependent scattering (Co 57). Our working assumption is that the residual interaction vanishes strongly. The more interesting and subtle problem is the operator O_a. This term may remain unchanged, may have an unexpected limit, or may have no limit. Different possibilities were realized in the discussion of the 2+1 body problem. The problem becomes trivial when O_a has some simple intertwining properties that permit the translation operator to be moved through O_a and canceled before the cluster limit is taken. In this case O_a, is its own limit. When used in this manner, algebraic clustering, coupled with the assumption that the residual interactions vanish strongly in the cluster limit, becomes a powerful tool for examining cluster properties of operators. In order to understand algebraic clustering in the context of the three-body problem, suppose that particle 1 is asymptotically separated from particles 2 and 3. We can always write

$$U_{BT}(\underline{\Lambda}, \underline{a}) = U^{BT}_{23,1}(\underline{\Lambda}, \underline{a}) + U^{23,1}_{BT}(\underline{\Lambda}, \underline{a}), \tag{7.25}$$

where $U^{BT}_{23,1}(\underline{\Lambda}, \underline{a})$ is the representation constructed using the kinematic generators, the kinematic front-form spin and the mass operator

$$M^{BT}_{23,1} \to A^{\dagger}_{23,1} M_{23,1} A_{23,1} + I M_0 I + I M_0 I - 2M_0 + 0 = A^{\dagger}_{23,1} M_{23,1} A_{23,1}. \tag{7.26}$$

This operator is constructed from Eq. (7.23) by turning off all interactions involving particle 1

and using $A_{ij,k} \to I$ for $k \neq 1$. The residual operator $U_{BT}^{23,1}(\underline{\Lambda}, \underline{a})$ is defined as the difference between $U_{BT}(\underline{\Lambda}, \underline{a})$ and $U_{12,3}^{BT}(\underline{\Lambda}, \underline{a})$.

Equation (7.26) implies the following: $U_{BT}(\underline{\Lambda}, \underline{a})$ can be expressed as follows:

$$U_{BT}(\underline{\Lambda}, \underline{a}) = A_{23,1}^{\dagger}[U_1(\underline{\Lambda}, \underline{a}) \otimes U_{23}(\underline{\Lambda}, \underline{a})]A_{23,1} + U_{BT}(\underline{\Lambda}, \underline{a})^{23,1}. \tag{7.27}$$

Equation(7.27) shows that the term obtained by turning off the residual interactions misses being a tensor product (which is needed for macroscopic locality) by the limit of the unitary transformation $A_{23,1}$. In addition, the desired unitary transformation is different for each asymptotic limit.

We must find a way to modify the dynamical model given by $U_{BT}(\underline{\Lambda}, \underline{a})$, so that the modified dynamics satisfies macroscopic locality, but without destroying the group representation property. The simplest modification of $U_{BT}(\underline{\Lambda}, \underline{a})$ that preserves the group representation property is a unitary transformation:

$$U_{BT}(\underline{\Lambda}, \underline{a}) \to U(\underline{\Lambda}, \underline{a}) := AU_{BT}(\underline{\Lambda}, \underline{a})A^{\dagger}. \tag{7.28}$$

If we want to satisfy macroscopic locality, then we must also demand that

$$\lim U(\underline{\Lambda}, \underline{a}) \to \lim\big[AA_{jk}^{\dagger}[U_i(\underline{\Lambda}, \underline{a}) \otimes U_{jk}(\underline{\Lambda}, \underline{a})]A_{jk}(A^{\dagger})\big] = U_i(\underline{\Lambda}, \underline{a}) \otimes U_{jk}(\underline{\Lambda}, \underline{a}) \tag{7.29}$$

for each of two-cluster asymptotic limits. A sufficient condition is for the limits in Eq. (7.29) to be strong limits, and for

$$(\lim A)A_{i,jk}^{\dagger} = I \qquad \text{or} \qquad \lim A = A_{i,jk} \tag{7.30}$$

for each asymptotic limit. This reduces the problem of finding a three-body model satisfying macroscopic locality to the problem of constructing a unitary A satisfying the condition (7.30). The solution to this problem was first given by Sokolov (So 77). It uses the notion of the symmetrized product of non-commuting operators. Consider the ordinary product

$A = A_{12,3}A_{23,1}A_{31,2}$. This operator is unitary if each of the individual $A_{ij,k}$'s is unitary. If particle 1 is asymptotically separated from particles 2 and 3, we expect that in this limit, $A_{12,3} \to I$, $A_{31,2} \to I$ and thus $A \to \lim A_{23,1}$. Although this last limit may not exist, we expect that the limit of everything else vanishes. If these are strong limits, then this A will satisfy Eq. (7.30), independent of whether $A_{23,1}$ has a limit as particle 1 is separated form the (23) pair. Similar results hold any other partition. This operator A is known and can be constructed in terms of the two-body solutions using Eq. (6.71).. However, a different operator is obtained depending on the order in which the factors appear in the product. In particular, if the system has a symmetry under exchange of identical particles, it will be destroyed by this construction.

To remedy this, Sokolov introduced the notion of a symmetrized product. For the purpose of this paper a symmetrized product should be commutative, associative, preserve unitarity, and become one operator when the other is set equal to the identity. There are many possible symmetrized products. One that is easy to calculate uses Cayley transformations (Co 82). We define

$$\alpha_a := i\frac{1 - A_a}{1 + A_a} \tag{7.31}$$

for each two-cluster partition a. Note that the operators α_a are all self-adjoint and vanish in the limit $A_a \to 1$. Now define

$$\alpha = \alpha_{12,3} + \alpha_{23,1} + \alpha_{31,2} \tag{7.32}$$

and

$$A := \frac{1 + i\alpha}{1 - i\alpha}. \tag{7.33}$$

It follows from Eq. (6.72) that $A - A_{ij,k}$ vanishes in the limit that particle i is asymptotically separated form the (ij) pair. To see this, note that in the limit that the particles in different clusters of the partition a are asymptotically separated, all of the $\alpha_{a'}$ for $a \neq a'$ vanish. It follows that

$$A \to \frac{1 + i\alpha_a}{1 - i\alpha_a} = A_a, \tag{7.34}$$

which can seen by inverting Eq. (7.31). Since the operators α_a are self-adjoint, A_a must be unitary. The operator A can be constructed directly in terms of the $A_{i,jk}$'s by solving an integral

equation:

$$A = I + i \sum_a B_a; \tag{7.35}$$

$$B_a = i(1 - A_a) + \frac{1}{2}(A_a - 1) \sum_{b \neq a} B_b, \tag{7.36}$$

where the sum runs over two-cluster partitions and the operators A_a are given by Eq. (6.71). They are expressed in terms of the solution of the two-body problem. Note that it is not necessary to construct explicitly the α_a's to determine A. Equation (7.36) has a bounded kernel and is designed to be connected on one iteration. This equation uses only the known $A_{ij,k}$'s, which depend on the two-body solutions, as input.

Given the solution A to Eqs. (7.35)–(7.36), we define the three-body dynamics as follows:

$$U_{TP}(\underline{\Lambda}, \underline{a}) := A U_{BT}(\underline{\Lambda}, \underline{a}) A^\dagger \tag{7.37}$$

This is clearly a unitary representation of $\mathcal{P}$, since it is a unitary transformation of a different unitary representation of $\mathcal{P}$. It clusters algebraically to the correct tensor-product representations by Eqs. (7.29) and (7.34). Thus, for sufficiently short-range interactions, it will satisfy macroscopic locality. This completes the formal solution of the three-body problem.

At this point it is useful to analyze what has been done. We began by building a three-body dynamics using a Bakamjian-Thomas construction. The Bakamjian-Thomas construction for systems of more than two particles does not generally lead to a dynamics that satisfies macroscopic locality (Mu 78). Next, we transformed this three-body dynamics into a tensor-product representation with a unitary operator containing a three-body interaction term. The only constraints on this operator beyond unitarity is that it has the form

$$A = A_{12} + A_{23} + A_{31} - 2I + \Delta(123), \tag{7.38}$$

where $\Delta(123)$ vanishes in all asymptotic limits. Our specific choice of symmetrized product, using Cayley transforms, implies a specific $\Delta(123)$ which is dictated by our choice of symmetrized product. It appears that we have an *ad hoc* packing operator combined with an *ad hoc* dynamics.

What is important is that these two elements conspire together in such a way that the two-body operators that appear in the physical representation are *uniquely* fixed by the various subsystem dynamics, and are independent of choice of symmetrized product and Bakamjian-Thomas construction. The only thing that is not fixed independent of these choices is $\Delta(123)$, which contributes a three-body interaction. This is similar to the nonrelativistic case, where the Hamiltonian is uniquely determined up to an overall three-body interaction by the subsystem dynamics. In the relativistic case, however, we cannot maintain the unitarity of A if we choose $\Delta(123) = 0$. Thus, the *requirement* of a non-vanishing three-body interaction is a new feature in a relativistic problem. In this particular case, the three-body interaction is not of the Bakamjian-Thomas type, because it does not commute with the free front-form spin. The spin operator in the TP representation manifestly has three-body interactions.

Next, we consider the effect of the packing operator on the underlying dynamics. We show that $U(\underline{\Lambda}, \underline{a})$ and $U_{BT}(\underline{\Lambda}, \underline{a})$ have (1) identical bound state mass spectra and (2) identical scattering matrices.

To show that $U_{TP}(\underline{\Lambda}, \underline{a})$ and $U_{BT}(\underline{\Lambda}, \underline{a})$ have identical bound state mass spectra, assume that $|\Psi\rangle$ is a point eigenstate of M_{TP} and j_f^2. It follows by construction that $A^\dagger|\Psi\rangle$ is a point eigenstate of M_{BT} and j_{0f}^2 with the same eigenvalues. Eigenstates in the point spectrum of each mass operator are in one-to-one correspondence with each other, and the mass and spin eigenvalues are identical. This is an immediate consequence of the unitarity of A.

The equality of the scattering operators is most efficiently established in the time independent formulation of the scattering problem. The scattering matrix $\langle b|S|a\rangle$ for a transition between an initial state $|a\rangle$, whose asymptotic Hamiltonian is H_a, to a final state $|b\rangle$, whose asymptotic Hamiltonian is H_b, is given by

$$S_{ba} = \langle b|\Omega_{b+}^\dagger \Omega_{a-}|a\rangle, \tag{7.39}$$

where the partition wave operators are given by the strong limits:

$$\Omega_{a\pm}|a\rangle := \lim_{t\to\mp\infty} T(t)T_a(-t)|a\rangle \tag{7.40}$$

and $|a\rangle$ is a normalizable state in which the particles of each cluster of a are bound (if the cluster contains more than one particle).

The scattering matrix for the BT system is obtained by making the replacements:

$$T^{TP}(t) \to T^{BT}(t) = A^\dagger T^{TP}(t) A; \tag{7.41}$$

$$T_a^{TP}(-t) \to T_a^{BT}(-t) = A_a^\dagger T_a^{TP}(-t) A_a; \tag{7.42}$$

$$|a\rangle_{TP} \to |a\rangle_{BT} = A_a^\dagger |a\rangle_{TP} \tag{7.43}$$

in Eq. (7.40) for two-cluster partitions. For three-cluster partitions, the last two equations are replaced by

$$T_0(-t) \to T_0^{BT}(-t) = T_0(-t); \tag{7.44}$$

$$|0\rangle_{TP} \to |0\rangle_{BT} = |0\rangle_{TP}, \tag{7.45}$$

where $|0\rangle_{TP}$ is any normalizable vector representing the asymptotic state.

If we use the substitutions (7.41)–(7.43) in the expression for the wave operators we obtain, after some algebra,

$$\Omega_{a\pm}^{BT}|\bar{a}\rangle = \lim_{t\to\mp\infty} A^\dagger T^{TP}(t) A A_a^\dagger T_a^{TP}(-t)|a\rangle. \tag{7.46}$$

We can compute $AA_a^\dagger$ using Eq. (7.38), which, for $a = 12, 3$, is

$$AA_a^\dagger = I + (A_{23,1} - I + A_{31,2} - I + \Delta_{123}) A_{12,3}^\dagger. \tag{7.47}$$

The operators $(A_{23,1} - I)$, $(A_{31,2} - 1)$ and Δ_{123} all vanish as particle 3 is asymptotically separated relative to the (12) pair. This follows from Eq. (6.72) and the definition of Δ_{123}. This implies that for suitable short-range interactions, the following strong limit vanishes:

$$\lim_{t\to\mp\infty} (A_{23,1} - I + A_{31,2} - 1 + \Delta_{123}) A_{12,3}^\dagger T_{12,3}(-t)|a\rangle = 0. \tag{7.48}$$

For the purpose of proving the equality of the two scattering matrices, we assume that Eq. (7.48) holds. A similar relation can be derived for the case that a is $(1, 2, 3)$. Using Eq. (7.48) in

Eq. (7.46), we obtain the following relationship (Co 82):

$$\Omega_{a\pm}^{BT}|\bar{a}\rangle = \lim_{t\to\mp\infty} A^\dagger T^{TP}(t) T_a^{TP}(-t)|a\rangle = A^\dagger \Omega_{a\pm}^{TP}|a\rangle. \tag{7.49}$$

Because A is unitary and the individual two-body A_a's are independent of scattering asymptotic condition for asymptotically complete two-body problems, it follows that

$$\begin{aligned} {}_{BT}\langle b|S^{BT}|a\rangle_{BT} &= {}_{BT}\langle b|\Omega_{b+}^{BT\dagger}\Omega_{a-}^{BT}|a\rangle_{BT} \\ &= {}_{TP}\langle b|\Omega_{b+}^{TP\dagger} A A^\dagger \Omega_{a-}^{TP}|a\rangle_{TP} \\ &= {}_{TP}\langle b|S^{TP}|a\rangle_{TP}. \end{aligned} \tag{7.50}$$

At this point, it is interesting to ask what, if anything, is gained by using the packing operators to enforce macroscopic locality, since they do not affect any of the three-body observables. The difference between the BT (Bakamjian-Thomas) and the TP (macroscopically local) models becomes relevant when the three-body system is imbedded in a system of four or more particles. This is needed to formulate larger problems, or to consider electron scattering from a system like the triton. In this case, the BT and TP dynamics can lead to inequivalent results. These effects are important in principle, although they may be small in practice. The size of these corrections can be tested by examining the strength of the three-body interaction in M that is needed for macroscopic locality. It is important to note that the predictive power of these models depends in large part on the possibility that the many-body interactions generated by the packing operators are small perturbations to the dynamics of pairwise interactions. It was shown in (Co 82) that this these many-body interactions are of order $\|v_{BT}M_0^{-1}\|^{N-2}$ relative to the two-body interaction. This is small for nucleons in nuclei. For other applications, this must be investigated on a case-by-case basis.

Thus, we conclude that the BT representation is sufficient for the purpose of computing three-body binding energies and cross sections. It should not be used to imbed this system into a four-body system or for formulating scattering of electroweak probes from the three-body system without carefully checking the strength of the three-body interactions in the TP dynamics.

Note that it is fairly straightforward to repeat this analysis in any of the various forms of the dynamics or in models with arbitrary choices of spin and continuous variables. A generalized

Bakamjian-Thomas construction must be available for the two- and three-body systems, and we must be able to evaluate the appropriate Clebsch-Gordan coefficients for $\mathcal{P}$.

7.2. **Faddeev Equations**

Given the operators M_{TP} or M_{BT}, the solution of the three-body problem becomes a computational one. The problem is to find a complete set of eigenstates of M_{TP} or M_{BT} that transform irreducibly.

This problem is more difficult than the two-body problem because there are more asymptotic conditions needed to completely describe the initial or final state of a three-particle scattering experiment. The asymptotic conditions in the relativistic case are the same as they are in the nonrelativistic case, although there are some differences that increase the numerical complexity of the three-body problem in the relativistic case.

The bound states do not require any special machinery. It is sufficient to diagonalize M_{BT} in a sufficiently large basis of localized functions, and then look for eigenvalues below the threshold for the continuous spectrum. In practice, this is a very large calculation, but it can be used to obtain answers to any desired accuracy (Gl 86). In addition, if the problem is formulated carefully, all of the eigenvalues below the continuum provide variational bounds on physical eigenvalues.

The scattering problem is more complicated because of the different asymptotic conditions. One can force these asymptotic conditions by hand, or one can reformulate the scattering problem so that the asymptotic conditions are all built into the dynamical equation. The trick is to reformulate the problem as an integral equation with a kernel term that has the property that it can be approximated uniformly (*i.e.,* independent of initial and final states) by a finite dimensional matrix. This will generally happen if all of the asymptotic conditions are built into the kernel, because in that case the kernel only modifies the dynamics when the particles are *not* asymptotic. This is precisely what the Faddeev equations (Fa 65) do. In practice, however, the problem is still far from trivial: the matrices needed to treat realistic three-nucleon problems accurately may contain $10^8 - 10^9$ elements.

We now develop an equation for the three-body transition amplitudes that is convenient for relativistic calculations performed in momentum space. Before doing this, we relate the

transition matrix elements to the scattering matrix of our model. In what follows, we employ the BT representation. The TP representation can be obtained by solving Eqs. (7.35)–(7.36) for the packing operators A, and then using these to transform the BT representation.

Since P^+ is kinematic, the following exact relation holds:

$$e^{-iHt}e^{+iH_at} = e^{-iM^2\frac{t}{2P^+}}e^{iM_a^2\frac{t}{2P^+}}. \tag{7.51}$$

The strong limit as $t \to \infty$ is identical to the strong limit as $t/2P^+ \to \infty$. The Kato-Birman invariance principle (Bi 62, Ka 65, Ch 76, Ob 78) implies

$$\lim_{t\to\pm\infty} e^{-iM^2t}e^{iM_a^2t}\Pi_a = \lim_{t\to\pm\infty} e^{-iMt}e^{iM_at}\Pi_a, \tag{7.52}$$

where Π_a is the projection operator for a channel governed by the asymptotic Hamiltonian H_a. When Eq. (7.52) is combined with the previous equation, we get

$$\Omega_{a\pm} = \lim_{t\to\mp\infty} e^{-iHt}e^{+iH_at}\Pi_a = \lim_{t\to\mp\infty} e^{-iMt}e^{iM_at}\Pi_a. \tag{7.53}$$

The scattering operator is defined in terms of the wave operators as follows:

$$S_{ba} = \Omega_{b+}^{\dagger}\Omega_{a-}. \tag{7.54}$$

Combining Eqs. (7.53) and (7.54) leads to the following relation (see Appendix A for a derivation):

$$S = I - 2\pi i\delta(m_f - m_i)\langle f|T_{BT}^{fi}|i\rangle, \tag{7.55}$$

where

$$T_{BT}^{ab}(z) = V_{BT}^b + V_{BT}^a\frac{1}{z - M_{BT}}V_{BT}^b, \tag{7.56}$$

$$V_{BT}^a = M_{BT} - M_a^{BT}. \tag{7.57}$$

The mass operators act on the three-particle Hilbert space, and z must be evaluated $(m + i0^+)$, where $m = m_f = m_i$ is the value of the invariant mass of the initial and final states. A set

of coupled operator equations for the transition operators T^{ab} is obtained by using the second resolvent relations (Hi 57)

$$\frac{1}{z-M_{BT}}-\frac{1}{z-M_a^{BT}}=\frac{1}{z-M_a^{BT}}V_{BT}^a\frac{1}{z-M_{BT}}=\frac{1}{z-M_{BT}}V_{BT}^a\frac{1}{z-M_a^{BT}}. \tag{7.58}$$

With the assumption that $V_{123}^{BT}=0$, this can be used in Eq. (7.56) to obtain the equations

$$T_{BT}^{ab}(z)=V_{BT}^b+\sum_{c\neq a}V_c^{BT}\frac{1}{z-M_c^{BT}}T_{BT}^{cb}(z), \tag{7.59}$$

where

$$V_b^{BT}:=M_b^{BT}-M_0. \tag{7.60}$$

This is the main dynamical three-body equation. This kernel is connected on one iteration. For two-cluster initial states, the driving term contains no delta functions. For suitably well behaved interactions, a finite power of the kernel is compact on a suitable linear space. In practice, this means that the kernel can be uniformly approximated by a finite dimensional matrix (provided the matrix is constructed by expanding the kernel in a sufficiently nice set of functions). These details are important for demonstrating the existence of convergent algorithms to solve the dynamics (Fa 65), as discussed above. In practice, if this equation is formulated in momentum space it is a singular connected kernel integral equation that can be solved to any desired accuracy using known numerical methods.

Integral equations are formulated by writing an operator equation such as (7.59) in a particular basis. Of course, there is tremendous freedom in the choice of basis, and several choices have been used in practice for the nonrelativistic case. Certainly one such choice is a plane-wave basis. In the relativistic case, we note that it is frequently advantageous to employ bases which are simultaneous eigenstates of M_a^{BT}, $\tilde{\mathbf{P}}$, $\mathbf{j}_{f0}^2$ and j_{f0}^3. This assumes that the two-body problem has been solved. In the nonrelativistic case, this technique was first suggested by Karlsson and Zeiger (Ka 75) and used in (Gl 86). The advantage of this method is that the kernel and driving term of the integral equation can be expressed in terms of the two-body bound-state wave functions and the half-on-shell two-body transition operators. In a plane-wave basis, the evaluation

of matrix elements of the interactions is complicated because of the non-linear relation (7.15) between the two-body interaction v_{ij} and the interactions $V_{ij,k}^{BT}$. We present the Karlsson-Zeiger approach here. Nevertheless, the plane-wave basis has been used in applications, and, under certain conditions of the two-body interaction, can also be used efficiently (Ca 90).

In order to formulate integral equations with a concise notation, we use the following abbreviations for the different states that are used to formulate these equations:

$$|[a]q_a\, j; \tilde{\mathbf{p}}\, \mu^{(\pm)}\rangle = |[L_a\, S_a\, (k_a\, j_a\, l_a\, s_a)]\, q_a\, j; \tilde{\mathbf{p}}\, \mu^{(\pm)}\rangle_{BT}; \tag{7.61}$$

$$|[a_B]q_a\, j; \tilde{\mathbf{p}}\, \mu\rangle = |[L_a\, S_a\, (\lambda_a\, j_a)]\, q_a\, j; \tilde{\mathbf{p}}\, \mu\rangle_{BT} \tag{7.62}$$

for the eigenstates of M_a^{BT} with the interacting two-body state in a scattering and bound state, respectively. Taken together, Eqs. (7.61) (with either the + or − asymptotic condition) and (7.62) make up a complete set on the three-body Hilbert space.

Using these basis vectors, we construct the following reduced matrix elements for the amplitudes X_j^{ba} and Y_j^{ba}:

$$\langle [b_B]q_b'\, j'; \tilde{\mathbf{p}}'\mu'|T_{BT}^{ba}(m+i0^+)|[a_B]q_a\, j; \tilde{\mathbf{p}}\mu\rangle = \delta_{\mu'\mu}\delta_{j'j}\delta(\tilde{\mathbf{p}}'-\tilde{\mathbf{p}})\langle [b_B]q_b\|X_j^{ba}\|[a_B]q_a\rangle; \tag{7.63}$$

$$\langle [b]q_b'\, j'; \tilde{\mathbf{p}}'\mu'^{(+)}|T_{BT}^{ba}(m+i0^+)|[a_B]q_a\, j; \tilde{\mathbf{p}}\mu\rangle = \delta_{\mu'\mu}\delta_{j'j}\delta(\tilde{\mathbf{p}}'-\tilde{\mathbf{p}})\langle [b]q_b^{(+)}\|Y_j^{ba}\|[a_B]q_a\rangle, \tag{7.64}$$

for the kernel:

$$\langle [b_B]q_b'\, j'; \tilde{\mathbf{p}}'\mu'|V_a|[a_B]q_a\, j; \tilde{\mathbf{p}}\mu\rangle = \delta_{\mu'\mu}\delta_{j'j}\delta(\tilde{\mathbf{p}}'-\tilde{\mathbf{p}})\langle [b_B]q_b\|V_j^{ba}\|[a_B]q_a\rangle; \tag{7.65}$$

$$\langle [b]q_b'\, j'; \tilde{\mathbf{p}}'\mu'^{(+)}|V_a|[a_B]q_a\, j; \tilde{\mathbf{p}}\mu\rangle = \delta_{\mu'\mu}\delta_{j'j}\delta(\tilde{\mathbf{p}}'-\tilde{\mathbf{p}})\langle [b]q_b^{(+)}\|V_j^{b^{(+)}a}\|[a_B]q_a\rangle; \tag{7.66}$$

$$\langle [b_B]q_b'\, j'; \tilde{\mathbf{p}}'\mu'|V_a|[a]q_a\, j; \tilde{\mathbf{p}}\mu^{(+)}\rangle = \delta_{\mu'\mu}\delta_{j'j}\delta(\tilde{\mathbf{p}}'-\tilde{\mathbf{p}})\langle [b_B]q_b\|V_j^{ba^{(+)}}\|[a]q_a^{(+)}\rangle; \tag{7.67}$$

$$\langle [b]q_b'\, j'; \tilde{\mathbf{p}}'\mu'^{(+)}|V_a|[a]q_a\, j; \tilde{\mathbf{p}}\mu^{(+)}\rangle = \delta_{\mu'\mu}\delta_{j'j}\delta(\tilde{\mathbf{p}}'-\tilde{\mathbf{p}})\langle [b]q_b^{(+)}\|V_j^{b^{(+)}a^{(+)}}\|[a]q_a^{(+)}\rangle, \tag{7.68}$$

and the driving terms:

$$\langle [b_B]q'_b\, j'; \tilde{\mathbf{p}}'\mu'|V^a_{BT}|[a_B]q_a\, j; \tilde{\mathbf{p}}\mu\rangle = \delta_{\mu'\mu}\delta_{j'j}\delta(\tilde{\mathbf{p}}' - \tilde{\mathbf{p}})\langle [b_B]q_b\|D^{ba}_j\|[a_B]q_a\rangle; \tag{7.69}$$

$$\langle [b]q'_b\, j'; \tilde{\mathbf{p}}'\mu'^{(+)}|V^a_{BT}|[a_B]q_a\, j; \tilde{\mathbf{p}}\mu\rangle = \delta_{\mu'\mu}\delta_{j'j}\delta(\tilde{\mathbf{p}}' - \tilde{\mathbf{p}})\langle [b]q^{(+)}_b\|D^{b^{(+)}a}_j\|[a_B]q_a\rangle. \tag{7.70}$$

These operators act on the internal Hilbert space, which does not involve $\tilde{\mathbf{p}}$ and μ. If we use the bases given by Eqs. (7.61) and (7.62) in the three-body equation (7.59), we obtain the following integral equations for the reduced transition matrix elements:

$$X^{ba}_j(z) = D^{ba}_j + \sum_{c\neq b}\int V^{bc}_j \frac{q_c^2 dq_c}{z - m_{c_B} + i0^+} X^{ca}_j(z) + \sum_{c\neq b}\int V^{bc^{(+)}}_j \frac{q_c^2 dq_c k_c^2 dk_c}{z - m_c + i0^+} Y^{ca}_j; \tag{7.71}$$

$$Y^{ba}_j(z) = D^{b^{(+)}a}_j + \sum_{c\neq b}\int V^{b^{(+)}c}_j \frac{q_c^2 dq_c}{z - m_{c_B} + i0^+} X^{ca}_j(z) + \sum_{c\neq b}\int V^{b^{(+)}c^{(+)}}_j \frac{q_c^2 dq_c k_c^2 dk_c}{z - m_c + i0^+} Y^{ca}_j. \tag{7.72}$$

For $c = ij, k$, the mass m_c is

$$m_{ij,k} = \sqrt{m_k^2 + q_c^2} + [\left(\sqrt{k_{ij}^2 + m_i^2} + \sqrt{k_{ij}^2 + m_j^2}\right)^2 + q_c^2]^{\frac{1}{2}}, \tag{7.73}$$

and for $c = c_B$:

$$m_{c_B} = \sqrt{m_k^2 + q_c^2} + \sqrt{\lambda_B^2 + q_c^2}. \tag{7.74}$$

This is a set of coupled two-variable singular integral equations for each value of j. Sums over all of the remaining degeneracy parameters are implied.

Before we evaluate the kernel and driving term of this equation, let us interpret the amplitudes $X^{ba}_j(z)$ and $Y^{ba}_j(z)$. It follows directly from its definition that $X^{ba}_j(z)$ is the reduced transition matrix element for a system prepared in a two-cluster initial state associated with the partition a, to be detected in a two-cluster final state associated with partition b. In order to

interpret $Y_j^{ba}(z)$, note that

$$\begin{aligned} Y_j^{ba}\delta_{\mu'\mu}\delta_{j'j}\delta(\tilde{\mathbf{p}}'-\tilde{\mathbf{p}}) &= \langle [b]q_b'\,j';\tilde{\mathbf{p}}'\mu'^{(+)}|T_{BT}^{ba}(m+i0^+)|[a_B]q_a\,j;\tilde{\mathbf{p}}\mu\rangle \\ &= \lim_{z_b^*\to m_b+i0^+}\langle [b]q_b'\,j';\tilde{\mathbf{p}}'\mu'|(z_b^*-M_0)\frac{1}{z_b^*-M_b^{BT}} \\ &\quad\times\left[1+V_{BT}^b(z^0-M_{BT})^{-1}\right]V_{BT}^a|[a_B]q_a\,j;\tilde{\mathbf{p}}\mu\rangle, \end{aligned} \tag{7.75}$$

where $|[b]q_b\,j;\tilde{\mathbf{p}}\,\mu\rangle$ denotes the plane wave used to initiate the scattering. When this is evaluated for $z_b^*\to z_0$, the resolvent equations can be used to obtain

$$\begin{aligned} Y_j^{ba}\delta_{\mu'\mu}\delta_{j'j}\delta(\tilde{\mathbf{p}}'-\tilde{\mathbf{p}}) &\to \langle [b_0]q_b'\,j';\tilde{\mathbf{p}}'\mu'|(z_0-M_0)(z_0-M_{BT})^{-1}V_{BT}^a|[a_B]q_a\,j;\tilde{\mathbf{p}}\mu\rangle \\ &= \langle [b]q_b\,j';\tilde{\mathbf{p}}'\mu'|T^{0a}(z_0)|[a_B]q_a\,j;\tilde{\mathbf{p}}\mu\rangle, \end{aligned} \tag{7.76}$$

which is the physical breakup amplitude in the on-shell limit. Thus the solutions (7.63)–(7.64) are exactly the physical scattering amplitudes when they are evaluated on shell.

Now we consider the input to equations (7.69)–(7.70). The kernel and driving terms are constructed out the following basic objects:

$$\langle [a]q_a'\,j';\tilde{\mathbf{p}}'\mu'|[a_B]q_a\,j;\tilde{\mathbf{p}}\mu\rangle = \delta_{\mu'\mu}\delta_{j'j}\delta_{L'L}\delta_{S'S}\delta_{j_a'j_a}\delta(\tilde{\mathbf{p}}'-\tilde{\mathbf{p}})\frac{1}{q_a^2}\delta(q_a'-q_a)\langle k_a\,l_a\,s_a|\lambda_a\,j_a\rangle, \tag{7.77}$$

where $\langle k_a\,l_a\,s_a|\lambda_a\,j_a\rangle$ is the two-body bound state wave function, and

$$\begin{aligned} &\langle [a]q_a'\,j';\tilde{\mathbf{p}}'\mu'|[a]q_a\,j;\tilde{\mathbf{p}}\mu^{(+)}\rangle \\ &\quad= \delta_{\mu'\mu}\delta_{j'j}\delta_{L'L}\delta_{S'S}\delta_{j_a'j_a}\delta(\tilde{\mathbf{p}}'-\tilde{\mathbf{p}})\frac{1}{q_a^2}\delta(q_a'-q_a) \\ &\quad\times\left[\delta_{l_a'l_a}\delta_{s_a's_a}\frac{1}{k_a^2}\delta(k_a'-k_a)+\frac{\langle k_a'\,l_a'\,s_a'\|t^{j_a}(m_{2a}(k_a')-i0^+)\|k_a\,l_a\,s_a\rangle}{m_{2a}(k_a)-m_{2a}(k_a')-i0^+}\right], \end{aligned} \tag{7.78}$$

where $\langle k_a'\,l_a'\,s_a'\|t^{j_a}(m_{2a}(k_a'))\|k_a\,l_a\,s_a\rangle$ is the reduced two-body transition operator evaluated half off shell. Some care is needed at this point. In the section on the two-body problem, we constructed several dynamical equations by diagonalizing various functions of M_{ij}. All of these equations lead to the same wave functions. Nevertheless, they imply different transition operators. The right-hand side of Eq. (7.78) assumes a two-body transition operator of the form

$$t=(M_{ij}-m_{ij})+(M_{ij}-m_{ij})\frac{1}{(z-M_{ij})}(M_{ij}-m_{ij}), \tag{7.79}$$

where $m_{ij}=m_{2a}$ is the invariant mass of two non-interacting particles. It is possible to replace the transition matrix elements on the right-hand side of Eq. (7.78) by the corresponding matrix

elements of

$$t_f = (f(M_{ij}) - f(m_{ij})) + (f(M_{ij}) - f(m_{ij}))\frac{1}{(f(z) - f(M_{ij}))}(f(M_{ij}) - f(m_{ij})), \tag{7.80}$$

where $f(M)$ is any function satisfying the conditions of the Kato-Birman invariance principle, *provided the denominator in Eq.* (7.78) *is also replaced by*

$$m_{2a}(k_a) - m_{2a}(k'_a) - i0^+ \to f(m_{2a}(k_a)) - f(m_{2a}(k'_a)) - i0^+. \tag{7.81}$$

This has the advantage that we can use directly the solutions of any of the two-body eigenvalue equations (5.64)–(5.66) as input. In particular, it allows us to use solutions of the two-body eigenvalue equations for M^2. As discussed in Sections 2 and 5, we can utilize solutions of the *nonrelativistic* Lippmann-Schwinger equation fit to two-body scattering data directly as input, without any approximation or refitting.

To compute the kernel and driving term, we also need the Racah coefficients of $\mathcal{P}$, which relate two unitarily equivalent representations of $\mathcal{P}$. They are computed explicitly in Appendix C. In general, they have the form

$$\langle [b]q'_b\, j'; \tilde{\mathbf{p}}'\mu' || [a]q_a\, j; \tilde{\mathbf{p}}\mu\rangle = \delta_{\mu'\mu}\delta_{j'j}\delta(\tilde{\mathbf{p}}' - \tilde{\mathbf{p}})R^{ba}_j([b]q_b; [a]q_a). \tag{7.82}$$

We can now construct the input to the three-body scattering equation in terms of these three quantities. For the kernel, we have the following expressions:

$$\begin{aligned}
&\langle [b_B]q_b \| V^{ba}_j \| [a_B]q_a\rangle \\
&\quad = \int k_b^2 dk_b \int k_a^2 dk_a\, \langle \lambda_b\, j_b | k_b\, l_b\, s_b\rangle R^{ba}_j([b]q_b; [a]q_a)\langle k_a\, l_a\, s_a | \lambda_a\, j_a\rangle \\
&\qquad \times \left(\sqrt{\lambda_a^2 + q_a^2} - \sqrt{m_{2a}(k_a)^2 + q_a^2}\right);
\end{aligned} \tag{7.83}$$

$$\begin{aligned}
&\langle [b_B]' q_b \| V^{ba(+)}_j \| [a]^{(+)} q_a\rangle \\
&\quad = \int k_b'^2 dk'_b \int k_a^2 dk_a\, \langle \lambda_b\, j_b | k_b\, l_b\, s_b\rangle R^{ba}_j([b]q_b; [a]q_a) \\
&\qquad \times \langle k'_a\, l'_a\, s'_a | t^{j_a}(m_{2a}(k'_a) - i0^+) | k_a\, l_a\, s_a\rangle \\
&\qquad \times \left(\frac{\sqrt{m_{2a}(k_a)^2 + q_a^2} - \sqrt{m_{2a}(k'_a)^2 + q_a^2}}{m_{2a}(k_a) - m_{2a}(k'_a)}\right);
\end{aligned} \tag{7.84}$$

$$
\begin{aligned}
&\langle [b]' q_b^{(+)} \| V_j^{b(+)a} \| [a_B] q_a \rangle \\
&\quad = \int k_b'^2 dk_b' \int k_a^2 dk_a \\
&\qquad \times \left[\delta_{l_b' l_b} \delta_{s_b' s_b} \frac{1}{k_b^2} \delta(k_b' - k_b) + \frac{\langle k_b'\, l_b'\, s_b' | t^{j_a}(m_{2b}(k_b') + i0^+) | k_b\, l_b\, s_b \rangle}{m_{2b}(k_b') - m_{2b}(k_b) + i0^+} \right] \\
&\qquad \times R_j^{ba}([b] q_b; [a] q_a) \langle k_a\, l_a\, s_a | \lambda_a\, j_a \rangle \left(\sqrt{\lambda_a^2 + q_a^2} - \sqrt{m_{2a}(k_a)^2 + q_a^2} \right);
\end{aligned}
\tag{7.85}
$$

$$
\begin{aligned}
&\langle [b]' q_b^{(+)} \| V_j^{b^{(+)} a^{(+)}} \| [a^{(+)}] q_a \rangle \\
&\quad = \int k_b^2 dk_b \int k_a \prime 2 dk_a' \\
&\qquad \times \left[\frac{1}{k_b^2} \delta_{l_b' l_b} \delta_{s_b' s_b} \delta(k_b' - k_b) + \frac{\langle k_b'\, l_b'\, s_b' | t^{j_b}(m_{2b}(k_b') + i0^+) | k_b\, l_b\, s_b \rangle}{m_{2b}(k_b') - m_{2b}(k_b) + i0^+} \right) \\
&\qquad \times R_j^{ba}([b] q_b; [a] q_a) \langle k_a'\, l_a'\, s_a' | t^{j_a}(m_{2a}(k_a) - i0^+) | k_a\, l_a\, s_a \rangle \\
&\qquad \times \left(\frac{\sqrt{m_{2a}(k_a)^2 + q_a^2} - \sqrt{m_{2a}(k_a')^2 + q_a^2}}{m_{2a}(k_a)^2 - m_{2a}(k_a')^2} \right).
\end{aligned}
\tag{7.86}
$$

The driving terms are

$$
\begin{aligned}
&\langle [b_B] q_b | D_j^{ba} | [a_B] q_a \rangle \\
&\quad = \sum_{c \neq a} \int k_b^2 dk_b \int k_a^2 dk_a \int q_c'^2 dq_c' \int k_c'^2 dk_c' \int q_c^2 dq_c \int k_c^2 dk_c \\
&\qquad \times \langle \lambda_b\, j_b | k_b\, l_b\, s_b \rangle R_j^{bc}([b] q_b; [c] q_c) R_j^{ca}([c] q_c'; [a] q_a) \langle k_a\, l_a\, s_a | \lambda_a\, j_a \rangle \\
&\qquad \times \Bigg[\langle [c] q_c | [c_B] q_c' \rangle \left(\sqrt{\lambda_c^2 + q_c'^2} - \sqrt{m_{2c}(k_c)^2 + q_c'^2} \right) \\
&\qquad\quad + \langle [c] q_c | [c^{(-)}] q_c' \rangle \left(\sqrt{m_{2c}(k_c') + q_c'^2} - \sqrt{m_{2c}(k_c) + q_c' 2} \right) \Bigg];
\end{aligned}
\tag{7.87}
$$

$$
\begin{aligned}
&\langle [b] q_b^{(+)} | D_j^{ba} | [a_B] q_a \rangle \\
&\quad = \sum_{c \neq a} \int q_c \prime 2 dq_c' \int q_c^2 dq_c \int k_c^2 dk_c \\
&\qquad \times \langle [b] j_b^{(+)} | k_b\, l_b\, s_b \rangle R_j^{bc}([b] q_b; [c] q_c) R_j^{ca}([c] q_c'; [a] q_a) \langle k_a\, l_a\, s_a | \lambda_a\, j_a \rangle \\
&\qquad \times \Bigg[\langle [c] q_c | [c_B] q_c' \rangle \left(\sqrt{\lambda_c^2 + q_c'^2} - \sqrt{m_{2c}(k_c)^2 + q_c'^2} \right) \\
&\qquad\quad + \int k_c \prime 2 dk_c' \langle [c] q_c | [c] q_c'^{(+)} \rangle \left(\sqrt{m_{2c}(k_c')^2 + q_c'^2} - \sqrt{m_{2c}(k_c)^2 + q_c'^2} \right) \Bigg].
\end{aligned}
\tag{7.88}
$$

The input to this equation consists of half-off-shell two-body transition operators and two-body bound state wave functions. The major technical differences between the relativistic and non-

relativistic three-body equation occur in two places. The first is that the two-body interactions $V_a^{BT} = M_a^{BT} - M_0$ that appear in the three-body problem are different than those that occur in the two-body problem. This is not a serious problem if the equations are formulated using a basis in terms of the two-body eigenstates. The second lies in the Racah coefficients that are used to transform between the different natural plane-wave bases.

If one is only interested in computing three-body binding energies and scattering cross sections it is sufficient to solve Eqs. (7.71), (7.72) and the corresponding bound state equation . To compute electromagnetic observables, we must solve Eqs. (7.35)–(7.36) for the packing operators A, and then transform the three-body wave functions.

7.3. **Symmetric Coupling Schemes**

The dynamical equations (7.71) and (7.72) are similar to the corresponding nonrelativistic equations. The biggest new complications come from the Racah coefficients, which are needed to change between the natural representations used to express each interaction. One of the advantages of developing relativistic quantum models using Clebsh-Gordan coefficients of the Poincaré group, in contrast to focusing directly on Dirac's forms of the dynamics, which correspond to very special choices of Clebsch-Gordan coefficients, is that it is possible to use the generality to develop a larger class of models.

One useful property of the Poincaré group is that a tensor product of three (or N) irreducible representations can be reduced directly to a linear superposition of irreducible representations which treats all particles symmetrically. This allows us to formulate three-body equations in a symmetric manner without having to use Racah coefficients to change basis. These coupling schemes, like the nonsymmetric schemes, do not lead to models that satisfy macroscopic locality, but that can be restored with suitable packing operators, as discussed previously. The price paid for symmetry is a representation space with a more abstract scalar product.

This section serves only as a brief introduction to this construction. First, the tensor product of front-form irreducible representations is reduced directly to a direct integral of irreducible representations, using Mackey's theory of induced representations (Ma 66). This gives a model Hilbert space and a representation of the Poincaré group for three non-interacting particles. We then

determine the general form of the interaction needed to perform a generalized Bakamjian-Thomas construction in this representation, write the eigenvalue equation for the mass eigenstates, and determine the transformation properties of these states. What is done here is only one of an infinite number of ways to treat this problem.

We begin by letting the state vector $|\tilde{\mathbf{p}}_1\,\mu_1\,\tilde{\mathbf{p}}_2\,\mu_2\,\tilde{\mathbf{p}}_3\,\mu_3\rangle$ denote a tensor product of three one-body representations, each with front-form spin. We now wish to express this tensor-product state as a linear superposition of states that transform irreducibly under the action of $\mathcal{P}$. To do this, let P denote the total four-momentum of this noninteracting state, and consider the identity

$$\begin{aligned}
&|\tilde{\mathbf{p}}_1\,\mu_1\,\tilde{\mathbf{p}}_2\,\mu_2\,\tilde{\mathbf{p}}_3\,\mu_3\rangle \\
&\quad = \int\limits_{SU(2)} d\underline{R}\, U[\underline{L}_f(P)]U(\underline{R}')\delta(\underline{R}'-\underline{R})U^\dagger(\underline{R})U^\dagger[\underline{L}_f(P)]|\tilde{\mathbf{p}}_1\,\mu_1\,\tilde{\mathbf{p}}_2\,\mu_2\,\tilde{\mathbf{p}}_3\,\mu_3\rangle \\
&\quad = \int\limits_{SU(2)} d\underline{R}\,\left[\frac{k_1^+k_2^+k_3^+}{p_1^+p_2^+p_3^+}\right]^{\frac{1}{2}} U[\underline{L}_f(P)]U(\underline{R}')^\dagger\delta(\underline{R}'-\underline{R})U(\underline{R})|\tilde{\mathbf{k}}_1\,\mu_1\,\tilde{\mathbf{k}}_2\,\mu_2\,\tilde{\mathbf{k}}_3\,\mu_3\rangle,
\end{aligned} \tag{7.89}$$

where $k_i := L_f^{-1}(P)p_i$. The integral over $SU(2)$ is with respect to the invariant Haar measure for $SU(2)$. (The Haar measure is the unique $SU(2)$ invariant measure normalized to unity.) For applications, this integral can be expressed as follows:

$$\begin{aligned}
&\int\limits_{SU(2)} d\underline{R}\, f(\underline{R}) \\
&\qquad = \frac{1}{16\pi^2}\int\limits_0^{4\pi} d\lambda \int\limits_0^{2\pi} d\phi \int\limits_0^{\pi} \sin\theta d\theta f\left(e^{-i\frac{\lambda}{2}(\sin\theta\cos\phi\sigma_x+\sin\theta\sin\phi\sigma_y+\cos\theta\sigma_z)}\right).
\end{aligned} \tag{7.90}$$

To evaluate this integral, note that any $\underline{R}\in SU(2)$ can be expressed as

$$\underline{R} = e^{-i\frac{\lambda}{2}(\sin\theta\cos\phi\sigma_x+\sin\theta\sin\phi\sigma_y+\cos\theta\sigma_z)}, \tag{7.91}$$

for a unique $\lambda\in[0,4\pi)$, $\theta\in[0,\pi)$, $\phi\in[0,2\pi)$.

Using properties of the rotation group (Go 66), the delta function in Eq. (7.89) can be

expressed as

$$\delta(\underline{R}' - \underline{R}) = \sum_{j\mu\nu} (2j+1) D^{j*}_{\mu\nu}(\underline{R}') D^{j}_{\mu\nu}(\underline{R}). \tag{7.92}$$

Note that we will not need this integral in the dynamical equations themselves. If we use the representation (7.92) in Eq. (7.89), we obtain

$$\begin{aligned} |\tilde{\mathbf{p}}_1\,\mu_1\,\tilde{\mathbf{p}}_2\,\mu_2\,\tilde{\mathbf{p}}_3\,\mu_3\rangle = \sum_{j\mu\nu} (2j+1) D^{j}_{\mu\nu}(R) U[\underline{L}_f(P)] U^{\dagger}(\underline{R}) \int\limits_{SU(2)} d\underline{R}'\, U(\underline{R}') \\ \times \left[\frac{k_1^+ k_2^+ k_3^+}{p_1^+ p_2^+ p_3^+}\right]^{\frac{1}{2}} |\tilde{\mathbf{k}}_1\,\mu_1\,\tilde{\mathbf{k}}_2\,\mu_2\,\tilde{\mathbf{k}}_3\,\mu_3\rangle D^{j*}_{\mu\nu}(\underline{R}'). \end{aligned} \tag{7.93}$$

Since this equation must be independent of $\underline{R}$, we let $\underline{R} = \underline{I}$, which gives

$$\begin{aligned} |\tilde{\mathbf{p}}_1\,\mu_1\,\tilde{\mathbf{p}}_2\,\mu_2\,\tilde{\mathbf{p}}_3\,\mu_3\rangle = \sum_{j\mu\nu} (2j+1) \delta_{\mu\nu} U[\underline{L}_f(P)] \int\limits_{SU(2)} d\underline{R}'\, U(\underline{R}') \\ \times \left[\frac{k_1^+ k_2^+ k_3^+}{p_1^+ p_2^+ p_3^+}\right]^{\frac{1}{2}} |\tilde{\mathbf{k}}_1\,\mu_1\,\tilde{\mathbf{k}}_2\,\mu_2\,\tilde{\mathbf{k}}_3\,\mu_3\rangle D^{j*}_{\mu\nu}(\underline{R}'). \end{aligned} \tag{7.94}$$

With this equation, we have projected a rest eigenstate of the non-interacting system onto a given irreducible representation space of $SU(2)$, and then defined a vector by applying a kinematic boost to the result. We now define the following state vector:

$$\begin{aligned} &|[\nu; \tilde{\mathbf{k}}_1\,\mu_1 \cdots \tilde{\mathbf{k}}_3\,\mu_3] k j; \tilde{\mathbf{p}}\mu\rangle \\ &\quad := U[\underline{L}_f(P)] \int\limits_{SU(2)} d\underline{R}' \left[\frac{k_1^+ k_2^+ k_3^+}{p^+}\right]^{\frac{1}{2}} U(\underline{R}') |\tilde{\mathbf{k}}_1\,\mu_1\,\tilde{\mathbf{k}}_2\,\mu_2\,\tilde{\mathbf{k}}_3\,\mu_3\rangle D^{j*}_{\mu\nu}(\underline{R}'). \end{aligned} \tag{7.95}$$

In general, some of these vectors are identically zero. For example, if all three particles have half-integral spins, the only non-vanishing terms all will have half-integral j's. If we use Eq. (7.95) in Eq. (7.94), we obtain

$$|\tilde{\mathbf{p}}_1\,\mu_1\,\tilde{\mathbf{p}}_2\,\mu_2\,\tilde{\mathbf{p}}_3\,\mu_3\rangle = \sum_{j\mu} (2j+1) \left[\frac{p^+}{p_1^+ p_2^+ p_3^+}\right]^{\frac{1}{2}} |[\mu; \tilde{\mathbf{k}}_1\,\mu_1 \cdots \tilde{\mathbf{k}}_3\,\mu_3] k\, j; \tilde{\mathbf{p}}\,\mu\rangle. \tag{7.96}$$

Since the tensor-product states form a basis on the three-particle Hilbert space, and each of the tensor-product states can be expressed in the above form, it follows that the vectors $|[\nu; \tilde{\mathbf{k}}_1\,\mu_1 \cdots \tilde{\mathbf{k}}_3\,\mu_3] k\, j; \tilde{\mathbf{p}}\,\mu\rangle$

are also complete. In fact, they are overcomplete, because the right-hand side of Eq. (7.96) only has contributions from vectors with $\mu = \nu$. To exhibit the overcompleteness, we insert $I = U^\dagger(\underline{R})U(\underline{R})$ to the right of $U(\underline{R}')$ in Eq. (7.95), from which we find that these states satisfy the following constraints:

$$\begin{aligned} &|[\nu; \tilde{\mathbf{k}}_1\, \mu_1 \cdots \tilde{\mathbf{k}}_3\, \mu_3]k\, j; \tilde{\mathbf{p}}\, \mu\rangle \\ &\quad = |[\bar{\nu}; \tilde{\mathbf{k}}_{1R}\, \bar{\mu}_1 \cdots \tilde{\mathbf{k}}_{3R}\, \bar{\mu}_3]k\, j; \tilde{\mathbf{p}}\, \mu\rangle D^{j*}_{\bar{\nu}\nu}(\underline{R}) \prod_{i=1}^{3} D_{\bar{\mu}_i\mu_i}[R_f(\underline{R}, k_i)], \end{aligned} \tag{7.97}$$

which are independent of $\underline{R}$. Here, $\tilde{\mathbf{k}}_{iR}$ is the light-front component of $k_i' = Rk_i$. This overcompleteness increases the number of labels we can use for a given physical vector. The more labels we have for the same vector, the less we have to transform the labels. Stated another way: covariance is an increasing function of redundancy. The same thing happens in a *free* field theory, where covariance is achieved by requiring functions that are identical on shell to correspond to the same vector. Because of this covariance, one must be careful *not* to associate the value of redundant label as the eigenvalue of a self-adjoint operator. In this case, the labels are chosen in such a way that the labels in square brackets in the state vector $|[\nu; \tilde{\mathbf{k}}_1\, \mu_1 \cdots \tilde{\mathbf{k}}_3\, \mu_3]k\, j; \tilde{\mathbf{p}}\, \mu\rangle$ are Poincaré invariant. In fact, one can show that the vectors $|[\nu; \tilde{\mathbf{k}}_1\, \mu_1 \cdots \tilde{\mathbf{k}}_3\, \mu_3]k\, j; \tilde{\mathbf{p}}\, \mu\rangle$ transform irreducibly under the action of the Poincaré group. This can be shown by using the invariance of the Haar measure for $SU(2)$. The result of a direct calculation is

$$\begin{aligned} &U(\underline{\Lambda}, \underline{a})|[\nu; \tilde{\mathbf{k}}_1\, \mu_1 \cdots \tilde{\mathbf{k}}_3\, \mu_3]k\, j; \tilde{\mathbf{p}}\, \mu\rangle \\ &\quad = \sqrt{\frac{P^+_\Lambda}{P^+}} e^{i\Lambda P\cdot a} |[\nu; \tilde{\mathbf{k}}_1\, \mu_1 \cdots \tilde{\mathbf{k}}_3\, \mu_3]k\, j; \tilde{\mathbf{p}}_\Lambda\, \bar{\mu}\rangle D^j_{\bar{\mu}\mu}[R_f(\Lambda, p)]. \end{aligned} \tag{7.98}$$

None of the labels in square brackets change, and the mass of the representation is given by $m = k^+ = k_1^+ + k_2^+ + k_3^+$. This representation is clearly symmetric under interchange of identical particles. Equations (7.95) and (7.96) define the transformations between these two bases.

In order to use the state vectors $|[\nu; \tilde{\mathbf{k}}_1\, \mu_1 \cdots \tilde{\mathbf{k}}_3\, \mu_3]k\, j; \tilde{\mathbf{p}}\, \mu\rangle$ to construct a representation of a Hilbert space of square integrable functions, we write the identity operator in the tensor-product

representation, and use Eq. (7.96) to express it in terms of the irreducible overcomplete basis:

$$\begin{aligned} I &= \sum \int d\tilde{\mathbf{p}}_1 \int d\tilde{\mathbf{p}}_2 \int d\tilde{\mathbf{p}}_3 \, |\tilde{\mathbf{p}}_1 \, \mu_1 \, \tilde{\mathbf{p}}_2 \, \mu_2 \, \tilde{\mathbf{p}}_3 \, \mu_3\rangle\langle \tilde{\mathbf{p}}_1 \, \mu_1 \, \tilde{\mathbf{p}}_2 \, \mu_2 \, \tilde{\mathbf{p}}_3 \, \mu_3| \\ &= \sum_{j\mu\nu} (2j+1)\delta_{\mu\nu} \sum_{j'\mu'\nu'} (2j'+1)\delta_{\mu'\nu'} \int d\tilde{\mathbf{p}}_1 \int d\tilde{\mathbf{p}}_2 \int d\tilde{\mathbf{p}}_3 \frac{p^+}{p_1^+ p_2^+ p_3^+} \\ &\quad \times |[\nu; \tilde{\mathbf{k}}_1 \, \mu_1 \cdots \tilde{\mathbf{k}}_3 \, \mu_3] k^+ \, j; \tilde{\mathbf{p}} \, \mu\rangle\langle [\nu; \tilde{\mathbf{k}}_1 \, \mu_1 \cdots \tilde{\mathbf{k}}_3 \, \mu_3] k^+ \, j'; \tilde{\mathbf{p}} \, \mu'|. \end{aligned} \tag{7.99}$$

We make use of the following identities (Go 66):

$$\int\limits_{SU(2)} d\underline{R} \, (2j+1) D^{j*}_{\mu\nu}(\underline{R}) D^{j'}_{\mu'\nu'}(\underline{R}) = \delta_{jj'}\delta_{\mu\mu'}\delta_{\nu\nu'}; \tag{7.100}$$

$$\int d^4p \delta^4(p - \sum_{i=1}^{3} p_i) = 1; \tag{7.101}$$

Using these identities and the constraints (7.97), we can express the identity operator as follows:

$$\begin{aligned} I = &\sum_{j\mu_1\cdots\mu_3\nu} 2(2j+1) \int d\tilde{\mathbf{k}}_1 \int d\tilde{\mathbf{k}}_2 \int d\tilde{\mathbf{k}}_3 \int d\tilde{\mathbf{p}} \frac{\left(\sum_{i=1}^3 k_i^+\right)^2}{k_1^+ k_2^+ k_3^+} \\ &\times \delta\left(\sum_{i=1}^{3} \mathbf{k}_{i\perp}\right) \delta\left(\sum_{i=1}^{3} \{k_i^+ - \frac{\mathbf{k}_{i\perp}^2 + m_i^2}{k_i^+}\}\right) \\ &\times |[\nu; \tilde{\mathbf{k}}_1 \, \mu_1 \cdots \tilde{\mathbf{k}}_3 \, \mu_3] k \, j; \tilde{\mathbf{p}} \, \mu\rangle\langle [\nu; \tilde{\mathbf{k}}_1 \, \mu_1 \cdots \tilde{\mathbf{k}}_3 \, \mu_3] k \, j; \tilde{\mathbf{p}} \, \mu|. \end{aligned} \tag{7.102}$$

Equations (7.97) and (7.102) define the representation of the Hilbert space. We define $\mathcal{H}$ to be the space of functions

$$\psi([\nu; \tilde{\mathbf{k}}_1 \, \mu_1 \cdots \tilde{\mathbf{k}}_3 \, \mu_3] k \, j; \tilde{\mathbf{p}} \, \mu) := \langle [\nu; \tilde{\mathbf{k}}_1 \, \mu_1 \cdots \tilde{\mathbf{k}}_3 \, \mu_3] k \, j; \tilde{\mathbf{p}} \, \mu | \psi\rangle, \tag{7.103}$$

which satisfy the constraint

$$\begin{aligned} &\psi([\nu; \tilde{\mathbf{k}}_1 \, \mu_1 \cdots \tilde{\mathbf{k}}_3 \, \mu_3] k \, j; \tilde{\mathbf{p}} \, \mu) \\ &\quad = \psi([\nu'; \tilde{\mathbf{k}}_{1R} \, \mu_1' \cdots \tilde{\mathbf{k}}_{3R} \, \mu_3'] k \, j; \tilde{\mathbf{p}} \, \mu) D^{j*}_{\nu\nu'}(\underline{R}^{-1}) \prod_{i=1}^{3} D^{j_i}_{\mu_i\mu_i'}[R_f^{-1}(\underline{R}, k_i)], \end{aligned} \tag{7.104}$$

and have a finite norm with respect to the scalar product:

$$\begin{aligned}\langle\psi|\phi\rangle = &\sum_{j\mu_1\cdots\mu_3\nu} 2(2j+1)\int d\tilde{\mathbf{k}}_1\int d\tilde{\mathbf{k}}_2\int d\tilde{\mathbf{k}}_3\int d\tilde{\mathbf{p}}\,\frac{\left(\sum_{i=1}^3 k_i^+\right)^2}{k_1^+k_2^+k_3^+}\\ &\times\delta\left(\sum_{i=1}^3\mathbf{k}_{i\perp}\right)\delta\left(\sum_{i=1}^3\{k_i^+-\frac{\mathbf{k}_{i\perp}^2+m_i^2}{k_i^+}\}\right)\\ &\times\psi^*([\nu;\tilde{\mathbf{k}}_1\,\mu_1\cdots\tilde{\mathbf{k}}_3\,\mu_3]k\,j;\tilde{\mathbf{p}}\,\mu)\phi([\nu;\tilde{\mathbf{k}}_1\,\mu_1\cdots\tilde{\mathbf{k}}_3\,\mu_3]k\,j;\tilde{\mathbf{p}}\,\mu).\end{aligned}\tag{7.105}$$

The vectors in this space are not represented by functions which satisfy the constraint (7.104), but rather equivalence classes of functions satisfying Eq. (7.104) whose difference vanishes when the arguments are subject to the constraints in the delta functions in Eq. (7.105). In this representation, the noninteracting representation of the Poincaré group acts irreducibly:

$$\begin{aligned}&\langle[\nu;\tilde{\mathbf{k}}_1\,\mu_1\cdots\tilde{\mathbf{k}}_3\,\mu_3]k\,j;\tilde{\mathbf{p}}\,\mu|U_0(\underline{\Lambda},\underline{a})|\psi\rangle\\ &\quad=e^{ip\cdot a}\sqrt{\frac{p^+_{\Lambda^{-1}}}{p^+}}\sum\langle[\nu;\tilde{\mathbf{k}}_1\,\mu_1\cdots\tilde{\mathbf{k}}_3\mu_3]k\,j;\tilde{\mathbf{p}}_{\Lambda^{-1}}\,\bar{\mu}|\psi\rangle D^j_{\mu\bar{\mu}}[R(\Lambda,p_{\Lambda^{-1}})].\end{aligned}\tag{7.106}$$

Given this representation, it is straightforward to perform a generalized Bakamjian-Thomas construction. In this representation, the mass operator of the noninteracting system is the multiplication operator:

$$M_0=\sum_{i=1}^3 k_i^+.\tag{7.107}$$

Although these quantities are involved in the constraints, this particular linear combination does define an operator because it commutes with the constraints. In this Hilbert space, an operator $\mathcal{O}$ is defined by the way it acts on a vector. We write this as follows:

$$\begin{aligned}&\langle[\nu;\tilde{\mathbf{k}}_1\,\mu_1\cdots\tilde{\mathbf{k}}_3\,\mu_3]k\,j;\tilde{\mathbf{p}}\,\mu|\mathcal{O}|\psi\rangle\\ &\quad=\sum_{j'\mu_1'\cdots\mu_3'\nu'} 2(2j'+1)\int d\tilde{\mathbf{k}}_1'\int d\tilde{\mathbf{k}}_2'\int d\tilde{\mathbf{k}}_3'\int d\tilde{\mathbf{p}}'\,\frac{\left(\sum_{i=1}^3 {k'}_i^+\right)^2}{{k'}_1^+{k'}_2^+{k'}_3^+}\\ &\qquad\times\delta\left(\sum_{i=1}^3\mathbf{k}'_{i\perp}\right)\delta\left(\sum_{i=1}^3\{{k'}_i^+-\frac{{\mathbf{k}'_{i\perp}}^2+m_i^2}{{k'}_i^+}\}\right)\\ &\qquad\times\langle[\nu;\tilde{\mathbf{k}}_1\,\mu_1\cdots\tilde{\mathbf{k}}_3\,\mu_3]k\,j;\tilde{\mathbf{p}}\,\mu|\mathcal{O}|[\nu';\tilde{\mathbf{k}}_1'\,\mu_1'\cdots\tilde{\mathbf{k}}_3'\,\mu_3']k'\,j';\tilde{\mathbf{p}}'\,\mu'\rangle\\ &\qquad\times\langle[\nu';\tilde{\mathbf{k}}_1'\,\mu_1'\cdots\tilde{\mathbf{k}}_3'\,\mu_3']k'\,j';\tilde{\mathbf{p}}'\,\mu'|\psi\rangle.\end{aligned}\tag{7.108}$$

In order for this to be well defined when it acts on two equivalent functions, the result of each calculation must be the same. This means that the kernel of the operator $\mathcal{O}$ satisfies the following

rotational covariance constraint:

$$\begin{aligned}
&\langle[\nu;\tilde{\mathbf{k}}_{1R}\,\mu_1\cdots\tilde{\mathbf{k}}_{3R}\,\mu_3]k\,j;\tilde{\mathbf{p}}\,\mu|\mathcal{O}|[\nu';\tilde{\mathbf{k}}'_{1R}\,\mu'_1\cdots\tilde{\mathbf{k}}'_{3R}\,\mu'_3]k'\,j';\tilde{\mathbf{p}}'\,\mu'\rangle \\
&\quad= D^{j*}_{\nu\eta}(\underline{R})\prod_{i=1}^{3}D^{j_i}_{\mu_i\zeta_i}[R_f(\underline{R},k_i)D^{j*}_{\nu'\eta'}(\underline{R}^{-1})\prod_{i=1}^{3}D^{j'_i}_{\mu'_i\zeta'_i}[R_f^{-1}(\underline{R},k'_i)] \\
&\qquad\times\langle[\eta;\tilde{\mathbf{k}}_1\,\zeta_1\cdots\tilde{\mathbf{k}}_3,\zeta_3]k\,j;\tilde{\mathbf{p}}\,\mu|\mathcal{O}|[\eta';\tilde{\mathbf{k}}'_1\,\zeta'_1\cdots\tilde{\mathbf{k}}'_3,\zeta'_3]k'\,j';\tilde{\mathbf{p}}'\,\mu'\rangle.
\end{aligned}\tag{7.109}$$

This condition is easily satisfied. One starts with a rotationally covariant kernel, and then multiplies by unitary representations of Melosh rotations on the left and the right. The remaining constraints are those needed to perform a Bakamjian-Thomas construction. The interaction must commute with the kinematic subgroup and the front-form spin of the noninteracting system. The interaction must therefore have the form

$$\begin{aligned}
&\langle[\nu';\tilde{\mathbf{k}}'_1\,\mu'_1\cdots\tilde{\mathbf{k}}'_3,\mu'_3]k'\,j';\tilde{\mathbf{p}}'\,\mu'|V|[\nu;\tilde{\mathbf{k}}_1\,\mu_1\cdots\tilde{\mathbf{k}}_3\,\mu_3]k\,j;\tilde{\mathbf{p}}\,\mu\rangle \\
&\quad=\delta_{j'j}\delta_{\mu'\mu}\delta(\tilde{\mathbf{p}}'-\tilde{\mathbf{p}})\langle[\nu';\tilde{\mathbf{k}}'_1\,\mu'_1\cdots\tilde{\mathbf{k}}'_3\,\mu'_3]\|V^j\|[\nu;\tilde{\mathbf{k}}_1\,\mu_1\cdots\tilde{\mathbf{k}}_3\,\mu_3]\rangle,
\end{aligned}\tag{7.110}$$

where the reduced matrix element is constrained by the rotational covariance constraint. The simultaneous eigenstates of the four-momentum and the spin have the general form

$$\langle[\nu';\tilde{\mathbf{k}}'_1\,\mu'_1\cdots\tilde{\mathbf{k}}'_3,\mu'_3]k'\,j';\tilde{\mathbf{p}}'\,\mu'|\lambda\,j;\tilde{\mathbf{p}},\mu\rangle=\delta_{j'j}\delta_{\mu'\mu}\delta(\tilde{\mathbf{p}}'-\tilde{\mathbf{p}})\psi_{\lambda j}[\nu';\tilde{\mathbf{k}}'_1\,\mu'_1\cdots\tilde{\mathbf{k}}'_3,\mu'_3],\tag{7.111}$$

where the internal wave function satisfies the following equation:

$$\begin{aligned}
&(\lambda-\sum_{i=1}^{3}k_i^+)\psi_{\lambda j}[\nu;\tilde{\mathbf{k}}_1\,\mu_1\cdots\tilde{\mathbf{k}}_3,\mu_3] \\
&\quad=\sum_{j\mu'_1\cdots\mu'_3\nu'}2(2j+1)\int d\tilde{\mathbf{k}}'_1\int d\tilde{\mathbf{k}}'_2\int d\tilde{\mathbf{k}}'_3\,\frac{\left(\sum_{i=1}^{3}k'^{+}_i\right)^2}{k'^{+}_1+k'^{+}_2+k'^{+}_3} \\
&\qquad\times\delta\left(\sum_{i=1}^{3}\mathbf{k}'_{i\perp}\right)\delta\left(\sum_{i=1}^{3}\{k'^{+}_i-\frac{\mathbf{k}'^{\,2}_{i\perp}+m_i^2}{k'^{+}_i}\}\right) \\
&\qquad\times\langle[\nu;\tilde{\mathbf{k}}_1\,\mu_1\cdots\tilde{\mathbf{k}}_3\,\mu_3]\|V^j\|[\nu';\tilde{\mathbf{k}}'_1\,\mu'_1\cdots\tilde{\mathbf{k}}'_3\,\mu'_3]\rangle \\
&\qquad\times\psi_{\lambda j}[\nu';\tilde{\mathbf{k}}'_1\,\mu'_1\cdots\tilde{\mathbf{k}}'_3,\mu'_3].
\end{aligned}\tag{7.112}$$

In the three-body case, this equation is non-trivial solve, but it does allow one to avoid the use of Racah coefficients. The states defined in Eq. (7.111) transform irreducibly under the action of the

Poincaré group, and define the solution to the relativistic problem. Because the interaction must commute with the kernel of the scalar product, it is possible to find complete sets of solutions with the desired symmetry.

The most general interactions satisfying the covariance constraints have reduced kernels of the form

$$\begin{aligned}
&\langle[\nu;\tilde{\mathbf{k}}_1\,\mu_1\cdots\tilde{\mathbf{k}}_3\,\mu_3]\|V^j\|[\nu';\tilde{\mathbf{k}}'_1\,\mu'_1\cdots\tilde{\mathbf{k}}'_3\,\mu'_3]\rangle \\
&\quad=\prod_{i=1}^{3}D^{j_i}_{\mu_i\eta_i}[R_{cf}(k'_i)]\langle[\nu;\mathbf{k}_1\,\eta_1\cdots\mathbf{k}_3,\eta_3]\|\tilde{V}^j\|[\nu';\mathbf{k}'_1\,\eta'_1\cdots\mathbf{k}'_3,\eta'_3]\rangle \\
&\qquad\times\prod_{i=1}^{3}D^{j_i}_{\eta'_i\mu'_i}[R_{fc}(k_i)].
\end{aligned}\tag{7.113}$$

The reduced matrix element on the right-hand side of this equation satisfies the following rotational covariance constraint:

$$\begin{aligned}
&\langle[\nu;R\mathbf{k}_1\,\mu_1\cdots R\mathbf{k}_3,\mu_3]\|\tilde{V}^j\|[\nu';R\mathbf{k}'_1\,\mu'_1\cdots R\mathbf{k}'_3,\mu'_3]\rangle \\
&\quad=D^{j*}_{\nu\zeta}(\underline{R})\prod_{i=1}^{3}D^{j_i}_{\mu_i\eta_i}(\underline{R})\langle[\zeta;\mathbf{k}_1\,\eta_1\cdots\mathbf{k}_3,\eta_3]\|\tilde{V}^j\|[\zeta';\mathbf{k}'_1\,\eta'_1\cdots\mathbf{k}'_3,\eta'_3]\rangle \\
&\qquad\times\prod_{i=1}^{3}D^{j_i}_{\eta'_i\mu'_i}(\underline{R}^{-1})D^{j*}_{\zeta'\nu'}(\underline{R}^{-1}).
\end{aligned}\tag{7.114}$$

Kernels satisfying these rotational covariance conditions are easy to construct. Note that this construction generalizes immediately to N particles by the replacement $3\to N$ in all of the relations.

7.4. **Remarks**

The differences among equations corresponding to the different forms of the dynamics appear in the Racah coefficients used in the change of basis. These coefficients are constructed out of the appropriate Clebsch-Gordan coefficients. The use of any given set of Clebsch-Gordan coefficient has dynamical consequences. If we choose the *two-body* interactions so that the front- and instant-form dynamics are scattering equivalent (phase equivalent with the same two-body binding energies), the corresponding three-body solutions will not be phase equivalent. This is well known in the nonrelativistic case, where phase equivalent two-body interactions do not

lead to the same three-body dynamics. Note however, that having found three-body solutions in a front-form dynamics, there is a scattering equivalent instant-form three-body dynamics. It will differ from the one obtained directly from the scattering equivalent two-body dynamics by a three-body interaction. The question of which form is more appropriate is like the question of which on-shell equivalent two-body interaction is correct for describing the nucleus. Clearly, there is no "correct" answer to this. What is relevant is that each form implies a different type of three-body interaction. A desirable scheme is one in which these three-body interactions (as a function of the two-body interactions) remain weak. This in turn is very dependent on the two-body interactions. In Section 9, we will see that a front-form formulation of the dynamics has some special advantages in problems involving electromagnetic probes.

Finally, we note that this type of construction can be extended to the many-body problem. This is discussed briefly in (Co 82) and (So 77).

8. **Particle Production**

Reactions where particle number is not conserved account for a large fraction of the total cross section in medium and high energy experiments in nuclear and particle physics. Thus, even if a reaction does not change particle number, the underlying dynamics is likely to be strongly coupled to channels where particle number is not fixed.

Production of massive particles violates Galilean invariance. If momentum is conserved in one coordinate system, then it cannot be conserved in any other coordinate system related to the original system by a Galilean boost. To illustrate this, consider a reaction that has two nucleons of mass m_n in the initial state, and two nucleons and a pion mass m_π in the final state. Galilean invariance implies that the total momentum $\mathbf{p}$ of this system is conserved. Assume that the system has total momentum $\mathbf{p}$ in a fixed inertial coordinate system, and consider an observer moving with constant velocity $-\mathbf{v}$ relative to this coordinate system. Galilean invariance implies that to this observer, the initial momentum is $\mathbf{p} + \mathbf{v}(m_n + m_n)$, and the final momentum is $\mathbf{p} + \mathbf{v}(m_n + m_n + m_\pi)$. The initial and final momenta in this new coordinate system differ by $\mathbf{v}m_\pi$, in which case momentum is not conserved in the new coordinate system. Since momentum conservation is a consequence of Galilean invariance, we obtain a contradiction. The only assumptions we used in arriving at this conclusion were Galilean invariance and a non-zero pion mass. This limitation does not apply to Poincaré invariant models, and will be illustrated below by constructing an explicit counterexample.

There are two distinct points of view that can be taken in formulating models with particle production. One point of view is start with a model of bare particles, and then add interactions that can change particle number. In general, the physical (dressed) particle states will not be identical to the corresponding bare particle states. The quantum numbers of the bare particles are not directly measurable, but instead become parameters that must be varied so that physical observables have values consistent with experiment. These values fix the strength of the interaction between the physical particle states in this model.

The second point of view is to treat the particle degrees of freedom as physical, *i.e.,* to construct the model Hilbert space in terms of square integrable functions of eigenvalues of oper-

ators that can be measured in the laboratory. In this approach there are no "self interactions." Interactions can only occur when two or more particles are close together.

The first point of view is consistent with the approach taken in perturbative quantum field theory. It is essential for making phenomenological contact with QCD, where the color-carrying dynamical degrees of freedom associated with the quarks and gluons are not observable. The second is much closer in spirit to what is done in nonrelativistic nuclear physics. It is more appropriate in phenomenological approaches, where model interactions are determined by comparing few-body calculations with experiment. It may also a more appropriate starting point for constructing phenomenologies motivated by lattice gauge theory calculations.

Although in this review neither point of view is advocated over the other, it is important to be aware of the distinction between these points of view and to apply consistently one point of view to the problem of interest. The mechanics of constructing models is similar in both cases, but there are important differences. For models based on bare particles, the interactions are simpler in structure, but they cannot generally satisfy macroscopic locality for models limited to a *finite* number of degrees of freedom. Each time a new particle degree of freedom is added, everything must be refit. In addition, the bare interactions are not directly related to experiment. For models based on physical particles, the interactions are generally more complicated. The simplest interaction that can change particle number is a $2 \to 3$ interaction, rather than an elementary $1 \to 2$ vertex. On the other hand, these interactions are more directly related to experiment, and there do not appear to be any fundamental barriers to construction models with a *finite* number of degrees of freedom consistent with macroscopic locality.

Relativistic models of directly interacting particle that change particle number were discussed by Sokolov (So 76, So 77a). Instant form models of the $NN\pi$ system with a macroscopically local scattering operator were formulated by Betz and Coester (Be 80) and applied by Betz and Lee (Be 81). The relativistic Lee model (Le 59) has superselection rules that make it possible to construct models that satisfy macroscopic locality for arbitrary numbers of particles. The $(1,1)$ sector was discussed by de Dormale (Do 79), and the general case is discussed in (Co 82). Fuda (Fu 90) has considered all sectors of the relativistic Lee model in the front form. In the remainder of this section, we give an analytically solvable model of the $NN\pi$ system. This is similar in

spirit to the model of Betz and Coester, except that we formulate macroscopic locality in the generators rather than for the scattering operator.

8.1. **The Hilbert Space**

For the purpose of constructing a model Hilbert space, the physical particle point of view is taken. A representation for the Hilbert space is the space of square integrable functions of the eigenvalues of a complete set of commuting self-adjoint operators. For a medium energy scattering experiment, a complete set of commuting self-adjoint operators identifies the number, type and state of each particle.

The construction of a model Hilbert space requires specifying this maximal set of operators and the spectrum of eigenvalues of each operator. First, there are quantum numbers that identify the type of particle. These are the quantum numbers associated with isospin, strangeness, flavor, *etc.* Next, for a given type of particle, the specification of the mass and the spin of the particle fixes the spectrum of the remaining single-particle quantum numbers, following the construction in Section 3. The only remaining observables are the operators whose eigenvalues are the number of particles measured in each state.

In quantum field theory, the spectrum of the number operator is unbounded. In medium energy experiments, where the beam energy is bounded, only a finite number of these eigenvalues is accessible. Because of this, it makes sense to consider models where the spectrum of the model number operator is bounded. In this way it possible to consider models with a finite number of degrees of freedom with particle production. Such models are significantly simpler than models where the spectrum of the number operator is unbounded. Nevertheless, these models are non-trivial, because it is impossible to formulate models that produce particles in a manner consistent with Galilean relativity. What is relevant for the physical interpretation of these models is that for the type of experiment under consideration, the probability of measuring a particle number in the spectrum of the model number operator is identically or approximately unity.

An example of a relativistic model that does not conserve particle number is a nucleon-nucleon scattering process where the initial energy is sufficient to create one, but not two, pions. In this model, the pion and the nucleons are taken to be physical particles. This means that the

model cannot have an elementary pion-nucleon vertex: the only interaction that can produce a pion is a short-range interaction between *two* physical nucleons. This model will be formulated with a front-form symmetry for the purpose of illustration.

The Hilbert space $\mathcal{H}$ of this model is taken to be the direct sum of the two-nucleon space and the $NN\pi$ space:

$$\mathcal{H} := \mathcal{H}_{NN} \oplus \mathcal{H}_{NN\pi}, \tag{8.1}$$

where

$$\mathcal{H}_{NN} := \mathcal{H}_N \otimes \mathcal{H}_N \tag{8.2}$$

and

$$\mathcal{H}_{NN\pi} := \mathcal{H}_N \otimes \mathcal{H}_N \otimes \mathcal{H}_\pi. \tag{8.3}$$

A complete set of commuting self-adjoint operators that define the state of the single particles consists of the light-front components $\tilde{\mathbf{p}}_i$ of the four-momentum, the longitudinal component j^3_f of the front-form spin, and the z component I_z of the isospin. Following our notational convention, the eigenvalue ι of I_z is a degeneracy parameter. For the single-nucleon sector, we write

$$\mathcal{H}_N := \{\langle [\iota]\tilde{\mathbf{p}}\,\mu|\psi\rangle \mid \langle\psi|\psi\rangle_N < \infty\}, \tag{8.4}$$

where

$$\langle\phi|\psi\rangle_N := \sum_{\mu=-\frac{1}{2}}^{\frac{1}{2}} \sum_{\iota=-\frac{1}{2}}^{\frac{1}{2}} \int d\tilde{\mathbf{p}} \langle [\iota]\tilde{\mathbf{p}}\,\mu|\phi\rangle^* \langle [\iota]\tilde{\mathbf{p}}\,\mu|\psi\rangle. \tag{8.5}$$

For the pion sector,

$$\mathcal{H}_\pi := \{\langle [\iota]\tilde{\mathbf{p}}|\psi\rangle \mid \langle\psi|\psi\rangle_\pi < \infty\}, \tag{8.6}$$

where

$$\langle\phi|\psi\rangle_\pi := \sum_{\iota=-1}^{1} \int d\tilde{\mathbf{p}} \langle [\iota]\tilde{\mathbf{p}}|\phi\rangle^* \langle [\iota]\tilde{\mathbf{p}}|\psi\rangle. \tag{8.7}$$

A state vector in $\mathcal{H}$ has two components, with an associated wave function:

$$|\psi\rangle = \begin{pmatrix} |\psi_{NN}\rangle \\ |\psi_{NN\pi}\rangle \end{pmatrix} \to \begin{pmatrix} \langle [\iota_1]\mathbf{p}_1\, \mu_1\, [\iota_2]\tilde{\mathbf{p}}_2\, \mu_2 | \psi_{NN}\rangle \\ \langle [\iota_1]\mathbf{p}_1\, \mu_1 [\iota_2]\mathbf{p}_2\, \mu_2\, [\iota_\pi]\mathbf{p}_\pi | \psi_{NN\pi}\rangle \end{pmatrix}. \tag{8.8}$$

The normalization condition

$$1 = \langle \Psi | \Psi \rangle_{\mathcal{H}} = \langle \psi_{NN} | \psi_{NN} \rangle_{NN} + \langle \psi_{NN\pi} | \psi_{NN\pi} \rangle_{NN\pi} \tag{8.9}$$

implies that if the system is prepared in the state $|\Psi\rangle$, it will be found in a two-nucleon state with probability $\langle \psi_{NN} | \psi_{NN} \rangle_{NN}$, and a state with two nucleons and a pion with probability $\langle \psi_{NN\pi} | \psi_{NN\pi} \rangle_{NN\pi}$. The normalization condition (8.9) requires that the probability that this state will be found in either of these two states is unity. Such a model only makes sense if the probability of producing *more* than one pion is either identically zero or not significant. When this is not the case, a model with more degrees of freedom may be needed.

An interacting model can be formulated in a manner similar to what was done for the case of fixed number of particles. It is a three-step process. The first step is to use the single-particle representations to construct a representation of the Poincaré group for the non-interacting system on $\mathcal{H}$. The second step is to reduce this representation to a direct integral of irreducible representations using the appropriate Clebsch-Gordan coefficients for the Poincaré group. The third step is to add interactions using the Bakamjian-Thomas construction. This is done in the same manner as with a fixed number of particles. The new feature is that the constraints which the Bakamjian-Thomas construction imposes on the interactions are compatible with interactions which change particle number.

8.2. **Free-Particle Dynamics**

The free-particle dynamics on the model Hilbert space is defined completely by the single particle dynamics of each particle. A unitary representation of $\mathcal{P}$ on the model Hilbert space $\mathcal{H}$ for the non-interacting system is

$$U_0(\underline{\Lambda}, \underline{a}) := \begin{pmatrix} U_1(\underline{\Lambda}, \underline{a}) \otimes U_2(\underline{\Lambda}, \underline{a}) & 0 \\ 0 & U_1(\underline{\Lambda}, \underline{a}) \otimes U_2(\underline{\Lambda}, \underline{a}) \otimes U_\pi(\underline{\Lambda}, \underline{a}) \end{pmatrix} \tag{8.10}$$

where each of the one-body representation of $\mathcal{P}$ are front-form representations constructed in

Section 4. The infinitesimal generators of this representation can be expressed in terms of one-body generators as follows:

$$P_0^\mu := \begin{pmatrix} P_1^\mu \otimes I_2 + I_1 \otimes P_2^\mu & 0 \\ 0 & P_1^\mu \otimes I_2 \otimes I_\pi + I_1 \otimes P_2^\mu \otimes I_\pi + I_1 \otimes I_2 \otimes P_\pi^\mu \end{pmatrix} \tag{8.11}$$

and

$$J_0^{\alpha\beta} := \begin{pmatrix} J_1^{\alpha\beta} \otimes I_2 + I_1 \otimes J_2^{\alpha\beta} & 0 \\ 0 & J_1^{\alpha\beta} \otimes I_2 \otimes I_\pi + I_1 \otimes J_2^{\alpha\beta} \otimes I_\pi + I_1 \otimes I_2 \otimes J_\pi^{\alpha\beta} \end{pmatrix}. \tag{8.12}$$

The next step is to reduce the representation to a direct integral of irreducible representations. This is done with the Clebsch-Gordan coefficients for $\mathcal{P}$, treating the two- and three-body sectors of the $\mathcal{H}$ separately. We define the linear combinations of tensor product states that transform irreducibly on the two-body subspace, using front-form Clebsch-Gordan coefficients for $\mathcal{P}$ and $SU(2)$ Clebsch-Gordan coefficients for the isospin:

$$\begin{aligned} |[l\ s\ I\ \iota]k\ j;\tilde{\mathbf{p}}\ \mu\rangle := \sum \int |[\iota_1]\tilde{\mathbf{p}}_1\ \mu_1\ [\iota_2]\tilde{\mathbf{p}}_2\ \mu_2\rangle d\tilde{\mathbf{p}}_1 d\tilde{\mathbf{p}}_2 \\ \times \langle \tfrac{1}{2}\ \iota_1\ \tfrac{1}{2}\ \iota_2 | I\ \iota\rangle \langle [l\ s]k\ j;\tilde{\mathbf{p}}\ \mu | \tilde{\mathbf{p}}_1\ \mu_1\ \tilde{\mathbf{p}}_2\ \mu_2\rangle. \end{aligned} \tag{8.13}$$

These states transform irreducibly under the action of $U_{NN}(\underline{\Lambda}, \underline{a}) = U_1(\underline{\Lambda}, \underline{a}) \otimes U_2(\underline{\Lambda}, \underline{a})$. They can be coupled to the state with the pion by using an additional pair of Clebsch-Gordan coefficients of $\mathcal{P}$ and $SU(2)$ to construct linear combinations on the three-particle Hilbert space that transform irreducibly:

$$\begin{aligned} &|[L\ S\ k\ j_{NN}\ l\ s\ I_{NN}\ I_\pi\ I\ \iota]q\ j;\tilde{\mathbf{p}}\ \mu\rangle \\ &\quad := \sum \int d\tilde{\mathbf{p}}_{NN} \int d\tilde{\mathbf{p}}_\pi \int k'^2 dk'\ |[l\ s\ I_{NN}\ \iota_{NN}]k'\ j_{NN};\tilde{\mathbf{p}}_{NN}\ \mu_{NN}\rangle \otimes |[\iota_\pi]\ \tilde{\mathbf{p}}_\pi\rangle \\ &\qquad \times \langle I_{NN}\ \iota_{NN}\ 1\ \iota_\pi | I\ \iota\rangle \langle [L\ S\ k'\ j_{NN}]q\ j;\tilde{\mathbf{p}}\ \mu | k\ j_{NN}\ \tilde{\mathbf{p}}_{NN}\ \mu_{NN};\tilde{\mathbf{p}}_\pi\ 0\rangle. \end{aligned} \tag{8.14}$$

The operator $U_{NN\pi}(\underline{\Lambda}, \underline{a}) = U_1(\underline{\Lambda}, \underline{a}) \otimes U_2(\underline{\Lambda}, \underline{a}) \otimes U_\pi(\underline{\Lambda}, \underline{a})$ acts irreducibly on these linear combinations.

In order to simplify our notation, we denote the degeneracy quantum numbers by

$$[NN] := [l\, s\, I_{NN}\, \iota_{NN}], \tag{8.15}$$

and

$$[NN\pi] := [L\, S\, k\, j_{NN}\, l\, s\, I_{NN}\, I_\pi\, I\, \iota], \tag{8.16}$$

respectively. Taken together, these eigenstates form a basis on the model Hilbert space:

$$\begin{pmatrix} |[NN]k\, j; \tilde{\mathbf{p}}\, \mu\rangle \\ 0 \end{pmatrix} \qquad \begin{pmatrix} 0 \\ |[NN\pi]q\, j; \tilde{\mathbf{p}}\, \mu\rangle \end{pmatrix}, \tag{8.17}$$

with each of these states transforming irreducibly under the action of the free-particle dynamics:

$$U_0(\underline{\Lambda}, \underline{a}) \begin{pmatrix} |[NN]k\, j; \tilde{\mathbf{p}}\, \mu\rangle \\ 0 \end{pmatrix} = e^{i\Lambda p\cdot a} \sqrt{\frac{p^+_\Lambda}{p^+}} \sum \begin{pmatrix} |[NN]k\, j; \tilde{\mathbf{p}}_\Lambda\, \bar{\mu}\rangle \\ 0 \end{pmatrix} D^j_{\bar{\mu}\mu}[\underline{R}_f(\underline{\Lambda}, p/m)]; \tag{8.18}$$

$$U_0(\underline{\Lambda}, \underline{a}) \begin{pmatrix} 0 \\ |[NN\pi]q\, j; \tilde{\mathbf{p}}\, \mu\rangle \end{pmatrix} = e^{i\Lambda p\cdot a} \sqrt{\frac{p^+_\Lambda}{p^+}} \sum \begin{pmatrix} 0 \\ |[NN\pi]q\, j; \tilde{\mathbf{p}}_\Lambda\, \bar{\mu}\rangle \end{pmatrix} D^j_{\bar{\mu}\mu}[\underline{R}_f(\underline{\Lambda}, p/m)]. \tag{8.19}$$

In this representation, the front-form kinematic generators are:

$$\tilde{\mathbf{P}}_0 = \begin{pmatrix} \tilde{\mathbf{p}} & 0 \\ 0 & \tilde{\mathbf{p}}' \end{pmatrix}; \tag{8.20}$$

$$K^3_0 = \begin{pmatrix} -ip^+ \frac{\partial}{\partial p^+} & 0 \\ 0 & -ip'^+ \frac{\partial}{\partial p'^+} \end{pmatrix}; \tag{8.21}$$

$$E^i_0 = \begin{pmatrix} -ip^+ \frac{\partial}{\partial p^i} & 0 \\ 0 & -ip'^+ \frac{\partial}{\partial p'^i} \end{pmatrix}; \tag{8.22}$$

$$J^3_0 = j^3_{f0} - \frac{1}{P^+_0} \hat{\mathbf{z}} \cdot (\mathbf{P}_{\perp 0} \times \mathbf{E}_{\perp 0}), \tag{8.23}$$

where

$$j^3_{f0} = \begin{pmatrix} \mu & 0 \\ 0 & \mu' \end{pmatrix}. \tag{8.24}$$

The unprimed quantities are the quantum numbers of the NN system, and the primed quantities

are the quantum numbers of the $NN\pi$ system. The free mass operator is

$$m_0 = \begin{pmatrix} m_{NN} & 0 \\ 0 & m_{NN\pi} \end{pmatrix} := \begin{pmatrix} 2\sqrt{m_N^2 + k^2} & 0 \\ 0 & \sqrt{4(m_N^2 + k^2) + q^2} + \sqrt{m_\pi^2 + q^2} \end{pmatrix}. \tag{8.25}$$

The remaining generators are the functions of the kinematic generators $\tilde{\mathbf{P}}_0$, K_0^3, $\mathbf{E}_{\perp 0}$, J_0^3 and the free mass operator:

$$P_0^- := \frac{m_0^2 + \mathbf{P}_{\perp 0}^2}{p^+}; \tag{8.26}$$

$$\mathbf{J}_{\perp 0} := \frac{1}{2}\frac{(P_0^+ - P_0^-)}{P_0^+}\hat{\mathbf{z}} \times \mathbf{E}_{\perp 0} - \frac{\hat{\mathbf{z}} \times \mathbf{P}_{\perp 0}}{P_0^+}K_0^3 + \frac{\mathbf{P}_{\perp 0}}{P_0^+}j_{f0}^3 + \frac{m_0}{P^+}\mathbf{j}_{f\perp 0}. \tag{8.27}$$

8.3. **Interactions**

The next step in the construction of a relativistic model is the inclusion of interactions. We do this using a Bakamjian-Thomas construction. In the front form of the dynamics, this means that the interaction must commute with the kinematic generators and the free front-form spin. In this basis, matrix elements of the interaction should be diagonal in and independent of both $\tilde{\mathbf{p}}$ and μ. Charge conservation also demands that the interaction is diagonal in ι and, if the interaction is isospin independent, it will also be independent of ι. The interaction can be expressed as the difference

$$V = M^2 - m_0^2 \qquad \text{or} \qquad V' = M - m_0. \tag{8.28}$$

Since we employ the BT construction throughout this section, the BT subscripts are not shown explicitly, except for the discussion of packing operators which appears later. In the plane-wave basis, V has the matrix representation

$$\begin{aligned} &\langle [\cdots]' j'; \tilde{\mathbf{p}}'\, \mu' | V | [\cdots] j; \tilde{\mathbf{p}}\, \mu \rangle \\ &\quad = \delta_{\mu'\mu}\delta_{j'j}\delta_{\iota'\iota}\delta_{I'I}\delta(\tilde{\mathbf{p}}' - \tilde{\mathbf{p}}) \\ &\qquad \times \begin{pmatrix} \langle [NN]' k' | V^{jI\iota}_{NN\,NN} | [NN] k \rangle & \langle [NN]' k' | V^{jI\iota}_{NN\,NN\pi} | [NN\pi] q \rangle \\ \langle [NN] k | V^{jI\iota}_{NN\,NN\pi} | [NN\pi]' q' \rangle^* & \langle [NN\pi]' q' | V^{jI\iota}_{NN\pi\,NN\pi} | [NN\pi] q \rangle \end{pmatrix}. \end{aligned} \tag{8.29}$$

The ellipses $[\cdots]$ denote the degeneracy parameters. For physical particles, the off-diagonal matrix elements should be fully connected short-range interactions (*i.e.*, they should not contain

a vertex). The off-diagonal terms are the interactions which allows dynamical particle production. The delta functions in Eq. (8.29) ensure that the kinematic generators and the front-form spin of the NN space intertwine with the corresponding operators on the three-particle space. In general, the three-body sector will contain a nucleon-nucleon interaction and two pion-nucleon interactions which are imbedded in the three-particle Hilbert space in such a way that they commute with the kinematic subgroup and the free front-form spin of the three-body system. These interactions should lead separately to a 2+1 body dynamics that is scattering equivalent to the tensor product of the appropriate two- and one-body solutions.

With the interaction V as given in Eq. (8.29), the operator M (or M^2) commutes with the kinematic generators $\tilde{\mathbf{P}}_0$, K_0^3, $\mathbf{E}_{\perp 0}$, J_0^3 and the free front-form spin. We can now define the interacting generators in terms of the kinematic generators and M:

$$P^- := \frac{M^2 + \mathbf{P}_{\perp 0}^2}{P_0^+}; \tag{8.30}$$

$$\mathbf{J}_\perp := \frac{1}{2}\frac{(P_0^+ - P^-)}{P_0^+}\hat{\mathbf{z}} \times \mathbf{E}_{\perp 0} - \frac{\hat{\mathbf{z}} \times \mathbf{P}_{\perp 0}}{P_0^+}K_0^3 + \frac{\mathbf{P}_{\perp 0}}{P_0^+}j_{f0}^3 + \frac{M}{P_0^+}\mathbf{j}_{f\perp 0}. \tag{8.31}$$

Together with the kinematic generators, they define an interacting representation of the Lie algebra of $\mathcal{P}$. Representations of finite Poincaré transformations are constructed by diagonalizing M (or M^2) in the basis (8.17). These eigenstates can be chosen to be simultaneous eigenstates of the kinematic operators $\tilde{\mathbf{P}}_0$, j_{f0}^3 and $\mathbf{j0}_f^2$, which necessarily transform irreducibly. These eigenstates have two components, and are of the form

$$\begin{aligned}
&\langle [\cdots]' j' \, \tilde{\mathbf{p}}' \, \mu' | m \, j; \tilde{\mathbf{p}} \, \mu \rangle \\
&\quad = \delta_{j'j}\delta_{\mu'\mu}\delta_{I'I}\delta_{\iota'\iota}\delta(\tilde{\mathbf{p}}' - \tilde{\mathbf{p}}) \begin{pmatrix} \langle [l \, s \, I \, \iota]' k' | [I \, \iota] m \, j \rangle_{NN} \\ \langle [L \, S \, k \, j_{NN} \, l \, s \, I_{NN} \, I_\pi \, I \, \iota]' q' | [I \, \iota] m \, j \rangle_{NN\pi} \end{pmatrix} \\
&\quad := \delta_{j'j}\delta_{\mu'\mu}\delta_{I'I}\delta_{\iota'\iota}\delta(\tilde{\mathbf{p}}' - \tilde{\mathbf{p}}) \begin{pmatrix} \langle m_{NN} | m \, j \rangle_{NN} \\ \langle m_{NN\pi} | m \, j \rangle_{NN\pi} \end{pmatrix}.
\end{aligned} \tag{8.32}$$

The reduced wave functions, which depend on the degeneracy parameters and the free relative momentum variables, are solutions of the eigenvalue problem:

$$M|[I \, \iota] m \, j\rangle = m|[I \, \iota] m \, j\rangle. \tag{8.33}$$

For bound states, the solutions $M|[I \, \iota] m \, j\rangle$ are normalizable. The scattering solutions $|\psi^\pm\rangle$ of

the Schrödinger equation are defined by the asymptotic condition:

$$\lim_{\lambda\to\pm\infty} \|e^{-iM\lambda}|\psi^{\pm}\rangle - e^{-im_0\lambda}|\psi_0\rangle\| = 0. \tag{8.34}$$

These become eigenstates of M when $|\psi_0\rangle$ is a plane-wave eigenstate of m_0. In either case, the bound and scattering eigenstates transform as mass-m and spin-j irreducible representations under the dynamical representation associated with the generators $\tilde{\mathbf{P}}_0$, K_0^3, $\mathbf{E}_{\perp 0}$, J_0^3 and $P^-, \mathbf{J}_\perp$:

$$U(\underline{\Lambda}, \underline{a})|m\, j; \tilde{\mathbf{p}}\, \mu\rangle = e^{i\Lambda p\cdot a}\sqrt{\frac{p_\Lambda^+}{p^+}} \sum |m\, j; \tilde{\mathbf{p}}\, \bar{\mu}\rangle D^j_{\bar{\mu}\mu}[\underline{R}_f(\underline{\Lambda}, p/m)], \tag{8.35}$$

which assumes a delta function normalization in $\tilde{\mathbf{p}}$.

The solution of the eigenvalue problem requires diagonalizing the mass operator. We consider a solvable example to illustrate the general procedure. Take $V_{NN,NN} = V_{NN\pi,NN\pi} = 0$, and let the production interaction be separable:

$$\langle [NN\pi]' q' | V^{jI\iota}_{NN\pi\, NN} | [NN] k\rangle = F(q'\, k') f(k) := F^{jI\iota}_{L'\, S'\, j'_{NN}\, l'\, s'\, I'_{NN}}(q, k) f^{*jI\iota}_{ls}(k), \tag{8.36}$$

where $F^{jI\iota}_{L'\, S'\, j'_{NN}\, l'\, s'\, I_{NN}}(q', k')$ and $f^{*jI\iota}_{ls}(k)$ are square integrable functions of the relative momenta q', k' and k. This interaction is assumed to be diagonal in the quantum numbers j, I and ι. In what follows, the quantum numbers j, I and ι are parameters, and sums over the *remaining* discrete quantum numbers are implied. This type of interaction is not motivated by physical considerations, but a general $2 \to 3$ interaction consistent with charge conservation can always be expanded as a series in terms of this general structure.

The bound state satisfies the following equation:

$$\begin{pmatrix} |m_b\rangle_{NN} \\ |m_b\rangle_{NN\pi} \end{pmatrix} = \begin{pmatrix} (m_b^2 - m_{NN}^2)^{-1} & 0 \\ 0 & (m_b^2 - m_{NN\pi}^2)^{-1} \end{pmatrix} \begin{pmatrix} 0 & |f\rangle\langle F| \\ |F\rangle\langle f| & 0 \end{pmatrix} \begin{pmatrix} |m_b\rangle_{NN} \\ |m_b\rangle_{NN\pi} \end{pmatrix}, \tag{8.37}$$

where m_b^2 is the square of the mass eigenvalue. The solution of this equation is:

$$\begin{pmatrix} |m_b\rangle_{NN} \\ |m_b\rangle_{NN\pi} \end{pmatrix} = \begin{pmatrix} (m_b^2 - m_{NN}^2)^{-1}|f\rangle\tau_F(m_b^2)N \\ (m_b^2 - m_{NN\pi}^2)^{-1}|F\rangle N \end{pmatrix}, \tag{8.38}$$

where N is a normalization constant. The eigenvalue m_b is a positive solution of

$$\tau_F(m_b^2)\tau_f(m_b^2) = -1, \tag{8.39}$$

where

$$\begin{aligned}\tau_F(z) &= \langle F|\frac{1}{z - m_{NN\pi}^2}|F\rangle \\ &= \int_0^\infty q^2 dq \int_0^\infty k^2 dk \, \frac{|F(q\,k)|^2}{z - (\sqrt{4(m_N^2 + k^2) + q^2} - \sqrt{m_\pi^2 + q^2})^2}\end{aligned} \tag{8.40}$$

and

$$\tau_f(z) = \langle f|\frac{1}{z - m_{NN}^2}|f\rangle = \int_0^\infty \frac{|f(k)|^2 k^2 dk}{z - 4(m_N^2 + k^2)}. \tag{8.41}$$

Scattering solutions consistent with the incoming wave asymptotic conditions are superpositions of solutions of the Lippmann-Schwinger equation:

$$\begin{aligned}\begin{pmatrix} |m^-\rangle_{NN} \\ |m^-\rangle_{NN\pi} \end{pmatrix} &= \begin{pmatrix} |m_{NN}\rangle \\ 0 \end{pmatrix} \\ &+ \begin{pmatrix} (m^2 - m_{NN}^2 + i0^+)^{-1} & 0 \\ 0 & (m^2 - m_{NN\pi}^2 + i0^+)^{-1} \end{pmatrix} \\ &\times \begin{pmatrix} 0 & |f\rangle\langle F| \\ |F\rangle\langle f| & 0 \end{pmatrix} \begin{pmatrix} |m^-\rangle_{NN} \\ |m^-\rangle_{NN\pi} \end{pmatrix},\end{aligned} \tag{8.42}$$

where $|m_{NN}\rangle$ is an abbreviation for the invariant part of the plane-wave state

$$|[l\ s\ I\ \iota]k\,j;\tilde{\mathbf{p}}\,\mu\rangle$$

that fixes the asymptote as $t \to -\infty$. The equations for the two- and three-body components of the wave function decouple to give

$$|m^-\rangle_{NN} = |m_{NN}\rangle + \frac{1}{m^2 - m_{NN}^2 + i0^+}|f\rangle\langle F|\frac{1}{m^2 - m_{NN\pi}^2 + i0^+}|F\rangle\langle f|m^-\rangle_{NN}; \tag{8.43}$$

$$|m\rangle_{NN\pi} = \frac{1}{m^2 - m_{NN\pi}^2 + i0^+}|F\rangle\langle f|m^-\rangle_{NN}. \tag{8.44}$$

In this particular form, the solution of Eq. (8.43) can be expressed in terms of quadratures. Direct

calculation shows that the components of the reduced wave function are

$$\langle m_{NN}|m^-\rangle_{NN} = \langle m_{NN}|m'_{NN}\rangle + \frac{1}{m^2 - m^2_{NN} + i0^+} \frac{\langle m_{NN}|f\rangle\tau_F(m^2+i0^+)\langle f|m'_{NN}\rangle}{1-\tau_F(m^2+i0^+)\tau_f(m^2+i0^+)} \tag{8.45}$$

and

$$\langle m_{NN\pi}|m^-\rangle_{NN\pi} = \frac{1}{m^2 - m^2_{NN\pi} + i0^+} \frac{\langle m_{NN\pi}|F\rangle\langle f|m'_{NN}\rangle}{1-\tau_F(m^2+i0^+)\tau_f(m^2+i0^+)}. \tag{8.46}$$

In order to obtain a complete set of eigenstates, we also need to include solutions of the Lippmann-Schwinger equation that approach a state of two nucleons and a pion in the asymptotic past. These states satisfy Eq. (8.42), with the driving terms replaced as follows:

$$\begin{pmatrix} |m_{NN}\rangle \\ 0 \end{pmatrix} \to \begin{pmatrix} 0 \\ |m_{NN\pi}\rangle \end{pmatrix}. \tag{8.47}$$

The solutions to this equation are obtained from the solutions to Eq.(8.42) by replacing $F \leftrightarrow f$ and $NN \leftrightarrow NN\pi$ in Eqs. (8.45) and (8.46).

If these solutions are combined with the solutions Eqs. (8.38), (8.45) and (8.46), we obtain a complete set of states that transform irreducibly under $U(\underline{\Lambda}, \underline{a})$. For more general interactions, the calculation is similar, although it must be done numerically. When $V_{NN\pi;NN\pi} \neq 0$, the model has the numerical complexity of the three-body problem.

It is clear from this example that the formulation of relativistic models with particle production is very similar to the case of a fixed number of particles. We have formulated this model in the front-form; however, as before, this can be done in any Bakamjian-Thomas construction based on any given set of Clebsch-Gordan coefficients.

8.4. **Macroscopic Locality**

Particle production leads to new difficulties in the formulation of macroscopic locality that merits some discussion. For models with a finite number of degrees of freedom, it is important to distinguish models formulated in terms of physical particles from models formulated in terms of bare particles.

Let us begin by considering a model formulated in terms of bare particles. Consider a model with two nucleons and at most 10 pions. The physical nucleon is an eigenstate of a mass operator with different probabilities of having $0, 1, \ldots, 10$ bare pions. Returning to the original problem with two nucleons, consider what happens when the bare nucleons are separated by a large spacelike separation. Macroscopic locality suggests that a physical state of two asymptotically separated nucleons should be a tensor product of two physical one-nucleon states. Unfortunately, if the physical one-nucleon states have a finite probability of containing more than five bare pions, there should be a finite probability of finding more than 10 bare pions in the state of two asymptotically separated nucleons. Unfortunately, our model Hilbert space cannot accommodate this. The result is a violation of macroscopic locality which goes beyond the violations that arise in considering a system with a fixed number of particles.

One way to attack this problem is to add enough bare pions until the probability of finding more than N bare pions in a nucleon becomes vanishingly small. After this one must devise a limiting process which does not let these small corrections cause any difficulties when that particles are separated asymptotically.

The alternative is to work with physical particles. In this case, since particle number is not conserved, the nature of cluster properties must be somewhat different than in the case of a fixed number of particles. For the latter case, macroscopic locality allowed us to build up the Hamiltonian of the system by looking at properties of isolated subsystems. In particular, we obtained nucleon-nucleon interactions by considering two-body problems. Macroscopic locality then fixed the form of the two-body interactions in the three-body problem, which was completely decoupled from the two-body problem. Although we did not complete the argument, this process can be repeated inductively (Co 82) to show macroscopic locality implies that the dynamics of systems of less than N particles fix the dynamics of the N-particle system, up to a overall N-body interaction. What is relevant is that at each stage of this induction, the problems are not physically coupled, and this allows us to use the solutions for systems of less than N particles as input to the N-particle dynamics. When particle number is not conserved, this is no longer the case.

The first question to consider is there anything that we can do induction on that (1) re-

places particle number and (2) allows us to build the dynamics of bigger systems from smaller systems. One possible solution to this problem makes use of the energy available to the system. Consider the scattering operator associated with a system of massive particles. Assume that we are considering a state of a sharp asymptotic energy, and imagine dividing this system into two subsystems. If spatial translations are applied to different subsets of the initial and final particles in a scattering matrix element, momenta of the individual asymptotic particles do not change. Consequently, spatial translation does not change the energy of the initial or the final state. On the other hand, in the asymptotic limit, the scattering matrix should cluster (Re 79) into a product of two scattering matrix elements, one for each separated configuration. In this case, the two scattering matrix elements share the total energy of the system, in the sense that the sum of the subsystem energies is the total energy of the combined system. If one of the systems contains at least one pion in the initial or final state, then the energy of the other scattering matrix element must be at least the rest energy of the pion less than the energy of the original system. Thus, we obtain a constraint on the combined system corresponding to the physics of two-subsystems associated with strictly smaller total energy. This means that if we look at all possible ways to divide that system up into subsystems with lower total energy, the dynamics of these subsystems put constraints on the system dynamics. The natural question is how to turn this around so that one can include these constraints in the system dynamics to construct a many-body dynamics based on few-body input. Although the general answer to is problem is an open question, the pion nucleon model just discussed can be reformulated consistent with these constraints.

To do this, note that the mass operator M has the general form:

$$M = \begin{pmatrix} M_{NN,NN} & M_{NN\pi,NN} \\ M_{NN,NN\pi} & M_{NN\pi,NN\pi} \end{pmatrix}. \tag{8.48}$$

If we assume that the two-body dynamics below the threshold for the production of one pion has already been fixed, and has been used to determine the form of $M_{NN,NN}$ which is physically relevant for scattering below the threshold for pion production, then we do not want the addition of a pion to the system to modify this part of the dynamics. This can be done if we restrict the form of the interactions $M_{NN\pi,NN}$ and $M_{NN,NN\pi}$ by demanding that they do not couple to the invariant subspace of M_{NN} associated with the two-body continuum below the pion production

threshold. We define the projector

$$\Pi = \sum \int_{2m_N}^{2m_N+m_\pi} dm\, |m_{NN}^{(+)}\rangle\langle m_{NN}^{(+)}| = \sum \int_{2m_N}^{2m_N+m_\pi} dm\, |m_{NN}^{(-)}\rangle\langle m_{NN}^{(-)}|, \tag{8.49}$$

which is independent of asymptotic condition, provided the two-body scattering theory is asymptotically complete and the sum runs over all degeneracy parameters. The constraint on the interactions becomes

$$M_{NN\pi,NN}\Pi = M_{NN\pi,NN}; \tag{8.50}$$

$$\Pi M_{NN,NN\pi} = M_{NN,NN\pi}. \tag{8.51}$$

This is still a Bakamjian-Thomas type of dynamics, since Π commutes with the free front-form spin and the kinematic generators. In practice, we simply enforce this condition by hand by multiplying with the appropriate projector. This implies that we are restricting our consideration to mass operators of the form

$$M = \begin{pmatrix} M_{NN} & M_{NN\pi,NN}\Pi \\ \Pi M_{NN,NN\pi} & M_{NN\pi,NN\pi} \end{pmatrix}. \tag{8.52}$$

Next, we observe that the three-body mass operator $M_{NN\pi;NN\pi}$ is a normal mass operator for three interacting particles in a Bakamjian-Thomas formulation. It can be constructed from nucleon-nucleon and pion-nucleon interactions that are imbedded in the three particle Hilbert space, subject to the constraint that they commute with the three-body kinematic subgroup and the free three-body front form spin. We also require that these interactions lead to 2+1 body problems that are scattering equivalent to the tensor products of one and two-body dynamical models. Under these assumptions, M has the same relation to the 2+1 body operators associated with the tensor-product representation as those in Eq. (7.23):

$$\begin{aligned} {M^{BT}_{NN\pi;NN\pi}}^2 = A^\dagger_{NN',\pi} {M^{TP}_{NN',\pi}}^2 A_{NN',\pi} + A^\dagger_{N\pi,N'} {M^{TP}_{n\pi,n'}}^2 A_{N\pi,N'} \\ + A^\dagger_{N'\pi,N} {M^{TP}_{N'\pi,N}}^2 A_{N'\pi,N} - 2M_0^2 + V^{BT}_{NN'\pi}, \end{aligned} \tag{8.53}$$

where the $A'_{ij,k}s$ are the packing operators defined in Eq. (6.70). In the three-body sector, this fails to cluster properly for the same reason that this happens in the case of a fixed number of

particles. This can be fixed by solving Eqs. (7.35) and (7.36) for the packing operators $A_{NN\pi,NN\pi}$. Macroscopic locality will be restored to three- body sector of our pion-nucleon-nucleon model if we transform M with the unitary operator

$$A = \begin{pmatrix} I & 0 \\ 0 & A_{NN\pi,NN\pi} \end{pmatrix}, \tag{8.54}$$

which gives

$$\begin{aligned} M_{TP} &= AM_{BT}A^\dagger \\ &= \begin{pmatrix} M^{BT}_{NN} & A_{NN\pi,NN\pi}M^{BT}_{NN\pi,NN}\Pi_{BT} \\ \Pi_{BT}M^{BT}_{NN,NN\pi}A^\dagger_{NN\pi,NN\pi} & A_{NN\pi,NN\pi}M^{BT}_{NN\pi,NN\pi}A^\dagger_{NN\pi,NN\pi} \end{pmatrix}. \end{aligned} \tag{8.55}$$

This implies the following:

$$U_{TP}(\underline{\Lambda}, \underline{a}) = AU_{BT}(\underline{\Lambda}, \underline{a})A^\dagger; \tag{8.56}$$

$$G_{TP} = AG_{BT}A^\dagger \tag{8.57}$$

for the finite transformations and the generators. In this case, the BT and TP dynamics are scattering equivalent (provided the TP model is consistent with the constraints (8.50) and (8.51)). The BT model has the property that if the pion is separated from the nucleons so that the nucleons have total energy below the threshold for production of another pion, then the dynamics of the separated nucleons is the same as the dynamics of the two nucleons in the two-nucleon sector. Although it is not as interesting, if one of the nucleons is moved away the system, it clusters into a tensor product of a single-nucleon and a pion-nucleon subsystem. In this case, the projector takes care of one problem, while the packing operators take care of the three-body sector. Note that this model will not cluster properly for states with energy above the threshold for production of two pions.

Even if this method can be generalized, it still appears somewhat cumbersome. The advantage is that for systems of normal densities, these few-body problems will fix uniquely a large fraction of the many-body dynamics.

This construction has not yet been extended to larger systems.

9. **Electromagnetic Currents and Tensor Operators**

An important application of relativistic quantum mechanical models is the calculation of observables associated with the interaction of electrons, photons and other weak probes with hadronic targets.

Such calculations focus on the evaluation of matrix elements of a hadronic current operator $I_h^\mu(x)$ in applications of the one-photon-exchange approximation. In the first part of this section, we provide expressions for observables in terms of matrix elements of the hadronic electromagnetic current operator evaluated in eigenstates of the strong Hamiltonian. A complete measurement of the cross section can be used to determine these matrix elements up to an overall phase. We also exhibit the relation between these current matrix elements and observables in inclusive scattering, where only the final electron is measured.

Poincaré covariance, current conservation, and discrete symmetries imply relations between different current matrix elements. Because of these constraints, all of the matrix elements can be determined, using Poincaré covariance, current conservation, and discrete symmetries, from a maximal set of independent matrix elements. Alternatively, one can classify the matrix elements in terms of a set of Lorentz invariant form factors, which contain the same information as the maximal set of independent matrix elements. In the second part of this section, we present a general classification scheme for matrix elements of tensor operators, which amounts to a Wigner-Eckart theorem for the Poincaré group. We also show the reduction to a maximal set of independent matrix elements can be especially simple in the front form.

In general, the number of independent matrix elements is the same as the number of invariant form factors. Form factors are a conveniently chosen maximal set of independent functions of the independent matrix elements that also contain the same information as the form factors.

In the last part of this section, we discuss the actual computation of matrix elements of the hadronic current operator. The conditions of current covariance and continuity put dynamical constraints on the current operator. The problem is that the classical picture of a current being constructed from charges and convection currents of each constituent is not consistent with either current conservation or with relativity. The difficulty for theorists is that although current

conservation and relativity constrain the hadronic current operator, they do not fix this operator. These issues are not limited to phenomenological models of hadrons; the same problem occurs when the hadrons are described by local fields. In spite of these difficulties, it is possible to formulate a sensible invariant impulse approximation using front-form dynamics. This is illustrated for the case of the $\pi \to \rho$ transition form factor.

9.1. **Basic Formulas and Observables**

For electron scattering from strongly interacting systems, the complete Hamiltonian can be written as follows:

$$H = H_{\rm QED} + H_h + H_{\rm em}, \tag{9.1}$$

where $H_{\rm QED}$ is the Hamiltonian of photons and electrons, H_h the hadronic system, and $H_{\rm em}$ the interaction between radiation and matter:

$$H_{em} = e \int d^3x \, I^\mu(x) A_\mu(x), \tag{9.2}$$

where

$$I^\mu = I^\mu_h + I^\mu_e, \tag{9.3}$$

is the sum of the hadronic and electron current densities and we use units of electric charge where $\alpha = e^2/4\pi$ is the fine structure constant. We use a convention where the electron charge is factored out of the current density operators. The current density must transform covariantly:

$$U(\underline{\Lambda}, \underline{a}) I^\mu(x) U(\underline{\Lambda}, \underline{a})^\dagger = (\Lambda^{-1})^\mu{}_\nu I^\nu(\Lambda x + a), \tag{9.4}$$

and it must be conserved with respect to the four-momentum:

$$g_{\mu\nu}[P^\mu, I^\nu(x)] = 0. \tag{9.5}$$

The representation $U(\underline{\Lambda}, \underline{a})$ of $\mathcal{P}$ in Eq. (9.4) is associated with the interacting lepton-hadron system. In what follows, we employ the one-photon-exchange approximation, or more generally,

we consider QED at tree level. In this case, the representation $U(\underline{\Lambda}, \underline{a})$, of $\mathcal{P}$, in Eq. (9.4) can be replaced by

$$U(\underline{\Lambda}, \underline{a}) \to U_h(\underline{\Lambda}, \underline{a}) \otimes U_{\rm QED}(\underline{\Lambda}, \underline{a}). \tag{9.6}$$

In the one-photon-exchange approximation, the hadronic representation, $U_h(\underline{\Lambda}, \underline{a})$, has no radiative corrections (these go beyond the one-photon-exchange contribution), and the constraints (9.4) and (9.5) become

$$U_h(\underline{\Lambda}, \underline{a}) I_h^\mu(x) U_h(\underline{\Lambda}, \underline{a})^\dagger = (\Lambda^{-1})^\mu{}_\nu I_h^\nu(\Lambda x + a), \tag{9.7}$$

and

$$g_{\mu\nu}[P_h^\mu, I_h^\nu(x)] = 0. \tag{9.8}$$

For the treatment of electron scattering from large-Z nuclei, where one-photon exchange is not appropriate, the radiative corrections to $U_h(\underline{\Lambda}, \underline{a})$ must be considered. Such problems are not considered here.

The scattering cross section in the one-photon-exchange approximation can be determined using time ordered perturbation theory in the interaction representation. For a single-particle initial state, the calculation is summarized by the equations

$$d\sigma = \frac{(2\pi)^4}{|s\mathbf{v}|} |\langle \mathbf{k}'\eta'; \mathbf{p}_1'\mu_1'; \cdots; \mathbf{p}_N'\mu_N'^{(+)} \| T \| \mathbf{k}\eta; \mathbf{p}\mu \rangle|^2 d\Phi_N; \tag{9.9}$$

$$\langle f|S|i\rangle = 1 - (2\pi) i \delta^4(p' - p) \langle f \| T \| i \rangle, \tag{9.10}$$

$$\begin{aligned} S &= \mathcal{T} \exp(-i \int dt\, H_{\rm em}) \\ &\approx I - i \int d^4x \mathcal{H}_{\rm em}(x) + \frac{(-i)^2}{2!} \int d^4x_1 \int d^4x_2\, \mathcal{T}(\mathcal{H}_{\rm em}(x_1), \mathcal{H}_{\rm em}(x_2)), \end{aligned} \tag{9.11}$$

where $\mathbf{k}$ and $\mathbf{k}'$ are the initial and final electron momenta, $\mathbf{p}$ and $\mathbf{p}_j'$ are the initial and final hadron momenta, and $\mathcal{T}$ denotes time ordering. The phase space factor is

$$d\Phi_N = \prod_{i=1}^N d^3p_i' d^3k' \delta^4(\sum_{j=1}^N p_j' + k' - p - k), \tag{9.12}$$

where s is a statistical factor equal to the number of permutations of identical particles in the final state, and the relative speed of the electron and target can be expressed in the form (Mo 45,

Br 59):

$$\frac{1}{|\mathbf{v}|} = \frac{\omega_{m_e}(\mathbf{k})\omega_m(\mathbf{p})}{\sqrt{(p\cdot k)^2 - m_e^2 m^2}}. \tag{9.13}$$

The (+) superscript on the final hadron state indicates that this is an eigenstate of the hadronic Hamiltonian that asymptotically approaches a plane-wave state as $t \to +\infty$, where the plane-wave states have a delta function normalization:

$$\langle \mathbf{p}'\mu'|\mathbf{p}\mu\rangle = \delta_{\mu'\mu}\delta(\mathbf{p}'-\mathbf{p}). \tag{9.14}$$

The initial state is also an eigenstate of the hadronic Hamiltonian, but the target is usually bound (or elementary), and thus does not require a scattering asymptotic condition.

Spacetime translational invariance of the electron and hadron current matrix elements can be used to express the one-photon exchange contribution to the reduced matrix element $\langle f\|T^{fi}\|i\rangle$ in the form:

$$\begin{aligned}
&\langle \mathbf{k}'\,\eta'\,\mathbf{p}'_1\,\mu'_1\,;\cdots;\,\mathbf{p}'_N\,\mu'_N{}^{(+)}\|T\|\mathbf{k}\,\eta\,\mathbf{p}\,\mu\rangle \\
&\quad = -ie^2(2\pi)^3\int d^4x e^{-i(k'-k)\cdot x}\langle 0|\mathcal{T}[A_\mu(0)A_\nu(x)]|0\rangle \\
&\qquad \times \langle \mathbf{k}'\,\eta'|I_e^\mu(0)|\mathbf{k}\,\eta\rangle\langle \mathbf{p}'_1\,\mu'_1\,\cdots;\mathbf{p}'_N\,\mu'_N{}^{(+)}|I_h^\nu(0)|\mathbf{p}\,\mu\rangle.
\end{aligned} \tag{9.15}$$

The photon propagator is

$$\frac{g_{\mu\nu}}{-q^2+i0^+} = -i\int d^4x\, e^{iq\cdot x}\langle 0|\mathcal{T}[A_\mu(0)A_\nu(x)]|0\rangle, \tag{9.16}$$

where

$$q = k - k' = p' - p \tag{9.17}$$

is the momentum transferred to the target. This gives the following expression for the reduced matrix element:

$$\begin{aligned}
&\langle \mathbf{k}'\,\eta'\,\mathbf{p}'_1\,\mu'_1\,;\cdots;\mathbf{p}'_N\,\mu'_N{}^{(+)}\|T\|\mathbf{k}\,\eta\,\mathbf{p}\,\mu\rangle \\
&\quad = e^2(2\pi)^3\frac{g_{\mu\nu}}{-q^2+i0^+}\langle \mathbf{k}'\,\eta'|I_e^\mu(0)|\mathbf{k}\,\eta\rangle\langle \mathbf{p}'_1\,\mu'_1\,;\cdots;\mathbf{p}'_N\,\mu'_N{}^{(+)}|I_h^\nu(0)|\mathbf{p}\,\mu\rangle.
\end{aligned} \tag{9.18}$$

The electron current matrix elements can be expressed in terms of a free electron field Ψ:

$$\langle \mathbf{k}'\,\eta'|I_e^\mu(0)|\mathbf{k}\,\eta\rangle = e\langle 0|a(\mathbf{k}',\eta') : \bar\Psi(0)\gamma^\mu\Psi(0) : a^\dagger(\mathbf{k},\eta)|0\rangle. \tag{9.19}$$

The electron field satisfies canonical equal time anticommutation relations, and the electron

creation and annihilation operators have a delta function normalization. They are discussed further in Section 10.

Using Eq. (9.18), we can express the differential cross section in terms of known quantities and model dependent matrix elements of the hadronic current operator:

$$\begin{aligned} d\sigma = e^4 \frac{(2\pi)^4 \omega_{m_e}(\mathbf{k})\omega_m(\mathbf{p})}{s\sqrt{(k\cdot p)^2 - m_e^2 m^2}}\, \delta^4(\sum_{i=1}^{N} p_i' + k' - p - k) d^3k' d^3p_1' \cdots d^3p_N' \\ \times \left| (2\pi)^3 \frac{g_{\mu\nu}}{-q^2 + i0^+} \langle \mathbf{k}'\, \eta' | I_e^\mu(0) | \mathbf{k}\, \eta \rangle \langle \mathbf{p}_1'\, \mu_1' \,; \cdots ; \mathbf{p}_N'\, \mu_N'^{(+)} | I_h^\nu(0) | \mathbf{p}\, \mu \rangle \right|^2 . \end{aligned} \tag{9.20}$$

The statistical factor s accounts for identical particles in the final state. This expression can be put in a manifestly covariant form if the initial and final electron and hadron states are given a covariant normalization:

$$|\mathbf{p}\mu\rangle \to |\mathbf{p}\mu\rangle_{\rm cov} := (2\pi)^{\frac{3}{2}} \sqrt{2\omega_m(\mathbf{p})} |\mathbf{p}\mu\rangle. \tag{9.21}$$

In this case, Eq. (9.20) becomes

$$\begin{aligned} d\sigma = \frac{e^4 (2\pi)^4}{4s\sqrt{(k\cdot p)^2 - m_e^2 m^2}}\, \delta^4(\sum_{i=1}^{N} p_i' + k' - p - k) \\ \times \frac{d^3k'}{(2\pi)^3 2\omega_{m_e}(\mathbf{k}')} \frac{d^3p_1'}{(2\pi)^3 2\omega_{m_1}(\mathbf{p}_1')} \cdots \frac{d^3p_N'}{(2\pi)^3 2\omega_{m_N}(\mathbf{p}_N')} \\ \times \left| \frac{g_{\mu\nu}}{-q^2 + i0^+} {}_{\rm cov}\langle \mathbf{k}'\eta' | I_e^\mu(0) | \mathbf{k}\eta \rangle_{\rm cov\ cov} \langle \mathbf{p}_1'\mu_1' \cdots \mathbf{p}_N'\, \mu_N'^{(+)} | I_h^\nu(0) | \mathbf{p}\mu \rangle_{\rm cov} \right|^2 . \end{aligned} \tag{9.22}$$

This is the form found in the *Review of Particle Properties* (Yo 88).

Electron scattering is limited to spacelike momentum transfers. Similar equations can be developed for incident photons which probe the current matrix elements for lightlike momentum transfers, while electron-positron annihilation probes hadronic matrix elements with timelike momentum transfers. Each type of reaction implies independent constraints on the hadronic current matrix elements.

For the case that only the final electron is observed, these expressions can be written in a more compact form by performing the sum over the unobserved final hadronic states. First, we

define an electron structure tensor:

$$\begin{aligned} L^{\mu\nu} &:= (2\pi)^6 \omega_{m_e}(\mathbf{k})\omega_{m_e}(\mathbf{k}') \sum_{\eta'} \langle \mathbf{k}\,\eta | I_e^\mu(0) | \mathbf{k}'\,\eta' \rangle \langle \mathbf{k}'\,\eta' | I_e^\nu(0) | \mathbf{k}\,\eta \rangle \\ &= \tfrac{1}{2}\left[k'^\mu k^\nu + k'^\nu k^\mu - g^{\mu\nu}(k'\cdot k) - i m_e \varepsilon^{\mu\nu}{}_{\alpha\beta} s_e^\alpha q^\beta \right], \end{aligned} \tag{9.23}$$

where $s_e^\alpha = W_e^\alpha / m_e$, W_e^α is the Pauli-Lubanski vector for the electron for the spin state η, and m_e is the electron mass. The hadron structure tensor is defined as follows:

$$\begin{aligned} W^{\mu\nu} &:= (2\pi)^6 \omega_m(\mathbf{p}) \sum \int d^3p'_1 \cdots \int d^3p'_N\, \delta^4\Big(\sum_{i=1}^N p'_i + k' - p - k\Big) \\ &\times \langle \mathbf{p}\,\mu | I_h^\mu(0) | \mathbf{p}'_1\,\mu'_1\ \cdots \mathbf{p}'_N\,\mu'_N{}^{(+)} \rangle \langle \mathbf{p}'_1\,\mu'_1\ \cdots \mathbf{p}'_N\,\mu'_N{}^{(+)} | I_h^\nu(0) | \mathbf{p}\,\mu \rangle. \end{aligned} \tag{9.24}$$

With these definitions, the differential cross section for inclusive electron scattering from a hadronic target has the form (Dr 64):

$$d\sigma = 4\alpha^2 \frac{d^3k'}{\epsilon'} \frac{1}{(q^2)^2} W^{\mu\nu} L_{\mu\nu} \frac{1}{[(k\cdot p)^2 - m_e^2 m^2]^{\frac{1}{2}}}. \tag{9.25}$$

In Eq. (9.25) and henceforth, the electron initial energy ϵ and final energy ϵ' are assumed to be much greater than m_e. The structure tensor $W^{\mu\nu}$ defined in Eq. (9.24) is a dimensionless second-rank tensor. The continuity equation for the current operator requires that

$$q_\mu W^{\mu\nu} = W^{\mu\nu} q_\nu = 0. \tag{9.26}$$

To illustrate the connection to observables, consider a spin-$\frac{1}{2}$ target. The available four-vectors for constructing $W^{\mu\nu}$ are the initial momentum p, the momentum transfer q and the spin vector S^μ of the initial state, where S^μ is related to the Pauli-Lubanski vector via $S^\mu := 2W^\mu/m$. $W^{\mu\nu}$ can be written as a combination of symmetric and antisymmetric terms (Dr 64, Bj 66):

$$W^{\mu\nu} = W_S^{\mu\nu} + W_A^{\mu\nu}, \tag{9.27}$$

where

$$\begin{aligned} W_S^{\mu\nu} &= -W_1(\nu, Q^2)\bar{g}^{\mu\nu} + W_2(\nu, Q^2)\frac{\bar{p}^\mu \bar{p}^\nu}{m^2}; \\ W_A^{\mu\nu} &= i\frac{q^\alpha}{m}\varepsilon^{\mu\nu\alpha\beta}\left\{ G_1(\nu, Q^2) S_\beta + G_2(\nu, Q^2)\frac{1}{m^2}\left[(p\cdot q) S_\beta - (S\cdot q) p_\beta \right] \right\}, \end{aligned} \tag{9.28}$$

where

$$\bar{g}^{\mu\nu} := g^{\mu\nu} - \frac{q^\mu q^\nu}{q^2}; \quad \bar{p}^\mu := \bar{g}^{\mu\nu} p_\nu. \tag{9.29}$$

The functions $W_{1,2}$ and $G_{1,2}$ depend upon the invariants

$$\begin{aligned} \nu &:= \frac{(p\cdot q)}{m}; \\ Q^2 &:= -q^2. \end{aligned} \tag{9.30}$$

The unpolarized differential inclusive cross section is

$$\frac{d^2\sigma}{d\Omega' d\epsilon'} = \frac{4\alpha^2 \epsilon'^2}{(Q^2)^2} \frac{1}{m} \left[W_2(\nu, Q^2) \cos^2 \tfrac{1}{2}\theta + 2W_1(\nu, Q^2) \sin^2 \tfrac{1}{2}\theta \right]. \tag{9.31}$$

To expose the structure functions G_1 and G_2 requires a measurement involving polarized particles. For example, if the initial electron is polarized along the beam direction, the difference between cross sections for initial hadron states polarized parallel and antiparallel to that of the electron is

$$\frac{d^2\sigma^{\uparrow\downarrow}}{d\Omega' d\epsilon'} - \frac{d^2\sigma^{\uparrow\uparrow}}{d\Omega' d\epsilon'} = \frac{4\alpha^2}{Q^2} \frac{\epsilon'}{\epsilon} \frac{1}{m^2} \left[G_1(\epsilon + \epsilon' \cos\theta) - G_2 \frac{Q^2}{m} \right]. \tag{9.32}$$

The difference between cross sections for initial hadron states polarized parallel and antiparallel to a unit vector in the scattering plane and perpendicular to the beam direction is

$$\frac{d^2\sigma^{\uparrow\rightarrow}}{d\Omega' d\epsilon'} - \frac{d^2\sigma^{\uparrow\leftarrow}}{d\Omega' d\epsilon'} = \frac{4\alpha^2}{Q^2} \frac{\epsilon'^2}{\epsilon m^2} \sin\theta (G_1 + 2\frac{\epsilon}{m} G_2). \tag{9.33}$$

The reader should be cautioned that various definitions of $W^{\mu\nu}$, $W_{1,2}$ and $G_{1,2}$ differ in the literature by factors of 2, $(2\pi)^3$, the target mass m and various minus signs due to the choice of metric, normalization, the sign convention of the tensor $\varepsilon^{\mu\nu\rho\sigma}$, and whether $W^{\mu\nu}$ is dimensionless. Equations (9.31)–(9.33) reflect our particular choices in this regard.

9.2. **Matrix Elements and Invariants**

As stated earlier, constraints of current continuity and covariance imply that the physical information for a given process is completely contained in a set of quantities whose number is generally less than the total number of current matrix elements. There is a variety of ways available for determining these independent quantities. One technique is to express such matrix elements in terms of Lorentz invariant form factors which describe the interaction dynamics with the external probe, multiplied by factors which describe the transformation properties of the matrix elements between inertial frames, and which contain implicitly the continuity constraints. There are two common ways of doing this: the elementary particle parameterization and the multipole expansion. They are analogous to using irreducible Cartesian tensors and spherical tensors, respectively, to describe the appropriate transformation properties. Another way is to choose a maximally independent set of specific matrix elements, and to use the constraints to compute explicitly any other matrix elements of choice.

The elementary particle parameterization is so called because it has been used extensively in the description of current matrix elements and vertex functions of hadrons (Sc 68a, Sc 68b). It has also seen application to a variety of nuclear process (Ki 79). The relevant matrix elements are described using combinations of four-vectors and Lorentz-covariant spinors or tensors, each multiplied by an invariant form factor. Each term separately satisfies the continuity equation and has the correct transformation properties. A typical example is the set of current matrix elements for a spin-$\frac{1}{2}$ particle. The covariance and continuity requirements for the current imply that there are only two matrix elements $\langle \mathbf{p}'\mu'|I^\mu(0)|\mathbf{p}\mu\rangle$ which are independent. For canonical spin, this can be expressed as follows:

$$ {}_c\langle \mathbf{p}'\mu'|I^\mu(0)|\mathbf{p}\mu\rangle_c = \bar{u}_c(\mathbf{p}'\mu')\left[\gamma^\mu F_1(Q^2) + i\frac{\sigma^{\mu\nu}q_\nu}{2m}F_2(Q^2)\right]u_c(\mathbf{p}\mu), \tag{9.34} $$

where $u_c(\mathbf{p}\mu)$ is a canonical Dirac spinor with the normalization

$$ (2\pi)^3 u_c^\dagger(\mathbf{p}\mu)u_c(\mathbf{p}\mu) = 1. \tag{9.35} $$

The electromagnetic structure of the target is described by the invariant Dirac and Pauli form factors F_1 and F_2, respectively. The Dirac spinors are transformation matrices that transform

Wigner rotations to finite dimensional representations of $SL(2,C)$. Equation (9.34) applies equally well to the ground state of a ^{3}H nucleus. For each value of Q^2, it is sufficient to calculate two independent matrix elements to fix the form factors $F_1(Q^2)$ and $F_2(Q^2)$. Given the form factors, Eq. (9.34) can be used to determine all remaining matrix elements corresponding to the same Q^2.

The multipole expansion has frequently been applied in nuclear physics, both for photon transitions (Bl 52), and for the study of inelastic electroexcitation of discrete states of the nucleus (De 66). The corresponding parameterization of the nucleon current matrix element employs the Sachs form factors. In the Breit frame of the nucleon,

$$\mathbf{p} = -\tfrac{1}{2}\mathbf{q}; \qquad \mathbf{p}' = \tfrac{1}{2}\mathbf{q}, \tag{9.36}$$

the matrix elements for canonical spin are

$$\begin{aligned} {}_c\langle \tfrac{1}{2}\mathbf{q}\,\mu'|I^0(0)|-\tfrac{1}{2}\mathbf{q}\,\mu\rangle_c &= \frac{G_E(Q^2)}{1+\tau}\delta_{\mu'\mu}; \quad \tau := \frac{Q^2}{4m^2}; \\ {}_c\langle \tfrac{1}{2}\mathbf{q}\,\mu'|\mathbf{I}(0)|-\tfrac{1}{2}\mathbf{q}\,\mu\rangle_c &= \frac{G_M(Q^2)}{1+\tau}\frac{1}{2m}\left[i\boldsymbol{\sigma}\times\mathbf{q}\right]_{\mu'\mu}. \end{aligned} \tag{9.37}$$

In the multipole expansion, matrix elements are expanded in terms of *rotational* invariants in a particular frame, and are related to matrix elements in other frame by the appropriate explicit transformation coefficients. While this approach is not *manifestly* Lorentz covariant, it is completely equivalent to the elementary particle approach, containing the same amount of physical information and having the correct transformation properties. In practice, the form factors can be determined by calculating two independent matrix elements.

Since the elementary particle approach is manifestly covariant and is discussed extensively elsewhere, we have chosen to examine the multipole expansion in this section.

Matrix Elements of Tensor Operators In this section, we discuss ways to represent matrix elements of Lorentz tensor operators, such as field operators or current operators, as products of transformation coefficients (which guarantee the covariance) with invariant form factors (which define the operator).

In quantum field theories, Hermitian operators corresponding to quantum mechanical observables are built out of local fields that transform covariantly under the Lorentz group. In the same way that any finite dimensional unitary representation of the rotation group can be built up out of spin-$\frac{1}{2}$ representations, it is possible to build operators that transform irreducibly under any finite dimensional representation of the Lorentz group in terms of basic building blocks. The result is that any Lorentz covariant field can be constructed out of these irreducible parts.

There are two differences between the Lorentz group and the rotation group. The first is that the finite dimensional representations of the Lorentz group are built out of representations of $SL(2,C)$ rather than $SU(2)$. The second is that there are two inequivalent representations of $SL(2,C)$ that play the role analogous to that of $SU(2)$ in the case of rotations. Up to a similarity transformation, these representations are given in terms of the $SL(2,C)$ matrices $\underline{\Lambda}$ and their inverse adjoints $(\underline{\Lambda}^{-1})^\dagger$. These matrices are not the same because they are not unitary, and there is no similarity transformation which relates them.

We introduce the following convention (St 64, Be 82) for $SL(2,C)$ spinors: an upper (undotted) spinor index transforms by right multiplication by $\underline{\Lambda}$, a lower dotted index, *i.e.,* $\dot{\beta}$ transforms by right multiplication with $(\underline{\Lambda}^{-1})^\dagger$, a lower undotted spinor index transforms by right multiplication with $(\underline{\Lambda}^{-1})^T$, where T denotes transpose, and an upper dotted index transforms by right multiplication by $\underline{\Lambda}^*$. The 'metric spinor'

$$\underline{\epsilon} := i\sigma_2. \tag{9.38}$$

is the similarity transformation relating the two dotted or two undotted representations:

$$\underline{\epsilon\Lambda\epsilon}^{-1} = (\underline{\Lambda}^{-1})^T; \qquad \underline{\epsilon\Lambda}^*\underline{\epsilon}^{-1} = (\underline{\Lambda}^{-1})^\dagger \tag{9.39}$$

Right multiplication by $\underline{\epsilon}$ lowers an index, and right multiplication by $\underline{\epsilon}^{-1}$ raises an index.

We are now in a position to define the basic building blocks from which one builds all Lorentz covariant fields that transform according to finite dimensional representations of the Lorentz group. An irreducible spinor field operator of rank (m,n) is an operator valued function

of the spacetime coordinates with rank (m, n) spinor indices:

$$\Psi^{\alpha_1 \cdots \alpha_m}_{\dot\beta_1 \cdots \dot\beta_n}(x). \tag{9.40}$$

It is symmetric in the m undotted and n dotted indices respectively, and transforms under the Poincaré group as follows:

$$\begin{aligned} &U(\underline{\Lambda}, \underline{a})\Psi^{\alpha_1 \cdots \alpha_m}_{\dot\beta_1 \cdots \dot\beta_n}(x)U^\dagger(\underline{\Lambda}, \underline{a}) \\ &\quad = \Psi^{\zeta_1 \cdots \zeta_m}_{\dot\eta_1 \cdots \dot\eta_n}(\Lambda x + a)\underline{\Lambda}_{\zeta_1 \alpha_1} \cdots \underline{\Lambda}_{\zeta_m \alpha_m}(\underline{\Lambda}^{-1})^\dagger_{\dot\eta_1 \dot\beta_1} \cdots (\underline{\Lambda}^{-1})^\dagger_{\dot\eta_n \dot\beta_n}. \end{aligned} \tag{9.41}$$

Indices on Ψ can be raised and lowered using $\underline{\epsilon}$. Upper and lower dotted index or upper and lower undotted index can be contracted with the result transforming like a pseudoscalar.

We can use these irreducible representations to construct any type of Lorentz covariant field. As will be shown below, four-vectors transform as a rank (1,1) spinors. The rank $(0, 0)$ representation corresponds to a scalar field. A Dirac spinor transforms like a $(1, 0) \oplus (0, 1)$ representation. The rank $(1, 0)$ and $(0, 1)$ spinors correspond to right- and left-handed spin-$\frac{1}{2}$ fields. The rank $(2, 0)$ and $(0, 2)$ representations correspond to antisymmetric second rank tensors that are self-dual and anti-self-dual, respectively (Ra 81). The Maxwell field strength tensor $F^{\mu\nu}$ transforms as a direct sum $(2, 0) \oplus (0, 2)$ representation.

Space reflection and time reversal can be expressed as follows:

$$P\Psi^{\alpha_1 \cdots \alpha_m}_{\dot\beta_1 \cdots \dot\beta_n}(x)P^{-1} = \Psi^{*\beta_1 \cdots \beta_n}_{\dot\alpha_1 \cdots \dot\alpha_m}(Px) \tag{9.42}$$

and

$$T\Psi^{\alpha_1 \cdots \alpha_m}_{\dot\beta_1 \cdots \dot\beta_n}(x)T^{-1} = (i)^{m+n}\Psi^{*\beta_1 \cdots \beta_n}_{\dot\alpha_1 \cdots \dot\alpha_m}(Tx). \tag{9.43}$$

Note also that for the special case where $\underline{\Lambda}$ is a rotation $\underline{R}$,

$$\underline{R} = (\underline{R}^\dagger)^{-1}. \tag{9.44}$$

This property is very useful in the discussion below.

Before we discuss matrix elements of tensor fields, it is useful to compare the transformation properties of these fields with particle states. In the case of the fields, the discrete indices transform under representations of the Lorentz group, while in the particle case they transform under a representation of a subgroup that leaves a given four-momentum invariant. For spacelike four-momentum, this subgroup is the rotation group and the resulting transformations are Wigner rotations. The transformations of the discrete indices of the particles and fields are not really different. They are equivalent ways of representing the same transformation. This point is not generally appreciated and often causes confusion regarding the need for four-component spinors to treat spin-$\frac{1}{2}$ particles. To understand the relation between Lorentz spinors and Poincaré spinors, let $\tilde{\Psi}^{\alpha}_{\dot{\beta}}(p)$ be the Fourier transform of a rank $(1,1)$ spinor field. Clearly, the spinor indices in the Fourier transformed field transform like the indices in Eq. (9.41). Let us assume for the moment that p is timelike, and consider the transformation properties of the linear combination with p dependent coefficients:

$$\tilde{\Psi}^{\zeta}_{\dot{\eta}}(p)\underline{L}_c(p)_{\zeta\alpha}\underline{L}_c^{-1\dagger}(p)_{\dot{\eta}\dot{\beta}}. \tag{9.45}$$

A direct calculation shows that all of the Lorentz transformations of the discrete indices are replaced Wigner rotations. What is more important is that rotations of dotted and undotted indices are the same. In this case, the transformed dotted and undotted indices are no longer inequivalent and can be related by linear transformations; there is no need for four-component objects to describe a spin $\frac{1}{2}$ system if done using irreducible representation of the Poincaré group. It is legitimate to ask: why use Lorentz covariant spinor densities to make fields rather than irreducible Poincaré covariant tensor operators? The reason is that when p changes from spacelike to lightlike to timelike, the indices of irreducible Poincare covariant operators transform under different "little groups" for each case (Mo 65). If these "little groups" are used explicitly, they obscure the analytic properties of the field, which are important in the proof of general properties of field operators. Irreducible representations of the Poincaré group are used to describe particles, however, because in that case the four-momentum is timelike (for particles with $m \neq 0$), and the representations are unitary. Note that the equivalence between these representations requires that the field operator has a spacetime argument in addition to the discrete indices. Nothing is lost in using Eq. (9.45) because the transformations therein are invertible.

The quantities of interest are matrix elements of an irreducible Lorentz tensor field evaluated between physical particle states. The particle states will be assumed to transform irreducibly under the action of the Poincaré group. To treat multiparticle final states the Clebsch-Gordan coefficients of the Poincaré group can be used to express the desired final states as a linear combination of states that transform irreducibly. Examples of these quantities are current matrix elements, Bethe-Salpeter amplitudes, Blankenbecler-Cook amplitudes (Bl 60), and N-quantum amplitudes (Gr 65a).

We are interested in matrix elements of tensor operators of the form

$$\langle m'j';\mathbf{p}'\mu'|\Psi^{\alpha_1\cdots\alpha_m}_{\dot\beta_1\cdots\dot\beta_n}(0)|mj;\mathbf{p}\mu\rangle.$$

As a first step, we consider matrix elements

$$\langle m'j';\mathbf{p}'\mu'|\Psi^{\alpha}_{\dot\beta}(0)|mj;\mathbf{p}\mu\rangle,$$

containing one index each of the two inequivalent representations of $SL(2,C)$. While there are $2\times 2\times(2j+1)\times(2j'+1)$ separate matrix elements for each fixed $\mathbf{p}$ and $\mathbf{p}'$, they are not all independent. The covariance properties of the operator provide constraints which relate various matrix elements to each other. There may be other constraint such as parity, time reversal and current continuity that further reduce the number of independent matrix elements. For now, we focus on the covariance requirements.

As a preliminary exercise for studying matrix elements of current operators, we consider the operator $\Psi^{\alpha}_{\dot\beta}(x)$ – more specifically, $\Psi^{\alpha}_{\dot\beta}(0)$, and the set of matrix elements ${}_c\langle m'j';\mathbf{p}'\mu'|\Psi^{\alpha}_{\dot\beta}(0)|mj;\mathbf{p}\mu\rangle_c$. It is useful to consider the Fourier transform:

$$\tilde\Psi^{\alpha}_{\dot\beta}(q) := \frac{1}{(2\pi)^4}\int d^4x\, e^{-iq\cdot x}\Psi^{\alpha}_{\dot\beta}(x). \tag{9.46}$$

By translational invariance, we get

$$\begin{aligned}
&{}_c\langle m'j';\mathbf{p}'\mu'|\tilde\Psi^{\alpha}_{\dot\beta}(q)|mj;\mathbf{p}\mu\rangle_c \\
&\qquad = \frac{1}{(2\pi)^4}\int d^4x\, e^{-iq\cdot x}\,{}_c\langle m'j';\mathbf{p}'\mu'|e^{ip'\cdot x}\Psi^{\alpha}_{\dot\beta}(0)e^{-ip\cdot x}|mj;\mathbf{p}\mu\rangle_c \\
&\qquad = \delta^4(p'-p-q)\,{}_c\langle m'j';\mathbf{p}'\mu'|\Psi^{\alpha}_{\dot\beta}(0)|mj;\mathbf{p}\mu\rangle_c.
\end{aligned} \tag{9.47}$$

We are interested in kinematics where $p' := p+q$ is timelike, corresponding to a massive

final state. This means that there exists a frame where $p'_0 = (m', 0)$. Let $p_0 + q_0 = p'_0$ be related to p and q with the same Lorentz transformation that relates p'_0 to p'. Consider the state vector $\int d\hat{\mathbf{p}}_0\, Y^l_{\mu_l}(\hat{\mathbf{p}}_0)\tilde{\Psi}^\alpha_{\dot\beta}(q_0)|mj;\mathbf{p}_0\mu\rangle_c$. Under a rotation R, this linear combination transforms as follows:

$$\begin{aligned} U(R)&\int d\hat{\mathbf{p}}_0\, Y^l_{\mu_l}(\hat{\mathbf{p}}_0)\tilde{\Psi}^\alpha_{\dot\beta}(q_0)|mj;\mathbf{p}_0\mu\rangle_c \\ &= \sum\int d\hat{\mathbf{p}}_0\, Y^l_{\mu_l}(\hat{\mathbf{p}}_0)\tilde{\Psi}^\zeta_{\dot\eta}(Rq_0)|mj;R\mathbf{p}_0\bar\mu\rangle_c\, \underline{R}_{\zeta\alpha}\underline{R}_{\dot\eta\dot\beta}D^j_{\bar\mu\mu}(R) \\ &= \sum\int d\hat{\mathbf{p}}_0\, Y^l_{\bar\mu_l}(\hat{\mathbf{p}}_0)\tilde{\Psi}^\zeta_{\dot\eta}(q_0)|mj;\mathbf{p}_0\bar\mu\rangle_c\, \underline{R}_{\zeta\alpha}\underline{R}_{\dot\eta\dot\beta}D^l_{\bar\mu_l\mu_l}(R)D^j_{\bar\mu\mu}(R). \end{aligned} \tag{9.48}$$

In Eq. (9.48), we have made use of the fact that $(\underline{R}^\dagger)^{-1} = \underline{R}$, that is, dotted and undotted indices transform in the same way under rotations. Since all three indices transform with the same argument R, they can be combined to construct the following state:

$$\begin{aligned} |[\tilde\Psi ls\mathcal{J}]m'j';0\mu'\rangle_c &:= \langle\tfrac12\alpha\tfrac12\dot\beta|s\mu_s\rangle\langle l\mu_l s\mu_s|\mathcal{J}\mu_\mathcal{J}\rangle\langle j\mu\mathcal{J}\mu_\mathcal{J}|j'\mu'\rangle \\ &\times\int d\hat{\mathbf{p}}_0\, Y^l_{\mu_l}(\hat{\mathbf{p}}_0)\tilde{\Psi}^\alpha_{\dot\beta}(q_0)|mj;\mathbf{p}_0\mu\rangle_c, \end{aligned} \tag{9.49}$$

where repeated indices are summed, and $m'^2 = -p'^2$. This linear combination can be boosted to define a new state,

$$\begin{aligned} |[\tilde\Psi ls\mathcal{J}]m'j';\mathbf{p}'\mu'\rangle_c &:= \langle\tfrac12\alpha\tfrac12\dot\beta|s\mu_s\rangle\langle l\mu_l s\mu_s|\mathcal{J}\mu_\mathcal{J}\rangle\langle j\mu\mathcal{J}\mu_\mathcal{J}|j'\mu'_j\rangle \\ &\times\sqrt{\frac{m'}{\omega_{m'}(\mathbf{p}')}}U[L_c(p')]\int d\hat{\mathbf{p}}_0\, Y^l_{\mu_l}(\hat{\mathbf{p}}_0)\tilde{\Psi}^\alpha_{\dot\beta}(q_0)|mj;\mathbf{p}_0\mu\rangle_c, \end{aligned} \tag{9.50}$$

in the same manner as that of a particle of mass m' and spin j'. The extra square root factor gives this state the same transformation properties as a non-covariant normalized state vector.

From the above definitions and the transformation properties of the states, the inner product of this newly defined state vector with the final state $|m''j'';\mathbf{p}''\mu''\rangle_c$ must have the form

$$\begin{aligned} {}_c\langle m''j'';\mathbf{p}''\mu''|[\tilde\Psi ls\mathcal{J}]m'j';\mathbf{p}'\mu'\rangle_c &= \delta_{j''j'}\delta_{\mu''\mu'}\delta(m''-m')\delta^3(\mathbf{p}''-\mathbf{p}') \\ &\times\langle m''j''\|\Psi_{ls\mathcal{J}}(q^2)\|m'j'\rangle. \end{aligned} \tag{9.51}$$

The reduced matrix element $\langle m'j'\|\Psi_{ls\mathcal{J}}(q^2)\|mj\rangle$ is Poincaré invariant. We now make use of Eq. (9.51) and invert the definition (9.50) to obtain the desired matrix elements ${}_c\langle m'j';\mathbf{p}'\mu'|\tilde\Psi^\alpha_{\dot\beta}(q)|mj;\mathbf{p}\mu\rangle_c$.

Under Lorentz transformations,

$$U(\underline{\Lambda})\tilde{\Psi}^{\alpha}_{\dot{\beta}}(q)U^{\dagger}(\underline{\Lambda}) = \sum \tilde{\Psi}^{\zeta}_{\dot{\eta}}(\Lambda q)\underline{\Lambda}_{\zeta\alpha}(\underline{\Lambda}^{-1})^{\dagger}_{\dot{\eta}\dot{\beta}}; \tag{9.52}$$

$$U(\underline{\Lambda})|mj;\mathbf{p}\mu\rangle_c = \sqrt{\frac{\omega_m(\Lambda\mathbf{p})}{\omega_m(\mathbf{p})}}\sum |mj;\Lambda\mathbf{p}\bar{\mu}\rangle_c D^{j}_{\bar{\mu}\mu}[R_c(\Lambda,p)]. \tag{9.53}$$

From Eq. (9.52), (9.53) and the orthogonality and completeness properties of spherical harmonics, rotation matrices and rotational Clebsch-Gordan coefficients, we find

$$\begin{aligned}
&{}_c\langle m'j';\mathbf{p}'\mu'|\tilde{\Psi}^{\alpha}_{\dot{\beta}}(q)|mj;\mathbf{p}\mu\rangle_c \\
&= \delta(m''-m')\delta^3(\mathbf{p}''-\mathbf{p}')\sqrt{\frac{\omega_{m'}(\mathbf{p}')}{m'}}\sqrt{\frac{\omega_m(\mathbf{p}_0)}{\omega_m(\mathbf{p})}} \\
&\quad\times\sum\langle\tfrac{1}{2}\zeta\tfrac{1}{2}\dot{\eta}|s\mu_s\rangle\langle l\mu_l s\mu_s|\mathcal{J}\mu_{\mathcal{J}}\rangle\langle j\bar{\mu}\mathcal{J}\mu_{\mathcal{J}}|j'\mu'_j\rangle \\
&\quad\times Y^{l\,*}_{\mu_l}(\hat{\mathbf{p}}_0)[\underline{L}_c^{-1}(p')]_{\zeta\alpha}[\underline{L}_c(p')]^{\dagger}_{\dot{\eta}\dot{\beta}}D^{j\dagger}_{\bar{\mu}\mu}[R_c(\underline{L}_c(p'),p_0)]\langle m'j'\|\Psi_{ls\mathcal{J}}(q^2)\|mj\rangle,
\end{aligned} \tag{9.54}$$

where

$$p_0 := L_c^{-1}(p')p. \tag{9.55}$$

Since

$$\delta(m''-m') = \frac{m'}{p'^0}\delta(p''^0-p'^0), \tag{9.56}$$

we have

$$\begin{aligned}
&{}_c\langle m'j';\mathbf{p}'\mu'|\tilde{\Psi}^{\alpha}_{\dot{\beta}}(q)|mj;\mathbf{p}\mu\rangle_c \\
&\quad= \sqrt{\frac{m'}{\omega_{m'}(\mathbf{p}')}}\sqrt{\frac{\omega_m(\mathbf{p}_0)}{\omega_m(\mathbf{p})}}\delta^4(p''-p')\sum\langle\tfrac{1}{2}\zeta\tfrac{1}{2}\dot{\eta}|s\mu_s\rangle\langle l\mu_l s\mu_s|\mathcal{J}\mu_{\mathcal{J}}\rangle\langle j\bar{\mu}\mathcal{J}\mu_{\mathcal{J}}|j'\mu'\rangle \\
&\qquad\times Y^{l\,*}_{\mu_l}(\hat{\mathbf{p}}_0)[\underline{L}_c^{-1}(p')]_{\zeta\alpha}[\underline{L}_c(p')]^{\dagger}_{\dot{\eta}\dot{\beta}}D^{j\dagger}_{\bar{\mu}\mu}[R_c(\underline{L}_c(p'),p)]\langle m'j'\|\Psi_{ls\mathcal{J}}(q^2)\|mj\rangle.
\end{aligned} \tag{9.57}$$

Matrix elements of $\Psi^{\alpha}_{\dot{\beta}}(0)$ can be obtained via Eq. (9.45):

$$\begin{aligned}
&{}_c\langle m'j';\mathbf{p}'\mu'|\Psi^{\alpha}_{\dot{\beta}}(0)|mj;\mathbf{p}\mu\rangle_c \\
&\quad= \sqrt{\frac{m'}{\omega_{m'}(\mathbf{p}')}}\sqrt{\frac{\omega_m(\mathbf{p}_0)}{\omega_m(\mathbf{p})}}\sum\langle\tfrac{1}{2}\zeta\tfrac{1}{2}\dot{\eta}|s\mu_s\rangle\langle l\mu_l s\mu_s|\mathcal{J}\mu_{\mathcal{J}}\rangle\langle j\bar{\mu}\mathcal{J}\mu_{\mathcal{J}}|j'\mu'\rangle \\
&\qquad\times Y^{l\,*}_{\mu_l}(\hat{\mathbf{p}}_0)[\underline{L}_c^{-1}(p')]_{\zeta\alpha}[\underline{L}_c(p')]^{\dagger}_{\dot{\eta}\dot{\beta}}D^{j\dagger}_{\bar{\mu}\mu}[R_c(\underline{L}_c(p'),p_0)]\langle m'j'\|\Psi_{ls\mathcal{J}}(q^2)\|mj\rangle.
\end{aligned} \tag{9.58}$$

In a frame where the final state is at rest, Eq. (9.58) has an even simpler form:

$$
\begin{aligned}
{}_c\langle m'j';\mathbf{0}\mu'|\Psi^{\alpha}_{\dot\beta}(0)|mj;\mathbf{p}_0\mu\rangle_c = \sum \langle \tfrac{1}{2}\alpha\tfrac{1}{2}\dot\beta|s\mu_s\rangle\langle l\mu_l s\mu_s|\mathcal{J}\mu_{\mathcal{J}}\rangle\langle j\mu\mathcal{J}\mu_{\mathcal{J}}|j'\mu'\rangle \\
\times Y^{l}_{\mu_l}{}^{*}(\hat{\mathbf{p}}_0)\langle m'j'\|\Psi_{ls\mathcal{J}}(q^2)\|mj\rangle.
\end{aligned}
\tag{9.59}
$$

Although Eq. (9.58) does not appear covariant because of the appearance of explicit indices, it does in fact have the correct transformation properties. Put another way, what we have done is to obtain matrix elements of $\Psi^{\alpha}_{\dot\beta}(0)$ in an arbitrary frame by relating them to corresponding matrix elements in the final state rest frame.

Equation (9.58) also has the appearance of a nonrelativistic Wigner-Eckart theorem for tensor operators of this type, but with extra kinematic factors and Wigner rotations which disappear in the nonrelativistic limit. Note also that these expressions summarize the symmetry associated with Poincaré invariance, but not the additional constraints associated with parity, time reversal or current conservation which may further decrease the number of independent reduced matrix elements. Parity and time reversal constraint can be determined by using Eq. (9.42)- (9.43) with (9.58). We now sketch the necessary steps for constructing invariants associated with matrix elements of the general type:

$$
\langle m'j';\mathbf{p}'\mu'|\Psi^{\alpha_1\cdots\alpha_m}_{\dot\beta_1\cdots\dot\beta_n}(0)|mj;\mathbf{p}\mu\rangle.
$$

Consider the state vector $\int d\hat{\mathbf{p}}_0\, Y^{l}_{\mu_l}(\hat{\mathbf{p}}_0)\tilde\Psi^{\alpha_1\cdots\alpha_m}_{\dot\beta_1\cdots\dot\beta_n}(q_0)|mj;\mathbf{p}_0\mu\rangle_c$. Under a rotation R, this transforms as follows:

$$
\begin{aligned}
&U(R)\int d\hat{\mathbf{p}}_0\, Y^{l}_{\mu_l}(\hat{\mathbf{p}}_0)\tilde\Psi^{\alpha_1\cdots\alpha_m}_{\dot\beta_1\cdots\dot\beta_n}(q_0)|mj;\mathbf{p}_0\mu\rangle_c \\
&\quad=\sum\int d\hat{\mathbf{p}}_0\, Y^{l}_{\bar\mu_l}(\hat{\mathbf{p}}_0)\tilde\Psi^{\zeta_1\cdots\zeta_m}_{\dot\eta_1\cdots\dot\eta_n}(q_0)|mj;\mathbf{p}_0\bar\mu\rangle_c \\
&\qquad\times \underline{R}_{\zeta_1\alpha_1}\cdots\underline{R}_{\zeta_m\alpha_m}\,\underline{R}_{\dot\eta_1\dot\beta_1}\cdots\underline{R}_{\dot\eta_n\dot\beta_n}\,D^{l}_{\bar\mu_l\mu_l}(R)D^{j}_{\bar\mu\mu}(R).
\end{aligned}
\tag{9.60}
$$

Since all indices transform with the same argument R, they can be combined using rotational Clebsch-Gordan coefficients to form a state with indices $\mathcal{J}\mu_{\mathcal{J}}$. This then couples to the final state just like a particle with the same mass and spin assignments. The order of coupling is a matter of taste. The development then proceeds exactly as before, with the result similar to Eq. (9.58), except for the presence of extra rotational Clebsch-Gordan coefficients, factors of $\underline{L}^{-1}_c(\mathbf{p})$, $\underline{L}^{\dagger}_c(\mathbf{p})$ and Wigner rotations.

Front-Form Matrix Elements Matrix elements of tensor operators between front-form state vectors can be constructed directly from those given above in terms of canonical spin. The main difference is that the index μ in a state vector $|mj;\tilde{\mathbf{p}}\mu\rangle_f$ does not transform the same way in Eq. (9.48) as the corresponding index in a canonical-spin state vector. Under arbitrary rotations, $|mj;\tilde{\mathbf{p}}\mu\rangle_f$ experiences a Wigner rotation which is not necessary the same as the rotation itself. In order to make use of the development above for canonical spin, the front-form spins in Eq. (9.48) must first undergo a Melosh rotation before being coupled together. The result is

$$\begin{aligned}
&{}_f\langle m'j';\tilde{\mathbf{p}}'\mu'|\Psi^\alpha_{\dot\beta}(0)|mj;\tilde{\mathbf{p}}\mu\rangle_f \\
&\quad=\sqrt{\frac{m'}{p'^+}}\sqrt{\frac{p_0^+}{p^+}}\sum\langle\tfrac{1}{2}\zeta\tfrac{1}{2}\dot\eta|s\mu_s\rangle\langle l\mu_l s\mu_s|\mathcal{J}\mu_\mathcal{J}\rangle\langle j\bar\mu\mathcal{J}\mu_\mathcal{J}|j'\mu'\rangle \\
&\qquad\times Y^{l\,*}_{\mu_l}(\hat{\mathbf{p}}_0)[\underline{L}_f^{-1}(p')]_{\zeta\alpha}[\underline{L}_f(p')]^\dagger_{\dot\eta\dot\beta}D^{j\dagger}_{\bar\mu\mu}[R_{cf}(p_0)]\langle m'j'\|\Psi_{ls\mathcal{J}}(q^2)\|mj\rangle,
\end{aligned}\tag{9.61}$$

where

$$p_0:=L_f^{-1}(p')p,\tag{9.62}$$

in contrast to Eq. (9.55) for canonical-spin matrix elements. Also, the front-form spins do not undergo Wigner rotations for front-form boosts.

Example: Matrix Elements of Field Operators To illustrate the relations derived above, we consider the matrix element

$${}_c\langle m\tfrac{1}{2};\mathbf{p}\mu_p|\psi(0)|M_d1;\mathbf{P}_d\mu_d\rangle_c,$$

where $|m\frac{1}{2};\mathbf{p}\mu_p\rangle_c$ and $|M_d1;\mathbf{P}_d\mu_d\rangle_c$ are state vectors for a nucleon and a deuteron, respectively, and $\psi(x)$ is a four-component nucleon field operator. It is more instructive to examine the matrix element

$${}_c\langle M_d1;\mathbf{P}_d\mu_d|\psi^\dagger(0)|m\tfrac{1}{2};\mathbf{p}\mu_p\rangle_c$$

in a frame where $\mathbf{P}_d=0$. As mentioned above, a Dirac field transforms like a $(1,0)\oplus(0,1)$ representation, that is, a direct sum of a dotted and undotted spin-$\frac{1}{2}$ spinors. In place of the dotted and undotted spinors, we can use linear combinations $\psi^e(x)$ and $\psi^o(x)$, which have intrinsic

even and odd parity, respectively. They correspond to the usual u and v spinors (Bj 64) for free fermions. The field operator $\psi(x)$ can then be expressed as follows:

$$\psi(x) = \sum_{\mu_s = \pm\frac{1}{2}} \left[\psi^e_{\mu_s}(x) + \psi^o_{\mu_s}(x)\right]. \tag{9.63}$$

Although the dotted and undotted spinors transform differently under boosts, we only make use of the fact that they transform in the same way under rotations when $\mathbf{P}_d = 0$. From Eq. (9.59), we obtain

$$\begin{aligned} {}_c\langle M_d 1; 0\mu_d|\psi^\dagger(0)|m\tfrac{1}{2}; \mathbf{p}_0\mu_p\rangle_c \\ &= \sum_{\mu_s=\pm\frac{1}{2}} \langle l\mu_l \tfrac{1}{2}\mu_s|\mathcal{J}\mu_\mathcal{J}\rangle\langle\tfrac{1}{2}\mu_p\mathcal{J}\mu_\mathcal{J}|1\mu_d\rangle Y^{l\,*}_{\mu_l}(\hat{\mathbf{p}}_0) \\ &\quad\times\left[\langle M_d 1\|{\psi^e_{l\frac{1}{2}\mathcal{J}}}^\dagger(q^2)\|m\tfrac{1}{2}\rangle + \langle M_d 1\|{\psi^o_{l\frac{1}{2}\mathcal{J}}}^\dagger(q^2)\|m\tfrac{1}{2}\rangle\right]. \end{aligned} \tag{9.64}$$

In place of the coupling scheme $[\frac{1}{2}\otimes(l\otimes\frac{1}{2})\mathcal{J}]1$, it is instructive to use the coupling scheme $[l\otimes(\frac{1}{2}\otimes\frac{1}{2})S]1$, with the result:

$$\begin{aligned} {}_c\langle M_d 1; 0\mu_d|\psi^\dagger(0)|m\tfrac{1}{2}; \mathbf{p}_0\mu_p\rangle_c \\ &= \sum_{\mu_s=\pm\frac{1}{2}} \langle\tfrac{1}{2}\mu_p\tfrac{1}{2}\mu_s|S\mu_S\rangle\langle l\mu_l S\mu_S|1\mu_d\rangle Y^{l\,*}_{\mu_l}(\hat{\mathbf{p}}_0) \\ &\quad\times\left[\langle M_d 1\|{\psi^e_{l\frac{1}{2}S}}^\dagger(q^2)\|m\tfrac{1}{2}\rangle + \langle M_d 1\|{\psi^o_{l\frac{1}{2}S}}^\dagger(q^2)\|m\tfrac{1}{2}\rangle\right]. \end{aligned} \tag{9.65}$$

where the reduced matrix elements in Eq. (9.65) are related to those in Eq. (9.64) by a $6j$ coefficient. The parity of the spherical harmonic, together with the intrinsic parity of the field operators, imply that l must be even for ψ^e and odd for ψ^o. The allowed couplings are therefore ${}^3S_1$ and ${}^3D_1$ for ψ^e and ${}^3P_1$ and ${}^1P_1$ for ψ^o. The last two arise from the explicit use of a four-component field operator, and have been discussed extensively by Gross (Gr 65b).

Four-Vector Current Matrix Elements The current four-vector can be expressed as a tensor operator which transforms as a product of inequivalent spin-$\frac{1}{2}$ operators.

A 2×2 matrix $I(x)$ can be defined in terms of the current operator $I^\mu(x)$ via

$$\underline{I}_{\alpha\dot{\beta}}(x) := [I_\mu(x)\sigma^\mu]_{\alpha\dot{\beta}}. \tag{9.66}$$

which has the transformation property

$$U(\underline{\Lambda})\underline{I}(x)U^\dagger(\underline{\Lambda}) = \underline{\Lambda}^{-1}\underline{I}(\Lambda x)(\underline{\Lambda}^{-1})^\dagger. \tag{9.67}$$

It is more convenient to work with the matrix $\bar{I}^{\alpha}_{\dot{\beta}}(x) := [\underline{\epsilon I}(x)]^{\alpha}_{\dot{\beta}}(x)$. Under a Lorentz transformation,

$$U(\underline{\Lambda})\bar{I}^{\alpha}_{\dot{\beta}}(x)U^\dagger(\underline{\Lambda}) = \sum \bar{I}^{\zeta}_{\dot{\eta}}(\Lambda x)\underline{\Lambda}_{\zeta\alpha}(\underline{\Lambda}^{-1})^\dagger_{\dot{\eta}\dot{\beta}} \tag{9.68}$$

That is, $\bar{I}^{\alpha}_{\dot{\beta}}(x)$ is a realization of the example developed above, with

$$\Psi^{\alpha}_{\dot{\beta}}(x) \to \bar{I}^{\alpha}_{\dot{\beta}}(x). \tag{9.69}$$

To make a connection between $\bar{I}^{\alpha}_{\dot{\beta}}(x)$ and $I^\mu(x)$, we define

$$\hat{I}^s_{\mu_s}(x) := (-1)^s \frac{1}{\sqrt{2}} \sum_{\alpha\dot{\beta}} \langle \tfrac{1}{2}\alpha\tfrac{1}{2}\dot{\beta}|s\mu_s\rangle \underline{I}^{\alpha}_{\dot{\beta}}(x), \tag{9.70}$$

which has explicit components

$$\hat{I}^1_{\pm 1}(x) = \mp\frac{1}{\sqrt{2}}(I^1(x) \pm iI^2(x)); \qquad \hat{I}^1_0(x) = I^3(x); \tag{9.71}$$

$$\hat{I}^0_0(x) = I^0(x). \tag{9.72}$$

Note that Eq. (9.70) is simply a definition, and does not imply that $\hat{I}^s_{\mu_s}$ has the rotational properties of a rank-s tensor.

A multipole expansion which is similar in appearance to the expansion presented here forms the basis for an extensive review of elastic and inelastic electron scattering from nuclei by deForest and Walecka (De 66), which has often been used as a notational standard. Their treatment of nuclear matrix elements is nonrelativistic, and recoil of the initial and final target states has been ignored. Because of the latter assumption, it is difficult to make a direct term-by-term comparison with their series. Our expansion (9.59) of the matrix element in the rest frame of the final state has no explicit normalization factors or Wigner rotations: this serves to define the Lorentz invariant form factors, with relativistic kinematics becoming explicit for non-zero final-state momenta. This would be the natural place to make a comparison. However, because the deForest-Walecka approach neglects recoil, the orbital portion has different content in the two approaches. Thus, a direct comparison with the deForest-Walecka multipole matrix elements can only be approximate at best.

Symmetries and Constraints The reduced matrix elements $\langle m'j'\|\bar{I}_{l[\frac{1}{2}\frac{1}{2}]s\mathcal{J}}(q^2)\|mj\rangle$ are Lorentz invariant and contain all the dynamical information relevant to any electromagnetic transition matrix elements between two states with a given mass and spin. However, these reduced matrix elements are not all independent: beyond the requirement of Poincaré covariance of the matrix elements, there are additional symmetries which further constrain them.

The constraint of current continuity can be written as

$$q^0{}_c\langle m'j';\mathbf{p}'\mu'|I^0(0)|mj;\mathbf{p}\mu\rangle_c - \mathbf{q}\cdot{}_c\langle m'j';\mathbf{p}'\mu'|\mathbf{I}(0)|mj;\mathbf{p}\mu\rangle_c = 0. \tag{9.73}$$

This equation, together with Eq. (9.72), means that reduced matrix elements of $\hat{I}^0_0(0)$ can always be re-expressed in terms of matrix elements of $\hat{I}^1_\mu(0)$, or equivalently, matrix elements of the three-vector current $\mathbf{I}(0)$.

Under spatial inversion, the three-vector current transforms as follows:

$$P\mathbf{I}(0)P^{-1} = -\mathbf{I}(0). \tag{9.74}$$

The effect on a state vector is

$$P|mj;\mathbf{p}\mu\rangle_c = \Pi|mj;-\mathbf{p}\mu\rangle_c, \tag{9.75}$$

where Π is the intrinsic parity of the state. Combining this property with Eqs. (9.59), (9.71) and (9.72), we obtain the following constraint:

$$\Pi'\Pi(-1)^{l+s} = +1 \tag{9.76}$$

for non-vanishing $\bar{I}_{l[\frac{1}{2}\frac{1}{2}]s\mathcal{J}}(q^2)$. Equation (9.76) serves as a selection rule to eliminate certain combinations of l and s.

For the case of elastic scattering, time-reversal invariance may further reduce the number of independent matrix elements. The current operator transforms as follows:

$$TI^0(0)T^{-1} = I^0(0); \quad T\mathbf{I}(0)T^{-1} = -\mathbf{I}(0). \tag{9.77}$$

To obtain a constraint on the reduced matrix elements, we use time-reversal invariance to relate the matrix element in Eq. (9.62) for the *final* state at rest to the matrix element with the *initial* state at rest using Eq. (9.59). Not that there are no Wigner rotations for collinear p and p'. Taking $\mathbf{p}_0$ along the z axis, the constraint can be written as follows:

$$\begin{aligned}\langle m'j'\|\bar{I}_{l[\frac{1}{2}\frac{1}{2}]s\mathcal{J}}(q^2)\|mj\rangle = \sum_{\bar{l}\bar{s}}(-1)^{s+\mathcal{J}}\sqrt{\frac{2\bar{l}+1}{2l+1}} \\ \times \langle \tfrac{1}{2}\alpha\tfrac{1}{2}\dot{\beta}|s\mu_s\rangle\langle \tfrac{1}{2}\zeta\tfrac{1}{2}\dot{\eta}|\bar{s}\bar{\mu}_s\rangle\langle l0s\mu_s|\mathcal{J}\mu_{\mathcal{J}}\rangle\langle \bar{l}0\bar{s}\bar{\mu}_s|\mathcal{J}\mu_{\mathcal{J}}\rangle \\ \times [\underline{L}_c^{-1}(-\mathbf{p}_0/m)]_{\zeta\alpha}[\underline{L}_c(-\mathbf{p}_0/m)]^{\dagger}_{\dot{\eta}\dot{\beta}}\langle m'j'\|\bar{I}_{\bar{l}[\frac{1}{2}\frac{1}{2}]\bar{s}\mathcal{J}}(q^2)\|mj\rangle.\end{aligned} \tag{9.78}$$

In general, the time reversal constraint involves linear relations among the reduced matrix elements. In the limit that the $SL(2,C)$ matrices become unit matrices ($\mathbf{p}_0 \to 0$), the coupling collapses to yield the following constraint:

$$\mathbf{p}_0 \to 0: \qquad (-1)^{s+\mathcal{J}} = +1. \tag{9.79}$$

Front-Form Current Matrix Elements In the instant form, both the matrix elements of time component $I^0(0) = \rho(0)$ (charge density) and of the spatial components $\mathbf{I}(0)$ play a role in

determining the invariant form factors for spacelike momentum transfer. In the front form, all invariants can be determined using a *single* component $I^+(0)$ in an appropriate frame. We will prove this by demonstrating that the matrix elements of all other components can be computed from the matrix elements of $I^+(0)$. The component $I^+(0)$ has the property that it is mapped into itself under front-form boosts, up to a multiplicative factor. This feature implies that with our choice of normalization, the value of this matrix element is independent of all reference frames related by front-form boosts. Specifically,

$$ {}_f\langle m'j';\tilde{\mathbf{p}}'\mu'|I^+(0)|mj;\tilde{\mathbf{p}}\mu\rangle_f \tag{9.80}$$

is the same in all frames related by front-form boosts for each fixed set of spin indices.

For spacelike momentum transfer, it always possible to find a Breit frame in which $q^+ = q^- = 0$. In this frame, the matrix elements of $I^+(0)$ have the form

$$ {}_f\langle m'j';\tilde{\mathbf{p}}'\mu'|I^+(0)|mj;\tilde{\mathbf{p}}\mu\rangle_f,$$

where

$$\begin{aligned}
\mathbf{p}_\perp &= -\tfrac{1}{2}\mathbf{q}_\perp - \boldsymbol{\Delta}_\perp;\\
\mathbf{p}'_\perp &= \tfrac{1}{2}\mathbf{q}_\perp - \boldsymbol{\Delta}_\perp;\\
p^+ &= \sqrt{m^2 + (-\tfrac{1}{2}\mathbf{q}_\perp - \boldsymbol{\Delta}_\perp)^2};\\
p'^+ &= \sqrt{m'^2 + (\tfrac{1}{2}\mathbf{q}_\perp - \boldsymbol{\Delta}_\perp)^2} = p^+;\\
|\boldsymbol{\Delta}_\perp| &= \frac{|m'^2 - m^2|}{2Q}; \quad \boldsymbol{\Delta}_\perp \| \mathbf{q}_\perp; \quad Q^2 := \mathbf{q}_\perp^2.
\end{aligned} \tag{9.81}$$

We can also assume without loss of generality that $\mathbf{q}_\perp$ lies along the x axis. Any other orientation of $\mathbf{q}_\perp$ can be transformed to this via a kinematic front-form transformation which does not affect the form given above. Matrix elements of $I^1(0)$ are determined by the requirement of current continuity:

$$ {}_f\langle m'j';\tilde{\mathbf{p}}'\mu'|I^1(0)|mj;\tilde{\mathbf{p}}\mu\rangle_f = 0. \tag{9.82}$$

The matrix elements of the components I^- and I^2 can be obtained from I^+ by rotations about the x axis by π and $\frac{1}{2}\pi$, respectively, which affect the spins but not the momenta in the state

vectors:

$$\begin{aligned} &{}_f\langle m'j';\tilde{\mathbf{p}}'\mu'|I^-(0)|mj;\tilde{\mathbf{p}}\mu\rangle_f \\ &\qquad = {}_f\langle m'j';\tfrac{1}{2}\tilde{\mathbf{p}}'\mu'|U^\dagger[R_x(\pi)]I^+(0)U[R_x(\pi)]|mj;\tilde{\mathbf{p}}\mu\rangle_f, \end{aligned} \tag{9.83}$$

and

$$\begin{aligned} &{}_f\langle m'j';\tilde{\mathbf{p}}'\mu'|I^2(0)|mj;\tilde{\mathbf{p}}\mu\rangle_f \\ &\qquad = \tfrac{1}{2}\,{}_f\langle m'j';\tilde{\mathbf{p}}'\mu'|U^\dagger[R_x(\tfrac{1}{2}\pi)][I^+(0)+I^-(0)]U[R_x(\tfrac{1}{2}\pi)]|mj;\tilde{\mathbf{p}}\mu\rangle_f. \end{aligned} \tag{9.84}$$

Thus, it is only necessary to calculate matrix elements of $I^+(0)$ in order to determine all observable invariant form factors for spacelike momentum transfer.

In fact, the matrix elements of $I^+(0)$ are themselves not all independent. If the momentum transfer is directed along the x axis, then the combined transformations of spatial inversion and rotation by π about the y axis leave both $I^+(0)$ and the initial and final momenta unchanged, with the result:

$$\begin{aligned} {}_f\langle m'j';\tilde{\mathbf{p}}'\mu'|I^+(0)|mj;\tilde{\mathbf{p}}\mu\rangle_f = (-1)^{\Pi'\Pi}\sum D^{j'\dagger}_{\mu'\bar{\mu}'}[R_f(R_y(\pi),p')] \\ \times\, {}_f\langle m'j';\tilde{\mathbf{p}}'\bar{\mu}'|I^+(0)|mj;\tilde{\mathbf{p}}\bar{\mu}\rangle_f D^j_{\bar{\mu}\mu}[R_f(R_y(\pi),p)]. \end{aligned} \tag{9.85}$$

Note that the argument of the rotation matrices is not $R_y(\pi)$, but rather the front-form Wigner rotation corresponding to $R_y(\pi)$:

$$R_f(R_y(\pi),p) = L_f^{-1}[R_y(\pi)p]R_y(\pi)L_f(p). \tag{9.86}$$

While rotational symmetry was manifest in the classification of canonical-spin matrix elements above, it is a non-trivial constraint in the front form. One way to see this is to impose rotational symmetry on the charge operator $I^0(0) = \frac{1}{2}[I^+(0)+I^-(0)]$. It must be invariant with respect to arbitrary rotations. For definiteness, we consider a rotation of $\frac{1}{2}\pi$ about the x axis, which leaves the momenta unchanged in the frame we have chosen to use. The constraint is

$$\begin{aligned} &{}_f\langle m'j';\tilde{\mathbf{p}}'\mu'|[I^+(0)+I^-(0)]|mj;\tilde{\mathbf{p}}\mu\rangle_f \\ &\qquad = {}_f\langle m'j';\tilde{\mathbf{p}}'\mu'|U^\dagger[R_x(\tfrac{1}{2}\pi)][I^+(0)+I^-(0)]U[R_x(\tfrac{1}{2}\pi)]|mj;\tilde{\mathbf{p}}\mu\rangle_f. \end{aligned} \tag{9.87}$$

Equation (9.87), together with Eq. (9.83), which relates matrix elements of $I^-(0)$ to those of $I^+(0)$, represents a non-trivial constraint upon the matrix elements of $I^+(0)$. This constraint is

non-trivial because the coefficients of the linear relations between the current matrix elements depend on the mass eigenvalues. This condition, sometimes called an angular condition, cannot be satisfied in models which employ one-body operators only. However, as will be seen below, it is still possible to compute observables with one-body operators by using only a *subset* of matrix elements, the remainder of which are constrained by an angular condition, and contain many-body components by implication.

Finally, for the case of elastic scattering, the constraint imposed by time reversal can be seen by examining its combined effect with rotation by π about the y axis:

$$\begin{aligned} &{}_f\langle m'j';\tilde{\mathbf{p}}'\mu'|I^+(0)|mj;\tilde{\mathbf{p}}\mu\rangle_f \\ &\quad = (-1)^{\mu'-\mu}\sum D^{j\dagger}_{-\mu\bar{\mu}}[R_y(\pi)]\;{}_f\langle mj;\tilde{\mathbf{p}}\bar{\mu}|I^+(0)|m'j';\tilde{\mathbf{p}}'\bar{\mu}'\rangle_f\; D^{j'}_{\bar{\mu}'-\mu'}[R_y(\pi)]. \end{aligned} \tag{9.88}$$

Example: The $\pi \to \rho$ Transition Form Factor To illustrate the symmetry properties of matrix elements of $I^+(0)$, we consider the electromagnetic transition $\pi \to \rho$. A simple model for this transition will be presented below; the symmetry properties are discussed here because they are independent of any model.

Since the transition is not elastic, time-reversal constraints do not apply. To evaluate the constraint due to spatial inversion symmetry, we examine first the matrix elements for *canonical* spin. In that case, pure rotations are the same as their Wigner rotations. The only non-vanishing rotation matrices for $j = 1$ for a rotation by π about the y axis are $D^1_{1-1} = D^1_{-11} = D^1_{00} = 1$. Both the pion and the ρ meson have odd parity. The result is

$${}_c\langle m_\rho 1;\tilde{\mathbf{p}}'1|I^+(0)|m_\pi 0;\tilde{\mathbf{p}}0\rangle_c = {}_c\langle m_\rho 1;\tilde{\mathbf{p}}'-1|I^+(0)|m_\pi 0;\tilde{\mathbf{p}}0\rangle_c. \tag{9.89}$$

To obtain the corresponding constraints for front-form spin matrix elements, simply apply Melosh rotations to Eq. (9.89):

$$\begin{aligned} \sum D^1_{1\mu'}[R_{cf}(\tilde{\mathbf{p}}')]_f\langle m_\rho 1;\tilde{\mathbf{p}}'\mu'|I^+(0)|m_\pi 0;\tilde{\mathbf{p}}0\rangle_f \\ = \sum D^1_{-1\mu'}[R_{cf}(\tilde{\mathbf{p}}')]_f\langle m_\rho 1;\tilde{\mathbf{p}}'\mu'|I^+(0)|m_\pi 0;\tilde{\mathbf{p}}0\rangle_f. \end{aligned} \tag{9.90}$$

An angular condition can be seen by noting that for $\mu' = 0$:

$$ {}_c\langle m_\rho 1; \tilde{\mathbf{p}}' 0 | I^-(0) | m_\pi 0; \tilde{\mathbf{p}} 0 \rangle_c = {}_c\langle m_\rho 1; \tilde{\mathbf{p}}' 0 | I^+(0) | m_\pi 0; \tilde{\mathbf{p}} 0 \rangle_c. \tag{9.91} $$

A single non-vanishing front-form current matrix element can be used to compute all of the multipole reduced matrix elements. Since this is a $0^- \to 1^-$ transition, the only allowed $(ls\mathcal{J})$ combination is (111), corresponding to the $M1$ multipole used in nonrelativistic treatments. To relate the reduced multipole matrix element to those of I^+, we make use of Eqs. (9.59), (9.61) and (9.70) to obtain

$$ {}_f\langle m_\rho 1; \tilde{\mathbf{0}} \mu' | \hat{I}^s_{\mu_s}(0) | m_\pi 0; \tilde{\mathbf{p}}_0 0 \rangle_f = \delta_{s1} \sum \langle 1 \mu_l 1 \mu_s | 1 \mu' \rangle Y^{1\,*}_{\mu_l}(\hat{\mathbf{p}}_0) \langle m_\rho 1 \| \bar{I}_{111}(q^2) \| m_\pi 0 \rangle. \tag{9.92} $$

From the relations

$$ I^0(0) = \tfrac{1}{2}[I^+(0) + I^-(0)]; \qquad I^3(0) = \tfrac{1}{2}[I^+(0) - I^-(0)], \tag{9.93} $$

the explicit components are given as follows:

$$ {}_f\langle m_\rho 1; \tilde{\mathbf{0}} \mu' | \hat{I}^0_0(0) | m_\pi 0; \tilde{\mathbf{p}}_0 0 \rangle_f = 0; \tag{9.94} $$

$$ {}_f\langle m_\rho 1; \tilde{\mathbf{0}} \mu' | \hat{I}^1_0(0) | m_\pi 0; \tilde{\mathbf{p}}_0 0 \rangle_f = \langle 1 \mu' 1 0 | 1 \mu' \rangle Y^{1\,*}_{\mu'}(\hat{\mathbf{p}}_0) \langle m_\rho 1 \| \bar{I}_{111}(q^2) \| m_\pi 0 \rangle. \tag{9.95} $$

For this example, the continuity relation does not provide an extra constraint, but is in fact already satisfied. To see this, let $\hat{\mathbf{p}}_0$ lie along the z axis in the rest frame of the ρ meson. Since the matrix elements of $\hat{I}^0_0(0)$ all vanish via Eq. (9.94), continuity requires that the matrix elements of $I^3(0) = \hat{I}^1_0(0)$ must also vanish. The only non-zero spherical harmonic in Eq. (9.95) occurs for $\mu' = 0$, but the Clebsch-Gordan coefficient which multiplies it is $\langle 1010 | 10 \rangle = 0$.

9.3. **Computation of Composite Form Factors**

In this section, we consider the ingredients which enter a model calculation of the current matrix elements $\langle \mathbf{p}'\mu'|I^\mu(0)|\mathbf{p}\mu\rangle$ for composite systems of directly interacting particles. The treatment of currents runs parallel to the treatment of the direct interactions themselves. The most general form for the current operator $I^\mu(x)$ is

$$I^\mu(x) = \sum_i I_i^\mu(x) + \sum_{i<j} I_{ij}^\mu(x) + \sum_{i<j<k} I_{ijk}^\mu(x) + \cdots . \tag{9.96}$$

This expansion is analogous to the expansion of the strong-interaction Hamiltonian in terms of clusters of particles (Po 80). The first term in Eq. (9.96) corresponds to a one-body current, which acts on each constituent in turn. The second term is the two-body current: by definition it vanishes if particles i and j are widely separated. The current operator $I^\mu(x)$ must obey certain constraints dictated by special relativity and the continuity equation. These constraints taken together imply the existence of many-body components in the current operator.

For this approach to be of practical use, Eq. (9.96) should converge rapidly, so that reliable calculations can be done using one- and two-body current matrix elements only. It is tempting to go one step further and examine the possibility that observables can be calculated using one-body current operators alone. This idea has certainly been used extensively in nonrelativistic nuclear physics. However, as will be seen, the requirements of Poincaré invariance impose sufficient constraints upon electromagnetic current operators that it is impossible to satisfy them all without many-body components.

In this section, we will show how to construct current matrix elements for calculating electromagnetic observables. It turns out that the front form has special advantages for this task.

Basic Requirements of Current Operators The combined requirements of Poincaré invariance and current conservation imply that the current operator cannot be a sum of the currents of the single constituent particles. The need for exchange currents to maintain current conservation was first pointed out by Siegert (Si 37). Siegert assumed that the charge density is a one-body operator, and then used current conservation to determine a many-body correction to the vector part of the current to maintain current covariance. Poincaré covariance implies that the charge

density cannot be a one-body operator. The constraints of Poincaré invariance on current operators has been investigated by Close and Osborn (Cl 70, Cl 71), Coester and Osterbee (Co 75), Gross (Gr 65, Gr 66a, Gr 66b, Gr 69), Friar (Fr 73, Fr 76), Polyzou (Po 85b), Polyzou and Klink (Po 88), and Coester (Co 88), and others. This is well known, and has been reviewed extensively in the contex of exchange currents in (Ch 79, Fr 77, Fr 79).

To begin, we examine the constraints of continuity and covariance on the current operators. Since $I^\mu(0)$ is a four-vector operator, it has the same transformation properties as the four momentum under Lorentz transformations:

$$U(\underline{\Lambda},\underline{a})I^\mu(x)U^\dagger(\underline{\Lambda},\underline{a}) = [\Lambda^{-1}]^\mu{}_\nu I^\nu(\Lambda x + a). \tag{9.97}$$

This implies that

$$I^\mu(x) = e^{-iP\cdot x}I^\mu(0)e^{iP\cdot x}. \tag{9.98}$$

The commutation relations between $I^\mu(0)$ and the Lorentz generators is identical to the commutation relations between the Lorentz generators and the four-momentum:

$$[J^j(0), I^k(0)]_- = i\epsilon^{jkl}I^l(0); \qquad [J^j(0), I^0(0)]_- = 0; \tag{9.99}$$

$$[K^j(0), I^k(0)]_- = -i\delta^{jk}I^0(0); \qquad [K^j(0), I^0(0)]_- = -iI^j(0); \tag{9.100}$$

In addition to these equations, the current must satisfy the continuity equation :

$$[P^\mu, I_\mu(0)]_- = 0. \tag{9.101}$$

Equations (9.99)–(9.101) summarize the constraints on the current operator. To understand the need for exchange currents, we express the current as a sum of one-body and many-body terms: Assume that the one-body part of the current is covariant with respect to the dynamics of the

noninteracting system:

$$I^\mu(0) = I_1^\mu(0) + \delta I_{ex}^\mu(0) \tag{9.102}$$

We assume also that each generator can be expressed as the sum of a non-interacting generator and an interaction. For example,

$$\mathbf{K} = \mathbf{K}_1 + \boldsymbol{\delta}\mathbf{K}, \tag{9.103}$$

and so on for each generator. Inserting these expressions into Eqs. (9.99) and (9.100), we obtain

$$-i\delta^{jk}\delta I^0(0) = [\delta K^j(0), I^k(0)]_- + [K_1^j(0), \delta I^k(0)]_-; \tag{9.104}$$

$$-i\delta I^j(0) = [\delta K^j(0), I^0(0)]_- + [K_1^j(0), \delta I^0(0)]_-. \tag{9.105}$$

If $\boldsymbol{\delta}\mathbf{K} \neq 0$, then the right-hand side of both of these equations will not vanish, and thus the current operator must contain interaction terms. This conclusion holds for instant-form dynamics. In a front-form dynamics, the interactions come from the rotational commutation relations.

All of these equations can be reduced to a set of equivalent constraints on the charge density operator. Using Eqs. (9.99) and (9.100), we find that the four-current is uniquely determined by a knowledge of $I^0(0) := \rho(0)$:

$$I^\mu(0) = \left(\rho, -i[\mathbf{K}, \rho]_-\right). \tag{9.106}$$

This equation is a little misleading; it includes a closure sum and thus requires all current matrix elements. Nevertheless, it is sufficient for our discussion to concentrate on matrix elements of $\rho(0)$.

Equations (9.99)–(9.101) taken together imply the relations

$$\begin{aligned} [H, \rho(0)]_- + i\sum_{j=1}^{3}[P^j, [K^j, \rho(0)]_-]_- &= 0; \\ [J^1(0), \rho(0)]_- = [J^2(0), \rho(0)]_- &= 0; \\ [K^i, [K^i, \rho(0)]_-]_- + \rho(0) &= 0. \end{aligned} \tag{9.107}$$

where there is no sum over repeated indices in (9.107).

If we use the translation property (9.98) of the current in the continuity relation (9.101), we obtain

$$\nabla \cdot \mathbf{I}(x) - i[H, \rho(x)]_- = 0. \tag{9.108}$$

This is the form of the continuity equation often encountered in nonrelatistic nuclear physics (although no nonrelativistic approximation has been made at this point). If the Hamiltonian has charge-changing pieces (as is the case in nuclear physics with pion exchange) and/or is velocity dependent, then $[H, \rho(x)]_-$ has two-body components, in which case $\mathbf{I}(x)$ must also have two-body components. This fact has been known for many years. It has often been dealt with by means of the Siegert hypothesis (Si 37), under which it is assumed that $\rho(x)$ has negligible two-body matrix elements, and that matrix elements of the commutator in Eq. (9.108) can be evaluated explicitly to determine any needed matrix elements of $\mathbf{I}(x)$.

For Poincaré invariant systems, however, the relations (9.107) imply that there must be two-body currents *even if the Hamiltonian is charge neutral and velocity independent.* This is a consequence of the fact that the current is a four-vector, which must transform properly under boosts and rotations, some of which are interaction dependent. Nevertheless, as will be shown below, there remain circumstances under which one can calculate matrix elements of certain components of the current at least approximately, and still maintain a level of consistency with current continuity and Poincaré invariance.

The construction of a suitable current operator is equivalent to finding a charge density operator which satisfies simultaneously the three conditions (9.107). It is possible to show the these condition are compatible (Po 85b). They have been manipulated in a variety of ways to determine leading order relativistic corrections to the current. Many of these attempts involve expansions in unbounded operators, which are difficult to justify. Moreover, a solution to these conditions, however difficult it may be to obtain in practice, is not unique. Even if one also includes the additional constraints of macroscopic locality, it is possible to show with an explicit counterexample that a solution is not unique.

Another way to understand the relationship between the strong-interaction Hamiltonian and the current operator is to examine the constraints imposed upon the matrix elements themselves.

For canonical spin, the initial hadronic state has the following transformation property:

$$U(\underline{\Lambda},\underline{a})|\mathbf{p}\,\mu\rangle = e^{i\Lambda p\cdot a}\sqrt{\frac{\omega_m(\mathbf{p}_{\Lambda i})}{\omega_m(\mathbf{p})}}\sum|\mathbf{p}_\Lambda\,\bar{\mu}\rangle D^s_{\bar{\mu}\mu}[R_c(\underline{\Lambda},p)], \tag{9.109}$$

and the multi-particle final state transforms as follows:

$$\begin{aligned} &U(\underline{\Lambda},\underline{a})|\mathbf{p}'_1\,\mu'_1\,;\cdots;\mathbf{p}'_N\,\mu'_N{}^{(+)}\rangle \\ &\quad= e^{i\sum_i \Lambda p'_i\cdot a}\sum|\mathbf{p}_{\Lambda f_1}\,\bar{\mu}'_1\,;\cdots;\mathbf{p}_{\Lambda f_N}\,\bar{\mu}'_N{}^{(+)}\rangle \\ &\quad\times\prod_{i=1}^N\left[\sqrt{\frac{\omega_{m'_i}(\mathbf{p}'_{\Lambda i})}{\omega_{m'_i}(\mathbf{p}'_i)}}D^{s'_i}_{\bar{\mu}'_i\mu'_i}[R_g(\underline{\Lambda},p'_i)]\right], \end{aligned} \tag{9.110}$$

for the final state. In Eq. (9.110), the interacting final state undergoes the same transformation as a free-particle state. Formally, this is expressed as follows:

$$U(\underline{\Lambda},\underline{a})\Omega_+ = \Omega_+\bigotimes_{i=1}^N U_{0i}(\underline{\Lambda},\underline{a}) \tag{9.111}$$

where $U_{0i}(\underline{\Lambda},\underline{a})$ is the free-particle transformation for the ith particle, and

$$|\mathbf{p}'_1\,\mu'_1\,;\cdots;\mathbf{p}'_N\,\mu'_N{}^{(+)}\rangle = \Omega_+|\mathbf{p}'_1\,\mu'_1;\cdots;\mathbf{p}'_N\,\mu'_N\rangle. \tag{9.112}$$

Equation (9.111) is a consequence of macroscopic locality (Co 82). The covariance condition (9.4) for current matrix elements becomes:

$$\begin{aligned} &\langle\mathbf{p}'_1\,\mu'_1\,;\cdots;\mathbf{p}'_N\,\mu'_N{}^{(+)}|I^\mu(x)|\mathbf{p}\,\mu\rangle \\ &\quad= e^{i\Lambda(p-\sum_i p'_i)\cdot a}(\Lambda^{-1})^\mu{}_\nu\langle\mathbf{p}_{\Lambda f_1},\bar{\mu}'_1,;\cdots;\mathbf{p}_{\Lambda f_N}\,\bar{\mu}'_N{}^{(+)}|I^\nu(\Lambda x+a)|\mathbf{p}_{\Lambda i},\bar{\mu}\rangle \\ &\quad\times\prod_{i=1}^N\left[\sqrt{\frac{\omega_{m'_i}(\mathbf{p}_{\Lambda i})}{\omega_{m'_i}(\mathbf{p}_i)}}D^{*s'_i}_{\bar{\mu}'_i\mu'_i}[R_g(\underline{\Lambda},p'_i)]\right]\sqrt{\frac{\omega_m(\mathbf{p}_\Lambda)}{\omega_m(\mathbf{p})}}D^s_{\mu\bar{\mu}}[R_g(\underline{\Lambda},p)]. \end{aligned} \tag{9.113}$$

Current conservation gives an additional constraint:

$$g_{\mu\nu}(\sum_{i=1}^N p'^\mu_i - p^\mu)\langle\mathbf{p}_1\,\mu'_1\,;\cdots;\mathbf{p}'_N\,\mu'_N{}^{(+)}|I^\nu(x)|\mathbf{p}_i\,\mu\rangle = 0. \tag{9.114}$$

While these equations look complicated, they are linear relations. Nevertheless, they represent a non-trivial set of constraints. The potential difficulty is that the square-root factors and the

Wigner rotations depend upon the mass eigenvalue, which is interaction dependent. As we have already seen, only a subset of current matrix elements is truly independent. All others are related to a particular subset via the continuity and covariance constraints just given. The problem is that there are many choices of independent sets of matrix elements, each of which will predict a complete set of observables for that process. In general, two such sets will predict identical sets of observables only if the current matrix elements satisfy all of the constraints. A related conclusion is that the exchange current terms, although always required, are not uniquely fixed by current conservation or current covariance.

These observations apply to manifestly covariant theories in which the constituents are assigned form factors (*i.e.,* non-pointlike vertex functions). The set of form factors must satisfy a non-trivial set of constraints in order to ensure the overall continuity of the current, but these constraints do not in turn have a unique solution (Gr 87). These observations apply to other forms of dynamics as well. In the front form, the non-trivial covariance constraints appear in the properties of matrix elements under rotations about an axis in the $\perp$ plane. Nevertheless, as will be shown below, there remain circumstances under which one can calculate matrix elements of certain components of the current at least approximately, and still maintain a level of consistency with current continuity and Poincaré invariance.

Similar arguments can be applied to a Dirac spinor field operator, or any other covariant field operator. The transformation properties of any such operator is not fully fixed until the dynamics is defined, and it will in general have many-body or interaction-dependent contributions.

Impulse Approximation As stated above, a common first step in describing currents in composite systems is to describe I^μ as a sum of one-body current operators, *i.e.,* to take only the first term in Eq. (9.96). While we have already seen that this contribution alone cannot satisfy all of the combined requirements of continuity and covariance, it is still possible to make a consistent first approximation using one-body currents only, *but only in the front form.*

Consider the case of electron scattering from nuclei. In an impulse approximation, the relevant nuclear matrix element is

$$\langle p'|\sum_i I_i^\mu(0)|p\rangle \approx \sum_i \int d^3p_i' \int d^3p_i \, \langle p'|p_i'\rangle\langle p_i'|I_i^\mu(0)|p_i\rangle\langle p_i|p\rangle. \tag{9.115}$$

For simplicity of illustration, spin indices have been suppressed, and the momentum integrals have been written generically to represent any form of dynamics. The amplitudes of the form $\langle p_i|p\rangle$ describe the wave function and the current matrix elements are taken between *physical* one-nucleon states, which are in turn connected to the physically observable nucleon form factors. The calculation is then a matter of integrating wave functions over form factors, just as in the corresponding nonrelativistic treatment.

The problem is that Eq. (9.115) is not internally consistent in the instant form. This can be seen as follows. The one-nucleon current information can be summarized as a set of Breit-frame matrix elements $\langle \frac{1}{2}\mathbf{q}|I^\mu(0)|-\frac{1}{2}\mathbf{q}\rangle$. In the integral in Eq. (9.115), the necessary matrix elements $\langle \mathbf{p}'|I_i^\mu(0)|\mathbf{p}\rangle$ are related to the Breit-frame matrix elements by a Lorentz transformation. In the instant form, Lorentz transformations are interaction dependent, which implies that a transformed one-body current matrix element will acquire two-body components. One could try to avoid this difficulty by using the free-particle matrix elements for *all* values of $\mathbf{p}'$ and $\mathbf{p}$, but then the set of matrix elements corresponding to a particular value of q^2 will not transform properly into each other with the interaction-dependent Lorentz transformation. Thus, *one-body currents in the instant form cannot be formulated consistently with respect to Lorentz covariance.*

In the front form, this consistency problem can be avoided. As we have seen, for spacelike momentum transfer, it is possible to find frames in which all invariants can computed from matrix elements of $I^+(0)$ alone. These matrix elements are independent of the choice of frame for a large class of frames related by the subgroup of front-form boosts. Because the front-form boosts form a subgroup, the spins remain invariant under front-form boosts. As we will show, if the light front is oriented so that $q^+ = 0$, then the integral over the Fermi motion only picks up contributions from subsystem matrix elements related by these kinematic front-form boosts. The result is that an impulse approximation can be made consistently in any frame with $q^+ = 0$. This can be demonstrated explicitly in a simple model. Consider a matrix element of $I^+(0)$ in a many-body system:

$$\begin{aligned}\langle \tilde{\mathbf{p}}'|I^+(0)|\tilde{\mathbf{p}}\rangle = &\int d\tilde{\mathbf{p}}'_1 \cdots \int d\tilde{\mathbf{p}}'_n \int d\tilde{\mathbf{p}}_1 \cdots \int d\tilde{\mathbf{p}}_n \langle \tilde{\mathbf{p}}'|\tilde{\mathbf{p}}'_1 \cdots \tilde{\mathbf{p}}'_n\rangle \\ &\times \langle \tilde{\mathbf{p}}'_1 \cdots \tilde{\mathbf{p}}'_n|I^+(0)|\tilde{\mathbf{p}}_1 \cdots \tilde{\mathbf{p}}_n\rangle \langle \tilde{\mathbf{p}}_1 \cdots \tilde{\mathbf{p}}_n|\tilde{\mathbf{p}}\rangle.\end{aligned} \tag{9.116}$$

The spin labels have been suppressed, since they do not affect the basic conclusion, as will be

discussed below. The initial and final state vectors have the form

$$\langle \tilde{\mathbf{p}}_1 \cdots \tilde{\mathbf{p}}_n | \tilde{\mathbf{p}} \rangle = \delta(\tilde{\mathbf{p}} - \sum_{i=1}^{n} \tilde{\mathbf{p}}_i) \psi(\tilde{\mathbf{p}}_1 \cdots \tilde{\mathbf{p}}_n), \tag{9.117}$$

where ψ is an n-particle wave function which may differ between initial and final states. The impulse approximation can be written as follows:

$$\langle \tilde{\mathbf{p}}'_1 \cdots \tilde{\mathbf{p}}'_n | I_i^+(0) | \tilde{\mathbf{p}}_1 \cdots \tilde{\mathbf{p}}_n \rangle \approx \sum_{i=1}^{n} \prod_{j \neq i} [\delta(\tilde{\mathbf{p}}'_j - \tilde{\mathbf{p}}_j)] \langle \tilde{\mathbf{p}}'_i | I_i^+(0) | \tilde{\mathbf{p}}_i \rangle. \tag{9.118}$$

The integrals over each of the $\tilde{\mathbf{p}}'_i$ contain a delta function and are trivially eliminated. The net effect is that each struck particle carries all of the momentum transfer:

$$\begin{aligned} \langle \tilde{\mathbf{p}}' | I^+(0) | \tilde{\mathbf{p}} \rangle \approx \int d\tilde{\mathbf{p}}_1 \cdots \int d\tilde{\mathbf{p}}_n \sum_{i=1}^{n} \psi^*(\tilde{\mathbf{p}}_1 \cdots (\tilde{\mathbf{p}}_i + \tilde{\mathbf{q}}) \cdots \tilde{\mathbf{p}}_n) \\ \times \langle \tilde{\mathbf{p}}_i + \tilde{\mathbf{q}} | I_i^+(0) | \tilde{\mathbf{p}}_i \rangle \psi(\tilde{\mathbf{p}}_1 \cdots \tilde{\mathbf{p}}_i \cdots \tilde{\mathbf{p}}_n), \end{aligned} \tag{9.119}$$

where $\tilde{\mathbf{q}} = \tilde{\mathbf{p}}' - \tilde{\mathbf{p}}$. In general, the matrix elements $\langle \tilde{\mathbf{p}}_i + \tilde{\mathbf{q}} | I_i^+(0) | \tilde{\mathbf{p}}_i \rangle$ appearing inside the integral depend upon the internal momenta of the target. However, for a specific choice of coordinates, this matrix element can be factored out of the integral. To see this, consider a front-form boost for which $\tilde{\mathbf{p}}_i \to \tilde{\mathbf{0}}$. The final momentum transforms as follows:

$$\begin{aligned} (p_i^+ + q^+) &\to \frac{m}{p_i^+}(p_i^+ + q^+); \\ (\mathbf{p}_{i\perp} + \mathbf{q}_\perp) &\to (\mathbf{p}_{i\perp} + \mathbf{q}_\perp) - \frac{p_i^+ + q^+}{p_i^+} \mathbf{p}_{i\perp}. \end{aligned} \tag{9.120}$$

In general, the right-hand side of Eq. (9.120) depends upon the Fermi momentum $\tilde{\mathbf{p}}_i$. However, *for the special case* $q^+ = 0$, we have

$$\begin{aligned} (p_i^+ + q^+) &\to m; \\ (\mathbf{p}_{i\perp} + \mathbf{q}_\perp) &\to \mathbf{q}_\perp. \end{aligned} \tag{9.121}$$

The combined effects of the transformation of the operator $I_i^+(0)$ and the normalization changes in the state vectors cancel exactly, with the result:

$$\langle \tilde{\mathbf{p}}_i + \tilde{\mathbf{q}}_i | I_i^+(0) | \tilde{\mathbf{p}}_i \rangle = \langle m\mathbf{q}_\perp | I_i^+(0) | m\mathbf{0}_\perp \rangle \qquad (q^+ = 0). \tag{9.122}$$

The current matrix element thus factors out of the Fermi motion integral. This means that for each momentum transfer q, the impulse approximation depends upon a *single* one-body current

matrix element $\langle m\mathbf{q}_\perp|I_i^+(0)|m\mathbf{0}_\perp\rangle$, or, via a different front-form boost, the Breit-frame matrix element

$$\langle p^+\tfrac{1}{2}\mathbf{q}_\perp|I_i^+(0)|p^+-\tfrac{1}{2}\mathbf{q}_\perp\rangle, \qquad p^+=\sqrt{m^2+\frac{1}{4}\mathbf{q}_\perp^2}.$$

If $q^+\neq 0$, the one-body matrix elements do not factor out of the integral, and are related to each other by transformations which include transverse rotations, which are interaction dependent. Nevertheless, as shown earlier, it is always possible for spacelike momentum transfers to find a frame in which the condition $q^+=0$ is satisfied.

In this example, the spin indices have been suppressed. Had they been included, there would in general be Wigner rotations associated with the transformation of the one-body current matrix element, and the argument of the rotation matrices would depend upon the Fermi momentum. However, *for front-form boosts,* there are no Wigner rotations, which means that spin-dependent one-body matrix elements will also factor out of the integral over Fermi motion.

This result marks a major departure from a corresponding calculation in the instant form. First, the factorization of the matrix element does not occur inside the integral. Second, while the momentum transfer to the constituent particle in the one-body matrix element is the same, it must also be related to a Breit-frame matrix element by a Lorentz transformation. However, the transformations are interaction-dependent, which implies that the set of one-body matrix elements inside the integral cannot be related to each other in a consistent manner without introducing two-body current operators. This means that the impulse approximation depends non-trivially on the choice of frame in which one formulates it.

Example: The $\pi\to\rho$ Transition Form Factor We conclude this section with an illustrative example. Consider a constituent quark model of the sort discussed in the introductory sections of this review. We assume that a model of the quark-antiquark system is available with satisfactory fits to the spectra which include π and ρ mesons. We now obtain an expression for the one-body current matrix element for the $\pi\to\rho$ transition. As shown above, we need only compute matrix elements of $I^+(0)$. While the angular condition in Eq. (9.91) can only be satisfied through the use of two-body currents, there is still only one form factor. We therefore *assume* that matrix elements of $I^+(0)$ have only one-body components, and use them (in fact, we only need one non-

vanishing matrix element, as shown in Eq. (9.90)) to calculate the observable. This approach implies that $I^-(0)$ has non-vanishing two-body components, but they need never be calculated.

The starting point is

$$\begin{aligned}
&\langle m_\rho 1;\tilde{\mathbf{p}}'\mu'|I^+(0)|m_\pi 0;\tilde{\mathbf{p}}0\rangle \\
&\quad= \int d^2p_{1\perp}dp_1^+ \int d^2p_{2\perp}dp_2^+ \int d^2p'_{1\perp}dp'^+_1 \int d^2p'_{2\perp}dp'^+_2 \\
&\qquad\times \langle m_\rho 1;\tilde{\mathbf{p}}'\mu'|\tilde{\mathbf{p}}'_1\mu'_1\,\tilde{\mathbf{p}}'_2\mu'_2\rangle\langle\tilde{\mathbf{p}}'_1\mu'_1\,\tilde{\mathbf{p}}'_2\mu'_2|I^+(0)|\tilde{\mathbf{p}}_1\mu_1\,\tilde{\mathbf{p}}_2\mu_2\rangle \\
&\qquad\times \langle\tilde{\mathbf{p}}_1\mu_1\,\tilde{\mathbf{p}}_2\mu_2|m_\pi 0;\tilde{\mathbf{p}}\mu\rangle.
\end{aligned} \tag{9.123}$$

Only one-body matrix elements of the current are kept:

$$\begin{aligned}
\langle\tilde{\mathbf{p}}'_1\mu'_1\,\tilde{\mathbf{p}}'_2\mu'_2|I^+(0)|\tilde{\mathbf{p}}_1\mu_1\,\tilde{\mathbf{p}}_2\mu_2\rangle \approx{}& \delta_{\mu'_2\mu_2}\delta(\tilde{\mathbf{p}}'_2-\tilde{\mathbf{p}}_2)\langle\tilde{\mathbf{p}}'_1\mu'_1|I_1^+(0)|\tilde{\mathbf{p}}_1\mu_1\rangle \\
&+\delta_{\mu'_1\mu_1}\delta(\tilde{\mathbf{p}}'_1-\tilde{\mathbf{p}}_1)\langle\tilde{\mathbf{p}}'_2\mu'_2|I_2^+(0)|\tilde{\mathbf{p}}_2\mu_2\rangle.
\end{aligned} \tag{9.124}$$

We now make use of the following state vectors:

$$|\tilde{\mathbf{P}}\mathbf{k}\rangle := \left|\frac{\partial(\tilde{\mathbf{p}}_1\tilde{\mathbf{p}}_2)}{\partial(\tilde{\mathbf{P}}\mathbf{k})}\right|^{\frac{1}{2}} |\tilde{\mathbf{p}}_1\mu_1\,\tilde{\mathbf{p}}_2\mu_2\rangle, \tag{9.125}$$

The bound-state wave function for the pion is

$$\begin{aligned}
\langle\tilde{\mathbf{P}}\mathbf{k}\mu_1\mu_2|m_\pi 0;\tilde{\mathbf{p}}0\rangle ={}&\delta(\tilde{\mathbf{P}}-\tilde{\mathbf{p}})D^{(\frac{1}{2})\dagger}_{\mu_1\bar{\mu}_1}[R_{cf}(k_1)]\,D^{(\frac{1}{2})\dagger}_{\mu_2\bar{\mu}_2}[R_{cf}(k_2)] \\
&\times\langle\tfrac{1}{2}\bar{\mu}_1\tfrac{1}{2}\bar{\mu}_2|1\mu_S\rangle\langle l\mu_l 1\mu_S|00\rangle Y^l\mu_l(\hat{\mathbf{k}})w^l_\pi(k),
\end{aligned} \tag{9.126}$$

and the wave function for the ρ meson is

$$\begin{aligned}
\langle\tilde{\mathbf{P}}'\mathbf{k}'\mu'_1\mu'_2|m_\rho 1;\tilde{\mathbf{p}}'\mu'\rangle ={}&\delta(\tilde{\mathbf{P}}'-\tilde{\mathbf{p}}')D^{(\frac{1}{2})\dagger}_{\mu'_1\bar{\mu}'_1}[R_{cf}(k'_1)]\,D^{(\frac{1}{2})\dagger}_{\mu'_2\bar{\mu}'_2}[R_{cf}(k'_2)] \\
&\times\langle\tfrac{1}{2}\bar{\mu}'_1\tfrac{1}{2}\bar{\mu}'_2|1\mu'_S\rangle\langle l'\mu'_l 1\mu'_S|1\mu'\rangle Y^{l'}\mu'_l(\hat{\mathbf{k}}')w^{l'}_\rho(k')
\end{aligned} \tag{9.127}$$

Taking into account the factorization of the one-body current matrix element, as discussed above,

we get

$$
\begin{aligned}
&\langle m_\rho 1;\tilde{\mathbf{p}}'\mu'|I^+(0)|m_\pi 0;\tilde{\mathbf{p}}0\rangle \\
&\quad\approx \frac{e_1}{e}\langle p^+\ \tfrac{1}{2}\mathbf{q}_\perp\mu_1'|I^+(0)|p^+\ -\tfrac{1}{2}\mathbf{q}_\perp\mu_1\rangle\int\frac{d^2p_{2\perp}dp_2^+}{(2\pi)^3}\left|\frac{\partial(\tilde{\mathbf{p}}_1'\tilde{\mathbf{p}}_2)}{\partial(\tilde{\mathbf{p}}'\mathbf{k}')}\right|^{\frac{1}{2}}\left|\frac{\partial(\tilde{\mathbf{p}}_1\tilde{\mathbf{p}}_2)}{\partial(\tilde{\mathbf{p}}\mathbf{k})}\right|^{\frac{1}{2}} \\
&\qquad\times D^{(\frac{1}{2})}_{\bar{\mu}_1\mu_1}\left[R_{cf}(k_1')\right]D^{(\frac{1}{2})}_{\bar{\mu}_2\mu_2}\left[R_{cf}(k_2')\right] \\
&\qquad\times\langle\tfrac{1}{2}\bar{\mu}_1'\tfrac{1}{2}\bar{\mu}_2'|1\mu_S'\rangle\langle l\mu_l'1\mu_S'|1\mu'\rangle Y^{l'}\mu_l'(\hat{\mathbf{k}}')w_\rho^l(k') \qquad (9.128)\\
&\qquad\times D^{(\frac{1}{2})\dagger}_{\mu_1\bar{\mu}_1}\left[R_{cf}(k_1)\right]D^{(\frac{1}{2})\dagger}_{\mu_2\bar{\mu}_2}\left[R_{cf}(k_2)\right] \\
&\qquad\times\langle\tfrac{1}{2}\bar{\mu}_1\tfrac{1}{2}\bar{\mu}_2|1\mu_S\rangle\langle l\mu_l1\mu_S|00\rangle Y^l\mu_l(\hat{\mathbf{k}})w_\pi^l(k) \\
&\qquad\quad+[1\leftrightarrow 2];\qquad (p^+=\sqrt{m^2+\frac{1}{4}\mathbf{q}_\perp^2}).
\end{aligned}
$$

10. **Relation to Covariant Theories**

The purpose of this section is to exhibit the relationship between relativistic particle dynamics formulated on a Hilbert space of square integrable functions, manifestly covariant quantum mechanical models, and covariant quantum field theory. For the case of a free particle, all three formulations are shown to be equivalent. Interactions can be considered as perturbations of non-interacting models. The problem of adding interactions in each of these cases is considered briefly.

To exhibit the relation between these three descriptions of a free particle, we construct transformations that relate state vectors in each of these representations, while preserving all transition probabilities. We do this for the case of a spin-$\frac{1}{2}$ Dirac particle.

We begin with a free particle of mass m and spin $\frac{1}{2}$, corresponding to the general construction in Section 4. The Hilbert space for this particle is an irreducible representation space of $\mathcal{P}$ corresponding to mass m and spin $\frac{1}{2}$, namely, the space of square integrable functions of three components of the linear momentum and a magnetic quantum number associated with spin $\frac{1}{2}$. The scalar product is

$$\langle f|g\rangle := \sum_{\mu=-\frac{1}{2}}^{\frac{1}{2}} \int d^3p \, \langle \mathbf{p}\,\mu|f\rangle^* \langle \mathbf{p}\,\mu|g\rangle, \tag{10.1}$$

where

$$\langle f|f\rangle < \infty. \tag{10.2}$$

These states transform irreducibly under $\mathcal{P}$:

$$U(\underline{\Lambda},\underline{a})|\mathbf{p}\,\mu\rangle = e^{i\Lambda p\cdot a}\sqrt{\frac{\omega_m(\mathbf{p}_\Lambda)}{\omega_m(\mathbf{p})}}\sum |\mathbf{p}_\Lambda\,\bar{\mu}\rangle D^{\frac{1}{2}}_{\bar{\mu}\mu}[\underline{R}_c(\underline{\Lambda},p)]. \tag{10.3}$$

To construct a field satisfying canonical equal-time (anti-) commutation relations, we must also introduce the corresponding antiparticle. Since the mass and spin fix the transformation properties of an irreducible representation of the Poincaré group, and the mass and spin of the antiparticle are the same as the mass and spin of the particle, we can assume that antiparticle states have

the same transformation properties under $U(\underline{\Lambda}, \underline{a})$ as particle states. We use the notation $|\overline{\mathbf{p}\,\mu}\rangle$ to denote a pure state of the antiparticle with momentum $\mathbf{p}$ and magnetic quantum number μ. The transformation properties of the antiparticle states are

$$U(\underline{\Lambda}, \underline{a})|\overline{\mathbf{p}\,\mu}\rangle = e^{i\Lambda p\cdot a}\sqrt{\frac{\omega_m(\mathbf{p}_\Lambda)}{\omega_m(\mathbf{p})}}\sum |\overline{\mathbf{p}_\Lambda\,\bar{\mu}}\rangle D^{\frac{1}{2}}_{\bar{\mu}\mu}[\underline{R}_c(\underline{\Lambda}, p)]. \tag{10.4}$$

To construct a field of spin $\frac{1}{2}$ that transforms covariantly under $U(\underline{\Lambda}, \underline{a})$, we introduce a Poincaré invariant vacuum $|0\rangle$, satisfying

$$U(\underline{\Lambda}, \underline{a})|0\rangle = |0\rangle \tag{10.5}$$

and

$$\langle 0|0\rangle = 1, \tag{10.6}$$

together with abstract creation operators $a^\dagger(\mathbf{p}, \mu)$ for particle states, and $b^\dagger(\mathbf{p}, \mu)$ for antiparticle states. These operators create the particle and antiparticle states from the vacuum:

$$a^\dagger(\mathbf{p}, \mu)|0\rangle = |\mathbf{p}\,\mu\rangle; \tag{10.7}$$

$$b^\dagger(\mathbf{p}, \mu)|0\rangle = |\overline{\mathbf{p}\,\mu}\rangle. \tag{10.8}$$

The creation operators and their adjoints are assumed to satisfy canonical anticommutation relations:

$$\{a(\mathbf{p}', \mu'), a^\dagger(\mathbf{p}, \mu)\} = \delta_{\mu'\mu}\delta(\mathbf{p}' - \mathbf{p}). \tag{10.9}$$

This fixes the statistics of multiparticle states and the normalization of the one-body states.

The vacuum is annihilated by the adjoints of the creation operators:

$$a(\mathbf{p}, \mu)|0\rangle = 0 \qquad b(\mathbf{p}, \mu)|0\rangle = 0. \tag{10.10}$$

The transformation properties of the creation operators are fixed by Eqs. (10.3), (10.4), (10.5), (10.7) and (10.8), and the property that $U(\underline{\Lambda}, \underline{a})$ acts irreducibly on the one-particle and

one-antiparticle subspaces. These properties are given as follows:

$$U(\underline{\Lambda},\underline{a})a^\dagger(\mathbf{p}\,\mu)U^\dagger(\underline{\Lambda},\underline{a}) = e^{i\Lambda p\cdot a}\sqrt{\frac{\omega_m(\mathbf{p}_\Lambda)}{\omega_m(\mathbf{p})}}\sum a^\dagger(\mathbf{p}_\Lambda\,\bar{\mu})D^{\frac{1}{2}}_{\bar{\mu}\mu}[\underline{R}_c(\underline{\Lambda},p)]; \tag{10.11}$$

$$U(\underline{\Lambda},\underline{a})b^\dagger(\mathbf{p}\,\mu)U^\dagger(\underline{\Lambda},\underline{a}) = e^{i\Lambda p\cdot a}\sqrt{\frac{\omega_m(\mathbf{p}_\Lambda)}{\omega_m(\mathbf{p})}}\sum b^\dagger(\mathbf{p}_\Lambda\,\bar{\mu})D^{\frac{1}{2}}_{\bar{\mu}\mu}[\underline{R}_c(\underline{\Lambda},p)]. \tag{10.12}$$

The transformation properties of $a(\mathbf{p},\mu)$ and $b(\mathbf{p},\mu)$ follow by taking adjoints of Eqs. (10.11) and (10.12):

$$U(\underline{\Lambda},\underline{a})a(\mathbf{p}\,\mu)U^\dagger(\underline{\Lambda},\underline{a}) = e^{-i\Lambda p\cdot a}\sqrt{\frac{\omega_m(\mathbf{p}_\Lambda)}{\omega_m(\mathbf{p})}}\sum a(\mathbf{p}_\Lambda\,\bar{\mu})D^{\frac{1}{2}*}_{\bar{\mu}\mu}[\underline{R}_c(\underline{\Lambda},p)] \tag{10.13}$$

$$U(\underline{\Lambda},\underline{a})b(\mathbf{p}\,\mu)U^\dagger(\underline{\Lambda},\underline{a}) = e^{-i\Lambda p\cdot a}\sqrt{\frac{\omega_m(\mathbf{p}_\Lambda)}{\omega_m(\mathbf{p})}}\sum b(\mathbf{p}_\Lambda\,\bar{\mu})D^{\frac{1}{2}*}_{\bar{\mu}\mu}[\underline{R}_c(\underline{\Lambda},p)]. \tag{10.14}$$

It is useful to rewrite Eqs. (10.13) and (10.14) using

$$D^{\frac{1}{2}*}_{\bar{\mu}\mu}[\underline{R}_c(\underline{\Lambda},p)] = D^{\frac{1}{2}}_{\mu\bar{\mu}}[\underline{R}_c^{-1}(\underline{\Lambda},p)], \tag{10.15}$$

which gives

$$U(\underline{\Lambda},\underline{a})a(\mathbf{p}\,\mu)U^\dagger(\underline{\Lambda},\underline{a}) = e^{-i\Lambda p\cdot a}\sqrt{\frac{\omega_m(\mathbf{p}_\Lambda)}{\omega_m(\mathbf{p})}}\sum D^{\frac{1}{2}}_{\mu\bar{\mu}}[\underline{R}_c^{-1}(\underline{\Lambda},p)]a(\mathbf{p}_\Lambda\,\bar{\mu}); \tag{10.16}$$

$$U(\underline{\Lambda},\underline{a})b(\mathbf{p}\,\mu)U^\dagger(\underline{\Lambda},\underline{a}) = e^{-i\Lambda p\cdot a}\sqrt{\frac{\omega_m(\mathbf{p}_\Lambda)}{\omega_m(\mathbf{p})}}\sum D^{\frac{1}{2}}_{\mu\bar{\mu}}[\underline{R}_c^{-1}(\underline{\Lambda},p)]b(\mathbf{p}_\Lambda\,\bar{\mu}). \tag{10.17}$$

We now use the property that the conjugate representations of $SU(2)$ are equivalent:

$$D^{s*}_{\mu'\mu}(\underline{R}) = \sum D^{s}_{\mu'\bar{\mu}'}(\underline{R}_y^{-1})D^{s}_{\bar{\mu}'\bar{\mu}}(\underline{R})D^{s}_{\bar{\mu}\mu}(\underline{R}_y), \tag{10.18}$$

where

$$\underline{R}_y = \exp(-\tfrac{1}{2}i\pi\sigma_2) = -i\sigma_2, \tag{10.19}$$

to transform $b^\dagger(\mathbf{p},\mu)$ so that it has the same transformation properties as $a(\mathbf{p},\mu)$. Relation (10.18) follows because a similarity transformation by any multiple of σ_2 is equivalent to a complex

conjugation of an $SU(2)$ matricex. This extends to the D functions because they are polynomials in the $SU(2)$ matrix elements with *real* coefficients, as defined in Eq. (4.14). It follows that the combination

$$\sum_{\mu'=-s}^{s} D^{\frac{1}{2}}_{\mu\mu'}(\underline{R}_y) b^\dagger(\mathbf{p},\mu') \tag{10.20}$$

has the same transformation properties as $a(\mathbf{p},\mu)$ and $b(\mathbf{p},\mu)$ under $U(\underline{\Lambda},\underline{a})$.

There are two ways to construct a covariant field operator. One method is to define the operator in terms of its matrix elements in a complete set of states that transform irreducibly under $\mathcal{P}$. The transformation properties of the states and the field allow us to use the Wigner-Eckart theorem for $\mathcal{P}$ to express these matrix elements of the field operator as a sum of Clebsch-Gordan coefficients (for $\mathcal{P}$) and some invariants, which can be fixed by local commutation relations, choice of spinor representation, and parity. This is exactly how we treated current operators in the previous section. The second is to assume that the field $\Psi(x)$ is a spinor-valued linear combination of $a(\mathbf{p},\mu)$ and $b^\dagger(\mathbf{p},\mu)$, and determine the coefficients from covariance, locality and parity. The two approaches are equivalent.

We begin by assuming the following decomposition:

$$\begin{aligned}\Psi_a(x) = \sum \int d^3p \Big[& u_a(\mathbf{p},\mu) a(\mathbf{p},\mu) e^{-i(\omega_m(\mathbf{p})t - \mathbf{x}\cdot\mathbf{p})} \\ & + v_a(\mathbf{p},\bar{\mu}) D^s_{\bar{\mu}\mu}(\underline{R}_y) b^\dagger(\mathbf{p},\mu) e^{i(\omega_m(\mathbf{p})t - \mathbf{x}\cdot\mathbf{p})} \Big],\end{aligned} \tag{10.21}$$

where $u_a(\mathbf{p},\mu)$ and $v_a(\mathbf{p},\mu)$ are to be determined. The function $D^{\frac{1}{2}}_{\bar{\mu}\mu}(\underline{R}_y)$ is included for convenience. It could easily be absorbed into the definition of $v_a(\mathbf{p},\bar{\mu})$ or $b^\dagger(\mathbf{p},\mu)$. We assume that this field transforms covariantly under the action of $U(\underline{\Lambda},\underline{a})$:

$$U(\underline{\Lambda},\underline{a}) \Psi_a(x) U^\dagger(\underline{\Lambda},\underline{a}) = S(\underline{\Lambda}^{-1})_{aa'} \Psi_{a'}(\Lambda x + a), \tag{10.22}$$

where $S(\underline{\Lambda})_{aa'}$ is a finite dimensional representation of $SL(2,C)$ which defines the type of field. For a Dirac particle, $S(\underline{\Lambda})$ is, up to a similarity transformation, the direct sum of a right- and

left-handed representation of $SL(2,C)$:

$$S_{\rm Dirac}(\underline{\Lambda})_{aa'} = \begin{pmatrix} \underline{\Lambda} & 0 \\ 0 & (\underline{\Lambda}^\dagger)^{-1} \end{pmatrix}_{aa'}. \tag{10.23}$$

In order to find equations that fix the spinor wave functions $u_a(\mathbf{p},\mu)$ and $v_a(\mathbf{p},\mu)$, we evaluate Eq. (10.22) using the transformation properties of $a(\mathbf{p},\mu)$ and $b^\dagger(\mathbf{p},\mu)$, and equate coefficients of $a(\mathbf{p},\mu)$ and $b^\dagger(\mathbf{p},\mu)$. The result is

$$\sum u_a(\mathbf{p}_\Lambda,\bar{\mu}) D^{\frac{1}{2}}_{\bar{\mu}\mu}[\underline{R}_c(\underline{\Lambda},p/m)] = \sum S(\underline{\Lambda})_{aa'} u_{a'}(\mathbf{p},\mu)\sqrt{\frac{\omega_m(\mathbf{p})}{\omega_m(\mathbf{p}_\Lambda)}}; \tag{10.24}$$

$$\sum v_a(\mathbf{p}_\Lambda,\bar{\mu}) D^{\frac{1}{2}}_{\bar{\mu}\mu}[\underline{R}_c(\underline{\Lambda},p/m)] = \sum S(\underline{\Lambda})_{aa'} v_{a'}(\mathbf{p},\mu)\sqrt{\frac{\omega_m(\mathbf{p})}{\omega_m(\mathbf{p}_\Lambda)}}. \tag{10.25}$$

Equations (10.24) and (10.25) show that these coefficients change finite-dimensional representations of $SL(2,C)$ into Wigner rotations associated with a given momentum and Lorentz transformation. The $\mathbf{p}$ dependence of the wave functions is fixed by covariance. To see this, let $(\underline{\Lambda},\mathbf{p}) \to (\underline{L}_c(p/m),\mathbf{0})$ in Eqs. (10.24) and (10.25). In this case, the Wigner rotation becomes the identity:

$$u_a(\mathbf{p},\mu) = S[\underline{L}_c(p/m)]_{aa'} u_{a'}(\mathbf{0},\mu)\sqrt{\frac{m}{\omega_m(\mathbf{p})}}; \tag{10.26}$$

$$v_a(\mathbf{p},\mu) = S[\underline{L}_c(p/m)]_{aa'} v_{a'}(\mathbf{0},\mu)\sqrt{\frac{m}{\omega_m(\mathbf{p})}}. \tag{10.27}$$

This fixes the $\mathbf{p}$ dependence of these wave functions in terms of their value when $\mathbf{p}=\mathbf{0}$. To determine the rest wave functions, we consider the case that $\underline{\Lambda}=\underline{R}$ is a rotation, which leaves $\mathbf{p}=\mathbf{0}$ invariant:

$$u_a(\mathbf{0},\mu) = S(\underline{R})_{aa'} u_{a'}(\mathbf{0},\mu') D^s_{\mu'\mu}(\underline{R}^{-1}); \tag{10.28}$$

$$v_a(\mathbf{0},\mu) = S(\underline{R})_{aa'} v_{a'}(\mathbf{0},\mu') D^s_{\mu'\mu}(\underline{R}^{-1}). \tag{10.29}$$

For an infinitesimal rotation about the z axis, these equations become eigenvalue relations:

$$\sum_{a'} \delta S^z_{aa'} u_{a'}(\mathbf{0},\mu) = -i\mu u_a(\mathbf{0},\mu); \tag{10.30}$$

$$\sum_{a'} \delta S^z_{aa'} v_{a'}(\mathbf{0},\mu) = -i\mu v_a(\mathbf{0},\mu), \tag{10.31}$$

where

$$\delta S^z_{aa'} = \frac{d}{d\theta}\, S_{aa'}(e^{\frac{i}{2}\theta\sigma_z})\Big|_{\theta=0}. \tag{10.32}$$

For the case of a Dirac spinor,

$$\delta S^z_{aa'} = -\frac{i}{2}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \tag{10.33}$$

which is already diagonal. In particular, we have

$$u_a(\mathbf{0},\tfrac{1}{2}) = v_a(\mathbf{0},\tfrac{1}{2}) = \begin{pmatrix} c \\ 0 \\ d \\ 0 \end{pmatrix}; \tag{10.34}$$

$$u_a(\mathbf{0},-\tfrac{1}{2}) = v_a(\mathbf{0},-\tfrac{1}{2}) = \begin{pmatrix} 0 \\ e \\ 0 \\ f \end{pmatrix}, \tag{10.35}$$

where c, d, e and f are independent constants. In $SL(2,C)$, space reflection implies that $\underline{\Lambda} \leftrightarrow (\underline{\Lambda}^\dagger)^{-1}$, which interchanges the upper and lower components of these spinors. This was discussed in the previous section on tensor operators. It also changes the sign of $D^{\frac{1}{2}}_{\mu\mu'}(\underline{R}_y)$, and does nothing to the creation or annihilation operators when $\mathbf{p} = \mathbf{0}$. Thus, if we want the Dirac field to be invariant under spatial reflection, we find that

$$\begin{pmatrix} c \\ 0 \\ d \\ 0 \end{pmatrix} = \begin{pmatrix} d \\ 0 \\ c \\ 0 \end{pmatrix}; \qquad \begin{pmatrix} 0 \\ e \\ 0 \\ f \end{pmatrix} = \begin{pmatrix} 0 \\ f \\ 0 \\ e \end{pmatrix} \tag{10.36}$$

for $u_a(\mathbf{0},\mu)$, and

$$\begin{pmatrix} c \\ 0 \\ d \\ 0 \end{pmatrix} = - \begin{pmatrix} d \\ 0 \\ c \\ 0 \end{pmatrix}; \qquad \begin{pmatrix} 0 \\ e \\ 0 \\ f \end{pmatrix} = - \begin{pmatrix} 0 \\ f \\ 0 \\ e \end{pmatrix} \tag{10.37}$$

for $v_a(\mathbf{0},\mu)$, since $v_a(\mathbf{0},\mu)$ is multiplied by $D^{\frac{1}{2}}_{\mu\mu'}(\underline{R}_y)$. If we combine Eqs. (10.34)–(10.37), we obtain

$$u_a(\mathbf{0},\tfrac{1}{2}) = C \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}; \qquad u_a(\mathbf{0},-\tfrac{1}{2}) = C \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}; \tag{10.38}$$

$$v_a(\mathbf{0},\tfrac{1}{2}) = C \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix}; \qquad v_a(\mathbf{0},-\tfrac{1}{2}) = C \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix}. \tag{10.39}$$

The constants are determined by the requirement

$$\{\Psi_a(x), \Psi^\dagger_b(y)\}_{|_{x^0=y^0}} = \delta_{ab}\delta(\mathbf{x}-\mathbf{y}). \tag{10.40}$$

This is the only part of the development which cannot be completed without including the antiparticle. The commutation relations (10.40) imply the normalization

$$C = \frac{1}{\sqrt{2}(2\pi)^{\frac{3}{2}}} \tag{10.41}$$

for each of the four spinor amplitudes. Thus, the conditions of covariance, parity and equal-time anticommutation relations fix the form of this field uniquely, up to phase and degeneracies. The

results are

$$u(\mathbf{p},\mu) = \frac{m - \gamma\cdot p}{\sqrt{2\omega_m(\mathbf{p})(m+\omega_m(\mathbf{p}))}} u(\mathbf{0},\mu); \tag{10.42}$$

$$v(\mathbf{p},\mu) = \frac{m + \gamma\cdot p}{\sqrt{2\omega_m(\mathbf{p})(m+\omega_m(\mathbf{p}))}} v(\mathbf{0},\mu), \tag{10.43}$$

where

$$\gamma^0 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \qquad \boldsymbol{\gamma} = \begin{pmatrix} 0 & -\boldsymbol{\sigma} \\ \boldsymbol{\sigma} & 0 \end{pmatrix}. \tag{10.44}$$

Note that the representation of the Dirac γ matrices was determined by the choice of representation of $S(\underline{\Lambda})_{aa'}$ in Eq. (10.25).

With these wave functions, the field operator $\Psi_a(x)$ is an ordinary Dirac field. In most modern treatments, the factor $D^{\frac{1}{2}}_{\bar{\mu}\mu}(\underline{R}_y^{-1})$ is absorbed in the definition of $b^\dagger(\mathbf{p},\mu)$. Our discussion follows that of Weinberg (We 65), where this factor appears explicitly.

Note that the index a on the spinor wave function transforms under a finite-dimensional representation of $SL(2,C)$, while the index μ transforms under a unitary representation of the little group ($SU(2)$) of a positive-energy, positive-mass irreducible representation of $\mathcal{P}$. In $SU(2)$, the fundamental representation is self-conjugate, while this is not the case for finite-dimensional representations of $SL(2,C)$. This is why four-component spinors are needed to construct Lorentz covariant fields, but not for the creation and annihilation operators.

To find the transformation that relates the field description to the particle description, it is convenient to compute the Fourier transform of the adjoint field applied to the vacuum:

$$\tilde{\Psi}^\dagger_a(p)|0\rangle = (2\pi)^{-2}\int d^4x\, e^{-ip\cdot x}\Psi^\dagger_a(x) = (2\pi)^2 u^*_a(\mathbf{p},\mu)a^\dagger(\mathbf{p},\mu)|0\rangle\delta(p^0-\omega_m(\mathbf{p})). \tag{10.45}$$

We use the relation

$$a^\dagger(\mathbf{p},\mu)|0\rangle\delta[p^0-\omega_m(\mathbf{p})] = 2\pi\tilde{\Psi}^\dagger(p)|0\rangle\gamma^0 u(\mathbf{p},\mu), \tag{10.46}$$

which follows from Eq. (10.45) by using the following:

$$u^\dagger(\mathbf{p},\mu)\gamma^0 u(\mathbf{p},\nu) = (2\pi)^{-3}\delta_{\mu\nu}. \tag{10.47}$$

Vectors in the field and particle case can be expressed in terms of functions $g_a(p)$ and $f(\mathbf{p},\mu)$,

respectively, by using

$$|\xi\rangle = \int d^4p\, \tilde{\Psi}_a^\dagger(p)|0\rangle g_a(p); \tag{10.48}$$

$$|\xi\rangle = \int d^3p\, |\mathbf{p}\,\mu\rangle f(\mathbf{p},\ \mu). \tag{10.49}$$

From Eqs. (10.45) and (10.46), we obtain the following relation between the functions that define the same vector in the particle and the field cases, respectively:

$$f(\mathbf{p},\mu) = (2\pi)^{-2} u_a^\dagger(\mathbf{p},\mu) g_a(\omega_m(\mathbf{p}),\mathbf{p}); \tag{10.50}$$

$$g_a(p)|_{p^0=\omega_m(\mathbf{p})} = (2\pi)(\gamma^0 u(\mathbf{p},\mu))_a f(\mathbf{p},\mu). \tag{10.51}$$

The correspondence $f(\mathbf{p},\mu) \leftrightarrow g_a(p)|_{p^0=\omega_m(\mathbf{p})}$ is a norm preserving correspondence between square integrable wave functions of the particle theory and the equivalence class of covariant wave functions that agree on the physical mass shell of the particle. Since covariant wave functions that vanish on the physical mass shell correspond to zero-norm states, by Eq. (10.45), the equivalence classes of covariant functions that agree on the mass shell correspond to physical states of the field theory. It follows that the correspondence $f(\mathbf{p},\mu) \leftrightarrow g_a(p)|_{p^0=\omega_m(\mathbf{p})}$ is a unitary correspondence between one-particle states of the particle theory and physical one-particle states of the field theory. There is a similar correspondence between the two representations of the antiparticles. We let W denote the unitary mapping that takes a particle state to the corresponding state in the field theory. This shows that free particles in a particle theory and a field theory differ only by a change of representation. *Antiparticles were not needed to construct covariant fields – only to obtain the canonical anticommutation relations.* Antiparticles are easily included in the particle theory if desired.

Next, we consider an equivalent quantum mechanical theory with covariant wave functions. This will be done using the field theory. The covariance of the field and the unitarity of $U(\underline{\Lambda}, \underline{a})$

implies the following:

$$\begin{aligned}
&\int d^4x \int d^4y\, g_a^*(x)\langle 0|\Psi_a(x)\Psi_b^\dagger(y)|0\rangle f_b(y) \\
&\quad = \sum \int d^4x \int d^4y\, g_a^*(x) S(\underline{\Lambda})_{aa'} \\
&\qquad \times \langle 0|\Psi_{a'}(\Lambda^{-1}(x-a))\Psi_{b'}^\dagger(\Lambda^{-1}(y-a))|0\rangle S^\dagger(\underline{\Lambda})_{b'b} f_b(y) \\
&\quad = \sum \int d^4x \int d^4y\, g_a^*(\Lambda x + a) S(\underline{\Lambda})_{aa'} \\
&\qquad \times \langle 0|\Psi_{a'}(x)\Psi_{b'}^\dagger(y))|0\rangle S^\dagger(\underline{\Lambda})_{b'b} f_b(\Lambda y + a).
\end{aligned} \tag{10.52}$$

Under the transformations

$$f_b(x) \to f'_b(y) = \sum S^\dagger(\underline{\Lambda})_{bb'} f_{b'}(\Lambda y + a); \tag{10.53}$$

$$g_a^*(x) \to g_a'^*(y) = \sum g_{a'}'^*(\Lambda y + a) S(\underline{\Lambda})_{a'a}, \tag{10.54}$$

the scalar product defined by Eq. (10.52) is invariant. Thus, $g_a(x)$ and $f_a(x)$ can be interpreted as covariant wave functions on a Hilbert space, where the scalar product is defined in terms of the kernel

$$K(x,y)_{ab} = \langle 0|\Psi_a(x)\Psi_b^\dagger(y)|0\rangle. \tag{10.55}$$

Direct computation of this kernel yields

$$K(x,y)_{ab} = \int \frac{d^4p}{(2\pi)^3}\, 2m\delta(p^2+m^2)e^{-ip\cdot(x-y)}\Theta(p^0)\Lambda^+(p)_{ab}, \tag{10.56}$$

where Λ^+_{ab} is an orthogonal projector on the subspace of the four-component spinor space spanned by the u's. In momentum space this kernel restricts two covariant wave functions to the positive-energy mass hyperbola, and projects the spinor indices on the space spanned by the u-spinors. How these functions behave away from the positive-energy mass hyperbola and in the subspace spanned by the v-spinors has no effect on this scalar product. Thus, although the wave functions $g_a(p)$ transform covariantly, a state corresponds to an equivalence class of these functions whose difference has zero norm in the semi-scalar product defined by Eq. (10.56). There is a similar kernel associated with the antiparticles.

The property of having a quantum mechanics with a non-trivial scalar product is characteristic of all relativistic quantum models with wave functions that (1) are probability amplitudes and (2) transform covariantly (Po 85a).

The correspondence (10.50) and (10.51) gives the connection between this covariant free particle quantum mechanics and the quantum theory of free particles that began this section. As in the case of fields, the correspondence can be realized as a unitary change of representation.

We have shown the equivalence of the description of a free particle in a particle theory, a field theory, and a theory with covariant wave functions.

In all three of these approaches, it is possible to add interactions. We have discussed interactions for the case of the particle theory. In a field theory, interactions are generally added in a manner which is designed to preserve microscopic locality, usually with a Lagrangian. It can also be done alternatively by directly adding interactions to the Poincaré generators of the free field (We 65). In a covariant quantum theory, interactions involve modifications of the scalar product. Although it is possible to develop the connections in the case that interactions are present, these connections are not really in the spirit of how the different models are used in practice. For instance, it is possible to formulate successive approximations to a local field theory, where each term is a particle model with a finite number of degrees of freedom satisfying macroscopic locality. If one took seriously the interactions derived in such an approach, it is unlikely that they would be as good as a purely phenomenological one, for the obvious reason that some of the physics contained in excluded degrees of freedom can be recovered by fitting parameters to data. A similar situation occurs if one applies the Bethe-Salpeter equation to a theory of nucleons and pions. It is only after one includes enough mesons, and fits all of the various parameters to phase shifts, that one obtains a sensible model.

In the case of a free particle, the connections between the three approaches can be made precise, and they provide a good starting point for a general understanding of the relation between these three approaches.

Models with covariant wave functions with probability interpretations arise in an alternative relativistic formulation of quantum mechanics called covariant constraint dynamics. This approach is discussed extensively elsewhere (Lo 87). We can think of a two-particle system with the

four-momentum of each particle restricted to its positive-energy mass hyperbloid. The two-body wave function is a covariant wave function in eight variables, subject to two covariant (mass shell) constraints. In this case, the constraints commute. They can be used to construct a kernel of a covariant scalar product (as above), or, equivalently, they can be realized as a pair of compatible equations that must be satisfied by the two-body wave function. To include interactions, either the constraints or the kernel of the scalar product is perturbed in a manner that preserves the compatibility of the constraints and the covariance, or, equivalently, the covariance and the Hermiticity of the kernel of the scalar product. The problem is to find covariant interactions which are compatible, satisfy cluster properties, and also satisfy the spectral condition.

The relation to a model satisfying macroscopic locality and not microscopic locality is an approximation at best. One way the treat this problem is to eliminate degrees of freedom from the generators in an manner that preserves the Lie algebra. This has been done perturbatively (Gl 81).

We close this section with some brief comments about "covariant wave functions" associated with the Bethe-Salpeter equation (Sa 51). In most treatments of quantum field theory, bound states are described by the Bethe-Salpeter equation or a related reduction. These have been both discussed and applied extensively in the literature (Bl 60, Bl 66, Gr 82a, Gr 82b). A discussion of this approach is not within the scope of this paper, but there are some important points that merit discussion. The first is that the "covariant wave functions" of the Bethe-Salpeter equation are not wave functions in the sense of probability amplitudes (*i.e.,* there is no associated scalar product that is positive semi-definite). Instead, they are more closely related to the current matrix elements discussed in the previous section. They are matrix elements of covariant field operators evaluated between physical states. They are sometimes confused with wave functions because it is possible to construct quadratic functions involving these amplitudes to evaluate matrix elements of any operator (Ma 55, Hu 75). Although the Bethe-Salpeter equation is motivated by field theories, in applications the kernel is approximated . When the approximate kernel does not have all of the properties of the exact theory, such as crossing symmetry, then these models violate microscopic locality. In addition, in realistic applications, the kernel is treated phenomenologically. Under these conditions, Bethe Salpeter methods resemble, in spirit, the methods discussed in this paper in the sense that they are phenomenologies that satisfy

macroscopic locality but not microscopic locality. As a practical matter, the dynamical equations are more difficult to solve, and a normalization condition is needed to extract physical matrix elements.

11. **Conclusion**

Our goal in this review has been to provide both a formal foundation and a set of tools for constructing relativistic quantum mechanical models in nuclear and particle physics. Rather than simply summarizing a paper of this size, we attempt here to highlight what we believe to be the most important points.

We have devoted considerable space to the topic of relativistic invariance and its implications for the construction of quantum mechanical models. A key element of this discussion is that there are *many* ways of making relativistic models. While relativistic invariance can certainly eliminate models on the basis of some sort of inconsistency, it serves more as a constraint than as a pointer to some specific theory. Thus, the choice of model must in general be made on other grounds, such as the complexity of the interaction and/or the number of parameters needed to describe a set of observables, or the ability to perform accurate and convergent calculations.

One consequence of the variability of models is that it is impossible to isolate a unique "relativistic correction." Different models may have the same nonrelativistic limit, or different limits, depending in detail on how such a limit is taken.

Another element which distinguishes relativistic models is that of locality. The models featured in this review can be made to satisfy *macroscopic* locality–a property seen in experiments and therefore, in our view, necessary for any model. To go beyond this and require a theory to satisfy *microscopic* locality, as in a local field theory, is certainly sufficient, but not necessary. The price paid for this in some instances can be a theory with an infinite number of degrees of freedom and a large coupling constant, for which few (if any) accurate and/or convergent calculations can be made. Furthermore, attempts to approximate such a theory can easily result in the loss of microscopic locality, in addition to macroscopic locality, or even the relativistic invariance which motivated the theory in the first place. Thus, while local quantum field theory has had certain highly visible successes, it does not represent the *required* path to combining relativity and quantum mechanics, and in some cases, particularly those involving strong interactions and/or phenomenology, it may not be the *desired* path.

Light-front dynamics occupies a special place in our discussion. The fact that it is implemented with only three out of ten interacting Poincaré generators (as opposed to four out of ten in the instant form) makes it possible to do a number of practical calculations with Lorentz transformations involving only kinematic generators. For instance, it permits one to determine the spin transformation properties of purely phenomenological wave functions under front-from boosts. It is especially useful for the evaluation of current matrix elements. For spacelike momentum transfer, it is possible in light-front dynamics to make a consistent impulse approximation (which is not possible in the instant form) whereby all observables can be consistently calculated from matrix elements of a single component of the relevant current operator in set of frames related by a subgroup of front-form Lorentz transformations.

We have not provided detailed discussion of specific applications, except as illustrative examples. As stated above, our goal has been to provide enough tools for the interested reader to digest the literature and to perform new calculations which advance the field. While citations to specific published studies have been provided (see the end of Section 2.1), we provide here a general view of the current status of the literature.

Most calculations to date which use direct interations have centered on systems involving two or three nucleons, or two or three quarks. While there have been calculations which make explicit comparisions to data, there are many more observables yet to be studied. These involve both pure strong-interaction processes in few-body systems as well as reactions involving an electromagnetic or weak probe. With new high-intensity, high-duty-factor machines becoming a reality in energy regions where relativistic kinematics are clearly important, the need for detailed relativistic models is not an idle one. Models of the type presented in this review require calculations which are certainly more intensive than their nonrelativistic counterparts, but many are within the capabilities of modern supercomputers or even workstations.

On the formal side, the need to construct realistic models with particle production which also satisfy macroscopic locality has been partially satisfied, but there is more work to be done. The general problem of making models which allow the production of arbitrary numbers of particles must also be addressed by calculations based upon field theory: while one can make exact statements in the latter case, specific (truncated) models raise the same sort of questions.

There is also the need to isolate and understand ways to implement a reliable relativistic quantum many-body problem. While an explicit formulation satisfying macroscopic locality is available in the approach described in this review (see So 77 or Co 82), it is neither workable nor desirable for large numbers of particles, any more than a nonrelativistic version would be. A first approach would then be to try to utilize approximations which have proven reliable in nonrelativistic nuclear physics. Of course, the word "reliable" implies an accompanying justification that corrections are small, and not that a certain step was taken in the face of prohibitive computational complexity. Relativistic models based upon any other approach must also address this issue.

We have not attempted to discuss in any detail the other approaches to relativistic quantum mechanics which now exist in the literature. A comparative discussion of the distinctive features of each approach would require enough extra technical machinery as to make this review unmanageable. These different approaches sit at various stages of advancement. Beyond relativistic invariance, key elements that models must address successfully include macroscopic locality and the spectral condition. In addition, since many other implementations are formulated in terms of transition operators, it must also be shown that there is an underlying quantum theory. These questions have been worked out for certain approaches and/or certain applications, but there is certainly room for many contributions.

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APPENDIX A

Scattering Theory

In this Appendix, we provide an overview of various aspects of scattering theory as they relate to relativistic models. Time-dependent treatments of scattering employ time development operators involving the Hamiltonian. In this review, it was convenient to utilize operators other than the Hamiltonian (such as the mass M or its square, or the front-form Hamiltonian P^-) to formulate the scattering problem. We have therefore included a derivation of the relation among the S and T operators and the differential cross section, along with a discussion of the invariance principle (Bi 62, Ka 65, Ch 76, Ob 78), and a discussion concerning the construction of phenomenological interactions fit to nucleon-nucleon scattering data. Our purpose is to provide the justification for these alternate formulations of the scattering problem. Although all of this material can be found in the literature, it is difficult to find it in one place.

In addition, we show that for *any* of the forms of dynamics covered in this review, it is possible to take advantage of two-body potentials which have been fit to scattering data using the Schrödinger equation. This property was discussed in the front-form examples in Section 2, but it is in fact more general

A.1. The Relation Between S and T

For models with a finite number of degrees of freedom, the multichannel scattering operator is defined in terms of channel wave operators as:

$$S_{\beta\alpha} := \Omega^{\dagger}_{\beta+}\Omega_{\alpha-} \tag{A.1}$$

where a is a partition of N particles into n_a clusters, and the partition wave operators are defined by

$$\lim_{t\to\infty} \|\Omega_{\alpha\pm}|\Psi_\alpha\rangle - e^{\pm iHt}e^{\mp iH_a t}|\Psi_\alpha\rangle\| = 0. \tag{A.2}$$

In equation (A.2), α denotes a channel in which the particles in each cluster of the partition a are separately bound. H_a is the *partition Hamiltonian,* which is obtained from the original

Hamiltonian by turning off all interactions between particles in *different* clusters of the partition a. The partition Hamiltonian is also the sum of the subsystem Hamiltonians for each cluster in a:

$$H_a = \sum_{i=1}^{n_a} H_{a_i}. \tag{A.3}$$

There is a scattering channel α associated with the partition a if each of the H_{a_i} has a bound state. Let $|\alpha_i\, \mathbf{p}_i\rangle$ denote a bound state of H_{a_i} corresponding to the channel α with total momentum $\mathbf{p}_i$. Now define the channel projection operator:

$$\Pi_\alpha := \int \prod_{i=1}^{n_a} d^3p_i |\mathbf{p}_1\, \alpha_1 \cdots \mathbf{p}_{n_a}\, \alpha_{n_a}\rangle\langle \mathbf{p}_1\, \alpha_1 \cdots \mathbf{p}_{n_a}\, \alpha_{n_a}|. \tag{A.4}$$

The channel states $|\Psi_\alpha\rangle$ are normalizable vectors satisfying:

$$|\Psi_\alpha\rangle = \Pi_\alpha |\Psi_\alpha\rangle. \tag{A.5}$$

Note that a given partition a may have none or many associated scattering channels α.

To construct formulas for scattering matrix elements, let α and β denote scattering channels associated with the asymptotic partition Hamiltonians H_a and H_b, respectively. Let $|\alpha\rangle$ and $|\beta\rangle$ denote sharp eigenstates of the partition Hamiltonians:

$$H_a|\alpha\rangle = E_\alpha|\alpha\rangle; \qquad H_b|\beta\rangle = E_\beta|\beta\rangle. \tag{A.6}$$

Using Eqs. (A.1) and (A.2), the S-matrix elements can be evaluated as follows:

$$\begin{aligned}
&\langle\beta|S_{ba}|\alpha\rangle \\
&\quad = \lim_{t\to\infty} \langle\beta|e^{iH_bt}e^{-2iHt}e^{iH_at}|\alpha\rangle \\
&\quad = \langle\beta|\alpha\rangle + \lim_{t\to\infty} \int_0^t dt'\, \frac{d}{dt'}\langle\beta|e^{i(E_\beta+E_\alpha-2H)t'}|\alpha\rangle \\
&\quad = \langle\beta|\alpha\rangle + \lim_{\epsilon\to 0^+} i\int_0^\infty dt'\, \langle\beta|\Big[(E_\beta - H)e^{i(E_\beta+E_\alpha-2H+i\epsilon)t'} \\
&\qquad + e^{i(E_\beta+E_\alpha-2H+i\epsilon)t'}(E_\alpha - H)\Big]|\alpha\rangle \\
&\quad = \langle\beta|\alpha\rangle + \lim_{\epsilon\to 0^+} \tfrac{1}{2}\langle\beta|\left[(H-E_\beta)\frac{1}{\bar{E}-H+i\epsilon} + \frac{1}{\bar{E}-H+i\epsilon}(H-E_\alpha)\right]|\alpha\rangle,
\end{aligned} \tag{A.7}$$

where $\bar{E} = \frac{1}{2}(E_\alpha + E_\beta)$ is the average energy of the initial and final asymptotic states. Equation

(A.7) is interpreted as the kernel of an integral operator. S-matrix elements are obtained by integrating the sharp eigenstates in Eq. (A.7) over normalizable functions of the energy and any other continuous variables. To simplify this expression, we introduce a more compact notation. The residual interactions V^a and V^b are defined as follows:

$$V^a := H - H_a; \qquad V^b := H - H_b, \tag{A.8}$$

where

$$V^a|\alpha\rangle = (H - E_\alpha)|\alpha\rangle; \qquad V^b|\beta\rangle = (H - E_\beta)|\beta\rangle. \tag{A.9}$$

We introduce resolvent operators of the Hamiltonian and the partition Hamiltonian:

$$R(z) := \frac{1}{z - H} \qquad R_a(z) := \frac{1}{z - H_a}. \tag{A.10}$$

These operators satisfy the second resolvent relations (Hi 57):

$$R(z) - R_a(z) = R_a(z)V^a R(z) = R(z)V^a R_a(z). \tag{A.11}$$

If these identities are used in Eq. (A.7), we get

$$\begin{aligned}
&\langle\beta|S|\alpha\rangle \\
&\quad = \langle\beta|\alpha\rangle + \lim_{\epsilon\to 0^+} \tfrac{1}{2}\langle\beta|\Big[V^b\left(1 + R(\bar{E} + i\epsilon)V^a\right) R_a(\bar{E} + i\epsilon) \\
&\qquad + R_b(\bar{E} + i\epsilon)\left(1 + V^b R(\bar{E} + i\epsilon)\right) V^a\Big]|\alpha\rangle \\
&\quad = \langle\beta|\alpha\rangle \left[1 - \lim_{\epsilon\to 0^+} \frac{E_\beta - E_\alpha}{E_\beta - E_\alpha + 2i\epsilon}\right] \\
&\qquad + \lim_{\epsilon\to 0^+} \left[\frac{1}{E_\beta - E_\alpha + 2i\epsilon} + \frac{1}{E_\alpha - E_\beta + 2i\epsilon}\right] \langle\beta|\left(V^a + V^b R(\bar{E} + i\epsilon)V^a\right)|\alpha\rangle \\
&\quad = \langle\beta|\alpha\rangle \lim_{\epsilon\to 0^+} \left[\frac{2i\epsilon}{E_\beta - E_\alpha + 2i\epsilon}\right] \\
&\qquad + \lim_{\epsilon\to 0^+} \left[\frac{-4i\epsilon}{(E_\beta - E_\alpha)^2 + 4\epsilon^2}\right] \langle\beta|\left(V^a + V^b R(\bar{E} + i\epsilon)V^a\right)|\alpha\rangle.
\end{aligned} \tag{A.12}$$

It is now possible to evaluate the limit as $\epsilon \to 0$. It is important to remember that this is the kernel of an integral operator.

The first term in square brackets is unity when the initial and final energies are identical, and zero otherwise; however, the limit in the bracket is a Kronecker delta and not a Dirac delta function. For $a \neq b$, we expect that $\langle\beta(E')|\alpha(E)\rangle$ will be Lebesgue measurable in E' for fixed E, so there is no contribution from the first term in Eq. (A.12). For the case that $H_b = H_a$, we have $\langle\beta(E')|\alpha(E)\rangle \propto \delta(E' - E)$. The matrix elements vanish by orthogonality unless $E_\beta = E_\alpha$, but then coefficient is unity. Thus, the first term is $\langle\beta|\alpha\rangle$ if the initial and final channel are the same, but zero otherwise. Note that the matrix elements vanish by orthogonality for two different channels governed by the same asymptotic Hamiltonian with the same energy. The first term in (A.12) is therefore a *channel* delta function.

For the second term, the quantity in square brackets becomes $-2\pi i\delta(E_\beta - E_\alpha)$, which leads to the relation

$$\langle\beta|S|\alpha\rangle = \langle a|b\rangle\delta_{\beta\alpha} - 2\pi i\delta(E_\beta - E_\alpha)\langle\beta|T^{ba}(E_a + i0^+)|\alpha\rangle, \tag{A.13}$$

where

$$T^{ba}(z) = V^a + V^b R(z) V^a. \tag{A.14}$$

This is the general expression for the transition operator in a multichannel scattering theory. It applies both to relativistic and to nonrelativistic applications. The multichannel transition operator $T^{ab}(z)$ must be constructed in a dynamical model.

In models where the residual interactions and the resolvent commute with the total linear momentum operator, and if the sharp channel states $|\alpha\rangle$ and $|\beta\rangle$ are simultaneous eigenstates of the appropriate partition Hamiltonian and the linear momentum, then the delta function can be factored out of the matrix element:

$$\langle\beta|T^{ba}(E_a + i0^+)|\alpha\rangle = \delta^3(p_\beta - p_\alpha)\langle\beta\|T^{ba}(E_a + i0^+)\|\alpha\rangle. \tag{A.15}$$

If this is used in Eq. (A.13), then the S-matrix elements can be expressed in terms of the reduced transition operators as follows:

$$\langle\beta|S|\alpha\rangle = \langle a|b\rangle\delta_{\beta\alpha} - i(2\pi)\delta^4(p_\beta - p_\alpha)\langle\beta\|T^{ba}(E_a + i0^+)\|\alpha\rangle \tag{A.16}$$

This expression has the advantage that the delta function is manifestly invariant under Poincaré

transformations. The factor $(2\pi)^3$ is included by convention.

For relativistic models of the type discussed in this review, there are cases where the residual interactions and resolvent commute with the system four-velocity or the light-front components of the four-momentum, rather than the three-momentum. In general, let $\mathbf{g} = \mathbf{g}(\mathbf{p}; p^0)$ be three independent functions of the four-momentum that commute with the residual interactions and the resolvent. Then Eq. (A.15) can be replaced by the following:

$$\begin{aligned}&\langle\beta|T^{ba}(E_a + i0^+)|\alpha\rangle \\ &\quad =: \delta^3(g_\beta - g_\alpha)\left|\frac{\partial\mathbf{g}(\mathbf{p}, p^0_\beta)}{\partial\mathbf{p}}\right|^{\frac{1}{2}}\left|\frac{\partial\mathbf{g}(\mathbf{p}, p^0_\alpha)}{\partial\mathbf{p}}\right|^{\frac{1}{2}}\langle\beta\|T^{ba}(E_a + i0^+)\|\alpha\rangle.\end{aligned} \tag{A.17}$$

The Jacobian factors can always be absorbed into the definition of the reduced matrix element. Thus, we can write

$$\delta^4(p_\alpha - p_\beta) = \delta(E_\alpha - E_\beta)\delta^3(p_\alpha - p_\beta) = \delta(E_\alpha - E_\beta)\delta^3(g_\beta - g_\alpha)\left|\frac{\partial\mathbf{g}(\mathbf{p}, p^0_\alpha)}{\partial\mathbf{p}}\right|, \tag{A.18}$$

and this yields

$$\langle\beta|S|\alpha\rangle = \langle a|b\rangle\delta_{\beta\alpha} - i(2\pi)\delta^4(p_\beta - p_\alpha)\langle\beta\|T^{ba}(E_a + i0^+)\|\alpha\rangle. \tag{A.19}$$

In what follows, this convention is used to define the reduced matrix element.

A.2. **The Invariance Principle**

In relativistic problems, it is convenient to utilize a more general form of the transition operator. In applications, we have replaced H by a non-trivial function $f(H, \mathbf{p})$ of the four-momentum, such as M, M^2, P^-, $(\frac{1}{4}M^2 - m^2)$, and other operators, and then formulated the scattering problem. The justification for this is the Kato-Birman invariance principle (Bi 62, Ka 66, Ch 76, Ob 78) for wave operators, and the dominated convergence theorem. The invariance principle is a rigorous result, whereby it is possible to replace H, H_a and H_b in Eq. (A.2) by $f(H)$, $f(H_a)$ and $f(H_b)$ for suitable real functions f, without changing the left-hand side of (A.3). There are a variety of formulations of the theorem, the strongest of which can be found in the

paper of Obermann and Wollenberg (Ob 78). It states that the existence of the channel wave operators $\Omega_a(H, H_0)$ in the sense of Abelian limits:

$$\lim_{\epsilon \to 0^+} \|\epsilon \int_0^\infty e^{-st} [e^{\pm iHt} e^{\mp iH_a t} - \Omega_{\pm a}(H, H_a)] |\alpha\rangle\| = 0 \tag{A.20}$$

implies both the existence and equality of $\Omega_{\pm a}[f(H), f(H_a)]$, when f is piecewise continuously differentiable, and the derivative of f is positive and locally of bounded variation. When the wave operators exist as strong limits, then the Abelian limits exist and are equal to the strong limits. The dominated convergence theorem allows us to extend these conclusions to the case that the functions f may depend parametrically on $\mathbf{p}$. The condition for the invariance principle should then hold for all values of $\mathbf{p}$.

This theorem can be applied to the scattering operator, with the result:

$$S_{ba} = \Omega^\dagger_{b+}(H, H_b)\Omega_{a-}(H, H_a) = \Omega^\dagger_{b+}[f(H), f(H_b)]\Omega_{a-}[f(H), f(H_a)] = S_{f\,ba}. \tag{A.21}$$

To illustrate Eq. (A.21), but without providing a complete proof, we consider the matrix elements

$$\langle\beta|S_f|\alpha\rangle := \lim_{t\to\infty} \langle\beta|e^{if(H_b)t} e^{-2if(H)t} e^{if(H_a)t}|\alpha\rangle. \tag{A.22}$$

Following the same steps which lead to Eqs. (A.12) and (A.13), we obtain

$$\langle\beta|S_f|\alpha\rangle = \langle\beta|\alpha\rangle - 2\pi i\delta\left(f(E_\beta) - f(E_\alpha)\right) \langle\beta|T^{ba}_f\left(f(E_\alpha) + i0^+\right)|\alpha\rangle, \tag{A.23}$$

where

$$T^{ba}_f(z) := V^a_f + V^b_f R_f(z) V^a_f; \tag{A.24}$$

$$V^a_f := f(H) - f(H_a); \qquad V^b_f := f(H) - f(H_b); \tag{A.25}$$

$$R_f(z) := [z - f(H)]^{-1}. \tag{A.26}$$

We now show that Eq. (A.23) agrees with (A.13), that is, the S matrix is the same using E or

$f(E)$. First, we note that

$$\delta\left(f(E_\beta) - f(E_\alpha)\right) = \left|\frac{dE}{df}\right|_{E=E_\alpha} \delta(E_\beta - E_\alpha), \tag{A.27}$$

provided the argument of the delta function has one root. A sufficient condition is that the derivative is positive. This gives

$$\langle\beta|S_f|\alpha\rangle = \langle\beta|\alpha\rangle - 2\pi i\delta(E_\beta - E_\alpha)\left|\frac{dE}{df}\right|_{E=E_\alpha} \langle\beta|T_f^{ba}\left(f(E_\alpha) + i0^+\right)|\alpha\rangle. \tag{A.28}$$

The desired equivalence follows if it can be shown that

$$\left|\frac{dE}{df}\right|_{E=E_\alpha} \langle\beta|T_f^{ba}\left(f(E_\alpha) + i0^+\right)|\alpha\rangle = \langle\beta|T^{ba}(E_\alpha + i0^+)|\alpha\rangle. \tag{A.29}$$

Note that $E_\alpha = E_\beta = E$ in this matrix element. Considering the definition of the transition operator T^{ba}, we get

$$\begin{aligned}
&\left|\frac{dE}{df}\right|_{E=E_\alpha} \langle\beta|T_f^{ba}\left(f(E_\alpha) + i0^+\right)|\alpha\rangle \\
&\quad= \left|\frac{dE}{df}\right|_{E=E_\alpha} \langle\beta|\left(f(H) - f(E)\right) \\
&\qquad + \left(f(H) - f(E)\right)\frac{1}{f(E) - f(H) + i0^+}\left(f(H) - f(E)\right)|\alpha\rangle \\
&\quad= \lim_{\epsilon\to 0^+} |\frac{dE}{df}|_{E=E_\alpha} \langle\beta|i\epsilon\frac{f(H) - f(E)}{f(E) - f(H) + i\epsilon}|\alpha\rangle \\
&\quad= \lim_{\epsilon\to 0^+} |\frac{dE}{df}|_{E=E_\alpha} \langle\beta|i\epsilon\frac{\frac{df}{dE}(H-E) + O(H-E)^2}{\frac{df}{dE}(E-H) + O(H-E)^2 + i\epsilon}|\alpha\rangle \\
&\quad= \lim_{\epsilon\to 0^+} |\frac{dE}{df}|_{E=E_\alpha} \frac{df}{dE}\langle\beta|i(\epsilon\frac{dE}{df})\frac{(H-E) + O(H-E)^2}{(E-H) + O(H-E)^2 + i(\epsilon\frac{dE}{df})}|\alpha\rangle \\
&\quad= \lim_{\epsilon'\to 0^+} \langle\beta|i\epsilon'\frac{(H-E) + O(H-E)^2}{(E-H) + O(H-E)^2 + i\epsilon'}|\alpha\rangle \\
&\quad= \langle\beta|T^{ba}(E_\alpha + i0^+)|\alpha\rangle,
\end{aligned} \tag{A.30}$$

where $\epsilon' := \epsilon\frac{dE}{df}$. In order to obtain Eq. (A.30), we have assumed that $\frac{df}{dE} > 0$ and that higher order terms in $(E - H)$ do not contribute in the limit $\epsilon \to 0$. The sign of the derivative ensures that ϵ' has the same sign as ϵ. In addition, some regularity conditions are needed so that higher

order terms do not contribute in the limit. An analysis based on time dependent scattering theory shows that the desired result holds rigorously for suitable interactions if f has positive derivative and the second derivative is locally integrable (Bi 62, Ka 66, Ch 75, Ob 78). When the function f depends parametrically on $\mathbf{p}$, the dominated convergence theorem justifies the interchange in the order of the limits and the integral over the total momentum.

The important property of the invariance principle is that it allows for a greater flexibility in formulating scattering problems. S-matrix elements can thus be expressed as follows:

$$\langle\beta|S|\alpha\rangle = \langle\beta|S_f|\alpha\rangle, \tag{A.31}$$

where

$$\langle\beta|S_f|\alpha\rangle = \langle\beta|\alpha\rangle\delta_{\beta\alpha} - 2\pi i\delta(E_\beta - E_\alpha)\frac{dE}{df}\langle\beta|T_f^{ba}\left(f(E_\alpha) + i0^+\right)|\alpha\rangle. \tag{A.32}$$

If this equation is combined with a reduced matrix element normalized according to Eq. (A.17), we get

$$\langle\beta|S_f|\alpha\rangle = \langle\beta|\alpha\rangle\delta_{\beta\alpha} - i(2\pi)\delta^4(p_\beta - p_\alpha)\langle\beta\|T_f^{ba}(f(E_a) + i0^+)\|\alpha\rangle. \tag{A.33}$$

This formula also holds when V commutes with the light-front components of the four-momentum or with the four-velocity, provided the reduced matrix element is defined with the same conventions used in Eq. (A.17).

This shows how the relation between the scattering operator and the transition operator is modified in time independent scattering theory when one (1) replaces H by $f(H)$ and/or (2) changes kinematic subgroups.

A.3. **Cross Sections**

We now discuss the relation between the scattering matrix elements and scattering cross sections The content of this section follows the classic references of Möller (Mo 45) and Brenig and Haag (Br 59).

Plane-wave states will be used with the normalization

$$\langle \mathbf{p}' | \mathbf{p} \rangle = \delta(\mathbf{p}' - \mathbf{p}). \tag{A.34}$$

With this choice, the square of the magnitude of the wave functions, $|\langle \mathbf{p} | \phi \rangle|^2$, will have the interpretation of probability per unit volume in momentum space. The Fourier transforms $|\langle \mathbf{r} | \phi \rangle|^2$ then have the interpretation of probability per unit volume.

Consider a reaction initiated by the collision of a projectile and target cluster. Assume that the target and projectile are described by normalizable wave functions with very sharp momentum distributions centered about $\bar{\mathbf{p}}_t$ and $\bar{\mathbf{p}}_p$, respectively. With this choice of normalization, the probability density that the system prepared with this initial state will be found in a state of N particles with momenta centered about $\mathbf{p}_1 \cdots \mathbf{p}_N$ is given by

$$|\langle \mathbf{p}_1 \cdots \mathbf{p}_N | \phi_n \rangle|^2, \tag{A.35}$$

where

$$\langle \mathbf{p}_1 \cdots \mathbf{p}_N | \phi_n \rangle := \int d^3 p_t \int d^3 p_p \, \langle \mathbf{p}_1 \cdots \mathbf{p}_N | S | \mathbf{p}_t \mathbf{p}_p \rangle \langle \mathbf{p}_t | \phi_t \rangle \langle \mathbf{p}_p | \phi_p \rangle. \tag{A.36}$$

If there are any identical particles in the final state, Eq. (A.36) must be multiplied by the square root of the statistical factor:

$$\frac{1}{s} = \prod_{i=1}^{k} \frac{1}{n_i!}, \tag{A.37}$$

where there is a factor $1/n_i!$ for each group of n_i identical particles in the final state. Note that for n identical particles, the resolution of the identity is

$$I = \frac{1}{n!} \int d^3 p_1 \cdots \int d^3 p_n | \mathbf{p}_1 \cdots \mathbf{p}_n \rangle \langle \mathbf{p}_1 \cdots \mathbf{p}_n |, \tag{A.38}$$

if the single particle states are normalized as in Eq. (A.34). In what follows, a factor of $1/s$ will be included.

This expression can be simplified by using the property that the initial wave packets are sharply peaked, and the assumption that the transition matrix elements are smooth functions of the momenta. If the scattering is not elastic or, in the case of elastic scattering, the detector is not along the beam line, then the scattering operator can be replaced by the transition operator term alone:

$$\begin{aligned}
&\langle \mathbf{p}_1 \cdots \mathbf{p}_N | \phi_n \rangle \\
&\quad = -2\pi i \delta(E_N - E_i) \frac{dE}{df} \int d^3p_t \int d^3p_p \, \langle \mathbf{p}_1 \cdots \mathbf{p}_N | T_f^{ba} \left(f(E_i) + i0^+ \right) | \mathbf{p}_t \mathbf{p}_p \rangle \langle \mathbf{p}_t \mathbf{p}_p | \phi \rangle \\
&\quad \approx \ -(2\pi)^4 i \frac{dE}{df} \langle \mathbf{p}_1 \cdots \mathbf{p}_N \| T_f^{ba} \left(f(E_i) + i0^+ \right) \| \bar{\mathbf{p}}_t \bar{\mathbf{p}}_p \rangle \\
&\qquad \times \int d^3p_t \int d^3p_p \, \langle \mathbf{p}_t \mathbf{p}_p | \phi \rangle \delta(E_N - E_i) \delta(\mathbf{P}_N - \mathbf{P}_t - \mathbf{p}_i).
\end{aligned} \tag{A.39}$$

Using Eq. (A.39) in (A.35), we get

$$\begin{aligned}
&|\langle \mathbf{p}_1 \cdots \mathbf{p}_N | \phi_n \rangle|^2 \\
&\quad = \frac{4\pi^2}{s} \left| \frac{dE}{df} \langle \mathbf{p}_1 \cdots \mathbf{p}_N \| T_f^{ba} \left(f(E_i) + i0^+ \right) \| \bar{\mathbf{p}}_t \bar{\mathbf{p}}_p \rangle \right|^2 \\
&\qquad \times \int d^3p'_t \int d^3p'_p \int d^3p_t \int d^3p_p \, \langle \mathbf{p}'_t | \phi_t \rangle^* \langle \mathbf{p}'_p | \phi_p \rangle^* \langle \mathbf{p}_t | \phi_t \rangle \langle \mathbf{p}_p | \phi_p \rangle \\
&\qquad \times \delta(E_N - E'_t - E'_p) \delta(\mathbf{P}_N - \mathbf{p}'_t - \mathbf{p}'_p) \delta(E_N - E_t - E_p) \delta(\mathbf{P}_N - \mathbf{p}_t - \mathbf{p}_p).
\end{aligned} \tag{A.40}$$

The integral can be expressed in terms of position-space wave functions as follows:

$$\begin{aligned}
&\int d^3p'_t \int d^3p'_p \int d^3p_t \int d^3p_p \, \langle \mathbf{p}_t | \phi_p \rangle \langle \mathbf{p}_p | \phi_p \rangle \langle \mathbf{p}'_t | \phi_p \rangle^* \langle \mathbf{p}'_p | \phi_p \rangle^* \\
&\qquad \times \delta(E_N - E_t - E_p) \delta(\mathbf{P}_N - \mathbf{p}_t - \mathbf{p}_p) \delta(E_t + E_p - E'_t - E'_p) \delta(\mathbf{p}_t + \mathbf{p}_p - \mathbf{p}'_t - \mathbf{p}'_p) \\
&\quad = (2\pi)^{-4} \int d^3r \int dt \, e^{i(\mathbf{p}_t + \mathbf{p}_p - \mathbf{p}'_t - \mathbf{p}'_p) \cdot \mathbf{r} - i(E_t + E_p - E'_t - E'_p)t} \\
&\qquad \times \int d^3p'_t \int d^3p'_p \int d^3p_t \int d^3p_p \, \langle \mathbf{p}_t | \phi_p \rangle \langle \mathbf{p}_p | \phi_p \rangle \langle \mathbf{p}'_t | \phi_p \rangle^* \langle \mathbf{p}'_p | \phi_p \rangle^* \\
&\quad = \delta(E_N - \bar{E}_t - \bar{E}_p) \delta(\mathbf{P}_N - \bar{\mathbf{p}}_t - \bar{\mathbf{p}}_p) (2\pi)^2 \int d^3r \int dt \, \left| \langle (\mathbf{r}, t) | \phi_t \rangle \langle (\mathbf{r}, t) | \phi_p \rangle \right|^2 .
\end{aligned} \tag{A.41}$$

We can now write the following expression for the differential probability:

$$\begin{aligned}
dW = &\frac{(2\pi)^4}{s} \left| \frac{dE}{df} \langle \mathbf{p}_1 \cdots \mathbf{p}_N \| T_f^{ba} \left(f(E_i) + i0^+ \right) \| \bar{\mathbf{p}}_t \bar{\mathbf{p}}_p \rangle \right|^2 \\
&\times \int d^3r \int dt \, \left| \langle \mathbf{r}, t | \phi_t \rangle \langle \mathbf{r}, t | \phi_p \rangle \right|^2 \delta(E_N - \bar{E}_t - \bar{E}_p) \delta(\mathbf{P}_N - \bar{\mathbf{p}}_t - \bar{\mathbf{p}}_p) \prod_{i=1}^{N} d^3p_i.
\end{aligned} \tag{A.42}$$

This expression is a distribution over all energies and momenta, and is integrated over all space and time. The distribution with respect to total energy and momenta (which are conserved) can be integrated out. To do this, we introduce the phase space element $d\Phi_N$ as follows:

$$\prod_{i=1}^{N} d^3p_i = dE_n d^3P_N d\Phi_N. \tag{A.43}$$

Since

$$1 = \int dE_n d^3P \delta^4(P - P(\mathbf{p}_1 \cdots \mathbf{p}_N)), \tag{A.44}$$

it is possible to write

$$d\Phi_N = \int \prod_{i=1}^{N} d^3p_i \delta^4(P(\bar{\mathbf{p}}_t \bar{\mathbf{p}}_p) - P(\mathbf{p}_1 \cdots \mathbf{p}_N)), \tag{A.45}$$

where the integral is over any four variables that eliminate the delta functions. The remaining quantities are independent measurable quantities. After integrating out the total energy and momentum, the integrand with respect to space and time represents the probability per unit time and volume that a particle will be detected in the phase space element $d\Phi_N$, independent of the specific energy-momentum distribution of the initial state. This quantity will be denoted by dw, and has been shown to be

$$dw = \frac{(2\pi)^4}{s} \left| \frac{dE}{df} \langle \mathbf{p}_1 \cdots \mathbf{p}_N \| T_f^{ba} \left(f(E_i) + i0^+ \right) \| \bar{\mathbf{p}}_t \bar{\mathbf{p}}_p \rangle \right|^2 |\langle (\mathbf{r}, t) | \phi_t \rangle \langle (\mathbf{r}, t) | \phi_p \rangle|^2 \, d\Phi_N. \tag{A.46}$$

The differential cross section is the ratio of the transition rate per unit volume to the product of the incident probability current and the target density:

$$\begin{aligned} d\sigma &:= \frac{dw}{v_{p-t} |\langle (\mathbf{r}, t) | \phi_t \rangle \langle (\mathbf{r}, t) | \phi_p \rangle|^2} d\Phi_N \\ &= \frac{(2\pi)^4}{s v_{p-t}} \left| \frac{dE}{df} \langle \mathbf{p}_1 \cdots \mathbf{p}_N \| T_f^{ba} \left(f(E_i) + i0^+ \right) \| \bar{\mathbf{p}}_t \bar{\mathbf{p}}_p \rangle \right|^2 d\Phi_N. \end{aligned} \tag{A.47}$$

This expression is valid both relativistically and nonrelativistically. In the relativistic case, $d\sigma$ is also relativistically invariant (Mo 45). To show this, we redistribute the momentum dependent

factors so that the phase space factors, the transition matrix elements, and the velocity factors are separately invariant. The first step is to change the single particle plane wave normalization to a covariant normalization:

$$|\mathbf{p}_i\rangle \to |\mathbf{p}_i\rangle_{\rm cov} := \mathcal{K}_{p_i}|\mathbf{p}_i\rangle. \tag{A.48}$$

Covariance requires that

$$\mathcal{K}_{p_i} = C \times \sqrt{\omega_{m_i}(\mathbf{p}_i^2)}; \qquad \omega_{m_i}(\mathbf{p}_i^2) := \sqrt{m_i^2 + \mathbf{p}_i^2} \tag{A.49}$$

The constant C is arbitrary. We now define an invariant reduced transition matrix element:

$$\begin{aligned}\langle \mathbf{p}_1 \cdots \mathbf{p}_n \| M^{ab} \| \mathbf{p}_t \mathbf{p}_p \rangle &:= \frac{dE}{df} {\rm cov}\langle \mathbf{p}_1 \cdots \mathbf{p}_N \| T_f^{ba} \left(f(E_i) + i0^+ \right) \| \bar{\mathbf{p}}_t \bar{\mathbf{p}}_p \rangle_{\rm cov} \\ &:= \frac{dE}{df} \langle \mathbf{p}_1 \cdots \mathbf{p}_N \| T_f^{ba} \left(f(E_i) + i0^+ \right) \| \bar{\mathbf{p}}_t \bar{\mathbf{p}}_p \rangle (\mathcal{K}_t \mathcal{K}_p \prod_{i=1}^N \mathcal{K}_{p_i}).\end{aligned} \tag{A.50}$$

This entire quantity is invariant (ignoring spins) because S is invariant, the basis in which S is evaluated is covariant, and the reduced transition operator is obtained from S by factoring out a four-momentum conserving delta function. This definition introduces the factor $(\mathcal{K}_t \mathcal{K}_p \prod_{i=1}^N \mathcal{K}_{p_i})^2$ into the expression for the cross section. The factors associated with the final state can be included in the phase space factor

$$dL_N := \frac{d\Phi_N}{\prod_{i=1}^N \mathcal{K}_{pi}^2} = \prod_{i=1}^N \frac{d^3 p_i}{\mathcal{K}_{pi}^2} \delta^4(P - P(\mathbf{p}_1 \cdots \mathbf{p}_N)), \tag{A.51}$$

which is invariant. What remains is the factor $(v_{p-t}\mathcal{K}_t^2 \mathcal{K}_p^2)^{-1}$. Direct evaluation shows that (Mo 45)

$$v_{p-t}\omega_{m_t}(\mathbf{p}_t^2)\omega_{m_p}(\mathbf{p}_p^2) = v_{p-t}\frac{\mathcal{K}_t^2}{C^2}\frac{\mathcal{K}_p^2}{C^2} = \sqrt{(p_t \cdot p_p)^2 - m_t^2 m_p^2}. \tag{A.52}$$

Thus, $(v_{p-t}\mathcal{K}_t^2\mathcal{K}_p^2)^{-1}$ is invariant, and we denote it by $C^4 F$, where C is defined in Eq. (A.49), and F is invariant. The differential cross section can now be expressed as follows:

$$d\sigma := \frac{(2\pi)^4}{s}\frac{1}{C^4}\frac{|\langle \mathbf{p}_1 \cdots \mathbf{p}_n \| M^{ab} \| \mathbf{p}_t \mathbf{p}_p \rangle|^2}{F} dL_N. \tag{A.53}$$

We have not yet considered spin degrees of freedom. The suppressed spin variable lead to

the modification

$$d\sigma \to \frac{(2\pi)^4}{s}\frac{1}{C^4}\frac{|\langle \mathbf{p}_1\mu_1\cdots\mathbf{p}_n\mu_n\|M^{ab}\|\mathbf{p}_t\nu_t\mathbf{p}_p\nu_p\rangle|^2}{F}dL_N. \tag{A.54}$$

In general, the magnetic quantum numbers can be associated with any type of spin. In applications, any type of spin can be used. However, when one finally compares a calculation to experiment, one has to know how the given spin is coupled to the device that separates different spin states. For electron scattering, one measures invariant form factors, which can be used to extract current matrix elements with different types of spins. The relation of the current matrix element to the form factor will be different for each type of spin.

The initial and final states are prepared from measured ensembles, which are properly described by a density matrix in the spin degrees of freedom. For the initial state, the target and projectile are described by the density matrices $\rho_t(\nu_t\nu_t')$ and $\rho_b(\nu_b\nu_b')$, which are positive Hermitian matrices with unit trace. The differential cross section becomes an unnormalized density matrix in the final spins:

$$\begin{aligned} d\rho_f := \frac{(2\pi)^4}{s}\frac{1}{C^4F}&\langle \mathbf{p}_1\mu_1'\cdots\mathbf{p}_n\mu_n'\|M^{ab}\|\mathbf{p}_t\nu_t'\mathbf{p}_p\nu_p'\rangle\rho_t(\nu_t'\nu_t) \\ &\times \rho_p(\nu_p'\nu_p)\langle \mathbf{p}_t\nu_t\mathbf{p}_p\nu_p\|M^{ab\dagger}\|\mathbf{p}_1\mu_1\cdots\mathbf{p}_n\mu_n\rangle dL_N. \end{aligned} \tag{A.55}$$

The expectation value of a spin observable $\mathcal{O}$ in this ensemble is computed by taking the trace with respect to the renormalized density matrix:

$$\langle\mathcal{O}\rangle := \frac{\mathrm{Tr}(\mathcal{O}d\rho_f)}{\mathrm{Tr}(d\rho_f)}, \tag{A.56}$$

where the trace is over the spins. The kinematic factors cancel in the computation of the renormalized density matrix.

A.4. **Phenomenological Interactions**

One case of special interest is nucleon-nucleon scattering, where phenomenological interactions have been determined by fitting solutions of the Lippmann-Schwinger equation:

$$|\psi^{(-)}\rangle = |\mathbf{k}_i \mu_{1i} \mu_{2i}\rangle + \frac{1}{\mathbf{k}_i^2/2m_r - \mathbf{k}^2/2m_r + i0^+} V_{NN} |\psi^{(-)}\rangle \tag{A.57}$$

to the invariant cross section:

$$d\sigma_{NR} = (2\pi)^4 \frac{1}{k/\mu} |\langle \mathbf{k}_f \mu_{1f} \mu_{2f} \| H - H_0 \| \psi^{(-)} \rangle|^2 k^2 \frac{\mu}{k} d\Omega(\hat{\mathbf{k}}_f), \tag{A.58}$$

where μ is the reduced mass of the two-body system:

$$\mu = m_1 m_2/(m_1 + m_2), \tag{A.59}$$

and

$$H = \frac{\mathbf{p}^2}{2(m_1 + m_2)} + \frac{\mathbf{k}^2}{2\mu} + V_{NN}. \tag{A.60}$$

The double bars indicate the reduced matrix element. Normally, the procedure is to transform the results of measurements to the center-of-momentum frame using rotationless Lorentz transformations, and then to adjust the parameters of V_{NN} to fit phase shifts as a function of the relative momentum. The key to the connection between fitting procedures based upon the nonrelativistic Hamiltonian H as defined in Eq. (A.60) and those using a mass operator in a two-body construction of the sort considered in the review is that a relative momentum variable enters the problem in the same way in both approaches.

Consider a relativistic two-body problem involving interacting and non-interacting mass operators M and M_0, respectively. We introduce functions $f(H)$ and $f(H_0)$, where H is defined in Eq. (A.60) and H_0 is the non-interacting Hamiltonian. We also introduce functions $g(M)$ and $g(M_0)$, which can also be kinematic functions of the four-momentum (*i.e.,* the three-momentum

in an instant form dynamics, the light-front component of the four momentum in a front-form dynamics, or the four-velocity in a point-form dynamics). We now require that

$$W = f(H) - f(H_0) = g(M) - g(M_0) \tag{A.61}$$

and

$$f(H_0) = g(M_0) = k^2. \tag{A.62}$$

An examples is

$$f(\alpha) = m(\alpha - \mathbf{p}^2/4m); \qquad g(\beta) = (\frac{1}{4}\beta^2 - m^2) \tag{A.63}$$

for nucleons of mass m. The differential cross section in the relativistic case is

$$d\sigma = (2\pi)^4 \frac{1}{k/\eta} |\langle \mathbf{k}_f \mu_{1f} \mu_{2f} \| M - M_0 \| \psi^{(-)} \rangle|^2 k^2 \frac{\eta}{k} d\Omega(\hat{\mathbf{k}}_f), \tag{A.64}$$

where

$$\eta = \frac{\omega_{m_1}(\mathbf{k})\omega_{m_2}(\mathbf{k})}{\omega_{m_1}(\mathbf{k}) + \omega_{m_2}(\mathbf{k})}. \tag{A.65}$$

We now show that $d\sigma_{NR} = d\sigma$, provided M and H are related as follows:

$$\begin{aligned} W &= f(H) - f(H_0) = g(M) - g(M_0)) \\ &= f(H) - \mathbf{k}^2 = g(M) - \mathbf{k}^2. \end{aligned} \tag{A.66}$$

To show the equality of the cross sections, let $|\mathbf{p}\mathbf{k}_i \mu_{1i} \mu_{2i}\rangle$ and $|\psi^{(-)}\rangle$ be eigenstates of H_0 and H with the same eigenvalues. The relation (A.66) implies that these are also eigenstates of M and M_0, respectively, but with different eigenvalues.

The two cross sections can be formally expressed as

$$d\sigma_{NR} = (2\pi)^4 \frac{1}{k/\mu} \left| \langle \mathbf{k}_f \mu_{1f} \mu_{2f} \| \lim_{h \to h_0} \frac{h - h_0}{g(h) - g(h_0)} [g(H) - g(H_0)] \| \psi^{(-)} \rangle \right|^2 k^2 \frac{\mu}{k} d\Omega(\hat{\mathbf{k}}_f); \tag{A.67}$$

$$d\sigma = (2\pi)^4 \frac{1}{k/\eta} \left| \langle \mathbf{k}_f \mu_{1f} \mu_{2f} \| \lim_{m \to m_0} \frac{m - m_0}{f(m) - f(m_0)} [f(M) - f(M_0)] \| \psi^{(-)} \rangle \right|^2 k^2 \frac{\eta}{k} d\Omega(\hat{\mathbf{k}}_f), \tag{A.68}$$

where the lower-case quantities m, m_0, h and h_0 are the eigenvalues of M, M_0, H and H_0, respectively. We have suppressed the total momentum variable, which is only a parameter in the

reduced matrix element. Using Eq. (A.66) and taking the appropriate limits, the cross sections become

$$d\sigma_{NR} = (2\pi)^4 \frac{1}{k/\mu}\frac{dh_0}{dg}\left|\langle \mathbf{k}_f \mu_{1f}\mu_{2f}\|W\|\psi^{(-)}\rangle\right|^2 k^2 \left(\frac{\mu}{k}\frac{dh_0}{dg}\right) d\Omega(\hat{\mathbf{k}}_f); \tag{A.69}$$

$$d\sigma = (2\pi)^4 \frac{1}{k/\eta}\frac{dm_0}{df}\left|\langle \mathbf{k}_f \mu_{1f}\mu_{2f}\|W\|\psi^{(-)}\rangle\right|^2 k^2 \left(\frac{\eta}{k}\frac{dm_0}{df}\right) d\Omega(\hat{\mathbf{k}}_f). \tag{A.70}$$

Note that

$$\frac{1}{k/\mu}\frac{dh_0}{dk} = 1 = \frac{1}{k/\eta}\frac{dm_0}{dk}. \tag{A.71}$$

Equation (A.62) also implies that

$$\frac{df}{dk} = \frac{dg}{dk} = 2k. \tag{A.72}$$

If these relations are used with the chain rule in the expression for the cross section, we find

$$d\sigma_{NR} = d\sigma = (2\pi)^4/over4\left|\langle \mathbf{k}_f \mu_{1f}\mu_{2f}\|W]\|\psi^{(-)}\rangle\right|^2 d\Omega(\hat{\mathbf{k}}_f). \tag{A.73}$$

To understand the implication of this result, we construct a relativistic model whose cross section has the form

$$\begin{aligned} d\sigma &:= \frac{dw}{v_{p-t}|\langle(\mathbf{r},t)|\phi_t\rangle\langle(\mathbf{r},t)|\phi_p\rangle|^2} d\Phi_2 \\ &= \frac{(2\pi)^4}{sv_{p-t}}\left|\frac{dE}{dF}\langle \mathbf{p}_1\mu_1\mathbf{p}_2\mu_2\|T_F\left(f(E_i)+i0^+\right)\|\bar{\mathbf{p}}_t\bar{\mathbf{p}}_p\rangle\right|^2 d\Phi_2, \end{aligned} \tag{A.74}$$

where

$$T_F^{ba}(z) := V_F + V_F R_F(z) V_F. \tag{A.75}$$

The interaction $V_F := F(M) - F(M_0)$ can be chosen in such a way the that corresponding $W = f(M(F)) - \mathbf{k}^2$ is given by $2\mu V_{NN}$, The model defined by Eqs. (A.74) and (A.75) is then

equivalent to a model with

$$|\psi^{(-)}\rangle = |\psi_0\rangle = \frac{1}{z - k^2/2\mu} V_{NN} |\psi^{(+)}\rangle \tag{A.76}$$

and

$$d\sigma = \frac{(2\pi)^4}{sk/\mu} |\langle \psi_0 | V_{NN} | \psi^{(-)} \rangle|^2 k\mu d\Omega(\hat{\mathbf{k}}), \tag{A.77}$$

where the plane wave and scattering states in Eq. (A.77) are evaluated on shell. Equations (A.76) and (A.77) are fully relativistic, but are identical in form to the nonrelativistic equations that relate interactions to data. This result applies to any of the forms of dynamics discussed in this review, because it is possible in each case to find a function $g(M)$ which satisfies the necessary conditions.

APPENDIX B

Front-Form Kinematics

The front-form formulation of relativistic quantum mechanics has a distinct advantage because it has the largest kinematic subgroup. The kinematic subgroup includes three independent spacetime translations and a four-parameter subgroup of Lorentz transformations. This Appendix is intended to provide some general reference for the front form, including its generators and their commutation relations.

The kinematic subgroup of the front form is the subgroup of Poincaré transformations that leave the light front $x^+ = 0$ invariant. Note that the zero is important here: if this condition is replaced by $x^+ = \text{const} \neq 0$, a six-parameter proper subgroup of the kinematic subgroup is obtained.

The kinematic subgroup is the subgroup of elements $(\underline{\Lambda}, \underline{a})$ of $\mathcal{P}$ which preserve the condition $x^+ = 0$ in

$$\underline{X'} = \underline{\Lambda} \underline{X} \underline{\Lambda}^\dagger + \underline{a}, \tag{B.1}$$

where $X = \sigma_\mu x^\mu$, $X' = \sigma_\mu x'^\mu$. Direct computation shows that the required matrices must have the form

$$\underline{\Lambda} = \begin{pmatrix} a_{11} & 0 \\ a_{21} & 1/a_{11} \end{pmatrix}; \qquad \underline{a} = \begin{pmatrix} 0 & b_{12} \\ b_{12}^* & b_{22} \end{pmatrix}. \tag{B.2}$$

This is a seven-parameter subgroup. The parameters can be taken to be the real and imaginary parts of a_{11}, a_{21}, b_{12}, as well as b_{22}, which is real. This subgroup is generated by seven basic transformations. These transformation and the associated generators of the kinematic subgroup

are given below:

$$
\begin{aligned}
K^3 &:= i\frac{\partial}{\partial u}U(\underline{\Lambda}_1(u),\underline{0})_{|_{u=0}}; \qquad \underline{\Lambda}_1(u) := \begin{pmatrix} e^{u/2} & 0 \\ 0 & e^{-u/2} \end{pmatrix} = e^{u\sigma_3/2};\\
J^3 &:= i\frac{\partial}{\partial \phi}U(\underline{\Lambda}_2(\phi),\underline{0})_{|_{\phi=0}}; \qquad \underline{\Lambda}_2(\phi) := \begin{pmatrix} e^{-i\phi/2} & 0 \\ 0 & e^{i\phi/2} \end{pmatrix} = e^{-i\phi\sigma_3/2};\\
E^1 &:= i\frac{\partial}{\partial \lambda}U(\underline{\Lambda}_3(\lambda),\underline{0})_{|_{\lambda=0}}; \qquad \underline{\Lambda}_3(\lambda) := \begin{pmatrix} 1 & 0 \\ \lambda & 1 \end{pmatrix} = e^{-i\lambda(\sigma_2+i\sigma_1)/2};\\
E^2 &:= i\frac{\partial}{\partial \lambda}U(\underline{\Lambda}_4(\lambda),\underline{0})_{|_{\lambda=0}}; \qquad \underline{\Lambda}_4(\lambda) := \begin{pmatrix} 1 & 0 \\ i\lambda & 1 \end{pmatrix} = e^{-i\lambda(i\sigma_2-\sigma_1)/2};
\end{aligned}
\tag{B.3}
$$

$$
\begin{aligned}
P^1 &:= i\frac{\partial}{\partial b_1}U(\underline{I},\underline{a}_1(b_1))_{|_{b_1=0}}; \qquad \underline{a}_1(b_1) := \begin{pmatrix} 0 & b_1 \\ b_1 & 0 \end{pmatrix};\\
P^2 &:= i\frac{\partial}{\partial b_2}U(\underline{I},\underline{a}_2(b_2))_{|_{b_2=0}} \qquad \underline{a}_2(b_2) := \begin{pmatrix} 0 & -ib_2 \\ ib_2 & 0 \end{pmatrix};\\
P^+ &:= 2i\frac{\partial}{\partial b_-}U(\underline{I},\underline{a}_3(b_-))_{|_{b_-=0}}; \qquad \underline{a}_3(b_-) := \begin{pmatrix} 0 & 0 \\ 0 & b_- \end{pmatrix}.
\end{aligned}
\tag{B.4}
$$

In addition to these generators, there are three other independent generators which are not kinematic:

$$
\begin{aligned}
F^1 &:= i\frac{\partial}{\partial \lambda}U(\underline{\Lambda}_5(\lambda),\underline{0})_{|_{\lambda=0}}; \qquad \underline{\Lambda}_5(\lambda) := \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix} = e^{-i\lambda(i\sigma_1-\sigma_2)/2};\\
F^2 &:= i\frac{\partial}{\partial \lambda}U(\underline{\Lambda}_6(\lambda),\underline{0})_{|_{\lambda=0}}; \qquad \underline{\Lambda}_6(\lambda) := \begin{pmatrix} 1 & -i\lambda \\ 0 & 1 \end{pmatrix} = e^{-i\lambda(\sigma_1+i\sigma_2)/2};
\end{aligned}
\tag{B.5}
$$

$$
P^- := 2i\frac{\partial}{\partial b_+}U(\underline{I},\underline{a}_4(b_+))_{|_{b_+=0}}; \qquad \underline{a}_4(b_+) := \begin{pmatrix} b+ & 0 \\ 0 & 0 \end{pmatrix}. \tag{B.6}
$$

By explicit differentiation, the 45 commutation relations can be determined. The generators of the kinematic subgroup form a closed Lie subalgebra:

$$
[P^+,\mathbf{E}]_- = E^j,E^k]_- = [J^3,P^+]_- = [P^+,P^j]_- = 0; \tag{B.7}
$$

$$
[J^3,\mathbf{E}]_- = -i\hat{\mathbf{z}}\times\mathbf{E}; \qquad [K^3,\mathbf{E}]_- = -i\mathbf{E}; \tag{B.8}
$$

$$[P^j, E^k]_- = i\delta^{jk}P^+; \qquad [K^3, P^+]_- = -iP^+. \tag{B.9}$$

Commutators which involve at least one non-kinematic generator are:

$$[F^j, F^k]_- = [\mathbf{F}, P^-]_- = [P^-, J^3]_- = [P^-, P^k]_- = [P^+, P^-]_- = 0; \tag{B.10}$$

$$[F^j, E^j]_- = 2iK^3; \qquad [F^j, E^k]_- = -i\epsilon^{jk3}J^3; \qquad [\mathbf{F}, K^3]_- = -i\mathbf{F}; \tag{B.11}$$

$$[F^j, J^3]_- = i\epsilon^{j3k}F^k; \qquad [F^j, P^k]_- = -i\delta^{jk}P^-; \qquad [\mathbf{F}, P^+]_- = -2i\mathbf{P}; \tag{B.12}$$

$$[P^-, \mathbf{E}]_- = 2i\mathbf{P}; \qquad [P^-, K^3]_- = iP^+. \tag{B.13}$$

The generators $\mathbf{E}_\perp$ and $\mathbf{F}_\perp$ are related to the generators of transverse boosts and rotations by:

$$\mathbf{E}_\perp = \mathbf{K}_\perp - \hat{\mathbf{z}} \times \mathbf{J}_\perp; \tag{B.14}$$

$$\mathbf{F}_\perp = \mathbf{K}_\perp + \hat{\mathbf{z}} \times \mathbf{J}_\perp, \tag{B.15}$$

while P^+ and P^- are related to P^0 and P^3 by

$$P^+ = P^0 + P^3; \qquad P^- = P^0 - P^3. \tag{B.16}$$

Sometimes the dynamic generators F^1 and F^2 are replaced by the generators J^1 and J^2 of transverse rotations.

The mass operator is the following function of the generators:

$$M^2 = P^+P^- - \mathbf{P}_\perp^2. \tag{B.17}$$

The generators E^1, E^2, and K^3 (the so-called "front-form boost" generators) form a closed

Lie subalgebra. The corresponding subgroup of $ISL(2,C)$ is the group of matrices of the form

$$\begin{pmatrix} a & 0 \\ b_1 + ib_2 & 1/a \end{pmatrix} \tag{B.18}$$

where all three of these parameters are real. The following alternate parameterization is also useful:

$$\begin{aligned} \underline{\Lambda}_f(Q) &= \underline{\Lambda}_3(Q_1/Q^+)\underline{\Lambda}_4(Q_2/Q^+)\underline{\Lambda}_1(\ln Q^+) = \underline{\Lambda}_1(\ln Q^+)\underline{\Lambda}_3(Q_1)\underline{\Lambda}_4(Q_2) \\ &= \begin{pmatrix} (Q^+)^{\frac{1}{2}} & 0 \\ (Q_1 + iQ_2)(Q^+)^{-\frac{1}{2}} & (Q^+)^{-\frac{1}{2}} \end{pmatrix}. \end{aligned} \tag{B.19}$$

These transformations have the following property

$$\underline{Q} = \underline{\Lambda}_f(Q) I \underline{\Lambda}(Q)^\dagger, \tag{B.20}$$

where

$$\underline{Q} = \begin{pmatrix} Q^+ & Q_1 - iQ_2 \\ Q_1 + iQ_2 & Q^- \end{pmatrix}, \tag{B.21}$$

and

$$Q^- := \frac{1 + \mathbf{Q}^2}{Q^+}. \tag{B.22}$$

Multiplying Eq. (B.20) by m on both sides shows that the effect of $\underline{\Lambda}_f(Q)$ is to transform a timelike rest four-vector to a frame where it has momentum $\mathbf{P} = m\mathbf{Q}$. Since this boost is an element of the kinematic subgroup, it follows that these transformations take the light front into itself. The front-form boosts have two important distinctions from canonical or rotationless boosts. The first is that

$$\underline{L}_f(\mathbf{Q})^{-1} \neq \underline{L}_f(-\mathbf{Q}), \tag{B.23}$$

and the second is that these transformations form a subgroup, whereas the rotationless boosts do not.

The front-form spin operator is constructed by applying the inverse of a front-form boost to the Pauli Lubanski operator:

$$(0, \mathbf{j}_f) := \frac{1}{M} L_f^{-1}(Q)^\mu{}_\nu W^\nu, \tag{B.24}$$

where

$$L_f^{-1}(Q)^\mu{}_\nu := \tfrac{1}{2} \operatorname{Tr}(\sigma_\mu A_f(P/M)^{-1} \sigma_\nu (A_f(P/M)^{-1})^\dagger), \tag{B.25}$$

where P and M are the components of the four-momentum *operator* and the mass *operator.*

Direct computation of Eq. (B.25) in terms of the generators yields expressions for the components of the front form spin in terms of the generators:

$$j_f^3 = \frac{W^+}{P^+} = \frac{1}{P^+} \left[P^+ J^3 + \hat{\mathbf{z}} \cdot (\mathbf{E}_\perp \times \mathbf{P}_\perp) \right]; \tag{B.26}$$

$$\mathbf{j}_{f\perp} = \frac{1}{M} \left\{ \hat{\mathbf{z}} \times \left[\tfrac{1}{2}(P^- \mathbf{E}_\perp - P^+ \mathbf{F}_\perp) + \mathbf{P}_\perp K^3 \right] - \frac{\mathbf{P}}{P^+} \left[P^+ J^3 + \hat{\mathbf{z}} \cdot (\mathbf{E}_\perp \times \mathbf{P}_\perp) \right] \right\}. \tag{B.27}$$

These can be inverted to express the dynamical generators in terms of kinematic generators, the components of the front-form spin operator, and the mass operator:

$$P^- = \frac{M^2 + \mathbf{P}_\perp^2}{P^+}; \tag{B.28}$$

$$\mathbf{F}_\perp = 2\frac{K^3}{P^+}\mathbf{P}_\perp + \frac{P^-}{P^+}\mathbf{E}_\perp + \frac{2}{P^+}(j_f^3 \mathbf{P}_\perp + M \mathbf{j}_{f\perp}). \tag{B.29}$$

A complete set of dynamical variables for a single particle consists of the operators P^+, P^1, P^2 and j_f^3, along with the total spin and mass of the particle. For a many-body system it is convenient to define the total momentum operators

$$P^+ := \sum_{i=1}^N P_i^+; \tag{B.30}$$

$$\mathbf{P}_\perp := \sum_{i=1}^N P_{i\perp}, \tag{B.31}$$

along with the internal variables

$$k_i := (k_i^+, \mathbf{k}_\perp); \qquad k_i^\mu = L_f^{-1}(P/M_0)^\mu{}_\nu P_i^\nu. \tag{B.32}$$

If k_i had been defined with a rotationless boost in place of the front-form boost in Eq. (B.32), it would undergo Wigner rotations with a general Lorentz transformation of the system. However, because the front-form boosts form a subgroup, the front-form vectors k_i are invariant (*i.e.,* they do not undergo Wigner rotations) under front-form boosts. The components of k_i satisfy the relations

$$\sum_{i=1}^N k_i^+ = M_0; \tag{B.33}$$

$$\sum_{i=1}^N \mathbf{k}_{i\perp} = 0. \tag{B.34}$$

In applications is useful to replace the + component of k by its momentum fraction:

$$\xi_i := \frac{P_i^+}{\sum_{i=1}^N P_i^+} = \frac{k_i^+}{\sum_{i=1}^N k_i^+}. \tag{B.35}$$

The Hamiltonian P^- for a system of N free particles has the form

$$P^- = \sum_{i=1}^N P_i^- = \sum_{i=1}^N \frac{m_i^2 + \mathbf{P}_{i\perp}^2}{P_i^+} = \frac{M_0^2 + \mathbf{P}_\perp^2}{P^+}. \tag{B.36}$$

The noninteracting mass operators can be written as follows:

$$M_0 = \sum_{i=1}^N \frac{m_i^2 + \mathbf{k}_{i\perp}^2}{k_i^+}; \tag{B.37}$$

$$M_0^2 = \sum_{i=1}^N \frac{m_i^2 + \mathbf{k}_{i\perp}^2}{\xi_i}. \tag{B.38}$$

APPENDIX C

Racah Coefficients

This Appendix has explicit expressions for the coefficients of the unitary transformations which connect three-body irreducible representations of the Poincaré group with different orders of pairwise coupling. These coefficients form a particular set of Racah coefficients. In general, if $|[a]m\,j;\mathbf{p}\,\mu\rangle$ and $|[b]m\,j;\mathbf{p}\,\mu\rangle$ denote different unitarily equivalent irreducible representations of $\mathcal{P}$, we must have

$$\begin{aligned}\langle[a]m'\,j';\mathbf{p}'\,\mu'|[b]m\,j;\mathbf{p}\,\mu\rangle &= \delta_{j'j}\delta_{\mu'\mu}\delta(\mathbf{p}'-\mathbf{p})\left\{\begin{array}{c}\delta(m'-m)\\ \\ \delta_{m'm}\end{array}\right\}\hat{R}^{jm}([b],\,[a])\\ &= \delta_{j'j}\delta_{\mu'\mu}\delta(\mathbf{p}'-\mathbf{p})R^{jm}([b],\,[a]).\end{aligned} \tag{C.1}$$

We call $R^{jm}([b],\,[a])$ the Racah coefficient connecting these two representations. Note that it is diagonal in m and includes a delta function in the case of a continuous mass spectrum. From group theoretic considerations, it must have the form

$$R^{jm}([b],\,[a]) = \frac{1}{2j+1}\sum_{\mu=-j}^{j}\int d\mathbf{p}\,\frac{m}{\omega_m(\mathbf{p})}\langle[a]m'\,j';\mathbf{p}\,\mu|[b]m\,j;\mathbf{0}\,\mu\rangle. \tag{C.2}$$

In a front-form basis, these equations are replaced by

$$\begin{aligned}\langle[a]m'\,j';\tilde{\mathbf{p}}'\,\mu'|[b]m\,j;\tilde{\mathbf{p}}\,\mu\rangle &= \delta_{j'j}\delta_{\mu'\mu}\delta(\tilde{\mathbf{p}}'-\tilde{\mathbf{p}})\left\{\begin{array}{c}\delta(m'-m)\\ \\ \delta_{m'm}\end{array}\right\}\hat{R}^{jm}([b],\,[a])\\ &= \delta_{j'j}\delta_{\mu'\mu}\delta(\tilde{\mathbf{p}}'-\tilde{\mathbf{p}})R^{jm}([b],\,[a]),\end{aligned} \tag{C.3}$$

where

$$R^{jm}([b],\,[a]) = \frac{1}{2j+1}\sum_{\mu=-j}^{j}\int d\tilde{\mathbf{p}}\,\frac{m}{p^+}\langle[a]m'\,j';\mathbf{p}\,\mu|[b]m\,j;\tilde{\mathbf{0}}\,\mu\rangle, \tag{C.4}$$

and $\tilde{\mathbf{0}} := (m,0,0)$ is the light-front component of the four-vector left invariant by the rotation subgroup in this representation.

For the three-body problem, we are interested in the relation between the representations

$$\langle [ij,k]q'\, j';\mathbf{p}'\mu|[jk,i]q\, j;\mathbf{p}\mu\rangle. \tag{C.5}$$

It is a product of four Clebsch-Gordan coefficients:

$$\begin{aligned}
&\langle [ij,k]q'\, j';\mathbf{p}'\mu|[jk,i]q\, j;\mathbf{p}\mu\rangle \\
&\qquad = \sum \int d\mathbf{p}'_{ij} \int k'^2 dk' \int d\mathbf{p}_i \int d\mathbf{p}_j \int d\mathbf{p}_k \int d\mathbf{p}_{jk} \int k^2 dk \\
&\qquad\quad \times \langle [L'\, S']q'\, j';\mathbf{p}'\,\mu'|m_{ij}\, j_{ij}\, \mathbf{p}_{ij}\, \mu_{ij};m_k\, j_k\, \mathbf{p}_k\, \mu_k\rangle \\
&\qquad\quad \times \langle [l'\, s']k'\, j_{ij};\mathbf{p}_{ij}\, \mu_{ij}|m_i\, j_i\, \mathbf{p}_i\, \mu_i;m_j\, j_j\, \mathbf{p}_j\, \mu_j\rangle \\
&\qquad\quad \times \langle m_j\, j_j\, \mathbf{p}_j\, \mu_j;m_k\, j_k\, \mathbf{p}_k\, \mu_k|[l\, s]k\, j_{jk};\mathbf{p}_{jk}\, \mu_{jk}\rangle \\
&\qquad\quad \times \langle m_{jk}\, j_{jk}\, \mathbf{p}_{jk}\, \mu_{jk};m_i\, j_i\, \mathbf{p}_i\, \mu_i|[L\, S]q\, j;\mathbf{p}\,\mu\rangle \\
&\qquad = \delta_{j'j}\delta_{\mu'\mu}\delta(\mathbf{p}'-\mathbf{p})R^{jm}([ij,k],\,[jk,i]).
\end{aligned} \tag{C.6}$$

The Racah coefficient $R^{jm}([ij,k],\,[jk,i])$ can be obtained by factoring out the momentum and spin conserving delta functions in Eq. (C.6). For a general type of spin, the result is

$$\begin{aligned}
&\langle [ij,k]q'\, j';\mathbf{p}'\mu|[jk,i]q\, j;\mathbf{p}\mu\rangle \\
&\quad = \delta_{j'j}\delta_{\mu'\mu}\delta(\mathbf{p}'-\mathbf{p}) \\
&\qquad \times \frac{1}{2j+1}\sum_{\mu=-j}^{j} \frac{8\pi^2\delta(M(q',k')-M(q,k))m_{ij}^2 m_{ki}^2\omega_i(\mathbf{q}+\mathbf{q}')}{kk'qq'\omega_i(k')\omega_j(k')\omega_k(k)\omega_i(k)\omega_{ij}(q')\omega_{ki}(q)} \\
&\qquad \times \left[\left|\frac{\partial(\mathbf{p}_{ij}\mathbf{p}'_k)}{\partial(\mathbf{p}'\mathbf{q}')}\right|\left|\frac{\partial(\mathbf{p}'_{ij}\mathbf{k}')}{\partial(\mathbf{p}_i\mathbf{p}j)}\right|\left|\frac{\partial(\mathbf{p}_{jk}\mathbf{k})}{\partial(\mathbf{p}_j\mathbf{p}_k)}\right|\left|\frac{\partial(\mathbf{p}_{jk}\mathbf{p}_i)}{\partial(\mathbf{pq})}\right|\right]^{\frac{1}{2}}\Bigg|_{\mathbf{p}=\mathbf{0}} \\
&\qquad \times \langle j\,\mu|L'\,\mu'_L\,S'\,\mu'_S\rangle\langle S'\,\mu'_S|j_{ij}\,\mu'_{ij}\,j_k\,\mu'_k\rangle D^{j_{ij}}_{\mu'_{ij}\mu_{ij}}[\underline{R}_{cg}(-\mathbf{q}'_t/m_{ij})] \\
&\qquad \times Y^{L'*}_{\mu'_L}(\hat{\mathbf{q}}'_g)\langle j_{ij}\,\mu_{ij}|l'\,\mu'_l\,s'\,\mu'_s\rangle\langle s'\,\mu'_s|j_i\,\mu'_i\,j_j\,\mu'_j\rangle Y^{l'*}_{\mu'_l}(\hat{\mathbf{k}}'_g) \\
&\qquad \times D^{j_i}_{\mu'_i\mu_i}[L_c^{-1}(\mathbf{k}'_g/m_i)L_g^{-1}(-\mathbf{q}'_g/m_{ij})L_c(\mathbf{q}_g/m_i)] \\
&\qquad \times D^{j_j}_{\mu'_j\mu_j}[L_c^{-1}(-\mathbf{k}'_g/m_j)L_g^{-1}(-\mathbf{q}'_g/m_{ij})L_g^{-1}(-\mathbf{q}_g/m_{jk})L_c(\mathbf{k}_g/m_j)] \\
&\qquad \times D^{j_k}_{\mu'_k\mu_k}[L_c^{-1}(\mathbf{q}_g/m_k)L_g(-\mathbf{q}_g/m_{jk})L_c(\mathbf{k}/m_k)] \\
&\qquad \times Y^{l}_{\mu_l}(\hat{\mathbf{k}}_g)\langle j_j\mu_j j_k\mu_k|s\mu_s\rangle\langle l\mu_l s\mu_s|j_{jk}\mu'_{jk}\rangle \\
&\qquad \times D^{j_{jk}}_{\mu'_{jk}\mu_{jk}}[\underline{R}_{tc}(-\mathbf{q}_g/m_{jk})]Y^{L}_{\mu'_L}(\hat{\mathbf{q}}_g)\langle j_i\mu_i j_{jk}\mu_{jk}|S\mu_S\rangle\langle L\mu''_L S\mu_S|j\mu\rangle.
\end{aligned} \tag{C.7}$$

To avoid confusion, we have used the three-components of the four-velocity arguments in all boosts appearing in both the boosts and the Melosh rotations.

The front-form Racah coefficients are:

$$
\begin{aligned}
&\langle [ij,k] q'\, j'; \tilde{\mathbf{p}}'\mu | [jk,i] q\, j; \tilde{\mathbf{p}}\mu\rangle \\
&\quad = \delta_{j'j}\delta_{\mu'\mu}\delta(\tilde{\mathbf{p}}'-\tilde{\mathbf{p}}) \\
&\qquad \times \frac{1}{2j+1}\sum_{\mu=-j}^{j} \frac{8\pi^2\delta(M(q',k')-M(q,k))m_{ij}^2 m_{ki}^2\omega_i(\mathbf{q}+\mathbf{q}')}{kk'qq'\omega_i(k')\omega_j(k')\omega_k(k)\omega_i(k)\omega_{ij}(q')\omega_{ki}(q)} \\
&\qquad \times \left[\left|\frac{\partial(\tilde{\mathbf{p}}_{ij}\tilde{\mathbf{p}}'_k)}{\partial(\tilde{\mathbf{p}}'\mathbf{q}')}\right| \left|\frac{\partial(\tilde{\mathbf{p}}'_{ij}\mathbf{k}')}{\partial(\tilde{\mathbf{p}}_i\tilde{\mathbf{p}}j)}\right| \left|\frac{\partial(\tilde{\mathbf{p}}_{jk}\mathbf{k})}{\partial(\tilde{\mathbf{p}}_j\tilde{\mathbf{p}}_k)}\right| \left|\frac{\partial(\tilde{\mathbf{p}}_{jk}\tilde{\mathbf{p}}_i)}{\partial(\tilde{\mathbf{p}}\mathbf{q})}\right|\right]^{\frac{1}{2}}\Bigg|_{\tilde{\mathbf{p}}=\tilde{\mathbf{0}}} \\
&\qquad \times \langle j\,\mu | L'\,\mu'_L\, S'\,\mu'_S\rangle\langle S'\,\mu'_S | j_{ij}\,\mu'_{ij}\, j_k\,\mu'_k\rangle D^{j_{ij}}_{\mu'_{ij}\mu_{ij}}[\underline{R}_{cf}(-\mathbf{q}'_t/m_{ij})] \\
&\qquad \times Y^{L'*}_{\mu'_L}(\hat{\mathbf{q}}'_f)\langle j_{ij}\,\mu_{ij} | l'\,\mu'_l\, s'\,\mu'_s\rangle\langle s'\,\mu'_s | j_i\,\mu'_i\, j_j\,\mu'_j\rangle Y^{l'*}_{\mu'_l}(\hat{\mathbf{k}}'_f) \\
&\qquad \times D^{j_i}_{\mu'_i\mu_i}[L_c^{-1}(\mathbf{k}'_f/m_i)L_f^{-1}(-\mathbf{q}'_f/m_{ij})L_c(\mathbf{q}_f/m_i)] \\
&\qquad \times D^{j_j}_{\mu'_j\mu_j}[L_c^{-1}(-\mathbf{k}'_f/m_i)L_f^{-1}(-\mathbf{q}'_f/m_{ij})L_f^{-1}(-\mathbf{q}_f/m_{jk})L_c(\mathbf{k}_f/m_j)] \\
&\qquad \times D^{j_k}_{\mu'_k\mu_k}[L_c^{-1}(\mathbf{q}_f/m_k)L_f(-\mathbf{q}_f/m_{jk})L_c(\mathbf{k}/m_k)] \\
&\qquad \times Y^{l}_{\mu_l}(\hat{\mathbf{k}}_f)\langle j_j\mu_j j_k\mu_k | s\mu_s\rangle\langle l\mu_l s\mu_s | j_{jk}\mu'_{jk}\rangle \\
&\qquad \times D^{j_{jk}}_{\mu'_{jk}\mu_{jk}}[\underline{R}_{tc}(-\mathbf{q}_f/m_{jk})]Y^{L}_{\mu''_L}(\hat{\mathbf{q}}_f)\langle j_i\mu_i j_{jk}\mu_{jk} | S\mu_S\rangle\langle L\mu''_L S\mu_S | j\mu\rangle.
\end{aligned}
\tag{C.8}
$$

Equations (C.7) and (C.8) are independent of the angles, and can be evaluated for any $\hat{\mathbf{k}}$, $\hat{\mathbf{k}}'$, $\hat{\mathbf{q}}$ and $\hat{\mathbf{q}}'$ consistent with the kinematics.

APPENDIX D

Local Fields

The models presented in this review are relativistic quantum mechanical models but are not local relativistic field theories. The goal of this section is to derive direct interaction models as approximations to field theories. A connection to a field theory can provide constraints on model interactions and consistent models of current operators.

To do this successfully we must devise a systematic procedure the that preserves Poincaré invariance, the quantum mechanical structure, the spectral condition, macroscopic locality, and converges to the solution of the field theory in a precise sense. This goal is beyond the scope of this paper. The discussion in this section will be limited to the first step, which is to formulate a first approximation that leads to an instant form two-body model. It is important to note that there is nothing special about the instant form. This could have been done equally as well in the point or front-form.

In both relativistic and nonrelativistic quantum theories there are many types of "approximations" available. In many cases the approximation may look very different from a model with a standard quantum mechanical interpretation. For instance, a standard optical potential, which is used to reformulate elastic scattering, has an energy dependence due to eliminated degrees of freedom which is not permitted in an ordinary Galilean invariant dynamics.

The models in this paper were designed to have most of the properties of a local relativistic field theory. The only notable exception is property of microscopic locality which was replaced by macroscopic locality. In formulating approximations it is desirable to preserve most of the fundamental properties of the original theory. One type of approximation that retains most of the properties of the original theory is an approximation where the exact Hamiltonian, H, is replaced by a projected Hamiltonian, $H_\pi := \Pi H \Pi$, for a suitable orthogonal projection operator Π. This type of approximation will preserve conservation laws if the projectors commute with the same operators that commute with the full Hamiltonian. The projection operators also preserve the spectral condition. In order to interpret H_π as an approximation to H we need an infinite

collection of projectors, Π_n, that converge to the identity as $n \to \infty$ and a way of specifying how the approximate models converge to the exact model.

We discuss approximation based on projection operators first for a model of a fixed number of particles satisfying the constraints of Galilean relativity, then for a fixed number of particles in a Poincaré invariant theory, and finally for the case of a local field theory. This allows us to separate the issues that relevant in treating scattering theory from those that are relevant to Poincaré invariance and those that are special to field theories.

D.1. **Fixed Number of Particles - Galilean Invariance:**

Consider a nonrelativistic quantum mechanical model of a system of N particles. The dynamics of this system is governed by a Hamiltonian H, which in a Galilean invariant model has the form $\mathbf{P}^2/2M + h$ where h commutes with $\mathbf{X}$, $\mathbf{j}$, $\mathbf{P}$, and M for the system (see (3.34)). The Hilbert space for this system can be represented as the tensor product of the space of square integrable functions of the total momentum with an internal Hilbert space, $\hat{\mathcal{H}}$:

$$\mathcal{H} = L^2(R^2, d^3P) \otimes \hat{\mathcal{H}}. \tag{D.1}$$

Let Π be an orthogonal projection operator that commutes with $\mathbf{X}$, $\mathbf{J}$, $\mathbf{P}$, and M. In general it has the form

$$\Pi = I \otimes \hat{\Pi} \tag{D.2}$$

where $\hat{\Pi}$ projects on a rotationally invariant subspace of the internal Hilbert space, $\hat{\mathcal{H}}$. For such a projection operator define the projected Hamiltonian:

$$H_\pi := \Pi H \Pi. \tag{D.3}$$

If we define

$$\mathbf{K} = M\mathbf{X} \tag{D.4}$$

then $\mathbf{K}$, $\mathbf{P}$, $\mathbf{J}$, M, and H_π satisfy the commutation relations for the central extension of the Galilean Group. This defines a Galilean invariant approximate dynamics. To interpret this as

an approximation one has to specify a sequence Π_n of projection operators on successively larger subspaces such that H_n is well defined and converges to H. Since H has an unbounded energy spectrum, the type of convergence needed is called strong resolvent convergence, which means that

$$\lim_{n\to\infty} \|[(z-H)^{-1} - (z-H_n)^{-1}]|\Psi\rangle\| = 0 \tag{D.5}$$

for all $|\Psi\rangle$ and z not an eigenvalue of H or H_n. This is a useful mathematical way of saying that any bounded function of H_n strongly converges to the same bounded function of H.

When condition (D.5) holds we obtain a sequence of Galilean invariant approximations, $\{H_\pi$, $\mathbf{P}$, $\mathbf{J}$, $\mathbf{K}$, $M\}$, to the exact dynamics, $\{H$, $\mathbf{P}$, $\mathbf{J}$, $\mathbf{K}$, $M\}$. When the internal Hamiltonian, h, has only discrete spectra, the eigenvalues of $h_n := \hat{\Pi}_n h \hat{\Pi}_n$ become variational bounds on the eigenvalues of h.

If the projection operators have finite dimensional range on the internal Hilbert space, $\hat{\mathcal{H}}$, these approximate models will not allow scattering.

It is possible (Ch 80) to construct a sequence of projection operators $\{\Pi_n\}_{n=1}^{\infty}$ with infinite dimensional range on the internal Hilbert space such that H_n is the Hamiltonian of a Galilean invariant model that has a unitary scattering matrix, S_n, with the property that S_n weakly converges to the exact scattering matrix as $n \to \infty$. What is needed to be able to do this is that H_n should cluster into a projector of a channel Hamiltonian onto an invariant subspace of that channel Hamiltonian. The weak convergence of the approximate scattering operator requires additional restrictions on the projection operators which can always be realized in applications.

The conclusion is that by choosing suitable sequences of projection operators, $\{\Pi_n\}_{n=1}^{\infty}$, that converge strongly to the identity it is possible to construct a sequence of Galilean invariant models that converge to the Galilean invariant dynamics in the following sense. The unitary representations of the central extension of the Galilean group, $U_n(g)$, of the approximate dynamical models converge strongly to the exact $U(g)$ as $n \to \infty$ and the approximate scattering matrices, S_n, converge weakly to the exact scattering matrix, S. The proof of the weak convergence of

the scattering operators can be found in (Ch 80) while the strong convergence of the group representations follows from strong resolvent convergence of the approximate Hamiltonians on the internal Hilbert space. Note that the strong convergence of the approximate unitary representations of the Galilean group implies nothing about the existence S_n or the convergence of the S_n's if they exist.

D.2. **Fixed Number of Particles - Poincaré Invariance:**

Next we consider what happens to the framework discussed above if Galilean invariance is replaced by Poincaré invariance. In the Galilean invariant case it was possible to find a projection operator that commuted with all of the generators of Galilean transformations except for the Hamiltonian. This is impossible in the Poincaré invariant case because the Hamiltonian appears on the right hand side of some of the commutators. To see the problem compute the commutator of Π with :

$$[K^j, P^k]_- = -i\delta^{jk} H. \tag{D.6}$$

The Jacobi identity implies:

$$[K^j, [\Pi, P^k]_-]_- + [P^k, [K^j, \Pi]_-]_- = -i\delta^{jk}[\Pi, H]_-. \tag{D.7}$$

The left hand side of (D.7) vanishes if Π commutes with all generators except H, while the right hand side will not vanish under these conditions. This requires that if we want to preserve the Poincaré symmetry that several generators need to approximated simultaneously. The simplest way to do this is to replace the generators with M and nine independent functions of the generators. We can always include generators of any kinematic subgroup in this collection of nine operators, and this can be done independent of the existence of a true *kinematic* subgroup.

Thus, to construct the desired type of projections we let H, $\mathbf{P}$, $\mathbf{J}$, and $\mathbf{K}$ be the generators of the system and construct both the Newton-Wigner position operator $\mathbf{X}$ and the mass operator M. We introduce a sequence of Projection operators Π_n that converge strongly to the identity satisfying

$$M_n := \Pi_n M \Pi_n \to M \tag{D.8}$$

in the sense of strong resolvent convergence and

$$[\Pi_n, \mathbf{P}]_- = [\Pi_n, \mathbf{J}]_- = [\Pi_n, \mathbf{X}]_- = 0. \tag{D.9}$$

For this sequence of projection operators the operators $\mathbf{P}$, $\mathbf{J}$,

$$H_\pi = \sqrt{M_\pi^2 + \mathbf{P}^2} \tag{D.10}$$

and

$$\mathbf{K}_\pi = -\frac{1}{2}\{H_\pi, \mathbf{X}\}_+ - \frac{1}{H_\pi + M_\pi}(\mathbf{P} \times \mathbf{j}). \tag{D.11}$$

where $\mathbf{j} = \mathbf{J} - \mathbf{X} \times \mathbf{P}$ satisfy the commutation relations for the Poincaré Lie algebra. At this point, the analysis reduces to that of the non-relativistic case. The only complication is constructing a suitable set of projection operators that permit scattering and commute with $\mathbf{X}$. This difficulty is that $\mathbf{X}$ can be a complicated dynamical operator that may depend on interactions.

One way to avoid these complications is to construct a sequence of Euclidean invariant projectors satisfying:

$$[\mathbf{X}, \Pi]_- \neq 0. \tag{D.12}$$

This leads to a alternative sequence of approximations which are suitable, but not as elegant. First we note, since the projectors commute with the Euclidean group, the model mass operator remains Euclidean invariant with respect to the full dynamics. This means that it has the following representation:

$$\langle \mathbf{P}' \cdots | M_\pi | \mathbf{P} \cdots \rangle =$$

$$\delta(\mathbf{P}' - \mathbf{P}) \langle (\cdots)' \| \hat{M}_\pi(\mathbf{P}) \| \cdots \rangle \tag{D.13}$$

where the reduced mass operator, $\hat{M}_\pi(\mathbf{P})$, is rotationally invariant.

In the case that $\mathbf{X}$ commutes with Π the $\hat{M}_\pi(\mathbf{P})$'s for different values of $\mathbf{P}$ are scattering equivalent. This is a consequence of the Poincaré invariance of the approximate scattering matrices. The construction of the unitary operator that makes the equivalence is outlined below. The Poincaré invariance of the scattering matrix implies

$$\langle \mathbf{P'} \cdots | S | \mathbf{P} \cdots \rangle = \delta(\mathbf{P'} - \mathbf{P}) \langle (\cdots)' \| \hat{S} \| \cdots \rangle \tag{D.14}$$

with $\hat{S}$ independent of $\mathbf{P}$. Using this we have

$$\hat{S} = \hat{\Omega}_+^\dagger(\mathbf{P}) \hat{\Omega}_-(\mathbf{P}) = \hat{\Omega}_+^\dagger(\mathbf{P'}) \hat{\Omega}_-(\mathbf{P'}). \tag{D.15}$$

Asymptotic completeness can be used to write this as

$$\hat{B}(\mathbf{P'};\mathbf{P}) = |b(\mathbf{P'})\rangle\langle b(\mathbf{P})| + \hat{\Omega}_-(\mathbf{P}) \hat{\Omega}_-^\dagger(\mathbf{P}) = |b(\mathbf{P'})\rangle\langle b(\mathbf{P})| + \hat{\Omega}_+(\mathbf{P'}) \hat{\Omega}_+^\dagger(\mathbf{P}) \tag{D.16}$$

where $|b(\mathbf{P})\rangle$ are the point eigenstates of $\hat{M}(\mathbf{P})$ which may in principle depend on $\mathbf{P}$. The Euclidean invariance and intertwining properties of the wave operators can then be used to show that $\hat{B}(\mathbf{P'};\mathbf{P})$ is maps $\hat{M}(\mathbf{P})$ to $\hat{M}(\mathbf{P'})$. Equation (D.16) shows that map preserves the scattering operator. The multichannel generalization of the above construction can be found in (Co 82).

If instead we assume (D.12) then the $\hat{M}_\pi(\mathbf{P})$'s for different values of $\mathbf{P}$ are not equivalent. The reason for this is that the projection breaks the Poincaré invariance of the full theory as was illustrated in (D.7). These approximations still preserve the Euclidean invariance of the full theory.

It is possible to restore Poincaré invariance by changing the model mass operator, $M_\pi \to \bar{M}_\pi$, to

$$\langle \mathbf{P}' \cdots | \bar{M}_\pi | \mathbf{P} \cdots \rangle :=$$

$$\delta(\mathbf{P}' - \mathbf{P}) \langle (\cdots)' \| \hat{M}_\pi(\mathbf{0}) \| \cdots \rangle. \tag{D.17}$$

It follows the if we take $\mathbf{X}_\pi = i\nabla_p$, and define

$$\bar{H}_\pi = \sqrt{\bar{M}_\pi^2 + \mathbf{P}^2} \tag{D.18}$$

and

$$\bar{\mathbf{K}}_\pi = -\frac{1}{2}\{\bar{H}_\pi, \mathbf{X}_\pi\}_+ - \frac{1}{\bar{H}_\pi + \bar{M}_\pi}(\mathbf{p} \times \mathbf{j}_\pi). \tag{D.19}$$

where $\mathbf{j}_\pi = \mathbf{J} - \mathbf{X}_\pi \times \mathbf{P}$, then $\bar{H}_\pi$, $\bar{\mathbf{K}}_\pi$, $\mathbf{P}$, $\mathbf{J}$ satisfy the Poincaré commutation relations. They can be used to construct a unitary representation, $\bar{U}_\pi(\underline{\Lambda}, \underline{a})$, of the Poincaré group corresponding to these approximate generators.

In this case, as the projection operators converge to the identity, the operators $\bar{M}_\pi$ no longer converge to M; instead we expect that if the exact mass operator has the representation

$$\langle \mathbf{P}', \cdots | M | \mathbf{P}, \cdots \rangle =$$

$$\delta(\mathbf{P}' - \mathbf{P}) \langle (\cdots)' \| \hat{M}(\mathbf{P}) \| \cdots \rangle. \tag{D.20}$$

the projected operators will converge to an operator $\bar{M}$ with kernel

$$\langle \mathbf{P}', \cdots | \bar{M}_n | \mathbf{P}, \cdots \rangle \to \langle \mathbf{P}', \cdots | \bar{M} | \mathbf{P}, \cdots \rangle =$$

$$\delta(\mathbf{P}' - \mathbf{P}) \langle (\cdots)' \| \hat{M}(\mathbf{0}) \| \cdots \rangle. \tag{D.21}$$

If the exact model has an asymptotically complete, Poincaré invariant scattering operator then the right hand side of (D.21) is related to the right hand side of (D.20) by a scattering equivalent unitary transformation of the form (D.16) :

$$B\bar{M}B^\dagger = M \tag{D.22}$$

where

$$\langle \mathbf{P}', \cdots | \bar{B} | \mathbf{P}, \cdots \rangle =$$

$$\delta(\mathbf{P}' - \mathbf{P}) \langle (\cdots)' \| \hat{B}(\mathbf{P}; \mathbf{0}) \| \cdots \rangle. \tag{D.23}$$

Thus in this case we expect that with a suitable set of projectors that

$$\bar{U}_n(\underline{\Lambda}, \underline{a}) \to B^\dagger U(\underline{\Lambda}, \underline{a}) B \tag{D.24}$$

and

$$\bar{S}_n \to \bar{S} = S \tag{D.25}$$

where the convergence in (D.24) will be strong and the convergence in (D.25) will be weak. In the relativistic case neither of these statements have been proved, but they are suggested by the non-relativistic results.

It is worth noting that in this example the approximations were purely Bakamjian-Thomas models which we argued in section 6 violated cluster properties. We learned restore macroscopic locality with packing operators in section 7. If we let

$$U_n(\underline{\Lambda}, \underline{a}) = A_n \bar{U}_n(\underline{\Lambda}, \underline{a}) A_n^\dagger \tag{D.26}$$

where A_n are packing operators in the model system, then we expect that these macrolocal approximations will converge to

$$U_n(\underline{\Lambda}, \underline{a}) \to AB^\dagger U(\underline{\Lambda}, \underline{a}) BA^\dagger \tag{D.27}$$

where $A = \lim A_n$. If all of the limits exist and behave as expected then $AB^\dagger$ is a Euclidean invariant unitary operator that preserves the scattering operator and cluster properties. It is

simply a change of representation that preserves all of the physics. In this case the convergence of the scattering matrix is not affected.

D.3. **Fields**

We are now ready to generalize what was discussed above for fixed N to the case of a local field theory. To keep the discussion as simple as possible consider the construction of an instant form two-body model based on a local relativistic field theory with a single Hermitian scalar field $\phi(x)$ of mass $m > 0$. Our discussion will be limited to the construction of a projection operator on a "two-body" subspace of the field theory on which a model scattering theory can be formulated.

One projection operator does not by itself define an approximation. To complete the task started here and construct a convergent set of projectors that allow scattering is a research problem. The justification for this first step should be understood in the context of the previous section. Our limited goal in this section is to construct the analog of $\bar{M}_\pi$ in the case of a field theory. The main problem involves identifying a one particle subspace of the physical Hilbert space, and using the operators that create the one particle subspace to construct a subspace that behaves enough like a two particle subspace to allow scattering.

It is clear that one could treat this problem using combinations of perturbative and Fock space methods. This does not provide a justification of these models in the context of quantum field theory, but instead only in the limited scope of perturbative field theory. Because of this we discuss the formulation of approximations in a manner that does not require perturbation theory. Having said this we need some assumptions to start our construction. To this end we assume

1. The field $\phi(x)$ satisfies the Wightman axioms for a Hermitian scalar field of mass $m > 0$.
2. The physical Hilbert space has a Poincaré invariant one particle subspace $\mathcal{H}_m$ corresponding to a particle of mass m and spin 0. The Poincaré group acts irreducibly on this subspace.
3. The field has non-vanishing matrix elements between the vacuum and the one particle subspace, $\mathcal{H}_m$.
4. The two point function of this field has a Lehmann representation of the form

$$\langle 0|\phi(x)\phi(y)|0\rangle =$$

$$-iD_m^-(x-y) - i\int_{4m^2}^{\infty} D_{\sqrt{\lambda}}^-(x-y)d\mu(\lambda). \tag{D.28}$$

Equation (D.28) implies a normalization of the field.

The first assumption defines properties of the field in the absence of a perturbative description. The second assumption implies that this model admits particles of mass m the behave in the expected way and assumes that there is only one type particle with mass m. The third assumption ensures the existence of a Lehmann representation for the two-point function. The fourth assumption is about the structure of this Lehmann representation; what is says is that on the subspace spanned by vectors of the form

$$\phi(f)|0\rangle = \int f(x)\phi(x)|0\rangle d^4x \tag{D.29}$$

that the one and many-body parts of this subspace can be easily isolated.

We show how to use the Heisenberg fields to construct 2-body projection operators that allow one construct models of a finite number of degrees of freedom that permit scattering. The model constructed corresponds to the dynamics generated by $\bar{U}_\pi(\underline{\Lambda}, \underline{a})$ in the previous section. Our development is limited to the two-body problem. The elements of this construction summarized below:

1. Construct a unitary mapping, Φ_1, from an irreducible representation space $\mathcal{H}_1$ of the Poincaré group with the same mass and spin as the particle to the invariant one particle subspace of the field theory, $\mathcal{H}_m$, that satisfies the intertwining relation:

$$\Phi_1 U_1(\Lambda, a) = U(\Lambda, a)\Phi_1 \tag{D.30}$$

where $U_1(\Lambda, a)$ is the irreducible representation of the Poincaré group on $\mathcal{H}_1$ and $U(\Lambda, a)$ it

the physical representation of the Poincaré group associated with the field theory. This is done using Hagg's (Ha 58) notion of a quasilocal field.

2. Construct a unitary mapping Φ_2 from the symmetric subspace of $\mathcal{H}_2 := \mathcal{H}_1 \otimes \mathcal{H}_1$ to a subspace of the physical Hilbert space that satisfies the intertwining relation for the *Euclidean* subgroup:

$$\Phi_2 U_1(R, \mathbf{a}) \otimes U_1(R, \mathbf{a}) = U(R, \mathbf{a})\Phi_2 \tag{D.31}$$

3. Construct a model two-body mass operator $\bar{M}_2$ using the approximation (D.21) in $M_2 = \Phi_2^\dagger M \Phi_2$.

This leads to a Bakamjian Thomas model with a mass operator determined entirely by the field.

D.4. **Quasilocal Fields**

The first step in the construction is to identify the physical one particle subspace of the field theory. In general if a smeared Heisenberg field is applied to the vacuum the resulting vector will have both one and many particle components. The second term in the Lehmann representation (D.28) comes from these many-body components. One way to ensure that the result of applying the smeared Heisenberg field to the vacuum lies in the physical one-particle subspace of the field theory is to smear the field with a function whose fourier transform is a function of the four momentum that is identically zero when p^2 is in the continuous (mass)^2 spectrum of the field theory and 1 when $p^2 = -m^2$. It will be shown, using the assumption about the Lehmann representation, that if such a smeared field is applied to the vacuum that the resulting vector is in the one particle Hilbert space of the field theory. The full one particle subspace is the closed linear subspace containing this vector and finite linear combinations of vectors related to this by Poincaré transformations. By assumption this subspace carries an irreducible representation of the Poincaré group.

To formalize this procedure, following Haag (Ha 58), introduce the notion of a quasilocal covariant field. To construct this field let $\tilde{h}(p)$ be a smooth function of p^2 with the following properties:

$$\tilde{h}(p) = \begin{cases} 1 & \text{if } p^2 = -m^2; \\ 0 & \text{if } |p^2 + m^2| > 2m^2 \end{cases} . \tag{D.32}$$

This means that it is a smooth function of p^2 that is identically 1 on the point mass spectrum and identically 0 in a neighborhood of the continuous spectrum. Denote the fourier transform of the field $\phi(x)$ by

$$\tilde{\phi}(p) := (2\pi)^{-2} \int d^4x e^{-ip\cdot x} \phi(x) \tag{D.33}$$

and define the following quasilocal field operator

$$\tilde{A}(p) := \tilde{h}(p)\tilde{\phi}(p). \tag{D.34}$$

The reason for doing this is that the energy spectrum of the one particle states and many particle states are not disjoint, although the mass spectrum is. The function $\tilde{h}(p)$ separates these two independent parts of the dynamics. It will be shown that if a smeared quasilocal field of this form is applied to the vacuum the resulting vector is an element of the physical one particle subspace of the Hilbert space. The fourier transform of $\tilde{A}(p)$ is denoted by $A(x)$:

$$A(x) := (2\pi)^{-2} \int d^4p e^{ip\cdot x} \tilde{A}(p). \tag{D.35}$$

which also has the form

$$A(x) = (2\pi)^{-2} \int d^4y h(x-y)\phi(y). \tag{D.36}$$

where

$$h(x) = (2\pi)^{-2} \int d^4 p e^{ip\cdot x} \tilde{h}(p). \tag{D.37}$$

The operator $A(x)$ has many of the properties of a field operator $\phi(x)$. These follow form the definitions and the axioms satisfied by the Heisenberg fields. These will be stated without proof.

It transforms covariantly under the action of Poincaré transformations:

$$U(\Lambda, a) A(x) U^\dagger(\Lambda, a) = A(\Lambda x + a) \tag{D.38}$$

or equivalently

$$U(\Lambda, a) \tilde{A}(p) U^\dagger(\Lambda, a) = e^{i\Lambda p\cdot a} \tilde{A}(\Lambda p). \tag{D.39}$$

The infinitesimal generators of $U(\Lambda, a)$ can be computed on vectors constructed from polynomials in the quasilocal fields applied to the vacuum using equations (D.38) or (D.39):

$$P^\mu A(x_1) \cdots A(x_n)|0\rangle =$$

$$-i \sum_{k=1}^{n} A(x_1) \cdots \frac{\partial}{\partial x_{k\mu}} A(x_k) \cdots A(x_n)|0\rangle \tag{D.40}$$

$$J^{\mu\nu} A(x_1) \cdots A(x_n)|0\rangle =$$

$$-i \sum_{k=1}^{n} A(x_1) \cdots \big(x_k^\mu \frac{\partial}{\partial x_{k\nu}} - x_k^\nu \frac{\partial}{\partial x_{k\mu}}\big) A(x_k) \cdots A(x_n)|0\rangle \tag{D.41}$$

or equivalently

$$P^\mu \tilde{A}(p_1) \cdots \tilde{A}(p_n)|0\rangle =$$

$$\big(\sum_{k=1}^{n} p_k^\mu\big) \tilde{A}(p_1) \cdots \tilde{A}(p_n)|0\rangle \tag{D.42}$$

$$J^{\mu\nu}\tilde{A}(p_1)\cdots\tilde{A}(p_n)|0\rangle =$$

$$-i\sum_{k=1}^{n}\tilde{A}(p_1)\cdots\big(p_k^{\mu}\frac{\partial}{\partial p_{k\nu}}-p_k^{\nu}\frac{\partial}{\partial p_{k\mu}}\big)\tilde{A}(p_k)\cdots\tilde{A}(p_n)|0\rangle. \tag{D.43}$$

The mass operator can be computed from (D.42):

$$M\tilde{A}(p_1)\cdots\tilde{A}(p_n)|0\rangle =$$

$$\big(-\sum_{k=1}^{n}p_k^2\big)^{1/2}\tilde{A}(p_1)\cdots\tilde{A}(p_n)|0\rangle \tag{D.44}$$

It also satisfies the same cluster properties as the Heisenberg field:

$$\lim_{|\mathbf{a}-\mathbf{b}|\to\infty}\big\{\langle 0|A(x+\mathbf{a})A(y+\mathbf{b})A(z+\mathbf{a})A(w+\mathbf{b})|0\rangle -$$

$$\langle 0|A(x)A(z)|0\rangle\langle 0|A(y)A(w)|0\rangle\big\} = 0 \tag{D.45}$$

etc. For the case of quasilocal fields this equation holds both if the fields are smeared over test functions of the space or spacetime variables. For the Heisenberg field it must generally be smeared over test functions of the spacetime coordinated for this to be true.

The operator $A(x)$ has two properties that are characteristic of free fields that are not shared by the Heisenberg field $\phi(x)$. First it has the same two point function as a *free* field operator:

$$\langle 0|A(x)A(y)|0\rangle = -iD_m^-(x-y). \tag{D.46}$$

where

$$-iD_m^-(x) =$$

$$(2\pi)^{-3}\int e^{-ik\cdot x}\theta(-k^0)\delta(k^2+m^2)d^4k =$$

$$(2\pi)^{-3}\int e^{ik\cdot x}\theta(k^0)\delta(k^2+m^2)d^4k \tag{D.47}$$

is the two-point Wightman function for a *free* field. Second it has the property that the "sharp

time" fields smeared over test functions of $\mathbf{x}$ are well defined operators:

$$a^\dagger(\mathbf{x}) := A(\mathbf{x}, 0) \tag{D.48}$$

$$\tilde{a}^\dagger(\mathbf{p}) = (2\pi)^{-3/2} \int d^3x e^{-i\mathbf{p}\cdot\mathbf{x}} a^\dagger(\mathbf{x}) \tag{D.49}$$

$$a^\dagger(f) := \int d^3x f(\mathbf{x}) a^\dagger(\mathbf{x}) := \int d^3p \tilde{f}(\mathbf{p}) \tilde{a}(\mathbf{p}). \tag{D.50}$$

In general the Heisenberg field must be smeared over test functions of both space and time before it becomes an operator. Thus, one of the fundamental advantages of the quasilocal fields is that is possible to restrict the field to a suitable three-dimensional surface, as in (D.48). We will see that the choice of $t = 0$ implies that the Euclidean subgroup of the model is exactly the Euclidean subgroup of the field. What is given up is that these quasi-local fields do not satisfy local commutation relations.

D.5. **The One-Body Subspace**

We now show that $a^\dagger(f)|0\rangle$ is an element of the one particle subspace of the field theory and that $\tilde{a}^\dagger(\mathbf{p})|0\rangle$ is a simultaneous eigenstate of the four momentum of the field theory.

To show that $a^\dagger(f)$ is an element of the one particle subspace we must show that is an eigenstate of the mass operator with eigenvalue m^2. To show this is to compute the expectation value of M^2 and its dispersion in the state $a(f)|0\rangle$. This can be done analytically using (D.31), (D.33), (D.43), (D.45), (D.46), and (D.49) with the result:

$$\langle M^2 \rangle = m^2 \qquad \langle M^4 \rangle = m^4 \tag{D.51}$$

from which we conclude

$$\langle (\langle M^2 \rangle - M^2)^2 \rangle = 0. \tag{D.52}$$

Since there is no dispersion we conclude that $a(f)|0\rangle$ is an eigenstate of M^2 with eigenvalue m^2. By the spectral condition the mass operator of the field theory is non-negative and must be equal

to the unique (Re 72) non-negative square root of M^2. It follows that $a^\dagger(f)|0\rangle$ is an eigenstate of the mass operator with eigenvalue m and necessarily an element of the one particle subspace of the field theory.

By direct computation using (D.42) we find

$$\mathbf{P}\,\tilde{a}^\dagger(\mathbf{p})|0\rangle = \mathbf{p}\,\tilde{a}^\dagger(\mathbf{p})|0\rangle \tag{D.53}$$

which shows that $\tilde{a}^\dagger(\mathbf{p})|0\rangle$ is a generalized eigenstate of the linear momentum operator with eigenvalue $\mathbf{p}$.

The irreducible representation space for a spinless particle of mass m is spanned by the simultaneous eigenstates of the three momentum. Thus we conclude that the vectors $a^\dagger(f)|0\rangle$ are dense in the single particle subspace of the physical Hilbert space.

Note although $\tilde{a}^\dagger(\mathbf{p})$ looks like an ordinary creation operator, we have made no assumptions about the commutation relations. In general the eigenstates need to be renormalized. Although in the one-body case the renormalization is trivial, it is useful to cast the renormalization in a general form that can also be applied to the two-body case.

To renormalize these states let $f(\mathbf{p})$ be a square integrable function of the three momentum with normalization:

$$\int d^3p|f(\mathbf{p})|^2 = 1. \tag{D.54}$$

We denote the Hilbert space of functions with the norm (D.54) by $\mathcal{H}_1$. Let $\hat{\Phi}_1$ denote the mapping from $\mathcal{H}_1$ to the one particle subspace of the Hilbert space defined by

$$\hat{\Phi}_1(f) := a(f)|0\rangle. \tag{D.55}$$

The operator $\hat{\Phi}_1$ clearly maps the momentum operator on $\mathcal{H}_1$ to the restriction of the momentum operator of the field theory to the one particle subspace of the field theory. This operator is not

a unitary mapping form $\mathcal{H}_1$ to $\mathcal{H}_m$. To renormalize the vectors on $\mathcal{H}_m$ we replace $\hat{\Phi}_1$ by an operator that

i. has the same range as $\hat{\Phi}_1$,

ii. preserves the three momentum, and

iii. preserves the $\mathcal{H}_1$ norm of f .

Such an operator is given by

$$\Phi_1 := Z^{-1/2}\hat{\Phi}_1 \tag{D.56}$$

where Z is a wave function renormalization operator given by

$$Z^{-1} = (\hat{\Phi}_1\hat{\Phi}_1^\dagger)^{-1}_{\text{mp}} \tag{D.57}$$

Note that in general $\hat{\Phi}_1\hat{\Phi}_1^\dagger$ does not have an inverse on the full physical Hilbert space, but there is a well defined inverse on the range of $\hat{\Phi}_1$ called the Moore-Penrose generalized inverse. The quantity $(\hat{\Phi}_1\hat{\Phi}_1^\dagger)^{-1/2}_{\text{mp}}$ is the Moore Penrose generalized inverse of the square root of $\hat{\Phi}_1\hat{\Phi}_1^\dagger$. The Moore Penrose generalized inverse X^{-1}_{mp} of a linear operator X is the unique solution to the four Penrose equations (Na 76):

$$X^{-1}_{mp}XX^{-1}_{mp} = X^{-1}_{mp} \tag{D.58}$$

$$XX^{-1}_{mp}X = X \tag{D.59}$$

$$XX^{-1}_{mp} = \Pi_{\overline{\mathcal{R}(X)}} \tag{D.60}$$

$$X^{-1}_{mp} X = \Pi_{\mathcal{N}(X)^\perp} \tag{D.61}$$

where $\Pi_{\overline{\mathcal{R}(X)}}$ and $\Pi_{\mathcal{N}(X)^\perp}$ are orthogonal projectors on the closure of the range of X and the orthogonal complement of the null space of X respectively.

In this particular case the range of both projection operators is the one particle subspace of the field theory. Although there are convergent iterative algorithms for computing the Moore-Penrose generalized inverse of an operator, in this case the unique solution of the 4-Penrose equations can be written down explicitly. From the definitions:

$$Z = \hat{\Phi}_1 \hat{\Phi}_1^\dagger = \int d^3p \tilde{a}^\dagger(\mathbf{p})|0\rangle d^3p \langle 0|\tilde{a}(\mathbf{p}) \tag{D.62}$$

and by direct computation:

$$(\hat{\Phi}_1 \hat{\Phi}_1^\dagger)^{1/2} = \int d^3p \tilde{a}^\dagger(\mathbf{p})|0\rangle d^3p \big(2\omega_m(\mathbf{p})\big)^{1/2} \langle 0|\tilde{a}(\mathbf{p}) \tag{D.63}$$

From these expressions it follows that

$$(\hat{\Phi}_1 \hat{\Phi}_1^\dagger)^{-1/2}_{\rm mp} = \int d^3p \tilde{a}^\dagger(\mathbf{p})|0\rangle d^3p \big(2\omega_m(\mathbf{p})\big)^{3/2} \langle 0|\tilde{a}(\mathbf{p}) \tag{D.64}$$

satisfies the 4 Penrose equations.

The renormalized injection operator Φ_1 is

$$\Phi_1(\mathbf{p}) := \big(2\omega_m(\mathbf{p})\big)^{1/2} \hat{\Phi}_1(\mathbf{p}) = \big(2\omega_m(\mathbf{p})\big)^{1/2} \tilde{a}^\dagger(\mathbf{p})|0\rangle = \tilde{a}^\dagger_r(\mathbf{p})|0\rangle. \tag{D.65}$$

Thus we see that in this special case the wave function renormalization factor is $Z^{-1/2} = \big(2\omega_m(\mathbf{p})\big)^{1/2}$ This result could have been obtained by directly computing $\langle 0|\tilde{a}^\dagger(\mathbf{p})\tilde{a}(\mathbf{p}')|0\rangle$ and including the factors so it has the normalization, $\delta^3(\mathbf{p}-\mathbf{p}')$, consistent with (D.54). The renormalization procedure defined by (D.56) and (D.57) has the virtue that is generalizes without change to the two-body case.

The renormalized injection operator Φ_1 is a unitary operator form $\mathcal{H}_1$ to $\mathcal{H}_m$. To see this note that the Penrose equations imply:

$$\Phi_1\Phi_1^\dagger = (\hat{\Phi}_1\hat{\Phi}_1^\dagger)^{-1/2}_{\rm mp}(\hat{\Phi}_1\hat{\Phi}_1^\dagger)(\hat{\Phi}_1\hat{\Phi}_1^\dagger)^{-1/2}_{\rm mp} \tag{D.66}$$

is the orthogonal projector on the one particle subspace of the physical Hilbert space and

$$\Phi_1^\dagger\Phi_1 = \hat{\Phi}_1^\dagger(\hat{\Phi}_1\hat{\Phi}_1^\dagger)^{-1}_{\rm mp}\hat{\Phi}_1 \tag{D.67}$$

which is the identity on $\mathcal{H}_1$. This follows because $\mathcal{H}_1$ is the range of $\hat{\Phi}_1^\dagger$, so any vector in $\mathcal{H}_1$ can be expressed in the form $\hat{\Phi}^\dagger(X)$ for some X in the one particle subspace of the Hilbert space.

By unitarity of Φ_1 it follows that

$$U_1(\Lambda, a) := \Phi_1^\dagger U(\Lambda, a)\Phi_1 \tag{D.68}$$

is an irreducible unitary representation of the Poincaré group on $\mathcal{H}_1$. Because $\hat{\Phi}$ intertwines with the momentum, (i.e. the momentum in $\mathcal{H}_1$ has the same value as the momentum of the particle in the field theory), it follows from the unitarity that this representation necessarily has the form

$$\big(U_1(\Lambda, a)f\big)(\mathbf{p}) = e^{-i(\omega_m(\mathbf{p})a^0 - \mathbf{p}\cdot\mathbf{a})}f(\mathbf{p}')\sqrt{\frac{\omega_m(\mathbf{p}')}{\omega_m(\mathbf{p})}} \tag{D.69}$$

where

$$p' := \Lambda^{-1}(\omega_m(\mathbf{p}), \mathbf{p}). \tag{D.70}$$

D.6. **The Two-Body Subspace**

Unlike the one particle sector of the physical Hilbert space of a local field theory, there is no Poincaré invariant two particle subspace of the physical Hilbert space (except in a free field theory). What we need is a subspace that allows us to construct a suitable two-body scattering theory. A minimal requirement for a two body scattering theory is that when the two "particles" are separated by asymptotic spacelike distances that the Hamiltonian becomes a sum of single particle Hamiltonians.

To begin this construction we define our "two-body" subspace by applying the operators used to construct the one particle space twice to the vacuum. Formally we construct the mapping $\hat{\Phi}_2$ from

$$\mathcal{H}_2 := \mathcal{H}_1 \otimes \mathcal{H}_1 \tag{D.71}$$

to the physical Hilbert space. Define the mapping $\hat{\Phi}_2$:

$$\hat{\Phi}_2(f_1, f_2) := \frac{1}{2}\big(a_r^\dagger(f_1)a_r^\dagger(f_2) + a_r^\dagger(f_1)a_r^\dagger(f_2)\big)|0\rangle. \tag{D.72}$$

Note that the renormalization factors can be absorbed into the definitions of the functions f_1 and f_2 using:

$$\tilde{f}_{ir}(\mathbf{p}) := \sqrt{2\omega_m(\mathbf{p})}\tilde{f}_i(\mathbf{p}) \tag{D.73}$$

which implies

$$a_f^\dagger(f) = \int d^3x f_r(\mathbf{x})A(\mathbf{x}, 0). \tag{D.74}$$

Using this we can show that $\hat{\Phi}_2$ intertwines the Euclidean group and satisfies cluster properties. To show the Euclidean invariance first note:

$$U(R,\mathbf{a})\hat{\Phi}_2(f,g) =$$

$$\frac{1}{2}\int d^3x d^3y \big(g_r(\mathbf{x})f_r(\mathbf{y}) + g_r(\mathbf{y})f_r(\mathbf{x})\big)\times$$

$$A(R\mathbf{x}+\mathbf{a},0)A(R\mathbf{y}+\mathbf{a},0)|0\rangle =$$

$$\frac{1}{2}\int d^3x d^3y \big[g_r(R^{-1}(\mathbf{x}-\mathbf{a}))f_r(R^{-1}(\mathbf{y}-\mathbf{a})) + g_r(R^{-1}(\mathbf{y}-\mathbf{a}))f_r(R^{-1}(\mathbf{x}-\mathbf{a}))\big]\times$$

$$A(\mathbf{x},0)A(\mathbf{y},0)|0\rangle. \tag{D.75}$$

Since the renormalized functions f and g involve multiplication of the original functions in momentum space by a rotationally invariant function, the configuration space functions are the convolution of the unrenormalized configurations space function with a rotationally invariant distribution. This means that the renormalization is Euclidean invariant - or equivalently Euclidean transformations and renormalizations can be done in any order. It follows that the Euclidean transformed renormalized functions in (D.75) can be replaced by renormalized Euclidean transformed functions:

$$f_r \rightarrow f_{r\,(R,\mathbf{a})}(\mathbf{x}) = f_r(R^{-1}(\mathbf{x}-\mathbf{a})) \tag{D.76}$$

$$g_r \rightarrow g_{r\,(R,\mathbf{a})}(\mathbf{x}) = g_r(R^{-1}(\mathbf{x}-\mathbf{a})). \tag{D.77}$$

This can be summarized concisely by the following equation:

$$U(R,\mathbf{a})\hat{\Phi}_2 = \hat{\Phi}_2 U(R,\mathbf{a}) \otimes U(R,\mathbf{a}) \tag{D.78}$$

which shows that $\hat{\Phi}_2$ intertwines the physical representation of the Euclidean subgroup with the tensor product representation of $\mathcal{H}_2$.

To show that $\hat{\Phi}_2$ has the desired cluster properties consider

$$\lim_{|\mathbf{a}-\mathbf{b}|\to\infty} \hat{\Phi}_2^\dagger(f_{(I,\mathbf{a})}, g_{(I,\mathbf{g})}) H \hat{\Phi}_2(f_{(I,\mathbf{a})}, g_{(I,\mathbf{g})}) H =$$

$$\lim_{|\mathbf{a}-\mathbf{b}|\to\infty} \int d^3x d^3y d^3x' d^3y' \times$$

$$\frac{1}{4} f_r^*(\mathbf{x}) g_r^*(\mathbf{y})) f_r(\mathbf{x}') g_r(\mathbf{y}') \times$$

$$\langle 0 | \{ A(\mathbf{x}+\mathbf{a},0) A(\mathbf{y}+\mathbf{b},0) + A(\mathbf{y}+\mathbf{b},0) A(\mathbf{x}+\mathbf{a},0) \} H \times$$

$$\{ A(\mathbf{x}'+\mathbf{a},0) A(\mathbf{y}'+\mathbf{b},0) + A(\mathbf{y}'+\mathbf{b},0) A(\mathbf{x}'+\mathbf{a},0) \} | 0 \rangle. \tag{D.79}$$

Using cluster properties of the quasilocal fields, (D.45) translational invariance of the vacuum, and the expression for H in (D.42) this becomes:

$$\int d^3x d^3y d^3x' d^3y' \times$$

$$f_r^*(\mathbf{x}) g_r^*(\mathbf{y})) f_r(\mathbf{x}') g_r(\mathbf{y}') \times$$

$$\{\langle 0|A(\mathbf{x},0)HA(\mathbf{x}',0)|0\rangle\,\langle 0|A(\mathbf{y},0)A(\mathbf{y}',0)|0\rangle +$$

$$\langle 0|A(\mathbf{x},0)A(\mathbf{x}',0)|0\rangle\,\langle 0|A(\mathbf{y},0)HA(\mathbf{y}',0)|0\rangle\} =$$

$$\{\Phi_1^\dagger(f)H\Phi_1(f)\Phi_1^\dagger(g)\Phi_1(g) + \Phi_1^\dagger(f)\Phi_1(f)\Phi_1^\dagger(g)H\Phi_1(g)\}. \tag{D.80}$$

This is the sum of the one-body Hamiltonians. In momentum space the right hand side of (D.80) has the familiar form:

$$\to \delta(\mathbf{p}_1-\mathbf{p}_1')\delta(\mathbf{p}_2-\mathbf{p}_2')\big(\omega_m(\mathbf{p}_1)+\omega_m(\mathbf{p}_2)\big). \tag{D.81}$$

In general, the operator $\hat{\Phi}_2$, is not a-priori a unitary mapping from the symmetric subspace of $\mathcal{H}_2$ to a Euclidean invariant subspace of the field theory. This can be fixed with one more wave function renormalization. The renormalization can be treated as in the one-body case. Define the renormalized injection operator:

$$\Phi_2 := Z_2^{-1/2}\hat{\Phi}_2 \tag{D.82}$$

where

$$Z_2^{-1} = (\hat{\Phi}_2\hat{\Phi}_2^\dagger)^{-1}_{\rm mp} \tag{D.83}$$

It follows from the Penrose equations that $\hat{\Phi}_2\hat{\Phi}_2^\dagger$ and $\hat{\Phi}_2^\dagger\hat{\Phi}_2$ are orthogonal projection operators. The first of these operators is the projection on the range of $\hat{\Phi}_2$, which is a Euclidean invariant subspace of the field theory, while the second of these is the orthogonal projector on the *symmetric* subspace of $\mathcal{H}_2$. It follows that Φ_2 is a unitary mapping between these subspaces.

Because $\hat{\Phi}_2$ intertwines with the Euclidean group it follows that Z_2 is Euclidean invariant and consequently that Φ_2 also intertwines with the Euclidean group.

To show that the renormalization does not interfere with cluster properties note first that

$$(\hat{\Phi}_2\hat{\Phi}_2^\dagger)_{\rm mp}^{-1/2}\hat{\Phi}_2 = \hat{\Phi}_2(\hat{\Phi}_2^\dagger\hat{\Phi}_2)_{\rm mp}^{-1/2} \tag{D.84}$$

Next consider $\hat{\Phi}_2^\dagger\hat{\Phi}_2$, which is a mapping on $\mathcal{H}_2$. Up to one-body renormalizations factors it is the 4-point function of the quasilocal field A with each factor restricted to time 0. The operator $\hat{\Phi}_2^\dagger\hat{\Phi}_2$ can be expanded in its truncated parts, using the property that $\langle 0|A(x)|0\rangle = 0$ because the range of A is in the one particle subspace which is orthogonal to the vacuum. By examining the properties of each of the truncated pieces with the one-body renormalization factors included it can be shown to have the asymptotic form

$$\hat{\Phi}_2^\dagger\hat{\Phi}_2 \to \frac{1}{2}\Pi_{\mathcal{H}_{2\rm symmetric}} \tag{D.85}$$

where $\Pi_{\mathcal{H}_{2\rm symmetric}}$ is the projector on the symmetric subspace of $\mathcal{H}_2$. This means that up to normalization it asymptotically becomes the identity when the particles are separated. Similarly $(\hat{\Phi}_2^\dagger\hat{\Phi}_2)_{\rm mp}^{-1/2}$ is asymptotically a different constant multiple of the identity. The unitarity ensures that if we evaluate the expectation value of the energy in a completely symmetric wave function corresponding to two asymptotically separated particles that the expectation value will be approximately the sum of the expectation values of the energies of each particle. What this means is that on the symmetric subspace of $\mathcal{H}_2$ that

$$\Phi_2^\dagger H\Phi_2 = \omega_m(\mathbf{p}_1) + \omega_m(\mathbf{p}_2) + V_2 \tag{D.86}$$

where the interaction V_2 is short ranged, Euclidean invariant, and has the form

$$\langle \mathbf{p}_1'' \, \mathbf{p}_2'' | V | \mathbf{p}_1' \, \mathbf{p}_2' \rangle =$$

$$\Phi_2^\dagger(\mathbf{p}_1'' \, \mathbf{p}_2'')(P^0 - \omega_m(\mathbf{p}_1) - \omega_m(\mathbf{p}_2))\Phi_2^\dagger(\mathbf{p}_1' \, \mathbf{p}_2') =$$

$$\delta(\mathbf{P}'' - \mathbf{P}')\langle \mathbf{p}_1'' \, \mathbf{p}_2'' \| \hat{V}_2(\mathbf{0}) \| \mathbf{p}_1' \, \mathbf{p}_2' \rangle \tag{D.87}$$

where $\mathbf{P} = \mathbf{p}_1 + \mathbf{p}_2$ is the total momentum. For the reasons discussed previously, the interaction for different values of $\mathbf{P}$ do not define scattering equivalent models.

A relativistic dynamics is obtained by replacing the interaction defined above by:

$$\langle \mathbf{p}_1' \, \mathbf{p}_2' | \bar{V}_2 | \mathbf{p}_1 \, \mathbf{p}_2 \rangle =$$

$$\delta(\mathbf{P}' - \mathbf{P})\langle \mathbf{k}_1' \, - \mathbf{k}_1' \| \hat{V}_2(\mathbf{0}) \| \mathbf{k}_1 \, - \mathbf{k}_1 \rangle \tag{D.88}$$

which involves factoring out the momentum conserving delta function and replacing the remaining momenta by $k_i = L_c^{-1}(P)p_i$ evaluated on the one-particle mass shell. The operators

$$\bar{M}_\pi = 2\omega_m(\mathbf{k}) + \bar{V}_2 \tag{D.89}$$

$$\bar{\mathbf{X}} = i\nabla_p \tag{D.90}$$

$$\bar{\mathbf{P}} := \Phi_2^\dagger \mathbf{P} \Phi_2 \tag{D.91}$$

$$\bar{\mathbf{J}} := \Phi_2^\dagger \mathbf{J} \Phi_2, \tag{D.92}$$

where the first two expressions are representation dependent; they can be used to construct a Bakamjian-Thomas dynamics. To define the model dynamics let

$$\bar{H} = \sqrt{\bar{M}^2 + \mathbf{P}^2} \tag{D.93}$$

and

$$\bar{\mathbf{K}} = -\frac{1}{2}\{\bar{H}, \mathbf{X}\}_+ - \frac{1}{\bar{H} + \bar{M}}(\mathbf{P} \times \bar{\mathbf{j}}). \tag{D.94}$$

where $\bar{\mathbf{j}} = \mathbf{J} - \mathbf{X} \times \mathbf{P}$. It follows that $\bar{H}$, $\bar{\mathbf{K}}$, $\mathbf{P}$, $\mathbf{J}$ satisfy the Poincaré commutation relations. They can be used to construct a model dynamics following the construction of section 5.

It is useful to write the desired model interaction directly in terms of the renormalized fields, $\phi_r(x) := Z_1^{-1/2}\phi(x)$, and the two-body renormalization operators:

$$\langle \mathbf{p}_1'' \, \mathbf{p}_2'' | V_2 | \mathbf{p}_1' \, \mathbf{p}_2' \rangle =$$

$$\frac{1}{(2\pi)^2} \int dp_1''^0 dp_2''^0 dp_1'^0 dp_2'^0 h(p_1'') h(p_2'') h(p_1') h(p_2') \langle 0 | \phi_r(p_2'') \phi_r(p_1'') (Z_2^{-1/2})^\dagger \times$$

$$\big(P^0 - \omega_m(\mathbf{p}_1) - \omega_m(\mathbf{p}_1)\big)(Z_2^{-1/2}) \phi_r(p_1') \phi_r(p_2') | 0 \rangle. \tag{D.95}$$

This formula gives the interaction (D.87) which must be restricted as in (D.88) to obtain the interaction $\bar{V}_2$. This construction applies to the two-body system. It gives explicit formulas for the interactions in terms of field theoretic quantities. These interactions can depend on the functions $h(x)$ used to define the quasilocal fields.

An analogous construction can be used to construct two-body current matrix elements:

$$\langle \mathbf{p}_1'' \, \mathbf{p}_2'' | I^\mu(x) | \mathbf{p}_1' \, \mathbf{p}_2' \rangle =$$

$$\frac{1}{(2\pi)^2} \int dp_1''^0 dp_2''^0 dp_1'^0 dp_2'^0 h(p_1'') h(p_2'') h(p_1') h(p_2') \langle 0 | \phi_r(p_2'') \phi_r(p_1'') (Z_2^{-1/2})^\dagger \times$$

$$I^\mu(x) (Z_2^{-1/2}) \phi_r(p_1') \phi_r(p_2') | 0 \rangle. \tag{D.96}$$

or matrix elements of any field combination of field operators. In this case because the projection operators are Euclidean invariant this projected current operator will satisfy Euclidean covariance, but will not generally be Lorentz covariant or satisfy current conservation. This can be restored, in a manner similar to the way in which Poincaré invariance was restored, by computing dynamically independent current matrix elements and using current covariance and current conservation to determine the remaining matrix elements. These computations should be done using wave functions associated with the barred representation. Similar remarks apply to any Lorentz tensor operators.

The construction in this section can be modified to treat forms of the dynamics other than the instant form. In principle one should be able to extend this to treat more complex systems, although in these cases other consideration such as cluster properties become relevant. The general principle is that as one systematically increases the range of the injections operators one should expect that the model can be systematically improved. The extension to system of confined quarks can be treated with some modifications of the starting assumptions. For nucleons, the quasilocal fields must be constructed out of three quark fields coupled to a color singlet. This combination of fields must have non-vanishing matrix elements between the vacuum and one particle (i.e. hadron) states. In this case there will be constraints on the structure of the six-point function as well as the two-point and four point functions.

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