Wavelet Notes

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Abstract

Wavelets are a useful basis for constructing solutions of the integral and differential equations of scattering theory. Wavelet bases efficiently represent functions with smooth structures on different scales, and the matrix representation of operators in a wavelet basis are well-approximated by sparse matrices. The basis functions are related to solutions of a linear renormalization group equation, and the basis functions have structure on all scales. Numerical methods based on this renormalization group equation are discussed. These methods lead to accurate and efficient numerical approximations to the scattering equations. These notes provide a detailed introduction to the subject that focuses on numerical methods. We plan to provide periodic updates to these notes.

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1 Introduction

Wavelets are versatile functions with a wide range of applications including time-frequency analysis, data compression, and numerical analysis. The objective of these notes is to provide an introduction to the properties of

wavelets which are useful for solving integral and differential equations by using the wavelets to represent the solution of the equations.

While there are many types of wavelets, we concentrate primarily on orthogonal wavelets of compact support, with particular emphasis on the wavelets introduced by Daubechies. The Daubechies wavelets have the additional property that finite linear combinations of the Daubechies wavelets provide local pointwise representations of low-degree polynomials. We also have a short discussion of continuous wavelets in the Appendix I and spline wavelets in Appendix II.

There notes are not intended to provide a complete discussion of the subject which can be found in the references given at the end of this section. Rather, we concentrate on the specific properties which are useful for numerical solutions of integral and differential equations. Our approach is to develop the wavelets as orthonormal basis functions rather than in terms of low- and high-pass filters, which is more common for time-frequency analysis applications.

The Daubechies wavelets have some properties that make them natural candidates for basis functions to represent solutions of integral equations. Like splines, they are functions of compact support that can locally pointwise represent low degree polynomials. Unlike splines, they are orthonormal. More significantly, only a relatively small number of wavelets are needed to represent smooth functions.

One of the interesting features of wavelets is that they can be generated from a single scaling function, which is the solution of a liner renormalization-group equation, by combinations of translations and scaling. This equation, called the scaling equation, expresses the scaling function on one scale as a finite linear combination of discrete translations of the same function on a smaller scale. The resulting scaling functions and wavelets have a fractal-like structure. This means that they have structure on all scales. This requires a different approach to the numerical analysis, which is provided by the scaling equation. These notes make extensive use of the scaling function.

Some of the references that we have found useful are:

[1] I. Daubechies, Orthonormal bases of compactly supported wavelets, Comm. Pure Appl. Math. **41**(1988)909.

[2] G. Strang, "Wavelets and Dilation Equations: A Brief Introduction," SIAM Review, 31:4, pp. 614–627, (Dec 1989).

[3] I. Daubechies, *Ten Lectures on Wavelets*, SIAM, Philadelphia, 1992.

[4] C. K. Chui *Wavelets - A tutorial in Theory and Applications*, Academic

Press, 1992.
[5] W.-C. Shann, "Quadrature rules needed in Galerkin-wavelets methods", Proceedings for the 1993 annual meeting of Chinese Mathematics Association, Chiao-Tung Univ, (Dec 1993).
[6] W.-C. Shann and J.-C. Yan, "Quadratures involving polynomials and Daubechies' wavelets", Technical Report 9301, Department of Mathematics, National Central University, (1993).
[7] G. Kaiser, *A Friendly Guide to Wavelets*, Birkhauser 1994.
[8] W. Sweldens and R. Piessens, "Quadrature Formulae and Asymptotic Error Expansions for wavelet approximations of smooth functions", SIAM J. Numer. Anal., 31, pp. 1240–1264, (1994).
[9] H. L. Resnikoff and R. O. Wells, *Wavelet Analysis, The Scalable Structure of Information*, Springer Verlag, NY.
[10] O. Bratelli and P. Jorgensen, *Wavelets through a Looking Glass*, Birkhauser, 2002.

In addition, some of the material in these notes is in our paper

[11] B. M. Kessler, G. L. Payne, and W. N. Polyzou, Scattering Calculations With Wavelets, Few Body Systems, 33,1-26(2003).

2 Haar Scaling Functions and Wavelets

Scaling functions play a central role in the construction of orthonormal bases of compactly supported wavelets. The scaling functions and wavelets are distinct bases related by an orthogonal transformation called the wavelet transform.

The concept of scaling functions is most easily understood using Haar wavelets (these are made out of simple box functions). The Haar functions are the simplest compactly supported scaling functions and wavelets.

The **Haar scaling function** is defined by

$$\phi(x) := \begin{cases} 0 & x \leq 0 \\ 1 & 0 < x \leq 1 \\ 0 & x > 1 \end{cases} . \tag{1}$$

It satisfies the normalization conditions:

$$(\phi, \phi) := \int_{-\infty}^{\infty} \phi^*(x)\phi(x)dx = \int_0^1 \phi(x)dx = 1. \tag{2}$$

The operations of discrete translation and dilatation are used extensively in the study of compactly supported wavelets. The **unit translation operator** T is defined by

$$(T\chi)(x) = \chi(x-1). \tag{3}$$

This operator translates the function $\chi(x)$ to the right by one unit. The unit translation operator has the property:

$$(T\psi, T\chi) = \int_{-\infty}^{\infty} \psi^*(x-1)\chi(x-1)dx = \tag{4}$$

$$\int_{-\infty}^{\infty} \psi^*(y)\chi(y)dy = (\psi, \chi) \tag{5}$$

where $y = x - 1$. This means that the unit translation operator preserves the scalar product:

$$(T\psi, T\chi) = (\psi, \chi). \tag{6}$$

If A is a linear operator its **adjoint** $A^\dagger$ is defined by the relation

$$(\psi, A^\dagger\chi) = (A\psi, \chi). \tag{7}$$

It follows that

$$(\psi, T^\dagger\chi) = (T\psi, \chi) = \int_{-\infty}^{\infty} \psi^*(x-1)\chi(x)dx. \tag{8}$$

Changing variables to $y = x - 1$ gives

$$(\psi, T^\dagger\chi) = \int_{-\infty}^{\infty} \psi^*(y)\chi(y+1)dy \tag{9}$$

or

$$(T^\dagger\chi)(x) = \chi(x+1) \tag{10}$$

which is a left shift by one unit. Since

$$(\psi, \chi) = (T\psi, T\chi) = (\psi, T^\dagger T\chi) \tag{11}$$

it follows that $T^\dagger = T^{-1}$. An operator whose adjoint is its inverse is called unitary. Unitary operators preserve inner products.

It follows from the definition of the Haar scaling function, $\phi(x)$, that

$$(T^m\phi, T^n\phi) = (\phi, T^{n-m}\phi) = \int_{-\infty}^{\infty} \phi^*(x)\phi(x-n+m)dx =$$

$$\int_0^1 \phi(x-n+m)dx = \delta_{nm} \tag{12}$$

This means the functions

$$\phi_n(x) := (T^n\phi)(x) = \phi(x-n) \tag{13}$$

are orthonormal. There are an infinite number of these functions for integers n satisfying $-\infty < n < \infty$.

The integer translates of the scaling function span a space, $\mathcal{V}_0$, which is a subspace of the space of square integrable functions. The elements of $\mathcal{V}_0$ are functions of the form

$$f(x) = \sum_{n=-\infty}^{\infty} f_n\phi_n(x) = \sum_{n=-\infty}^{\infty} f_n(T^n\phi)(x) = \sum_{n=-\infty}^{\infty} f_n\phi(x-n), \tag{14}$$

where the square integrability requires that the coefficients satisfy

$$\sum_{n=-\infty}^{\infty} |f_n|^2 < \infty. \tag{15}$$

For the Haar scaling function $\mathcal{V}_0$ is the space of square integrable functions that are piecewise constant on each unit-width interval. Note that while there are an infinite number of functions in $\mathcal{V}_0$, it is a small subspace of the space of square integrable functions.

In addition to translations T, the linear operator D, corresponding to **discrete scale transformations**, is defined by:

$$(D\chi)(x) = \frac{1}{\sqrt{2}}\chi(x/2). \tag{16}$$

When this is applied to the Haar scaling function it gives

$$(D\phi)(x) = \begin{cases} 0 & x \leq 0 \\ \frac{1}{\sqrt{2}} & 0 < x \leq 2 \\ 0 & x > 2 \end{cases}. \tag{17}$$

This function has the same box structure as the original Haar scaling function, except it is twice as wide as the original scaling function and shorter by a factor of $\sqrt{2}$. Note that the normalization ensures

$$(D\psi, D\chi) = \int_{-\infty}^{\infty} \frac{1}{2}\psi^*(x/2)\chi(x/2)dx \tag{18}$$

$$= \int_{-\infty}^{\infty} \psi^*(y)\chi(y)dy = (\psi, \chi) \tag{19}$$

where the variable in the integrand has been changed to $y = x/2$.

The adjoint of D is determined by the definition

$$(\psi, D^\dagger \chi) = (D\psi, \chi) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2}} \psi^*(x/2)\chi(x)dx. \tag{20}$$

Setting $y = x/2$ gives

$$\int_{-\infty}^{\infty} \psi^*(y)\sqrt{2}\chi(2y)dy \tag{21}$$

which gives

$$(D^\dagger \chi)(x) = \sqrt{2}\chi(2x). \tag{22}$$

This shows that $D^\dagger = D^{-1}$ or D is also unitary.

Define the functions constructed by n translations followed by m scale transformations

$$\phi_{mn}(x) = (D^m T^n \phi)(x) = (D^m \phi_n)(x) \tag{23}$$

$$= 2^{-m/2}\phi(2^{-m}x - n) = 2^{-m/2}\phi(2^{-m}(x - 2^m n)). \tag{24}$$

It follows that for a fixed scale m

$$(\phi_{mn}, \phi_{mk}) = (D^m \phi_n, D^m \phi_k) = (\phi_n, D^{m-m}\phi_k) = (\phi_n, \phi_k) = \delta_{nk}. \tag{25}$$

This shows that the functions $\phi_{mn}(x)$ for any fixed scale m are orthonormal.

We define the subspace $\mathcal{V}_m$ of the square integrable functions to be those functions of the form:

$$f(x) = \sum_{n=-\infty}^{\infty} f_n \phi_{mn}(x) = \sum_{n=-\infty}^{\infty} f_n (D^m T^n \phi)(x) \tag{26}$$

where the square integrability requires that the coefficients satisfy

$$\sum_{n=-\infty}^{\infty} |f_n|^2 < \infty. \tag{27}$$

These elements of $\mathcal{V}_m$ are square summable functions that are piecewise constant on intervals of width 2^m. The spaces $\mathcal{V}_m$ and $\mathcal{V}_0$ are related by m scale transformations $D^m \mathcal{V}_0 = \mathcal{V}_m$.

In general the scaling function $\phi(x)$ is defined as the solution of a scaling equation subject to a normalization condition. The scaling equation relates the scaled scaling function, $(D\phi)(x)$, to translates of the original scaling function. The general form of the **scaling equation** is

$$(D\phi)(x) = \sum_l h_l T^l \phi(x) \tag{28}$$

where h_l are fixed constants, and the sum may be finite or infinite. This equation can be expressed as

$$\frac{1}{\sqrt{2}}\phi(\frac{x}{2}) = \sum_l h_l \phi(x-l) \tag{29}$$

which is sometimes written as

$$\phi(x) = \sqrt{2}\sum_l h_l \phi(2x-l) = \sum_l c_l \phi(2x-l) \tag{30}$$

where $c_l = \sqrt{2}h_l$. Equation (28) is the most important equation in these notes.

In general the scaling equation cannot be solved analytically. In the special case of the Haar scaling function the solution is obtained by observing that the scaled box is stretched over two adjacent boxes with a suitable reduction in height. It follows that:

$$\begin{aligned} D\phi(x) &= \frac{1}{\sqrt{2}}\phi(x/2) = \frac{1}{\sqrt{2}}\phi(x) + \frac{1}{\sqrt{2}}T\phi(x) \\ &= \frac{1}{\sqrt{2}}\phi(x) + \frac{1}{\sqrt{2}}\phi(x-1). \end{aligned} \tag{31}$$

Here $h_0 = h_1 = 1/\sqrt{2}$. These coefficients are special to the Haar scaling function. The best way to think about the scaling function $\phi(x)$ is to note that the scaling function $\phi(x)$ is the solution of the scaling equation up to normalization. The normalization is fixed by

$$\int \phi(x)dx = 1.$$

An additional relation involving the translation T and dilatation operator D is useful for future computations. First note that

$$DT\psi(x) = D\psi(x-1) = \frac{1}{\sqrt{2}}\psi(x/2-1) = \frac{1}{\sqrt{2}}\psi(\frac{x-2}{2}) = T^2 D\psi(x), \quad (32)$$

which leads to the operator relation

$$DT = T^2 D. \quad (33)$$

It follows from this equation that

$$D\phi_n(x) = DT^n\phi(x) = T^{2n}D\phi(x) = T^{2n}(h_0\phi(x) + h_1 T\phi(x)). \quad (34)$$

This shows that all of the basis elements in $\mathcal{V}_1$ can be expressed in terms of basis elements in $\mathcal{V}_0$. For the case of the Haar scaling function this is obvious, but the argument above is more general.

Specifically if $\psi(x) \in \mathcal{V}_1$ then

$$\psi(x) = \sum_{n=-\infty}^{\infty} d_n \phi_{1n}(x) = \sum_{n=-\infty}^{\infty} d_n D\phi_n(x) \quad (35)$$

$$= \sum_{n=-\infty}^{\infty} [d_n h_0 \phi_{2n}(x) + d_n h_1 \phi_{2n+1}(x)] = \sum_{-\infty}^{\infty} e_n \phi_n(x) \quad (36)$$

where

$$e_{2n} = d_n h_0 \qquad e_{2n+1} = d_n h_1. \quad (37)$$

It is easy to show that

$$\sum_{n=-\infty}^{\infty} |e_n|^2 = \sum_{n=-\infty}^{\infty} |d_n|^2. \quad (38)$$

What we have shown, as a consequence of the scaling equation, is the inclusion property

$$\mathcal{V}_0 \supset \mathcal{V}_1. \quad (39)$$

Similarly, using the same method, it is possible to show the chain of inclusions

$$\cdots \mathcal{V}_{-k} \supset \mathcal{V}_{-k+1} \supset \cdots \supset \mathcal{V}_0 \supset \cdots \mathcal{V}_k \supset \mathcal{V}_{k+1} \cdots \quad (40)$$

These properties hold for the solution of any scaling equation. In the Haar example the spaces $\mathcal{V}_m$ are spaces of piecewise constant, square integrable functions that are constant on intervals of the real line of width 2^m.

The subspaces $\mathcal{V}_m$ are used as approximation spaces in applications. To understand how they are used as approximation spaces note that as $m \to -\infty$ the approximation to $f(x)$ given by

$$f_m(x) = \sum_{n=-\infty}^{\infty} f_{mn}\phi_{mn}(x) \tag{41}$$

with

$$f_{mn} = \int_{-\infty}^{\infty} \phi_{mn}(x) f(x) dx \tag{42}$$

is bounded by the upper and lower Riemann sums for steps of width 2^{-m}. This is because, up to a scale factor, the coefficients f_{mn} are just average values of the function on the appropriate sub-interval (to deal with the infinite interval it is best to first consider functions that vanish outside of finite intervals and take limits). Since the upper and lower Riemann sums converge to the same integral (when the function is integrable) it follows that

$$\int_{-\infty}^{\infty} |f_m(x) - f(x)| dx < \epsilon \tag{43}$$

for sufficiently large $-m$. A similar argument can be extended to get L^2 convergence .

Similarly, as $m \to +\infty$, the width of $\phi_{mn}(x)$ grows like 2^m while the height falls off like $2^{-m/2}$. Again, if the function vanishes outside of a bounded interval then for sufficiently large m there is only one (or two) $\phi_{mn}(x)$ that are non-vanishing where the function is non-vanishing. In the case that only one $\phi_{mn} = \phi_{mn_0}$ overlaps the support of $f(x)$

$$f_m(x) \sim 2^{-m/2}\phi_{mn_0}(x) \int_{-\infty}^{\infty} f(x) dx. \tag{44}$$

The integral of the square of this function $\sim 2^{-m} \to 0$ as $m \to \infty$.

Note that

$$\int_{-\infty}^{\infty} f_m(x) dx \to \int_{-\infty}^{\infty} f(x) dx \tag{45}$$

as $m \to \infty$. This shows that the limit of the integral of $f_m(x)$ as $m \to \infty$ is finite in L^1 but 0 in L^2.

It is useful to express some of these results in a more useful form. Define the projection operators

$$P_m f(x) = \sum_{n=-\infty}^{\infty} f_{mn}\phi_{mn}(x) \tag{46}$$

where

$$f_{mn} = \int_{-\infty}^{\infty} \phi^*_{mn}(x) f(x) dx. \tag{47}$$

The above conditions can be stated in terms of these projectors:

$$\lim_{m\to-\infty} P_m = I \tag{48}$$

$$\lim_{m\to+\infty} P_m = 0. \tag{49}$$

These results mean that the approximation space $\mathcal{V}_m$ approaches the space of square integrable functions as $m \to -\infty$. We have shown that (48) and (49) are valid for the Haar scaling function, but they are also valid for a large class of scaling functions,

We are now ready to construct **wavelets**. First recall the condition

$$\mathcal{V}_0 \supset \mathcal{V}_1. \tag{50}$$

Let $\mathcal{W}_1$ be the subspace of vectors in the space $\mathcal{V}_0$ that are orthogonal to the vectors in $\mathcal{V}_1$. We can write

$$\mathcal{V}_0 = \mathcal{V}_1 \oplus \mathcal{W}_1. \tag{51}$$

This notation means that any vector in $\mathcal{V}_0$ can be expressed as a sum of two vectors - one that is in $\mathcal{V}_1$ and one that is orthogonal to every vector in $\mathcal{V}_1$.

Note that the scaling equation implies that every vector in $\mathcal{V}_1$ can be expressed as a linear combination of vectors in $\mathcal{V}_0$ using

$$D\phi_n(x) = h_0\phi_{2n}(x) + h_1\phi_{2n+1}(x). \tag{52}$$

Clearly the functions that are orthogonal to these in $\mathcal{V}_1$ on the same interval can be expressed in terms of the difference functions

$$\psi_{1n}(x) := D\psi_n(x) = h_1\phi_{2n}(x) - h_0\phi_{2n+1}(x) = \frac{1}{\sqrt{2}}(\phi_{2n}(x) - \phi_{2n+1}(x)). \tag{53}$$

Direct computation shows that the $\psi_{1n}(x)$ are elements of $\mathcal{V}_0$ that satisfy

$$(D\psi_{1n}, D\phi_l) = 0. \tag{54}$$

and

$$(\psi_{1n}, \psi_{1k}) = \delta_{nk}. \tag{55}$$

Thus we conclude that $\mathcal{W}_1$ is the space of square integrable functions of the form

$$f(x) = \sum_{n=-\infty}^{\infty} f_n \psi_{1n}(x) \tag{56}$$

with

$$f(x) = \sum_{n=-\infty}^{\infty} |f_n|^2. \tag{57}$$

Similarly, we can decompose $\mathcal{V}_l = \mathcal{V}_{l+1} \oplus \mathcal{W}_{l+1}$ for each value of l. For the special case of $\mathcal{W}_0$ we define the Haar **mother wavelet** as

$$\psi(x) := D^{-1}(h_1\phi(x) - h_0 T\phi(x)) = \tag{58}$$

$$h_1\sqrt{2}\phi(2t) - h_0\sqrt{2}\phi(2(t-1)) = (\phi(2t) - \phi(2(t-1))) \tag{59}$$

which is manifestly orthogonal to the scaling function. Translates of the mother wavelet define a basis for $\mathcal{W}_0$

$$\psi_n(x) = T^n\psi(x) = T^n D^{-1}(h_1\phi(x) - h_0 T\phi(x)) = \tag{60}$$

$$D^{-1}(h_1\phi_{2n}(x) - h_0\phi_{2n+1}(x)). \tag{61}$$

If we decompose $\mathcal{V}_m$ we have:

$$\mathcal{V}_{-m} = \mathcal{W}_{-m+1} \oplus \mathcal{V}_{-m+1}$$

$$= \mathcal{W}_{-m+1} \oplus \mathcal{W}_{-m+2} \oplus \mathcal{V}_{-m+2}$$

$$= \mathcal{W}_{-m+1} \oplus \mathcal{W}_{-m+2} \oplus \cdots \oplus \mathcal{W}_l \oplus \mathcal{V}_l. \tag{62}$$

Note that unlike the $\mathcal{V}_m$ spaces, the $\mathcal{W}_m$ spaces are all mutually orthogonal, since if $m > n \to \mathcal{W}_m \subset \mathcal{V}_n$ which is orthogonal to $\mathcal{W}_n$ by definition.

If $f(x)$ is any square integrable function the conditions

$$\lim_{m\to-\infty} P_m = I \tag{63}$$

$$\lim_{m\to+\infty} P_m = 0 \tag{64}$$

mean that for sufficiently large m and any l that $f(x)$ can be well approximated by a function in

$$\mathcal{W}_{-m+1} \oplus \mathcal{W}_{-m+2} \oplus \cdots \oplus \mathcal{W}_l. \tag{65}$$

This means that the function can be approximated by a linear combination of basis functions (wavelets) from each of the spaces $\mathcal{W}_r$.

A **multiresolution analysis** is a set of subspaces $\mathcal{V}_m$ and $\mathcal{W}_m$ satisfying (62), (63), and (64). The condition (63) allows one to interpret the space $\mathcal{V}_m$, for sufficiently large $-m$, as an approximation space for numerical applications.

Basis functions for $\mathcal{W}_m$ are given by

$$\psi_{mn}(x) = D^m T^n \psi(x) = D^{m-1}(h_1\phi_{2n}(x) - h_0\phi_{2n+1}(x)). \tag{66}$$

That these are a basis with the required properties is easily shown by showing that these functions are orthogonal to $\mathcal{V}_m$ and can be used to recover the remaining vectors in $\mathcal{V}_{m-1}$.

The functions $\psi_{nl}(x)$, are called Haar wavelets. They satisfy the orthonormality conditions:

$$(\psi_{nl}, \psi_{n'l'}) = \delta_{nn'}\delta_{ll'} \tag{67}$$

where the $\delta_{nn'}$ follows from the orthogonality of the spaces $\mathcal{W}_n$ and $\mathcal{W}_{n'}$ for $n \neq n'$.

The $\delta_{ll'}$ follows from the unitarity of D and

$$(\psi, T^n\psi) = \delta_{n0}. \tag{68}$$

The important steps discussed above generalize to the case of a general scaling equation of the form:

$$D\phi(x) = \sum h_l T^l \phi(x). \tag{69}$$

This equation is solved to find the scaling function $\phi(x)$. This, along with translations and dilatations is used to construct the spaces $\mathcal{V}_l$. The scaling equation ensures the existence of spaces $\mathcal{W}_m$, satisfying $\mathcal{V}_{m+1} = \mathcal{W}_m \oplus \mathcal{V}_m$ that can be used to build discrete orthonormal bases. The mother wavelet

function is expressed in terms of the scaling function and the coefficients h_l as

$$\psi(x) = D^{-1} \sum_l g_l T^l \phi(x) \tag{70}$$

where we will see later that

$$g_l = (-)^k h_{k-l} \tag{71}$$

where k is any odd integer. In general the coefficients h_l must satisfy constraints for the solution to the scaling equation to exist. General wavelets can be expressed in terms of the mother wavelet using (66). In the next section the coefficients g_l will be expressed in terms of the scaling function.

3 Scaling Functions - General Considerations

This section extends the treatment of scaling equation to a more general class of scaling functions than the Haar functions. In general, a scaling function satisfies the following three conditions. First, the **scaling function** is the solution of the **scaling equation**

$$D\phi(x) = \sum_l h_l T^l \phi(x) \tag{72}$$

where h_l are numerical coefficients that define the scaling equation. Second, in addition to satisfying the scaling equations, **integer translates** of the scaling functions are required to be **orthonormal**

$$(\phi_n, \phi_m) = (T^n \phi, T^m \phi) = (\phi, T^{m-n} \phi) = \delta_{mn}. \tag{73}$$

Third, the initial scale is fixed by the **normalization condition**

$$\int \phi(x) dx = 1. \tag{74}$$

It might seem like the normalization conditions in (73) and (74) are not compatible. To see that this is not true note that condition (73) is invariant under unitary changes of scale of the form

$$D_s \chi(x) := \frac{1}{\sqrt{s}} \phi\left(\frac{x}{s}\right)$$

while condition (74) is not. It follows that condition (74) can be interpreted as setting a starting scale, s. The condition (73) is preserved independent of the starting scale.

We now investigate the consequences of these three conditions. Using the definitions of the operators D and T the scaling equation becomes:

$$\frac{1}{\sqrt{2}}\phi(\frac{x}{2}) = \sum h_l \phi(x-l). \tag{75}$$

As shown in section 1, it can be put in the useful form

$$\phi(x) = \sum_l \sqrt{2} h_l \phi(2x-l). \tag{76}$$

In general the sums may be from $-\infty \to \infty$. Finite sums are treated by assuming that only a finite number of the h_l's are non zero. All of the compactly supported scaling functions are solutions of scaling equations with a finite number of non-zero coefficients.

If the scaling equation has a solution, it is unique up to an overall normalization factor. To see this take the Fourier transform of both sides of equation (76) to get

$$\tilde{\phi}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx} \phi(x) dx = \sum_l \sqrt{2} h_l \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx} \phi(2x-l) dx. \tag{77}$$

Changing variables $x \to 2x - l$ on the right-hand side gives

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx} \phi(x) dx = \sum_l \frac{1}{\sqrt{2}} h_l \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i(k/2)(x+l)} \phi(x) dx \tag{78}$$

or

$$\tilde{\phi}(k) = \tilde{\phi}\left(\frac{k}{2}\right) \tilde{h}\left(\frac{k}{2}\right) \tag{79}$$

where

$$\tilde{h}(k) = \sum_l \frac{h_l}{\sqrt{2}} e^{-ikl}. \tag{80}$$

This form of the scaling equation can be iterated n times to get:

$$\tilde{\phi}(k) = \tilde{\phi}\left(\frac{k}{2^n}\right) \prod_{m=1}^{n} \tilde{h}\left(\frac{k}{2^m}\right) \tag{81}$$

This equation holds for any n provided the Fourier transforms exist. For a finite n, an approximation can be made by a finite number of iterations of the form

$$\tilde{\phi}_n(k) = \tilde{\phi}_{n-1}\left(\frac{k}{2}\right)\tilde{h}\left(\frac{k}{2}\right) \tag{82}$$

for any starting function $\tilde{\phi}_0(k)$. In the limit of large n the function $\tilde{\phi}_n(k)$ should converge to a solution to the scaling equation. The result of formally taking this limit is

$$\begin{aligned}\tilde{\phi}(k) &= \lim_{n\to\infty}\tilde{\phi}_0(\frac{k}{2^n})\prod_{l=1}^{n}\tilde{h}(\frac{k}{2^l}) \\ &= \tilde{\phi}(0)\prod_{l=1}^{\infty}\tilde{h}(\frac{k}{2^l}).\end{aligned} \tag{83}$$

If the limit exists as $n \to \infty$, and the scaling function is continuous in a neighborhood of zero, then the solution of the scaling equation is uniquely determined by the scaling coefficients h_l up to the overall normalization $\tilde{\phi}_0(0)$. The condition $\tilde{\phi}_0(0) = 1/\sqrt{2\pi}$ is equivalent to the standard normalization condition

$$\int_{-\infty}^{\infty}\phi(x)dx = 1.$$

The resulting solution of the scaling equation is independent of the choice of starting function provided it is normalized so $\tilde{\phi(0)} = 1/\sqrt{2\pi}$. Once the normalization is fixed, the limit only depends on the coefficients h_l.

Thus, if the infinite product converges, then we have an expression for the scaling function, up to normalization, which is fixed by assigning a value to $\tilde{\phi}(0)$. To show how this works we compute this limit for the Haar scaling equation.

For the Haar scaling equation the expression for the scaling function is

$$\frac{1}{\sqrt{2\pi}}\prod_{l=1}^{\infty}\frac{1}{2}(1+e^{-ik/2^l})$$

$$= \lim_{l\to\infty}\frac{1}{\sqrt{2\pi}}\frac{1}{2^l}(1+e^{-ik/2})(1+e^{-ik/4})\cdots(1+e^{-ik/2^l})$$

expanding this out in powers of $e^{-ik/2^l}$ gives

$$= \frac{1}{\sqrt{2\pi}}\lim_{l\to\infty}\frac{1}{2^l}\sum_{m=0}^{2^l-1}(e^{-ik/2^l})^m$$

$$= \frac{1}{\sqrt{2\pi}} \lim_{l\to\infty} \frac{1}{2^l} \frac{1-e^{-ik}}{1-e^{-ik/2^l}} =$$

$$= \frac{1}{\sqrt{2\pi}} e^{-ik/2} \frac{\sin(k/2)}{(k/2)}. \tag{84}$$

A direct calculation of the Fourier transform of the Haar scaling function gives

$$\tilde{\phi}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx} \phi(x) dx = \frac{1}{\sqrt{2\pi}} \int_0^1 e^{-ikx} dx$$

$$= \frac{1}{\sqrt{2\pi}} e^{-ik/2} \frac{\sin(k/2)}{(k/2)} \tag{85}$$

which agrees with (84).

The above analysis shows that the solution of the scaling equation depends on the choice of scaling coefficients h_l. The scaling coefficients h_l are not arbitrary. First note that setting $k = 0$ in (83) gives

$$1 = \prod_{l=0}^{\infty} \tilde{h}(0). \tag{86}$$

Now using (80) gives

$$\tilde{h}(0) = 1 = \sum_l \frac{h_l}{\sqrt{2}} \tag{87}$$

or

$$\sum_l h_l = \sqrt{2}. \tag{88}$$

This condition is satisfied by the Haar wavelets. This is a necessary condition on the scaling coefficients in order to have a solution to the scaling equation.

Another condition which constrains the scaling coefficients is the orthogonality of the unit translates, $(\phi_n, \phi_m) = \delta_{nm}$. This requires, using (76),

$$2 \sum_{lk} h_l h_k \int_{-\infty}^{\infty} \phi(2x - 2n - l) \phi(2x - 2m - k) dx$$

$$= 2 \sum_{lk} h_l h_k \int_{-\infty}^{\infty} \phi(2x) \phi(2x - 2(m-n) - (k-l)) dx$$

$$= \sum_{lk} h_l h_k \int_{-\infty}^{\infty} \phi(x) \phi(x - 2(m-n) - (k-l)) dx$$

$$= \sum_l h_l h_{l-2(m-n)} = \delta_{mn} \tag{89}$$

or equivalently

$$\sum_l h_{l-2m} h_l = \delta_{m0}. \tag{90}$$

This is trivially satisfied for the Haar wavelets. Here and in all that follows we restrict our considerations to the case that the scaling coefficients and scaling functions are real.

The orthogonality condition also requires that the number of non-scaling coefficients must be even. To see this assume by contradiction that there are $2N+1$ non-zero scaling coefficients, $h_0 \cdots h_{2N}$. Then setting $m = -N$ in (90) gives

$$\sum_l h_{l+2N} h_l = h_{2N} h_0 = \delta_{N0} = 0. \tag{91}$$

which means that either $h_0 = 0$ or $h_{2N} = 0$, which contradicts the assumption that there are $2N+1$ non-zero scaling coefficients. This shows that if the number of non-zero scaling coefficients are finite, then there must be an *even number*, $2K$, with $l = 0 \cdots 2K-1$.

Note that if there are only two non-vanishing scaling coefficients, h_0 and h_1, then the conditions (88) and (90) have a unique solution, which is the Haar scaling coefficients. In this case these equations become

$$h_0 + h_1 = \sqrt{2} \tag{92}$$

$$h_0 h_0 + h_1 h_1 = 1. \tag{93}$$

These equations have the unique solution $h_0 = h_1 = 1/\sqrt{2}$.

Conditions (88) and (90) are important constraints on the scaling coefficients.

For scaling equations with more than two non-zero scaling coefficients, additional conditions are needed to determine the scaling coefficients.

The number of non-zero scaling coefficients determines the support of the scaling function. The important property is that scaling functions that are solutions of a scaling equation with a finite number of non-zero scaling coefficients have compact support. The support is determined by the number of non-zero scaling coefficients.

To determine the support of the scaling function, consider a scaling equation with $N = 2K+1$ non-zero scaling coefficients. The scaling function is

given by

$$\phi(x) = \frac{\tilde{\phi}(0)}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ikx} \prod_{m=1}^{\infty} \tilde{h}(\frac{k}{2^m}) dk$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx} \prod_{m=1}^{\infty} \left(\sum_{n_m=0}^{N-1} \frac{h_{n_m}}{\sqrt{2}} e^{-ikn_m/2^m} \right) dk$$

$$= \lim_{m\to\infty} \sum_{n_1=0}^{N-1} \cdots \sum_{n_m=0}^{N-1} \left(\prod_{k=1}^{m} \frac{h_{n_k}}{\sqrt{2}} \right) \delta(x - \sum_{k=1}^{m} n_k/2^m).$$

This defines the scaling function as a distribution. This is not a useful representation for computation, however it indicates that if a scaling function has N non-zero coefficients h_l then the scaling function has support on

$$[0, (N-1)(\frac{1}{2} + \frac{1}{4} + \frac{1}{8} \cdots)] = [0, N-1]$$

where N is the number of non-zero scaling coefficients.

While the support condition depends only on the number of non-zero coefficients, there are many scaling functions with N non-zero scaling coefficients. Except for the constraints dictated by the scaling equation, orthonormality, and normalization, there is considerable freedom in choosing the coefficients h_l.

The scaling coefficients also determine the mother wavelet function. In the general case the spaces $\mathcal{V}_m$ are the spaces of square integrable functions spanned by the orthogonal basis functions $\phi_{mn}(x) := D^m T^n \phi(x)$ for integer n satisfying $-\infty < n < \infty$. As in the Haar case, the scaling equation implies that $\mathcal{V}_m \supset \mathcal{V}_{m+k}$ for $k > 0$. Wavelet spaces are defined by

$$\mathcal{W}_m : \mathcal{V}_{m-1} = \mathcal{V}_m \oplus \mathcal{W}_m.$$

It they also satisfy (63) and (64) they define a **multiresolution analysis**. The **mother wavelet function** lives in the space $\mathcal{W}_0$ which means that it has an expansion in $\mathcal{V}_{-1}$:

$$\psi(x) = \sum_n \sqrt{2} g_n \phi(2x - n) = \sum_n g_n D^{-1} T^n \phi(x). \tag{94}$$

This equation can be expressed in a form similar to the scaling equation:

$$D\psi(x) = \sum_n g_n T^n \phi(x). \tag{95}$$

The mother wavelet and all of its integer translates should be orthogonal to the scaling function, which is in $\mathcal{V}_0$. In terms of the coefficients this requirements is:

$$\begin{aligned}(\psi_m, \phi) &= \sum_{n,l} h_l g_n (\phi_{n+2m}, \phi_l) \\ &= \sum_{n,l} h_l g_n \delta_{n+2m,l} \\ &= \sum_n h_{n+2m} g_n = 0 \end{aligned} \tag{96}$$

for all m.

Orthonormality of the translated mother function requires

$$(\psi_m, \psi_n) = \sum_{l,k} g_l g_k (\phi_{l+2m}, \phi_{k+2n})$$

$$\sum_k g_{k+2(n-m)} g_k = \delta_{mn} \tag{97}$$

or equivalently

$$(\psi_m, \psi) = \sum_k g_{k+2m} g_k = \delta_{m0}. \tag{98}$$

The choice $g_k := (-1)^k h_{l-k}$ where l is any odd integer it satisfies (95) and (98):

$$\begin{aligned}\sum_k g_{k+2(n-m)} g_k &= \sum_k (-1)^{k+2(n-m)} h_{l-k-2(n-m)} (-1)^k h_{l-k} \\ &= \sum_{k'} h_{k'+2(n-m)} h_{k'} = \delta_{mn} \end{aligned} \tag{99}$$

where we have let $k' = l - k$ in the last term. It also follows that

$$\sum_n h_{n+2m} g_n = \sum_n h_{n+2m} (-1)^n h_{l-n}$$

$$= \sum_{n'} h_{l-n'} (-1)^{l-n'-2m} h_{n'+2m} = (-)^l \sum_{n'} h_{l-n'} (-1)^{n'} h_{n'+2m}$$

$$= (-1)^l \sum_{n'} g_{n'} h_{n'+2m}. \tag{100}$$

Since l is odd, the sum is equal to its negative which shows that it vanishes. The choice of l is arbitrary - changing l shifts the origin of the mother by an even number of steps. Since the mother is orthogonal to all integer translates of the scaling function, it does not matter where the origin is placed.

This shows that the coefficients h_l, satisfying

$$\sum_l h_l = \sqrt{2}. \tag{101}$$

$$\sum_l h_{l-2m} h_l = \delta_{m0} \tag{102}$$

with g_k defined by

$$g_k := (-1)^k h_{l-k} \qquad l \qquad \text{odd} \tag{103}$$

give a multi-resolution analysis, scaling function, and a mother function.

The Daubechies order-K wavelets are defined by the conditions

$$\int x^n \psi(x) dx = 0, \qquad n = 0, 1, \cdots, K-1. \tag{104}$$

These equations ensure that polynomials of degree $< K-1$ can be locally represented by finite linear combinations of scaling functions on a fixed scale. This is a useful property for numerical approximations. The order K-Daubechies scaling function has $2K$ scaling coefficients, with $K = 1$ corresponding to the Haar wavelets, and each additional value of K adds one more orthogonality condition.

The scaling equation (95) and the moment conditions (104) for the mother wavelet function gives the additional equations necessary to find the Daubechies scaling coefficients, h_l:

$$0 = (x^n, \psi) = (Dx^n, D\psi)$$

$$= \int dx x^n 2^{-n-1/2} \sum_m g_m \phi(x-m).$$

This gives

$$\sum_m \int dx (x+m)^n g_m \phi(x) = 0.$$

For $n = 0$ this gives (using the $n = 0$ equation)

$$\sum g_m = 0, \rightarrow \sum_m (-1)^m h_{l-m} = 0,$$

Table 1: Scaling Coefficients

h_l	K=1	K=2	K=3
h_0	$1/\sqrt{2}$	$(1+\sqrt{3})/4\sqrt{2}$	$(1+\sqrt{10}+\sqrt{5+2\sqrt{10}})/16\sqrt{2}$
h_1	$1/\sqrt{2}$	$(3+\sqrt{3})/4\sqrt{2}$	$(5+\sqrt{10}+3\sqrt{5+2\sqrt{10}})/16\sqrt{2}$
h_2	0	$(3-\sqrt{3})/4\sqrt{2}$	$(10-2\sqrt{10}+2\sqrt{5+2\sqrt{10}})/16\sqrt{2}$
h_3	0	$(1-\sqrt{3})/4\sqrt{2}$	$(10-2\sqrt{10}-2\sqrt{5+2\sqrt{10}})/16\sqrt{2}$
h_4	0	0	$(5+\sqrt{10}-3\sqrt{5+2\sqrt{10}})/16\sqrt{2}$
h_5	0	0	$(1+\sqrt{10}-\sqrt{5+2\sqrt{10}})/16\sqrt{2}$

for $n = 1$ this gives

$$\sum m g_m = 0, \rightarrow \sum_m m(-1)^m h_{l-m} = 0,$$

for $n = 2 \cdots k$ this gives

$$\sum m^2 g_m = 0, \rightarrow \sum_m m^2(-1)^m h_{l-m} = 0,$$

$$\vdots$$

$$\sum m^k g_m = 0, \rightarrow \sum_m m^k(-1)^m h_{l-m} = 0.$$

When coupled with

$$\sum h_l = \sqrt{2}$$

and the orthonormality constraints,

$$\sum_l h_l h_{l-2n} = \delta_{n0}$$

we get a system of equations that can be solved for the Daubechies-K scaling coefficients. The cases $K = 1, 2, 3$ have analytic solutions. These solutions are given in Table 1.

Scaling coefficients for other values of K are tabulated in the literature [1]. With the exception of the Haar case ($K = 1$), there are two solutions which are related by reversing the order of the coefficients.

Given the scaling coefficients, h_l, it is possible to use them to compute the the scaling function. While the Fourier transform method can be used to compute the Haar functions exactly, it is more difficult to use in the general case.

An alternative is to compute the scaling function exactly on a dense set of dyadic points. This construction starts from the scaling equation in the form:

$$\phi(x) = \sum_l \sqrt{2} h_l \phi(2x - l). \tag{105}$$

Let $x = n$ to get relations between the values of the scaling function at integer points

$$\phi(n) = \sum_l \sqrt{2} h_l \phi(2n - l). \tag{106}$$

Set $m = 2n - l$ to get

$$\phi(n) = \sum_m \sqrt{2} h_{2n-m} \phi(m) \tag{107}$$

This gives the equations

$$\phi(n) = \sum_m H_{nm} \phi(m) \tag{108}$$

for the non-zero $\phi(n)$ corresponding to $n = 1, \cdots, 2K - 2$ where

$$H_{nm} = \sqrt{2} h_{2n-m}. \tag{109}$$

Eigenvectors of the matrix H_{mn} with eigenvalue 1 are solutions of the scaling function at integer points - up to normalization.

Rather than solve the eigenvalue problem, one of the equations can be replaced by the condition

$$\sum_n \phi(n) = 1 \tag{110}$$

which follows from the assumption that $\int \psi(x) dx = 0$. (The proof of this statement uses the fact that the translates of the scaling function on a fixed scale and the wavelets on all smaller scales is a basis for square integrable functions. Since 1 is locally orthogonal to all of the wavelets by assumption, 1 can be expressed as a linear combination of translates of the scaling function. The normalization condition gives the coefficients of the expansion above.)

The support condition implies that only a finite number of the $\phi(n)$ are non-zero. This condition is independent of the orthonormality condition.

For the case of the $K = 2$ Daubechies wavelets these equations are

$$\phi(0) = \sqrt{2}h_0\phi(0)$$

$$\phi(1) = \sqrt{2}(h_0\phi(2) + h_1\phi(1) + h_2\phi(0))$$
$$\phi(2) = \sqrt{2}(h_1\phi(3) + h_2\phi(2) + h_3\phi(1))$$
$$\phi(3) = \sqrt{2}h_3\phi(3)$$
$$1 = \phi(0) + \phi(1) + \phi(2) + \phi(3).$$

The first and fourth equation give $\phi(0) = \phi(3) = 0$ (or $h_0 = h_1 = 1/\sqrt{2}$ which is the Haar solution). This also follows from the continuity of the wavelets, since 0 and 3 are the boundaries of the support. The second and third equations are eigenvalue equations

$$\begin{pmatrix} \phi(1) \\ \phi(2) \end{pmatrix} = \begin{pmatrix} \sqrt{2}h_1 & \sqrt{2}h_0 \\ \sqrt{2}h_2 & \sqrt{2}h_3 \end{pmatrix} \begin{pmatrix} \phi(1) \\ \phi(2) \end{pmatrix}. \tag{111}$$

Instead of solving the eigenvalue problem for an eigenvector with eigenvalue 1, use

$$\phi(1) + \phi(2) = 1 \tag{112}$$

with

$$\phi(1) = \sqrt{2}(h_0\phi(2) + h_1\phi(1))$$

to get

$$\phi(1) = \sqrt{2}(h_0(1 - \phi(1)) + h_1\phi(1))$$

which can be solved for

$$\phi(1) = \frac{\sqrt{2}h_0}{1 + \sqrt{2}(h_0 - h_1)} \tag{113}$$

and

$$\phi(2) = \frac{1 - \sqrt{2}h_1}{1 + \sqrt{2}(h_0 - h_1)}. \tag{114}$$

This gives exact values of the scaling function at integer points in terms of the scaling coefficients. This solution satisfies $\sum_n \phi(n) = 1$. In this case there are only two non-zero terms.

In order to construct the scaling function at an arbitrary point x the first step is to make a dyadic approximation to x. Let m be an integer that defines a dyadic resolution. This means that we want the dyadic approximation to satisfy the inequality $|x - x_{approx}| < 2^{-m}$. For any m it is possible to find an integer n such that

$$\frac{n}{2^m} \leq x < \frac{n+1}{2^m}. \tag{115}$$

Writing this as

$$n \leq 2^m x < n + 1 \tag{116}$$

immediately gives

$$n := [2^m x] = floor(2^m x) \tag{117}$$

where [] means greatest integer $\leq 2^m x$.

Since the scaling function is continuous, for any $\epsilon > 0$ we can find a large enough m so

$$|\phi(x) - \phi(\frac{n}{2^m})| < \epsilon.$$

In what follows we evaluate $\phi(n/2^m)$ exactly. Let $x = n/2^m$. We also assume that $0 < n < 2K - 1 \times 2^m$, otherwise $\phi(x) = 0$ by the support condition (in this example we consider the case $K = 2$) . In order to evaluate $\phi(x)$ note that the scaling equation gives:

$$\phi(x) = \phi\left(\frac{n}{2^m}\right) = \sqrt{2}D\phi\left(\frac{n}{2^{m-1}}\right) =$$

$$\sum_l \sqrt{2}h_{l_1}T^{l_1}\phi\left(\frac{n}{2^{m-1}}\right) = \sum_l \sqrt{2}h_{l_1}\phi\left(\frac{n}{2^{m-1}} - l_1\right)$$

$$= \sum_l \sqrt{2}h_{l_1}\phi\left(\frac{n - 2^{m-1}l}{2^{m-1}}\right) \tag{118}$$

Repeating this process a second time gives

$$\phi(x) = \sum_{l_1,l_2} 2h_{l_1}h_{l_2}\phi\left(\frac{n - 2^{m-1}l_1 - 2^{m-2}l_2}{2^{m-2}}\right). \tag{119}$$

Using the scaling equation m times gives

$$\phi(x) = \sum_{l_1,l_2\cdots l_m} 2^{m/2}h_{l_1}h_{l_2}\cdots h_m\phi(n - 2^{m-1}l_1 - 2^{m-2}l_2 - \cdots 2l_{m-1} - l_m). \tag{120}$$

In this case the last expression is evaluated at integer values which gives (for the Daubechies $K = 2$ case):

$$\phi(x) = \sum_{l_1, l_2 \cdots l_m} c_{l_1} c_{l_2} \cdots c_m \times \tag{121}$$

$$\left[\delta_{n-2^{m-1}l_1-2^{m-2}l_2-\cdots 2l_{m-1}-l_m,1} \frac{\sqrt{2}h_0}{1+\sqrt{2}(h_0-h_1)} + \right. \tag{122}$$

$$\left. \delta_{n-2^{m-1}l_1-2^{m-2}l_2-\cdots 2l_{m-1}-l_m,2} \frac{1-\sqrt{2}h_1}{1+\sqrt{2}(h_0-h_1)} \right] \tag{123}$$

where $c_k := \sqrt{2}h_k$.

This method generalizes to any value of K and any choice of scaling coefficients, h_l.

The scaling function and mother wavelet for the Daubechies wavelet are pictured in Figure 1.

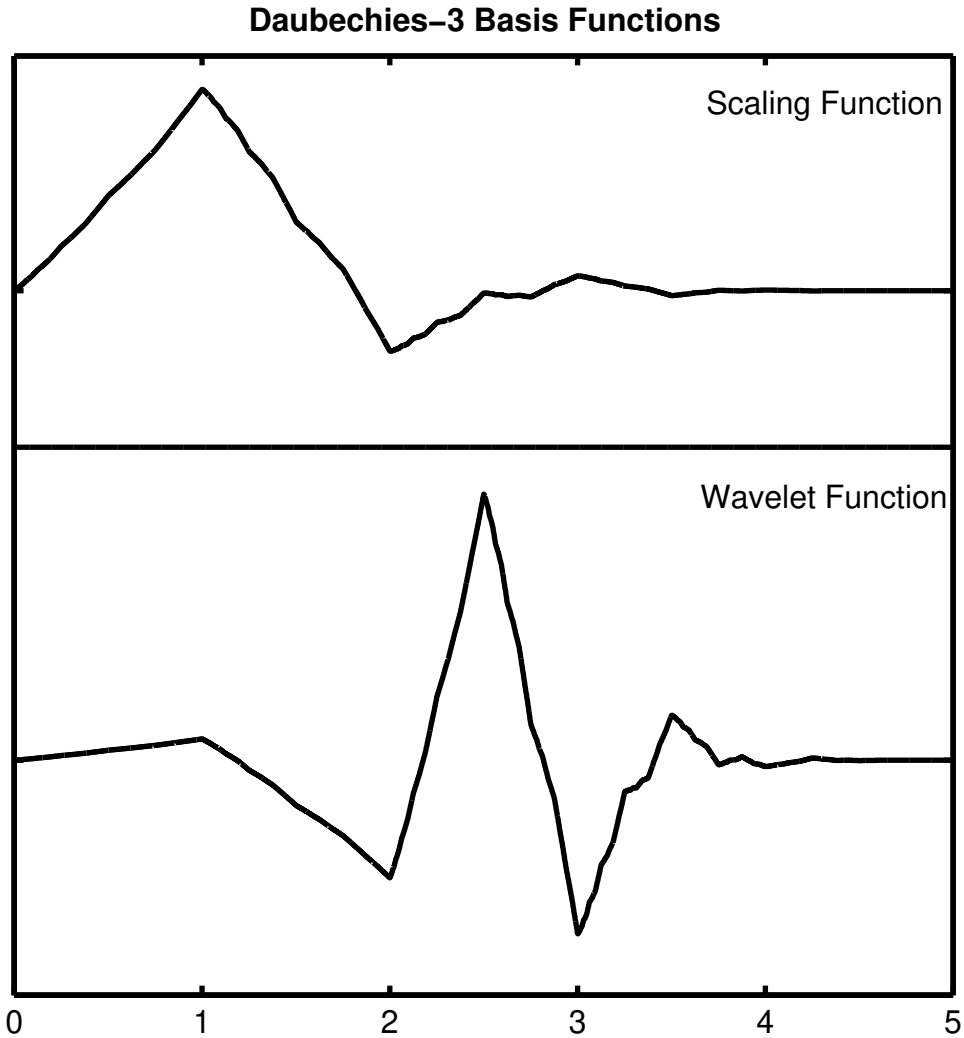

Figure 1.

4 Daubechies Wavelets

The Daubechies wavelets have two special properties. The first is that there are a finite number of non-zero scaling coefficients, h_l, which means that the scaling functions and wavelets have compact support. The order-K Daubechies scaling equation has $2K$ non-zero scaling coefficients, and the support of the scaling function and mother wavelet function is on the interval $[0, 2K-1]$. The second property of the order-K Daubechies wavelets is

that the first $K-1$ moments of the wavelets are zero.

The second property of the Daubechies wavelets is what makes them useful as basis functions. The expansion of a function $f(x)$ in a wavelet basis has the form

$$f(x) = \sum_{mn} f_{mn}\psi_{mn}(x) \qquad f_{mn} := \int f(x)\psi_{mn}(x)dx.$$

If $f(x)$ can be well-approximated by a low-degree polynomial on the support of $\psi_{mn}(x)$, then the vanishing of the low-order moments of $\psi_{mn}(x)$ means that the expansion coefficient f_{mn} will be small. On the other hand, as we will show in this section, the scaling function basis can be used to make local pointwise representation of low-degree polynomials. Since the scaling function basis on $\mathcal{V}_m$ is equivalent to the wavelet basis on all scales, $k > m$, this means that the wavelet basis provides an efficient representation of functions that can be accurately approximated by local polynomials on different scales. For integral equations with smooth kernels, this means that the matrix representation of the kernels in a wavelet basis will be represented by a sparse matrix.

The constraint on the moments of the Daubechies wavelets,

$$\int \psi(x)x^l dx = 0 \qquad l = 0 \cdots K-1, \tag{124}$$

has important consequences. Eq. (124) implies

$$\int \psi_{0m}(x)x^l dx = \int \psi(x-m)x^l dx = \int \psi(y)(y+m)^l dy$$

$$= \sum_{k=0}^{l} \frac{l!}{k!(l-k)!} m^{l-k} \int \psi(y)y^k dy = 0 \qquad l = 0 \cdots K-1, \tag{125}$$

which means that first $K-1$ moments of the unit translates of the mother wavelet function vanish. Similarly, changing scale gives

$$\int \psi_{10}(x)x^l = \int D\psi(x)x^l dx = \frac{1}{\sqrt{2}} \int \psi(x/2)x^l dx$$

$$= 2^{l+1/2} \int \psi(y)y^l dy = 0 \qquad l = 0 \cdots K-1. \tag{126}$$

It straight forward to proceed inductively to show for all m and n that

$$\int \psi_{nm}(x)x^l = 0 \qquad l = 0 \cdots K-1. \tag{127}$$

This means that every Daubechies wavelet basis function is orthogonal to all polynomials of degree less than K, where K is the order of the wavelet basis.

For the orthonormal basis of $L^2(\mathbb{R})$ consisting of

$$\{T^n\phi(x), D^m T^n \psi(x) : m \leq 0\} \tag{128}$$

the only basis functions with non-zero moments with $l < K$ are the scaling basis functions

$$\int \phi_m(x)x^l dx \neq 0 \qquad l = 0 \cdots K-1. \tag{129}$$

Although polynomials are not square integrable, we can multiply a polynomial by a box function $b(x)$ which is 1 between x_- and x_+ and zero elsewhere. The product of the box function and the polynomial is square integrable and is equal to the polynomial on the interval $[x_-, x_+]$. It follows that

$$p(x)b(x) = \sum_{mn} c_{mn}\psi_{mn}(x) = \sum_n d_n \phi_{kn}(x) + \sum_n \sum_{m \leq k} c_{mn}\psi_{mn}(x) \tag{130}$$

where $p(x)$ is a polynomial of degree less than K and

$$c_{mn} = \int_{x_-}^{x_+} \psi_{mn}(x)p(x)dx \tag{131}$$

$$d_n = \int_{x_-}^{x_+} \phi_n(x)p(x)dx. \tag{132}$$

The moment condition means that the coefficients $c_{mn} = 0$ whenever the support of the wavelet is completely contained inside of the interval $[x_-, x_+]$. Thus in the first expression the non-zero coefficients arise from end-point contributions and from many small contributions from wavelets with support that are much larger than the box.

If k is set to correspond to a sufficiently fine scale, so the support of all of the wavelets is much smaller than the support of the box, then the second sum in (130) has no wavelets with support larger than the width of the box. The endpoint contributions only affect the answer within a distance

Δ, equal to the width of the support of the scaling basis function, from the endpoints of the box. Inside this distance the only nonzero coefficient are due to the translates of the scaling functions. There are a finite number of these coefficients, and in this region they provide an exact representation of the polynomial. Specifically let

$$I(x) = b(x)p(x) - \sum_n d_n \phi_{kn}(x) + \sum_n \sum_{m \leq k} c_{mn} \psi_{mn}(x) \tag{133}$$

then we have

$$0 = \|I\|^2 = \int_{x_-}^{x_- + \Delta} I(x)^2 dx + \int_{x_+ - \Delta}^{x_+} I(x)^2 dx +$$

$$\int_{x_- + \Delta}^{x_+ - \Delta} |p(x) - \sum_n d_n \phi_{kn}(x)|^2 dx. \tag{134}$$

Since all three terms are non-negative we conclude that

$$\int_{x_- + \Delta}^{x_+ - \Delta} |p(x) - \sum_n d_n \phi_{kn}(x)|^2 dx = 0. \tag{135}$$

Since Δ is fixed by the choice of the support of the scaling function and $x_\pm$ is arbitrary we have

$$\int_a^b |p(x) - \sum_n d_n \phi_{kn}(x)|^2 dx = 0 \tag{136}$$

for any interval $[a, b]$. Since $p(x)$ and $\phi(x)$ are continuous (we did not prove this for $\phi(x)$) and the sum of translates has a finite number of non-zero terms, it follows that

$$p(x) = \sum_n d_{kn} \phi_n(x) \tag{137}$$

pointwise on every finite interval. Since the box support is arbitrary this holds for any k. This establishes the desired result, that polynomials of degree less than K can be represented exactly by the finite linear combinations of the scaling functions $\phi_n(x)$. Since both bases in (130) are equivalent, it follows that local polynomials can also be represented exactly in the wavelet basis.

Figure 2. shows integer translates of the Daubechies 2 scaling function. Note how the sum of the non zero wavelets at any point in identically one, in spite of the complex fractal nature of each individual scaling function.

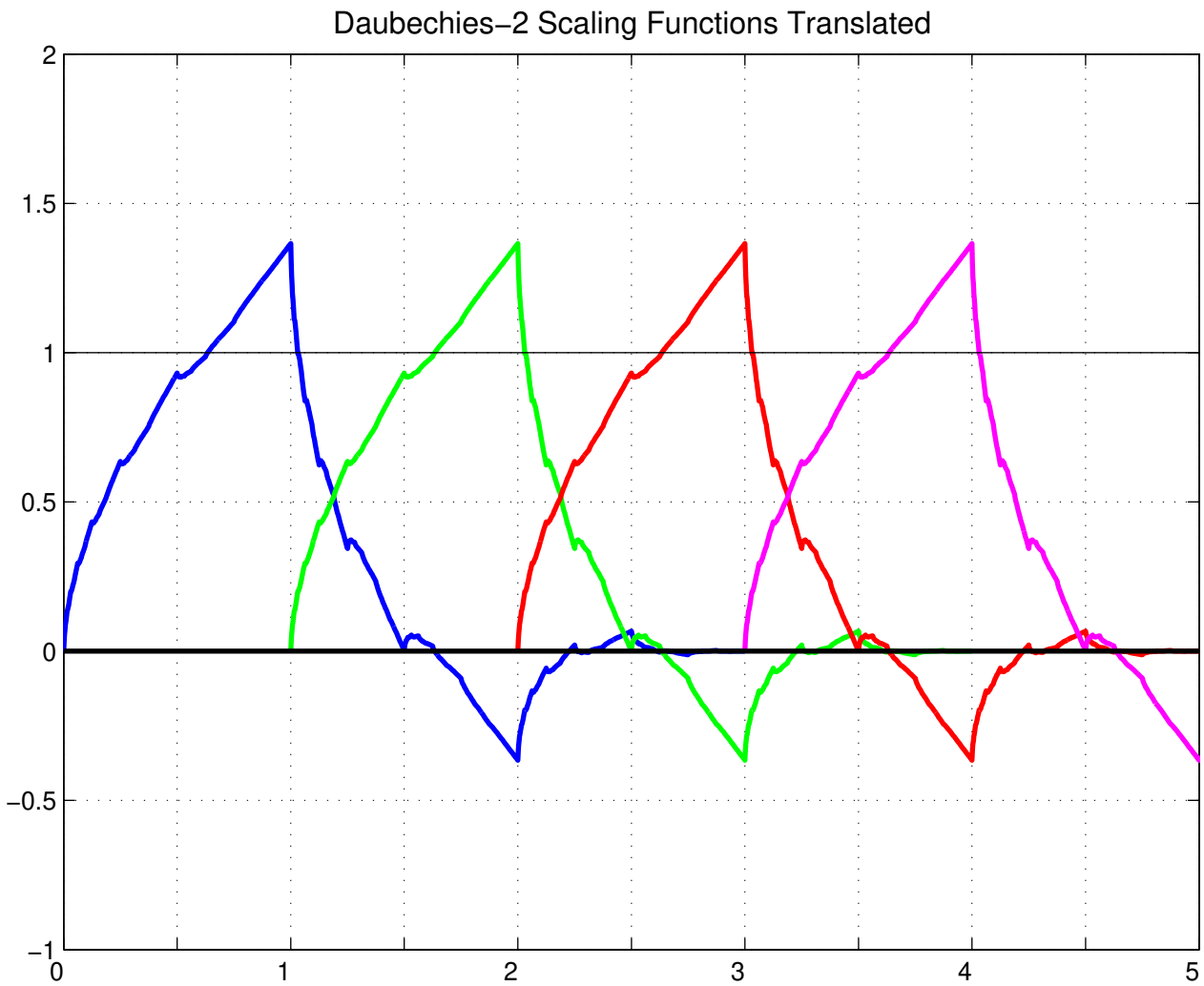

Figure 2.

Figure 2. shows a local representation of a constant function in terms of scaling fucntions.

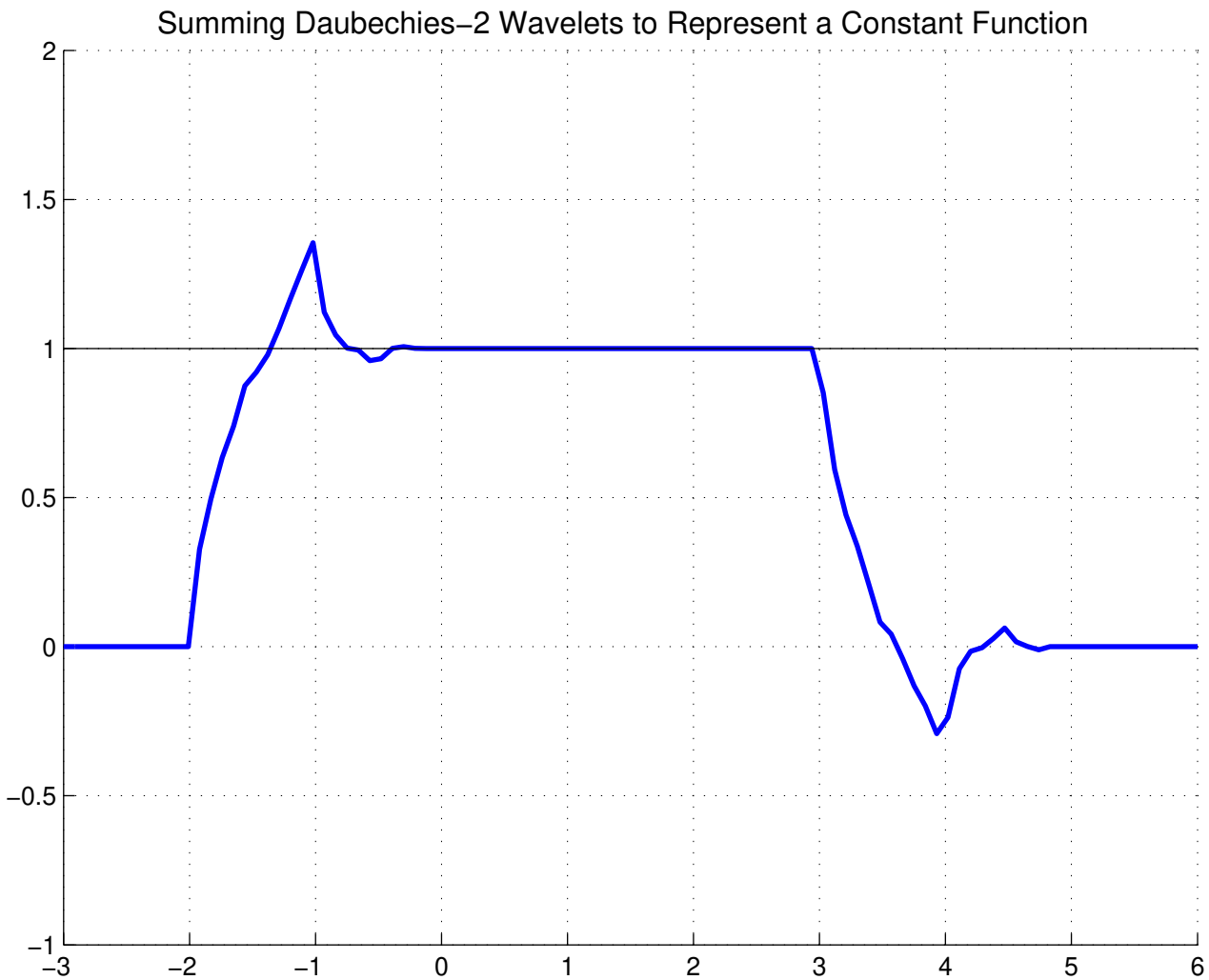

Figure 3.

Figure 3. shows a local representation of a linear function in terms of scaling functions, while Figure 4. shows a local representation of a linear function.

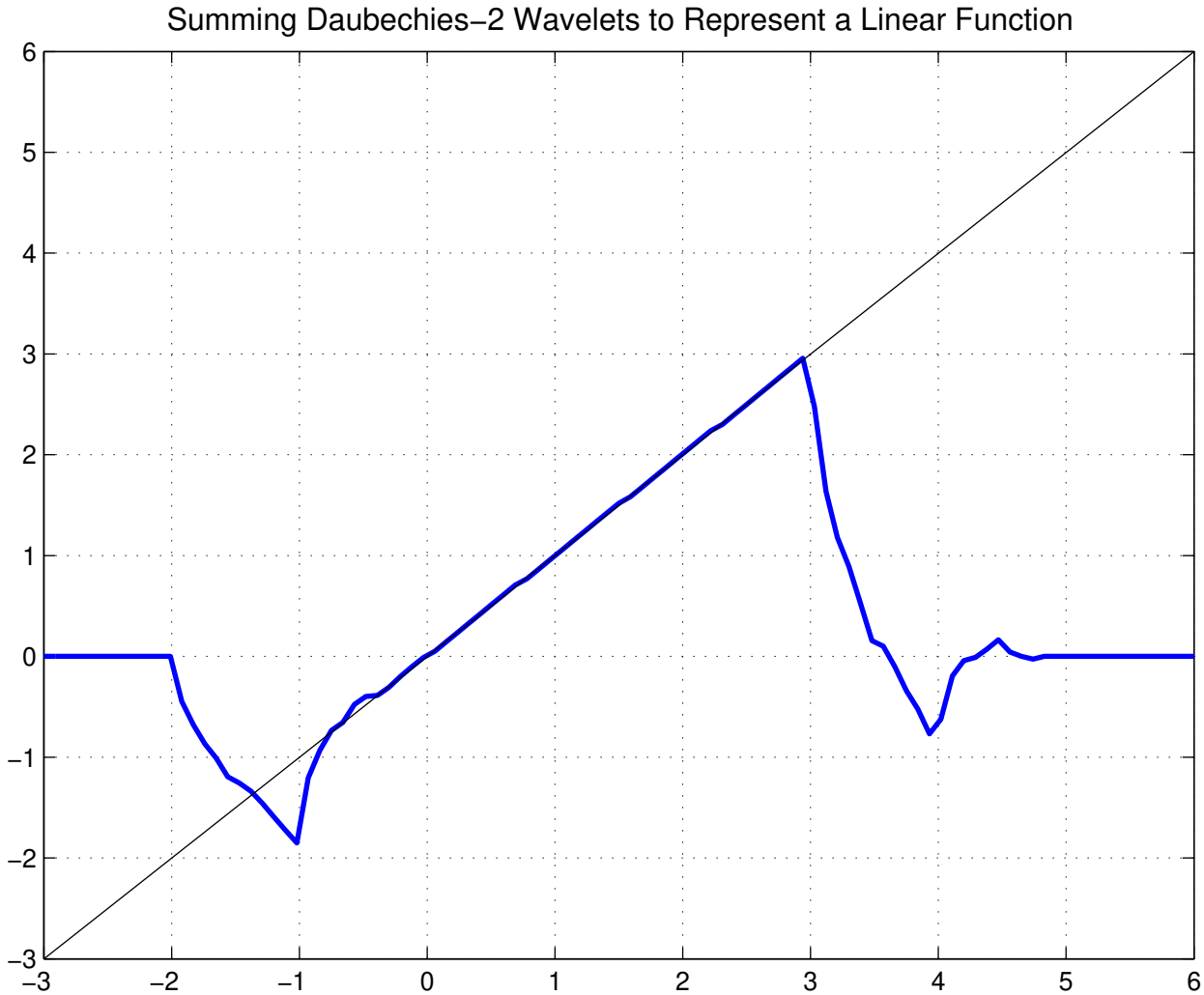

Figure 4.

Note that expansion in the wavelet basis gives all coefficients zero. This is not a contradiction because none of the polynomials are square integrable. The key point is that once one puts a box around a function, wavelets with very large support (large m) lead to many small contributions.

5 Moments and Quadrature Rules

One of the most important properties of the scaling equation is that it can be used to generate linear relations between integrals involving different scaling-function or wavelet basis elements. In this section we show how the scaling equation can be used to obtain exact expressions for moments of the scaling function and wavelet basis elements as functions of the scaling coefficients. These can be used to develop quadrature rules that can be applied to linear integral equations. In section 6 the scaling equation is used to obtain exact expressions for the inner products of the these functions and their derivatives, which are important for applications to differential equations. We also show that these same methods can be used to compute integrals of scaling functions and wavelets over the different types of integrable singularities encountered in singular integral equations.

Moments of the scaling function and mother wavelet function are defined by

$$< x^m >_\phi = \int \phi(x) x^m dx \qquad < x^m >_\psi = \int \psi(x) x^m dx. \tag{138}$$

Normally these are integrated over the real line. For compactly supported wavelets this is equivalent to integrating over the support of the wavelet.

A polynomial **quadrature rule** is a collection of N points $\{x_i\}$ and weights $\{w_i\}$ with the property

$$< x^m >_\phi = \int \phi(x) x^m dx = \sum_{i=1}^{N} x_i^m w_i \tag{139}$$

which hold for $0 \leq m \leq 2N-1$. By linearity this means that

$$\int \phi(x) P(x) dx = \sum_{i=1}^{N} P(x_i) w_i \tag{140}$$

is exact for all polynomials of degree up to $2N-1$.

In order to construct a quadrature rule we need to first compute the moments, and from these we can compute the points and weights. The moments can be constructed recursively from the normalization condition

$$< x^0 >_\phi = (x^0, \phi) = \int dx \phi(x) = 1 \tag{141}$$

using the scaling equation

$$< x^m >_\phi = (x^m, \phi) = (Dx^m, D\phi)$$

$$= \frac{1}{\sqrt{2}} \frac{1}{2^m} \sum_l h_l (x^m, T^l \phi)$$

$$= \frac{1}{\sqrt{2}} \frac{1}{2^m} \sum_l h_l ((x+l)^m, \phi)$$

$$= \frac{1}{\sqrt{2}} \frac{1}{2^m} \sum_l h_l \sum_{k=0}^{m} \frac{m!}{k!(m-k)!} l^{m-k} < x^k >_\phi .$$

Using $\sum_l h_l = \sqrt{2}$, and moving the $k = m$ term to the left side of the above equation gives the recursion relation:

$$< x^m >_\phi = \frac{1}{2^m - 1} \frac{1}{\sqrt{2}} \sum_{k=0}^{m-1} \frac{m!}{k!(m-k)!} \left(\sum_{l=1}^{2K-1} h_l l^{m-k} \right) < x^k >_\phi . \qquad (142)$$

Note that the right hand side of this equation involves moments with $k < m$. Similarly the moments of the mother wavelet function are expressed in terms of the moments of the scaling functions using eq. (95)

$$< x^m >_\psi = \frac{1}{2^m} \sum_{k=0}^{m} \frac{m!}{k!(m-k)!} \left(\sum_{l=0}^{2K-1} \frac{g_l}{\sqrt{2}} l^{m-k} \right) < x^k >_\phi . \qquad (143)$$

Since the scaling equation for the mother wavelet function relates the mother wavelet function to the scaling function there is no need to take the $k = m$ term to the left of the equation; it is known from the first recursion.

Equations (141), (142), and (143) give a recursive method for generating all non-negative moments of the scaling and mother wavelet function from the normalization integral of the scaling function.

The moments for $\phi_{kl} = D^k T^l \phi$ and $\psi_{kl} = D^k T^l \psi$ can be computed from the moments (142) and (143) using the unitarity of the D and T operators

$$< x^m >_{\phi_{kl}} = (x^m, D^k T^l \phi) = (T^{-l} D^{-k} x^m, \phi)$$

$$= 2^{k(m+1/2)} \sum_{n=0}^{m} \frac{m!}{n!(m-n)!} l^{m-n} < x^n >_\phi$$

and

$$< x^m >_{\psi_{kl}} = (x^m, D^k T^l \psi) = (T^{-l} D^{-k} x^m, \phi)$$
$$= 2^{k(m+1/2)} \sum_{n=0}^{m} \frac{m!}{n!(m-n)!} l^{m-n} < x^n >_{\psi} .$$

Thus, all moments of translates and scale transforms of both the mother wavelet and scaling functions can be computed exactly in terms of the scaling coefficients.

Partial Moments: Partial moments of the form

$$< x^m >_{\phi[0:n]} = \int_0^n \phi(x) x^m dx$$

and

$$< x^m >_{\phi[n:2K-1]} = \int_n^{2K-1} \phi(x) x^m dx = < x^m >_{\phi} - < x^m >_{\phi[0:n]}$$

for $n \in \{1, \cdots, 2K-2\}$ are also needed of numerical applications.

These are can be calculated recursively in terms of the full moments using the scaling equation. First consider the order $m = 0$ partial moments. Use the scaling equation

$$D\phi(x) = \frac{1}{\sqrt{2}} \phi(\frac{x}{2}) = \sum_l h_l \phi(x - l),$$

which gives

$$\phi(x) = \sum_l \sqrt{2} h_l \phi(2x - l),$$

in the definition of the $m = 0$ partial moment to obtain:

$$< x^0 >_{\phi[0:n]} = \int_0^n \phi(x) dx = \sum_l \sqrt{2} h_l \int_0^n \phi(2x - l) dx.$$

Substituting $y = 2x - l$ gives

$$< x^0 >_{\phi[0:n]} = \sum_l \frac{h_l}{\sqrt{2}} \int_{-l}^{2n-l} \phi(y) dy = \sum_l \frac{h_l}{\sqrt{2}} < x^0 >_{\phi[-l,2n-l]} .$$

The support condition of the scaling function $\phi(x)$ implies that the lower limit of all of the integrals can be taken as zero which gives the relations:

$$< x^0 >_{\phi[0:n]} = \sum_{k=2n-2K+1}^{2n} c_{2n-k} < x^0 >_{\phi[0:k]}$$

where

$$c_l := \frac{h_l}{\sqrt{2}}$$

are non-zero for $l = 0, \cdots, 2K-1$. These equations are a linear system for the non-trivial partial 0-moments, $m_n =< x^0 >_{\phi[0:n]}$, in terms of the full 0-moments $< x^0 >_\phi = 1$. These equations have the form

$$M_{mn} m_n = v_m \tag{144}$$

where $n, m : 1 \to 2K-2$ and

$$M_{mn} = \delta_{mn} - C_{mn}$$

with

$$C_{mn} = c_{2m-n} \qquad m < K-1, n = 1, \cdots, 2m$$
$$C_{mn} = 0 \qquad m < K-1, n = 2m+1, \cdots, 2K-2$$
$$C_{mn} = c_{2m-n} \qquad m = K-1, K$$
$$C_{mn} = c_{2m-n} \qquad m > K, n = 2m-2K+1, \cdots, 2K-2$$
$$C_{mn} = 0 \qquad m > K, n = 1, \cdots, 2m-2K$$

and

$$v_n = 0 \qquad n < K$$
$$v_n = \sum_{k=0}^{2(n-K)+1} c_k \qquad K \leq n \leq 2K-2.$$

The matrix $M := I - C$ has the form

$$I - C := \begin{pmatrix} 1-c_1 & -c_0 & 0 & 0 & \cdots & 0 \\ -c_3 & 1-c_2 & -c_1 & -c_0 & \cdots 0 & \\ \vdots & \vdots & \vdots & \vdots & & 0 \\ -c_{2K-3} & -c_{2K-4} & & \cdots & & -c_0 \\ -c_{2K-1} & -c_{2K-2} & & \cdots & & -c_2 \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 & -c_{2K-1} & 1-c_{2K-2} \end{pmatrix}$$

the vector m_n, of partial moments, has the form

$$m := \begin{pmatrix} < x^0 >_{\phi[0:1]} \\ < x^0 >_{\phi[0:2]} \\ \vdots \\ < x^0 >_{\phi[0:K-1]} \\ < x^0 >_{\phi[0:K]} \\ \vdots \\ < x^0 >_{\phi[0:2K-2]} \end{pmatrix}$$

and the driving term has the form

$$v' := \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ c_0 + c_1 \\ \vdots \\ c_0 + \cdots c_{2K-3} \end{pmatrix}.$$

The solution of this system of linear equations gives the partial moments of order zero:

$$m_n =< x^0 >_{\phi[0:n]} = (I - C)^{-1} v.$$

Complementary partial moments are given by:

$$< x^0 >_{\phi[n:2K+1]} = 1 - < x^0 >_{\phi[0,n]} .$$

Higher order partial moments can be constructed similarly

$$< x^k >_{\phi[0:n]} := \int_0^n \phi(x) x^k dx$$

$$= \sum_{l=0}^{2K-1} \sqrt{2} h_l \int_0^n \phi(2x - l) x^k dx$$

$$= \sum_{l=0}^{2K-1} \frac{h_l}{2^k \sqrt{2}} \int_{-l}^{2n-l} \phi(x)(x + l)^k dx$$

$$= \sum_{l=0}^{2K-1} \frac{h_l}{2^k\sqrt{2}} \sum_{m=0}^{k} \frac{k!}{m!(k-m)!} l^{k-m} \int_0^{2n-l} \phi(x) x^m dx.$$

Rearranging indices, putting terms with the partial moments of the highest power on the left gives the following equation for the order k partial moments:

$$\sum_{r=1}^{2K-2} (\delta_{mr} - \frac{1}{2^k} C_{mr}) < x^k >_{\phi[0:r]} = w_m \tag{145}$$

where the inhomogeneous term w_n in (145) is

$$w_n = \delta(n \geq K) \sum_{2n-2K+1}^{r=2n} \frac{c_{2n-r}}{2^k} < x^k >_\phi$$

$$+ \sum_{l=1}^{2K-1} \frac{c_l}{2^k} \sum_{m=0}^{k-1} \frac{k!}{m!(k-m)!} l^{k-m} < x^m >_{\phi[0:2n-l]}, \tag{146}$$

which can be expressed in terms of the full moments of order k and partial moments of order less than k. The desired order-k partial moments are obtained by inverting the matrix

$$< x^k >_{\phi[0:m]} := \sum_r (\delta_{mr} - \frac{1}{2^k} C_{mr})^{-1} w_r.$$

Note that C matrix is identical to the C matrix that appears in equation for the 0-order partial moments.

Having solved for the partial moments of the scaling function, it is possible to find the partial moments for the mother wavelet function $< x^m >_{\psi_k[0:n]}$ using

$$< x^m >_{\psi_k[0:n]} := \frac{1}{2^{m+1/2}} \sum_l g_l \sum_{k=0}^{m} \frac{m! l^{m-k}}{k!(m-k)!} < x^m >_{\phi_k[0:2n-l]}$$

where the $< x^m >_{\phi_k[0:2n-l]}$ vanish for $2n - l \leq 0$, are partial moments for $0 < 2n-l < 2K-1$, and are full moments for $2n-l \geq 2K-1$. This equation expresses the partial moments of the mother wavelet function directly terms of moments and partial moments of the scaling function.

Given the moments and partial moments of $\phi(x)$ and $\psi(x)$ we can solve for the partial moments of ϕ_{mn} and ψ_{mn} in terms of the partial moments of $\phi(x)$ and $\psi(x)$ by rescaling and translation.

Quadrature Rules: Given exact the expression for the moments it is possible to formulate quadrature rules for integrating the product smooth functions times wavelet or scaling basis functions. The simplest quadrature rule is the **one-point quadrature rule**. To understand this rule consider integrals of the form

$$\int f(x)\phi(x)dx. \tag{147}$$

The quadrature point is defined as the first moment of the scaling function

$$x_1 := \int x\phi(x)dx = \mu_1. \tag{148}$$

With this definition we have

$$\int (a+bx)\phi(x)dx = a + b\mu_1 = a + bx_1. \tag{149}$$

For orthogonal wavelets note that

$$k_m := \int x\phi(x)\phi(x-m)dx = \int (y+m)\phi(y+m)\phi(y)dy$$

$$= \int x\phi(x+m)\phi(x)dx = k_{-m}.$$

It follows that

$$\sum_m mk_m = \sum_m (-m)k_m = 0. \tag{150}$$

For the Daubechies order $K > 1$ scaling functions

$$\sum_m m\phi(x-m) = x - \mu_1 \tag{151}$$

which gives

$$0 = \sum_m m \int x\phi(x)\phi(x-m)dx = \sum_m \int \phi(x)x(x-\mu_1)dx = \mu_2 - \mu_1^2. \tag{152}$$

This means that $\mu_2 = \mu_1^2$ or

$$\int \phi(x)(a+bx+cx^2)dx = a + b\mu_1 + c\mu_2 = a + bx_1 + cx_1^2. \tag{153}$$

This gives a one-point quadrature rule with point $x_1 = \mu_1$ and weight $w_1 = 1$, that integrates the product of the scaling function and polynomials of degree 2 exactly. This is very useful and simple when used with scaling basis functions that have small support.

More generally, given a collection of $2N$ moments of the scaling function or mother wavelet we can construct quadrature points and weights using the following method [**?**]. If $\{x_i\}$ are the (unknown) quadrature points define the polynomial

$$P(x) = \prod_{i=1}^{N}(x - x_i) = \sum_{n=0}^{N} p_n x^n$$

where $p_N = 1$ and the other p_n's are unknown. Define the polynomials of degree $n + m$

$$Q_m(x) = x^m P(x)$$

for $m = 1, \cdots N - 1$. By construction, for each m and x_i , $Q_m(x_i) = 0$ because $P(x_i) = 0$.

If we require that the points $\{x_i\}$ and weights $\{w_i\}$ exactly reproduce $2N$ moments then it follows that

$$\int \phi(x) Q_m(x) dx = \sum_{i=1}^{N} Q_m(x_i) w_i = 0 \tag{154}$$

because $Q_m(x_i) = 0$. The condition that the weights reproduce the moments give the conditions

$$\int \phi(x) Q_m(x) dx = \sum_{n=0}^{N} p_n < x^{n+m} >_\phi,$$

and this must be equal to zero for each value m from $m = 0$ to $m = N - 1$. This gives N linear equations for the N unknowns $p_0 \cdots p_{N-1}$:

$$\sum_{n=0}^{N} p_n < x^{n+m} >_\phi = 0 \qquad m = 1 \cdots N;\ p_N = 1$$

or

$$\sum_{n=0}^{N-1} < x^{n+m} >_\phi p_n = - < x^{N+m} >_\phi p_N \qquad m = 1 \cdots N;\ p_N = 1.$$

Solving this linear system for the coefficients p_n , using $p_N = 1$, gives the polynomial $P(x)$.

Given the polynomial $P(x)$ the next step is to find its zeros. The N zeros of the polynomial $P(x)$ are the quadrature points x_i. Given the quadrature points, the weights are determined from the remaining N moments by solving the linear system

$$< x^n >_\phi = \sum_{i=1}^{N} x_i^n w_i \qquad n = 0, \cdots, N-1$$

for the weights, w_i.

This shows how to construct the quadrature points and weights from the moments. In applications the linear equations for the coefficients p_n are real . It follows that the zeros of $P(x)$ are either real or come in complex conjugate pairs. In general it is desirable that the points are real and lie in the support of the scaling function. When this fails to occur it is best to simply assign real quadrature points that lie on the support of the scaling function. In doing this some accuracy is sacrificed, but it is easy to go to a higher order. Generally quadrature rules are used to integrate over the support of a scaling function of a small scale; normally a small number of quadrature points and weights is sufficient.

For quadrature rules on a half-interval the partial moments, $< x^m >_{\phi_l[0:\infty]}$, need to be used near 0 to generate a quadrature rule.

Quadrature points are normally needed for different scaling basis functions, $\phi_{mn}(x)$. Points and weights for integrating $\phi_{mn}(x)$ can be generated by scaling and translation. To see this consider a set of points and weights $\{x_i, w_i\}$ that satisfy

$$(x^m, \phi) = \int x^m \phi(x) dx = \sum x_i^m w_i.$$

It follows that

$$(x^m, \phi_{nk}) = (x^m, D^n T^k \phi) = 2^{n(m+1/2)}(x^m, T^k \phi,)$$

$$= 2^{n(m+1/2)}((x+k)^m, \phi) = \sum 2^{nm+n/2} w_l (x_l + k)^m$$

$$= \sum (2^{n/2} w_l)(2^n (x_l + k))^m.$$

If we define the transformed points and weights by

$$w_l' = 2^{n/2} w_l \qquad x_l' = 2^n (x_l + k),$$

we get

$$(x^m, \phi_{nk}) = \sum_l w_l' (x_l')^m.$$

The new points and weights involve simple transformations of the original points and weights.

While it is possible to formulate quadrature rules for both the wavelet basis and scaling function bases, it makes more sense to develop the quadrature rules for the scaling function on a sufficiently fine scale. This is because the scaling basis functions have small support, which means that the quadrature rule will be accurate for functions that can be accurately represented by low-degree polynomials on the support of the scaling function.

Integral Equations: To use the quadrature rules to solve linear integral equations first consider the non-singular integral equation

$$f(x) = g(x) + \int K(x,y) f(y) dy.$$

Let

$$f(x) \approx \sum_n f_n \phi_{sn}(x)$$

where $\phi_{sn}(x)$ are translates of the scaling function on a sufficiently fine scale s. Inserting the approximate solution in the integral equation gives

$$\sum_n f_n \phi_{sn}(x) \approx g(x) + \sum_n \int K(x,y) f_n \phi_{sn}(y) dy.$$

Using the orthonormality of the $\phi_{sm}(x)$ for different m values and a suitable quadrature rule gives the equation for the coefficients f_m:

$$f_m = \sum_l g(x_{lm}) w_{lm} + \sum_n \sum_{l,k} w_{lm} K(x_{lm}, x_{kn}) w_{kn} f_n$$

or

$$\sum_n \left[\delta_{mn} - \sum_{l,k} w_{lm} K(x_{lm}, x_{kn}) w_{kn} \right] f_n = \sum_l g(x_{lm}) w_{lm}. \tag{155}$$

Note that no integrals need to be evaluated, except using the local quadrature rules. In addition the points and weights only have to be calculated for the scaling function on one scale - the rest can be obtained by simple transformations.

While the scaling function basis on the approximation space $\mathcal{V}_s$ is useful for deriving the matrix equation above, it is useful to use the multiresolution analysis to express the approximation space as

$$\mathcal{V}_s = \mathcal{W}_{s+1} \oplus \mathcal{W}_{s+2} \oplus \cdots \oplus \mathcal{W}_{s+r} \oplus \mathcal{V}_{s+r}.$$

The representation on the right has a natural basis consisting of scaling basis functions $\phi_{s+r,m}(x)$ on the larger scale, $s+r > s$, and wavelet basis functions $\psi_{mn}(x)$ on intermediate scales $s < m \leq s+r$. These two bases are related by a real orthogonal transformation called the **wavelet transform**. Normally the wavelet transform is an infinite matrix. In applications it is replaced by finite orthogonal transformation that uses some conventions about how to treat the boundary.

To solve the integral equation the last step is to use the wavelet transform on the indices m, n. This should give a sparse linear system that can be used to solve for f_n. While the precise form of the sparse matrix will depend on how the boundary terms are treated, the solution in the space $\mathcal{V}_m$ is independent of the choice of orthogonal transformation used to get the sparse-matrix form of the equations.

Given the solution f_n it is possible to construct $f(x)$ for any x using the interpolation

$$f(x) = g(x) + \sum_n \sum_k K(x, x_{kn}) w_{kn} f_n.$$

This method has the feature that the solution can be obtained without ever evaluating a wavelet or scaling function. The wavelet nature of this calculation appears in the quadrature points and weights, which are functions of the scaling coefficients.

Scattering integral equations have two complications. First the integral is over a half line. Second, the kernel has a $1/(x \pm i\epsilon)$ singularity.

The endpoint near $x = 0$ of the half line can be treated using special quadratures for the functions on the half interval. If there is a ϕ_n with support containing an endpoint, the δ_{mn} in (155) needs to be replaced by

$$\int_0^\infty \phi_m(x)\phi_n(x)dx = N_{mn} = N_{nm},$$

which is not a Kronecker delta when the support of ϕ_m and ϕ_n contain the origin. These integrals can be evaluated using the same methods that were used to calculate moments on the half interval. We use the scaling equations and the orthonormality when the support of both terms are in the half interval. Specifically for a and b integers

$$N_{i,j}^{a,b} = \int_a^b \phi_i(x)\phi_j(x)dx$$

$$= \int_{a-i}^{b-i} \phi(x)\phi(x+i-j)dx$$

$$= \sum_{l,l'} h_l h_{l'} \int_{a-i}^{b-i} \phi(2x-l)\phi(2x+2i-2j-l')2dx$$

$$= \sum_{l,l'} h_l h_{l'} \int_{2(a-i)-l}^{2(b-i)-l} \phi(x)\phi(x+2i-2j+l-l')dx$$

$$= \sum_{l,l'} h_l h_{l'} N_{0,-2i+2j-l+l'}^{2a-2i-l,2b-2i-l)}.$$

When either function has support inside the interval this is a Kronecker delta. These equations are linear equations that relate these known elements to the unknown elements where the support overlaps an upper or lower endpoint. These formulas simplify if one of the endpoints satisfy $a = \infty$ or $b = -\infty$. The final relations are

$$N_{0,j-i}^{a-i,b-i} = \sum_{l,l'} h_l h_l' N_{0,-2i+2j-l+l'}^{2a-2i-l,2b-2i-l}.$$

Note that $N_{0,k} = 1$ if $k = 0$, $N_{0,k} = 0$ if $k > 0$ or $k \leq -(2K-1)$. It is non-trivial for $-(2K-2) \leq k < 0$. This gives a linear system for the overlap coefficients, $N_{i,j}$.

For scaling functions that overlap $x = 0$ the equation becomes:

$$\sum_n \left[N_{0,n-m} - \sum_{l,k} \tilde{w}_{lm} K(\tilde{x}_{lm}, \tilde{x}_{kn}) \tilde{w}_{kn} \right] f_n = \sum_l g(\tilde{x}_{lm}) \tilde{w}_{lm}$$

where the $\tilde{x}_{lm}, \tilde{w}_{lm}$ indicate that for m satisfying $2K-2 \leq m < 0$ the quadrature points and weights need to be replaced by the ones for the half

interval. In this case overlap matrix elements N_{mn} need to be computed on the scale dictated by the approximation space $\mathcal{V}_k$.

Mapping techniques are valuable for transforming the equation to an equivalent equation on a finite interval and for treating singularities. For example, the mapping

$$x = -x_0 \frac{b}{a} \frac{y-a}{b-y}$$

transforms the domain of integration to $[a, b]$ and a singularity at $x = x_0$ to the origin. What remains is a mechanism for treating an integrable singularity. The first step is to use a mapping, like the one above, to place the singularity at the origin. After mapping, the relevant integrals for a $1/(x-x_0)$ singularity, when using the subtraction technique discussed below, are

$$I_m(n) := \int \frac{D^m T^n \phi(x)}{x} dx.$$

Using unitarity of D gives

$$I_m(n) := \int D(D^m T^n \phi(x)) D \frac{1}{x} dx =$$

$$\frac{2}{\sqrt{2}} \int \frac{D^{m+1} T^n \phi(x)}{x} dx =$$

$$\sqrt{2} \int \frac{D^m T^{2n} D\phi(x)}{x} dx =$$

$$\sum_{l=0}^{2K-1} \sqrt{2} h_l I_m(2n+l).$$

The equations

$$I_m(n) = \sum_{l=0}^{2K-1} \sqrt{2} h_l I_m(2n+l)$$

give linear equations relating the integrals with singularities to the integrals with no singularities. The singular terms for the order-K Daubechies scaling functions are

$$\phi_{m0}(-1), \phi_{m0}(-2), \cdots \phi_{m0}(-2K+2)$$

The endpoint terms, $\phi_{m0}(0)$ and $\phi_{m0}(-2K+1)$ are not singular because $\phi_m(x)$ must be continuous at the endpoints.

We found that these equations are ill-conditioned, but they can be supplemented by

$$0 = \sum_n \mathcal{P} \int_{-k}^{k} \frac{\phi_{mn}(x)}{x} dx, \tag{156}$$

which has the form

$$0 = \sum_n I_m^k(n) \tag{157}$$

where the integrals $I_m^k(n)$ include partial integrals when n is such that the support of $\phi_{mn}(x)$ contains the points k or $-k$. These linear relations relate the singular integrals to the non-singular ones. For $\phi_{mn}(x)$ with support far enough away from the origin and not containing k or $-k$ the integrals can be expressed in terms of the moments:

$$I_m(n) = \int \frac{D^m T^n \phi(x)}{x} dx = \int T^n \phi(x) D^{-m} \frac{1}{x} dx = 2^{-\frac{m}{2}} \int \phi(x) T^{-n} \frac{1}{x} dx$$

$$= 2^{-\frac{m}{2}} \int \phi(x) \frac{1}{x+n} dx = 2^{-\frac{m}{2}} \frac{1}{n} \sum_{l=0}^{\infty} \left(-\frac{1}{n}\right)^l \langle x^l \rangle_\phi.$$

For large values of n this series converges rapidly. Similar methods can be used for values of n where k or $-k$ is in the support of $\phi_{mn}(x)$. In this case the full moments need to be replaced by the appropriate partial moments.

For singularities of the form $1/(x \pm i\epsilon)$ equation (156) is replaced by

$$\int_{-m}^{m} \frac{dx}{x \pm i0^+} = \mp i\pi.$$

Using this with the wavelet expansion

$$\sum_n \phi(x) = 1$$

provides the needed additional equation,

$$\mp 2\pi i = \sum_n I_m^k(n),$$

which replaces (157). The result of solving these linear equations is accurate approximations to the integrals

$$\int \frac{\phi_n(x)}{x \pm i0^+}. \tag{158}$$

To use these to solve the integral equation consider the case $m = 0$:

$$\int \frac{K(x,y)}{y}\phi_n(y)dy$$

$$= \int \frac{K(x,y) - K(x,0)}{y}\phi_n(y)dy + K(x,0)\int \frac{\phi_n(y)}{y}dy$$

$$= \sum_l \frac{K(x,x_l) - K(x,0)}{x_l}w_l + K(x,0)I_0(n)$$

In applications the $I_0(m)$ should be computed for m values far from the singularity using the series expansion. The equations relating the $I_0(m)$ are used to calculate the remaining $I_0(n)$'s.

Similar methods can be used to treat other types of integrable singularities. For example, for logarithmic singularities define

$$I_0(n) := \int \phi_n(x)\ln(x)dx$$

The scaling equation gives the linear relations

$$I_0(n) = (T^n\phi, \ln) = (DT^n\phi, D\ln)$$

$$= \frac{1}{\sqrt{2}}[(T^{2n}D\phi, \ln) - \ln(2)(T^{2n}D\phi, 1)]$$

$$= \frac{1}{\sqrt{2}}\left(\sum_l h_l I_0(2n+l) - \ln(2)\right).$$

In this case, because the singularity is integrable and the value of the integral is unambiguous, we do not need an additional equation to specify the treatment of the singularity; however the function is multiply valued, so the computation of the input integrals far from the singularity should reflect the choice of Riemann sheet.

The linear equations above relate the integrals $I_0(n)$ that overlap the singularity to integrals far away from the singularity, which may or may not have support containing the endpoints $\pm a$. These terms serve as input to the linear system and can be computed with the moments and partial moments using expansion methods. For the case that the support of $\phi_n(x)$ does not

Table 2: Singular Integrals

$K=2$		
$n=-2$	$\int \phi(x-n)\ln\|x\|dx$	0.456927033732831
$n=-1$	$\int \phi(x-n)\ln\|x\|dx$	−1.64215549088219
$K=3$		
$n=-4$	$\int \phi(x-n)\ln\|x\|dx$	1.15737952417967
$n=-3$	$\int \phi(x-n)\ln\|x\|dx$	0.750468355278047
$n=-2$	$\int \phi(x-n)\ln\|x\|dx$	0.315624303943019
$n=-1$	$\int \phi(x-n)\ln\|x\|dx$	−1.83646456399118

contain $\pm a$ the following expansion can be used when the support of $\phi_n(x)$ contains positive values of x:

$$I_0(n) = \int \phi_n(x)\ln(x)dx = \int \phi(y)\ln(n(1+y/n))dy$$
$$= \ln(n) - \sum_{m=1}^{\infty} \frac{(-1)^m}{m} \frac{< x^m >_\phi}{n^m}. \tag{159}$$

This expansion converges rapidly for large n. When $|n|$ is large with $n<0$ we use

$$\ln(n) = ln|n| + i(2m+1)\pi$$

and m is fixed by the treatment of the integral near the origin. In the case that one of the endpoints $\pm a$ are contained in the support of ϕ_n the moments, a similar expression can be used to approximate the integrals $I_0^a(n)$, except the moments, including the 0-th moment multiplying $\ln(n)$, need to be replaced by partial moments.

For the case of the Daubechies $K=2$ and $K=3$ wavelets the solution of these equations in given in Table 2. The numbers in the table are for $\int \phi_n(x)\ln|x|dx$ for the $\phi_n(x)$ with support containing $x=0$.

6 Derivatives and Differential Equations

The scaling equation can also be used as a tool to solve differential equations. Consider the following approximation of the function $f(x)$ given by

$$f(x) \sim \sum_n f_n \phi_{mn}(x) \tag{160}$$

where $\phi_{mn}(x) = D^m T^n \phi(x)$ is the scaling function basis on the approximation space $\mathcal{V}_m$. As $m \to -\infty$ this representation becomes exact.

For the purpose solving differential equation we want to calculate approximate derivatives of $f(x)$ on the same approximation space

$$f'(x) := \sum_n d'_n \phi_{mn}(x).$$

$$f''(x) := \sum_n d''_n \phi_{mn}(x).$$

The orthonormality of the scaling basis functions can be used to find the expansion coefficients d'_n and d''_n:

$$d'_n := \int \phi_{mn}(x) f'(x) dx = -\int \phi'_{mn}(x) f(x) dx. \tag{161}$$

$$d''_n := \int \phi_{mn}(x) f''(x) dx = \int \phi''_{mn}(x) f(x) dx. \tag{162}$$

Using the expansion (160) in (161) gives

$$d'_n = -\sum_{n'} (\phi'_{mn}, \phi_{mn'}) f_{n'}$$

and

$$d''_n = \sum_{n'} (\phi''_{mn}, \phi_{mn'}) f_{n'}.$$

The coefficients $(\phi'_{mn}, \phi_{mn'})$ and $(\phi''_{mn}, \phi_{mn'})$ are needed to compute the expansion coefficients d'_n and d''_n for the derivatives in terms of the expansion coefficients $f_{n'}$ of the function.

Given these linear relations between the coefficients d'_n and d''_n, the solution of a linear differential equation can be reduced to solving a linear algebraic system for the coefficients f_n. The size of this system can be reduced by employing the wavelet transformation. The method of solution depends on the type of problem. Standard methods can be used to enforce boundary conditions; the only trick is that all of the basis functions with support that overlaps the boundary should be retained.

The goal of this section is to show that the scaling equation can be used to compute the needed overlap integrals.

To proceed we first consider the simplest case where the scale $m = 0$. Using unitarity of the translation operator gives the following relations

$$(\phi'_n, \phi_{n'}) = (\phi', \phi_{n'-n}) = (\phi'_{n-n'}, \phi)$$

$$(\phi''_n, \phi_{n'}) = (\phi'', \phi_{n'-n}) = (\phi''_{n-n'}, \phi).$$

In addition these coefficients have the following symmetry relations

$$(\phi', \phi_n) = \int \phi'(x)\phi(x-n)dx = -\int \phi(x)\phi'(x-n)dx =$$

$$= -\int \phi'(y)\phi_(y+n)dy = -(\phi', \phi_{-n})$$

and

$$(\phi'', \phi_n) = (\phi'', \phi_{-n})$$

which follow by integration by parts.

The overlap coefficients can be computed using the scaling equation and the derivatives of the scaling equation:

$$\phi(x) = \sqrt{2}\sum_{l=0}^{2K-1} h_l \phi(2x-l)$$

$$\phi'(x) = 2\sqrt{2}\sum_{l=0}^{2K-1} h_l \phi'(2x-l)$$

$$\phi''(x) = (2^2)\sqrt{2}\sum_{l=0}^{2K-1} h_l \phi''(2x-l).$$

We first consider the computation of the coefficients $(\phi'(x), \phi_n)$.

For a_n defined by

$$a_n := (\phi'(x), \phi_n)$$

this leads to the following linear relations among the overlap coefficients

$$a_n = 4\sum_{l,l'=0}^{2K-1} h_l h_{l'} \int \phi(2x-2n-l)\phi'(2x-l')dx$$

$$= 2\sum_{l,l'=0}^{2K-1} h_l h_{l'} \int \phi(y-2n-l+l')\phi'(y)dy = 2\sum_{l,l'=0}^{2K-1} h_l h_{l'} a_{2n+l-l'}. \qquad (163)$$

Since both $\phi(x)$ and $\phi'(x)$ have support for $0 \leq x \leq (2K-1)$, the non-zero terms in the sum are constrained by

$$-(2K-1) < 2n + l - l' < 2K - 1.$$

For the second derivative these equations are replaced by

$$a_n := (\phi''(x), \phi_n) = 8 \sum_{l,l'=0}^{2K-1} h_l h_{l'} a_{2n+l-l'} \tag{164}$$

These linear equation are homogeneous and must be supplemented by a normalization condition. For the Daubechies wavelets of order $K > 1$ we have the expansion

$$x = \sum_n b'_n \phi_n(x) \qquad x^2 = \sum_n b''_n \phi_n(x)$$

where the expansion coefficients are

$$b'_n = \int \phi_n(x) x dx = n + \int \phi(x-n)(x-n) dx = n + < x >_\phi$$

and

$$b''_n = \int \phi_n(x) x^2 dx = \int \phi(x)(x+n)^2 dx = n^2 + 2n < x >_\phi + < x^2 >_\phi$$

$$= n^2 + 2n < x >_\phi + < x >_\phi^2 = (n + < x >_\phi)^2.$$

Thus

$$x = < x >_\phi + \sum_n n \phi_n(x) \qquad x^2 = < x >_\phi^2 + 2(x - < x >_\phi) < x >_\phi + \sum_n n^2 \phi_n(x)$$

These equations can be differentiated to get

$$1 = \sum_n n \phi'_n(x) \qquad 2 = \sum_n n^2 \phi''_n(x)$$

Multiplying by $\phi(x)$ and integrating gives the desired inhomogeneous equation

$$1 = \sum_n n(\phi, \phi'_n) = \sum_n n(\phi_{-n}, \phi') =$$

$$-\sum_n n(\phi_n, \phi') = -\sum_n n a_n \tag{165}$$

and

$$2 = \sum_n n^2 a''_n \tag{166}$$

Equations (163) and (165) or (163) and (166) are linear systems that can be used to solve for the coefficients a'_n and a''_n.

In general, it is desirable to expand a function using a scaling basis associated with a sufficiently small scale m, In addition, for efficiency it also useful to use the basis on the approximation space $\mathcal{V}_m$ consisting or the scaling functions on the scale $m+k$ and wavelet basis functions on scales between $m+1$ and $m+k$. Finally, one needs to be able to treat higher derivatives. Generalizations of the methods can be used to find exact expression for all of these quantities expressed a solutions of linear equations involving the scaling coefficients. For the higher derivatives it is necessary to use a Daubechies wavelet of sufficiently high order. The number of derivative of the wavelet and scaling function basis increases with order.

A necessary condition for the solution of the scaling equation to have k derivatives can be obtained by differentiating the scaling equation k times, which gives

$$\phi^{(k)}(x) = \sqrt{2}2^k \sum_l h_l \phi^k(2x - l)$$

Letting $x = m$ and $n = 2m - l$ gives the equation

$$\phi^{(k)}(m) = \sqrt{2}2^k \sum_l h_{2m-n} \phi^k(n)$$

where the non-zero values of $\phi^k(n)$ satisfy $0 \leq n \leq 2K-1$. For this equation of have a solution the matrix $H_{mn} := h_{2m-n}$ must have eigenvalue $2^{-(k+1/2)}$. This is a necessary condition for the basis to have k derivatives. When the k-th derivative exists, it can be computed up to normalization by iterating the Fourier transform of equation (6). The method used to compute wavelets at dyadic points can also be used with the above equation to compute the k-th derivatives of scaling functions and wavelets at dyadic points.

The scaling equation can be used to exactly compute all of the expansion coefficients. In order to exhibit the key relations it is useful to use operators:

$$Df(x) = \frac{1}{\sqrt{2}} f(\frac{x}{2}) \tag{167}$$

$$Tf(x) = f(x-1) \tag{168}$$

$$\Delta f(x) = \frac{df}{dx}(x). \tag{169}$$

Direct computation shows the following intertwining relations

$$\Delta D = \frac{1}{2} D\Delta \tag{170}$$

$$DT = T^2 D \tag{171}$$

$$\Delta T = T\Delta \tag{172}$$

$$\Delta^\dagger = -\Delta \qquad T^\dagger = T^{-1} \qquad D^\dagger = D^{-1}. \tag{173}$$

We also have the scaling equations:

$$D\phi = \sum_l h_l T^l \phi \tag{174}$$

$$D\psi = \sum_l g_l T^l \phi. \tag{175}$$

Using the operator relations above give

$$D\Delta^r \phi = 2^r \sum_l h_l T^l \Delta^r \phi \tag{176}$$

$$D\Delta^r \psi = 2^r \sum_l g_l T^l \Delta^r \phi. \tag{177}$$

The different expansion coefficients can be expressed in terms of these operators as

$$(\phi_{m'n'}, \Delta^r \phi_{mn}) = (D^{m'} T^{n'} \phi, \Delta^r D^m T^n \phi) \tag{178}$$

$$(\psi_{m'n'}, \Delta^r \phi_{mn}) = (D^{m'} T^{n'} \psi, \Delta^r D^m T^n \phi) \tag{179}$$

$$(\phi_{k'n'}, \Delta^r \psi_{mn}) = (D^{k'} T^{n'} \phi, \Delta^r D^m T^n \psi) \tag{180}$$

$$(\psi_{k'n'}, \Delta^r \psi_{mn}) = (D^{k'} T^{n'} \psi, \Delta^r D^m T^n \psi). \tag{181}$$

The following steps are used to evaluate these coefficients:

1. Move all of the factors of D to a single side of the equation. Choose the side where the power of D is positive.
2. Move the D's through all derivatives.
3. Use the scaling equations to eliminate all of the D's.

4. Move all of the T's to the left side of the scalar product.

Using these steps all of the overlap coefficients can be expressed in terms of

$$(\phi_n, \Delta^r \phi) \tag{182}$$

$$(\psi_n, \Delta^r \phi) \tag{183}$$

$$(\phi_n, \Delta^r \psi,) \tag{184}$$

$$(\psi_n, \Delta^r \psi) \tag{185}$$

The scaling equation can be used to express all of the ψ terms in terms of the ϕ terms. The result is at all of the coefficients can be expressed in terms of the coefficients

$$(\phi_n, \Delta^r \phi) \tag{186}$$

We have shown how to compute these for $r = 1$ and 2. The coefficients for larger values of r can be obtained by solving the system:

$$r! = \sum_n n^r \phi_n^{(r)}(x)$$

$$a_n^{(r)} := (\phi''(x), \phi_n) = 2^{2r-1} \sum_{l,l'=0}^{2K-1} h_l h_{l'} a_{2n+l-l'}^{(r)} \tag{187}$$

7 Galerkin for Scattering

We want to find the solution to the s-wave Schrödinger equation

$$-\frac{\hbar^2}{2m}\frac{1}{r}\frac{d^2}{dr^2}[r\,\psi(r)] + V(r)\,\psi(r) = E\,\psi(r) \tag{188}$$

for a particle with mass m and energy $E = \hbar^2 k^2/2m$, where $\psi(r)$ has the asymptotic form

$$r\psi(r) \underset{r\to\infty}{\longrightarrow} \sin(kr + \delta)\,, \tag{189}$$

and $r\psi(r)$ is zero at the origin. Equation (188) can be rewritten in the form

$$-\frac{1}{r}\frac{d^2}{dr^2}[r\,\psi(r)] + U(r)\,\psi(r) = k^2\psi(r)\,, \tag{190}$$

where

$$U(r) = \frac{2m}{\hbar^2} V(r)\,. \tag{191}$$

To solve Equation (190) we choose a complete set of basis functions $\phi_n(r)$ and write

$$\psi(r) = \sum_{n=1}^{N} a_n\, \phi_n(r)\,. \tag{192}$$

To solve for the expansion coefficients a_n we first multiply Equation (190) by $r^2\phi_m(r)$ and integrate from 0 to R. This gives

$$-\int_0^R r\,\phi_m(r)\frac{d^2}{dr^2}[r\,\psi(r)]\,dr + \int_0^R \phi_m(r)\,U(r)\,\psi(r) r^2 dr = k^2 \int_0^R \phi_m(r)\,\psi(r)\,r^2 dr\,. \tag{193}$$

Using integration by parts, the first term in Equation (193) can be written as

$$-\int_0^R r\,\phi_m(r)\frac{d^2}{dr^2}[r\,\psi(r)]\,dr = -r\,\phi_m(r)\frac{d}{dr}[r\,\psi(r)]\Big|_0^R + \int_0^R \frac{d}{dr}[r\,\phi_m(r)]\frac{d}{dr}[r\,\psi(r)]\,dr\,. \tag{194}$$

Now we set

$$\frac{d}{dr}[r\,\psi(r)]\Big|_{r=R} = 1\,, \tag{195}$$

which corresponds to a change in the normalization of $\psi(r)$, and use $r\,\psi(r)$ is zero at $r = 0$ to write

$$-\int_0^R r\,\phi_m(r)\frac{d^2}{dr^2}[r\,\psi(r)]\,dr = -R\,\phi_m(R) + \int_0^R \frac{d}{dr}[r\,\phi_m(r)]\frac{d}{dr}[r\,\psi(r)]\,dr\,. \tag{196}$$

Thus, Equation (193) can be written in the form

$$\int_0^R \frac{d}{dr}[r\,\phi_m(r)]\frac{d}{dr}[r\,\psi(r)]\,dr + \int_0^R \phi_m(r)\,U(r)\,\psi(r) r^2 dr - k^2 \int_0^R \phi_m(r)\,\psi(r)\,r^2 dr = R\,\phi_m(R)\,. \tag{197}$$

Substituting the expansion for $\psi(r)$ in Equation (192) into Equation (197) gives

$$\sum_{n=1}^{N} a_n \left\{ \int_0^R \tfrac{d}{dr}[r\,\phi_m(r)]\tfrac{d}{dr}[r\,\phi_n(r)]\,dr + \textstyle\int_0^R \phi_m(r)\,U(r)\,\phi_n(r) r^2 dr - k^2 \textstyle\int_0^R \phi_m(r)\,\phi_n(r)\,r^2 dr \right\} = R\,\phi_m(R)\,, \tag{198}$$

which can be written as the matrix equation

$$\sum_{n=1}^{N} A_{mn}\, a_n = b_m \,. \tag{199}$$

Given the solution to Equation (199) we need to determine the normalization and the phase shift. For the new boundary condition given in Equation (195)

$$r\psi(r) \underset{r\to\infty}{\longrightarrow} A\, \sin(kr+\delta)\,, \tag{200}$$

Thus, we need an additional equation to use with Equation (195). We use the integral

$$\begin{aligned}
I &= \int_0^R \sin(kr)\, U(r)\, \psi(r)\, r dr \\
&= \int_0^R \sin(kr) \left\{ \frac{1}{r}\frac{d^2}{dr^2}[r\,\psi(r)] + k^2\,\psi(r) \right\} r dr \\
&= \sin(kr)\frac{d}{dr}[r\,\psi(r)]\Big|_0^R - k\cos(kr)[r\,\psi(r)]\Big|_0^R \\
&\qquad + \int_0^R \left[\left(\frac{1}{r}\frac{d^2}{dr^2} + k^2 \right) \sin(kr) \right] r\,\psi(r)\, dr \\
&= kA\,[\sin(kR)\cos(kR+\delta) - \cos(kR)\sin(kR+\delta)] \\
&= -kA\sin\delta\,.
\end{aligned} \tag{201}$$

From Equations (195) and (200) we get

$$kA\cos(kR+\delta) = kA\cos\delta\cos(kR) - kA\sin\delta\sin(kR) = 1\,, \tag{202}$$

which using Equation (201) can be written as

$$kA\cos\delta = \frac{1 - I\sin(kR)}{\cos(kR)}\,. \tag{203}$$

Finally, from Equations (201) and (203) we find

$$\tan\delta = \frac{-I\cos(kR)}{1 - I\sin(kR)}\,. \tag{204}$$

Given δ the value of A can be found using Equation (201).

Eckart Wave Function

To test the method we use the Eckart potential

$$V(r) = -\frac{\hbar^2}{2m}\frac{2\beta\lambda^2 e^{-\lambda r}}{(\beta e^{-\lambda r}+1)^2}\,, \tag{205}$$

which has the analytic solution

$$r\psi(r) = \frac{\left[(4k^2+\lambda^2)+(4k^2-\lambda^2)\beta e^{-\lambda r}\right]\sin(kr+\delta) - 4k\lambda\beta e^{-\lambda r}\cos(kr+\delta)}{(4k^2+\lambda^2)\,(\beta e^{-\lambda r}+1)}\,, \tag{206}$$

where the phase shift is given by

$$\delta = \arctan\left(\frac{\lambda}{2k}\right) + \arctan\left(\frac{\lambda(\beta-1)}{2k(\beta+1)}\right)\,. \tag{207}$$

If β is chosen to have the value

$$\beta = \frac{\lambda+2\kappa}{\lambda-2\kappa} \tag{208}$$

the potential will have a bound state with the energy

$$E_B = -\frac{\hbar^2\kappa^2}{2m}\,. \tag{209}$$

The boundstate wave function is given by

$$r\psi_B(r) = \frac{2\sqrt{2\kappa\beta}\sinh(\lambda r/2)\,e^{-\kappa r}}{e^{\lambda r/2}+\beta e^{-\lambda r/2}}\,. \tag{210}$$

8 Wavelet Filters

The wavelet transform is the orthogonal mapping between the scaling function basis on a fine scale and the equivalent basis consisting of scaling functions on a coarser scale and wavelets at all intermediate scales. The wavelet transform can be implemented by treating the scaling equation and the equation defining the wavelets as linear combinations of scaling functions for a finer scale, as low and high pass filters. This has the advantage when most of the high-frequency information is unimportant.

The wavelet filter is defined using the scaling relations

$$D\,\phi(x) = \frac{1}{\sqrt{2}}\phi\left(\frac{x}{2}\right) = \sum_{l=0}^{2k-1} h_l T^l \phi(x) \tag{211}$$

and

$$D\,\psi(x) = \frac{1}{\sqrt{2}}\psi\left(\frac{x}{2}\right) = \sum_{l=0}^{2k-1} g_l T^l \phi(x)\,, \tag{212}$$

where $g_l = (-1)^l h_{2k-1-l}$. Using Equations (211) and (212) we can write

$$\begin{aligned}
\phi_{j,m}(x) &= D^j T^m \phi(x) \\
&= \sum_{l=0}^{2k-1} h_l D^j T^m D^{-1} T^l \phi(x) \\
&= \sum_{l=0}^{2k-1} h_l D^{j-1} T^{2m} T^l \phi(x) \\
&= \sum_{l=0}^{2k-1} h_l \phi_{j-1,2m+l}(x)
\end{aligned} \tag{213}$$

and

$$\begin{aligned}
\psi_{j,m}(x) &= D^j T^m \psi(x) \\
&= \sum_{l=0}^{2k-1} g_l D^j T^m D^{-1} T^l \phi(x) \\
&= \sum_{l=0}^{2k-1} g_l D^{j-1} T^{2m} T^l \phi(x) \\
&= \sum_{l=0}^{2k-1} g_l \phi_{j-1,2m+l}(x)\,,
\end{aligned} \tag{214}$$

where we have used

$$\begin{aligned}
T\,D^{-1}\phi(x) &= T\,\sqrt{2}\phi(2x) \\
&= \sqrt{2}\phi(2x-2) \\
&= D^{-1}T^2\phi(x)
\end{aligned} \tag{215}$$

to write $T^m D^{-1} = D^{-1} T^{2m}$.

We use the orthonormality of the functions $\phi_{j,m}(x)$ and $\psi_{j,m}(x)$ to obtain the inverse relation. Since $\mathcal{V}_{j-1} = \mathcal{V}_j \oplus \mathcal{W}_j$, we can write

$$\phi_{j-1,m}(x) = \sum_n a_n \phi_{j,n}(x) + \sum_n b_n \psi_{j,n}(x)\,, \tag{216}$$

Using the scaling relations, we find

$$\begin{aligned} a_n &= \int \phi_{j,n}(x)\,\phi_{j-1,m}(x)\,dx \\ &= \sum_l h_l \int \phi_{j-1,2n+l}(x)\,\phi_{j-1,m}(x)\,dx \\ &= \sum_l h_l \delta_{2n+l,m} \qquad (217) \\ &= h_{m-2n} \qquad (218) \end{aligned}$$

and

$$\begin{aligned} b_n &= \int \psi_{j,n}(x)\,\phi_{j-1,m}(x)\,dx \\ &= \sum_l g_l \int \phi_{j-1,2n+l}(x)\,\phi_{j-1,m}(x)\,dx \\ &= \sum_l g_l \delta_{2n+l,m} \qquad (219) \\ &= g_{m-2n}\,. \qquad (220) \end{aligned}$$

Thus, we find

$$\phi_{j-1,m}(x) = \sum_n h_{m-2n} \phi_{j,n}(x) + \sum_n g_{m-2n} \psi_{j,n}(x)\,. \tag{221}$$

An alternate derivation is to use the orthonormality relations

$$\sum_l h_l h_{l+2m} = \delta_{m0} \tag{222}$$

$$\sum_l g_l g_{l+2m} = \delta_{m0} \tag{223}$$

$$\sum_l h_l g_{l+2m} = 0\,, \tag{224}$$

derived in the *Wavelet Notes*. Now using the scaling relations in Equation (216) gives

$$\phi_{j-1,m}(x) = \sum_n a_n \sum_l h_l \phi_{j-1,2n+l}(x) + \sum_n b_n \sum_l g_l \phi_{j-1,2n+l}(x) \,. \tag{225}$$

Taking the inner product with $\phi_{j-1,k}$ gives

$$\begin{aligned} \delta_{km} &= \sum_n \sum_l a_n \, h_l \, \delta_{2n+l,k} + \sum_n \sum_l b_n \, g_l \, \delta_{2n+l,k} \\ &= \sum_n a_n h_{k-2n} + \sum_n b_n g_{k-2n} \,. \end{aligned} \tag{226}$$

Multiplying Equation (226) by $h_{m-2n'}$ and summing over m gives

$$\begin{aligned} h_{k-2n'} &= \sum_n a_n \sum_k h_{k-2n'} h_{k-2n} + \sum_n b_n \sum_k h_{k-2n'} g_{k-2n} \\ &= \sum_n a_n \delta_{nn'} \\ &= a_{n'} \,, \end{aligned} \tag{227}$$

where we have used the relations (222) and (224). Multiplying Equation (226) by $g_{k-2n'}$ and summing over k gives

$$\begin{aligned} g_{k-2n'} &= \sum_n a_n \sum_k g_{k-2n'} h_{k-2n} + \sum_n b_n \sum_k g_{k-2n'} g_{k-2n} \\ &= \sum_n b_n \delta_{nn'} \\ &= b_{n'} \,, \end{aligned} \tag{228}$$

where we have used the relations (223) and (224).

Given the expansion of $f(x)$ in terms of the scaling functions

$$f(x) = \sum_{n=0}^{2^J-1} c_{j,n} \, \phi_{j,n}(x) \,, \tag{229}$$

we can use Equation (221) to write the expansion in the form

$$\begin{aligned} f(x) &= \sum_{n=0}^{2^J-1} c_{j,n} \sum_m h_{n-2m} \, \phi_{j+1,m}(x) + \sum_{n=0}^{2^J-1} c_{j,n} \sum_m g_{n-2m} \, \psi_{j+1,m}(x) \\ &= \sum_{m=0}^{2^{J-1}-1} c_{j+1,m} \, \phi_{j+1,m}(x) + \sum_{m=0}^{2^{J-1}-1} d_{j+1,m} \, \psi_{j+1,m}(x) \,, \end{aligned} \tag{230}$$

where

$$c_{j+1,m} = \sum_{n=2m}^{2m+2p-1} h_{n-2m}\, c_{j,n} \tag{231}$$

and

$$d_{j+1,m} = \sum_{n=2m}^{2m+2p-1} g_{n-2m}\, c_{j,n}\,, \tag{232}$$

where we use the periodic wrap-around condition $c_{j,2^J+i} = c_{j,i}$. Equations (231) and (232) can be written as a matrix equation, which for $J = 3$ has the form

$$\begin{pmatrix} c_{j+1,0} \\ c_{j+1,1} \\ c_{j+1,2} \\ c_{j+1,3} \\ d_{j+1,0} \\ d_{j+1,1} \\ d_{j+1,2} \\ d_{j+1,3} \end{pmatrix} = \begin{pmatrix} h_0 & h_1 & h_2 & h_3 & 0 & 0 & 0 & 0 \\ 0 & 0 & h_0 & h_1 & h_2 & h_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & h_0 & h_1 & h_2 & h_3 \\ h_2 & h_3 & 0 & 0 & 0 & 0 & h_0 & h_1 \\ g_0 & g_1 & g_2 & g_3 & 0 & 0 & 0 & 0 \\ 0 & 0 & g_0 & g_1 & g_2 & g_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & g_0 & g_1 & g_2 & g_3 \\ g_3 & g_4 & 0 & 0 & 0 & 0 & g_0 & g_1 \end{pmatrix} \begin{pmatrix} c_{j,0} \\ c_{j,1} \\ c_{j,2} \\ c_{j,3} \\ c_{j,4} \\ c_{j,5} \\ c_{j,6} \\ c_{j,7} \end{pmatrix} \tag{233}$$

for the Daubechies $p = 2$ wavelets. Repeated application of the filter transform to the remaining $c_{j+1,m}$ gives

$$\begin{pmatrix} c_{j,0} \\ c_{j,1} \\ c_{j,2} \\ c_{j,3} \\ c_{j,4} \\ c_{j,5} \\ c_{j,6} \\ c_{j,7} \end{pmatrix} \longrightarrow \begin{pmatrix} c_{j+1,0} \\ c_{j+1,1} \\ c_{j+1,2} \\ c_{j+1,3} \\ d_{j+1,0} \\ d_{j+1,1} \\ d_{j+1,2} \\ d_{j+1,3} \end{pmatrix} \longrightarrow \begin{pmatrix} c_{j+2,0} \\ c_{j+2,1} \\ d_{j+2,0} \\ d_{j+2,1} \\ d_{j+1,0} \\ d_{j+1,1} \\ d_{j+1,2} \\ d_{j+1,3} \end{pmatrix} \longrightarrow \begin{pmatrix} c_{j+3,0} \\ d_{j+3,0} \\ d_{j+2,0} \\ d_{j+2,1} \\ d_{j+1,0} \\ d_{j+1,1} \\ d_{j+1,2} \\ d_{j+1,3} \end{pmatrix}. \tag{234}$$

The reverse transform can be obtained by substituting Equations (213) and (214) into Equation (230). This gives

$$f(x) = \sum_{m=0}^{2^{J-1}-1} c_{j+1,m} \sum_{l=0}^{2k-1} h_l\, \phi_{j,2m+l}(x) + \sum_{m=0}^{2^{J-1}-1} d_{j+1,m} \sum_{l=0}^{2k-1} g_l\, \phi_{j,2m+l}(x)$$

$$
\begin{aligned}
&= \sum_n \sum_m h_{n-2m}\, c_{j+1,m}\, \phi_{j,n}(x) + \sum_n \sum_m g_{n-2m}\, d_{j+1,m}\, \phi_{j,n}(x) \qquad (235)\\
&= \sum_n c_{j,n}\, \phi_{j,n}(x)\,,
\end{aligned}
$$

where

$$
c_{j,n} = \sum_m h_{n-2m}\, c_{j+1,m} + \sum_m g_{n-2m}\, d_{j+1,m}\,. \qquad (236)
$$

For the example used in Equation (233), this gives

$$
\begin{pmatrix} c_{j,0} \\ c_{j,1} \\ c_{j,2} \\ c_{j,3} \\ c_{j,4} \\ c_{j,5} \\ c_{j,6} \\ c_{j,7} \end{pmatrix}
=
\begin{pmatrix}
h_0 & 0 & 0 & h_2 & g_0 & 0 & 0 & g_2 \\
h_1 & 0 & 0 & h_3 & g_1 & 0 & 0 & g_3 \\
h_2 & h_0 & 0 & 0 & g_2 & g_0 & 0 & 0 \\
h_3 & h_1 & 0 & 0 & g_3 & g_1 & 0 & 0 \\
0 & h_2 & h_0 & 0 & 0 & g_2 & g_0 & 0 \\
0 & h_3 & h_1 & 0 & 0 & g_3 & g_1 & 0 \\
0 & 0 & h_2 & h_0 & 0 & 0 & g_2 & g_0 \\
0 & 0 & h_3 & h_1 & 0 & 0 & g_3 & g_1
\end{pmatrix}
\begin{pmatrix} c_{j+1,0} \\ c_{j+1,1} \\ c_{j+1,2} \\ c_{j+1,3} \\ d_{j+1,0} \\ d_{j+1,1} \\ d_{j+1,2} \\ d_{j+1,3} \end{pmatrix}, \qquad (237)
$$

which is the transpose of the matrix in Equation (233).

9 Wavelet Transfrom

The wavelet transform is by its nature local at each level and therefore admits an implementation in which the data to be transformed can be placed in a buffer instead of storing the entire data set at once. This significantly reduces the amount of storage space required for applications involving compression.

In the one-dimensional case, the J-level wavelet transform can be computed by buffering $O(J)$ nonessential elements or the full transform can be computed buffering $O(\log(N))$ elements. The standard form for the two-dimensional transform of an $N \times N$ matrix can be performed by buffering only $O(N \log(N))$ elements. In general, a D-dimensional wavelet transform can be computed by only storing $O(N^{D-1} \log(N))$ elements. This buffered wavelet transform can be applied to any type of data that can be input or computed in series. Some notable examples include the compression of time-series data and applications to solutions of integral equations. Below, we will explain the exact implementation of the transform including the buffer

and the extension of the method to two dimensions. The extension to arbitrary dimension is straightforward from the two dimensional case. First, we will layout the terminology that will be used throughout. we adopt the filter viewpoint since it makes the explanation of the buffering procedure more clear. But this is equivalent to the linear algebra viewpoint and we will attempt to explain the procedure in this language as well.

From the filter viewpoint, the wavelet transform is a convolution of the data set and two vectors h and g followed by a decimation. This is equivalent to a convolution that proceeds by steps of two instead of one. For the Daubechies family of wavelets, both of these filters have a length $L = 2K - 1$. The convolution with h produces what is called the father set and the convolution with g produces the mother set. We denote the data set to be transformed as A and use brackets to denote subscripts. In all contexts below, the one-dimensional length of the data set will be $N = 2^n$ where n is an integer. That is the data runs from $A[0]$ to $A[N-1]$. The typical procedure to deal with a finite data set is to periodize the data over the boundary such that $A[N] = A[0], A[N+1] = A[1], \cdots A[N+L-3] = A[L-3]$.

Now, if we write out the first level of the transform as a matrix we can see that it is banded with a bandwidth L corresponding to the convolution operation. The second level of the transform is identical to the first except that it only acts on the father set of data, i.e. the transform on the mothers is the identity. This corresponds to the fact that all of the information about coarser levels is contained in the father functions. The mother functions form an orthogonal subspace to the fathers and mothers on all higher levels. Using this knowledge we can immediately perform a thresholding procedure on the mother set without affecting the rest of the data in any way. The father set can simultaneously be transformed and the resulting mothers thresholded as well. Iterating this procedure on all the relevant levels forms the basis for the buffered wavelet transform. Of course, one must have an a priori thresholding scheme to accomplish this. The simplest such example is an absolute threshold. In this scheme, one chooses an epsilon and all elements with a magnitude less than this epsilon. Other more sophisticated thresholding procedures exist as well, such as procedures based on the level on the transform. The important fact is that one cannot have a procedure that depends in any way on the final transformed data set. Examples of such procedures would be based on the relative size of the transformed elements or a threshold that keeps a certain number/percentage of the final coefficients. In many applications, the absolute thresholding is an acceptable method.

Now, we will explain the detailed implementation of the one-dimensional transform. One begins by computing the elements $A[0]$ to $A[L-3]$ of the data set. As noted above, these elements are necessary for the periodic boundary conditions and form a boundary buffer that must be saved until the end of the calculation. Now, elements can be added to a moving buffer of length L that constitutes the heart of the procedure. After the elements $A[L-2]$ and $A[L-1]$ are computed and placed in the moving buffer, one can begin transforming the data set. Convolving this data set, including the boundary terms, with h produces the first member of the father set $F[0]$ and convolving with g produces the first member of the mother set $M[0]$. As described previously, this mother element can be immediately thresholded and placed in the final output vector. The father element is considered the beginning of a new data set to be transformed and is placed in the boundary buffer corresponding to the next level of the transform. One then proceeds to compute two more elements and convolve $A[2]$ to $A[L+1]$ with h and g. This produces $F[1]$ and $M[1]$, which are treated the same way as before. We continue in this manner, calculating more elements and convolving, until we have computed the element $A[2L-3]$. The moving buffer is now full and we have reached the interior of the data set. When we compute the next element of the data set we can discard the last element of the moving buffer and shift all the elements of the buffer one place. The new element is then appended to the moving buffer. Discarding the last element is justified by the fact that all the information in that element is represented by the corresponding father and mother data sets due to the equivalence of the subspaces. The name moving buffer is clear since this buffer can be viewed as scanning the interior of the data set by moving over it. This process continues, the shifting and convolving, until the end of the data set is reached. When the end of the data set is reached we simply make the data set periodic using the boundary buffer. This process is simultaneously carried out at each level. Now counting the elements in each buffer we see that in each level we must store $L-2$ elements in the boundary buffer and L elements in the moving buffer. So, for J levels we must store $J(2L-2)$ elements. This gives us our size of $O(J)$ where the coefficient depends on the length L of the wavelet filter as is common with most wavelet algorithms.

In many wavelet applications, the data vector to be transformed will be of length $N = 2^n$ and a wavelet transform of level $J = n$ will be computed. In this case, the number of elements stored in the buffers will be of $O(\log(N))$. A minor point to note is that for wavelet filters of length > 2 the last few

levels will not be filled completely. As a programming point one can either fill the buffers periodically or just periodize the convolution. Both procedures are equivalent and consistent with the periodic wavelet transform. Also note that the number of operations has been increased by the shift operation, but is still of $O(N)$ which is the case for the standard wavelet transform. The standard procedure to perform the wavelet transform on a two dimensional data set is to first transform the rows of the matrix and then transform the columns. Alternatively, one could transform the columns and then the rows. Both are equivalent as can be seen by writing out the transform as a matrix multiply and noting the associativity of matrix multiplication.

To perform the buffered wavelet transform on a two-dimensional data set we calculate the data column by column. Each row has a separate set of buffers associated with it. We can view this as a strip that scans the matrix in much the same way as the moving buffer did in the one-dimensional case. Each of these buffers behaves in exactly the same way as the one-dimensional case, except the output is handled differently. The first output from the buffer associated with row one is placed in two vertical buffers. $FB[0]$ and $MB[0]$ the B stands for blank since these are internal buffers that have no outside significance. Both of these outputs must be saved because they contain information about the columns of the matrix. Row two then produces $FB[1]$ and $MB[1]$, and so on continuing down the rows. The transform procedure is applied to the vertical buffers, which produce output FF, FM, MF, and MM. The output MM can be thresholded immediately. The output FM is placed in an array of row buffers of height $N/2$ that transform the rows and filter immediately the MB's produced. The MF output is placed in another vertical buffer where the traditional one-dimensional transform procedure is enacted. The FF output is placed in an array of row buffers identical to the original configuration, except that it is only $N/2$ tall. The same procedure is enacted on this data set that is half as small. Now this proceeds across the matrix in a similar manner as the one-dimensional case, except that the vertical buffers can be completely purged as the next column is reached. To count the number of elements that are necessary we can ignore the vertical buffers, which are subdominant. At the first level we note that there are $N * (2L - 2)$ elements in the row buffer and $(N/2) * (2L - 2)$ in the FF output and FM output. Hereafter we will drop the $2L - 2$ to simplify the counting since we are just looking for order. So we have N and $N/2$ and $N/2$. The FM output will proceed like the one-dimensional case. Therefore it will produce $(N/2)\log(N)$ elements.

The FF out put will produce $N/4FF$ and $N/4FM$. So we can see that the total number of elements will be $N(1+\log(N))\sum_{j=0}^{J}\frac{1}{2}^{j}$. The sum is a simple geometric sum that in the limit that J goes to infinity is bounded by 2. So the final tally of the necessary elements is $O(N\log(N))$. The generalization to D dimensions is straightforward. One begins with a data structure of dimensions $(N^{D-1})(2L-2)$. One then performs a transform to produce two N^{D-2} data structures. One performs a transform on these two structures to produce four N^{D-3} structures. This process continues until the final transform where one has a single dimension. This transform is enacted and the M^D elements are filtered. Appropriate lower dimensional transforms are applied to the mixed output, $MMMFF\cdots MF, FFFFFF\cdots M$, etc. The process is repeated for the $FFF\cdots FFF$ data set. In higher dimensions the algorithm becomes more complicated, but the idea is the same. And the leading order number of elements that need to be saved is $O(N^{D-1}\log(N))$.

10 Appendix I: Continuous Wavelets

We begin by considering the continuous wavelet transform. The continuous wavelet transform is an alternate representation of a function, like a Fourier transform. Both continuous and discrete wavelets are built from a single function called a **mother function**. The notation, $\psi(x)$, is used to denote the mother function of a wavelet.

Wavelets are built from translations and scale transformations of the mother function. Translations and scale transformations of $\psi(x)$ are defined by:

$$\psi_{t,s}(x) := |s|^{-p}\psi(\frac{x-t}{s}). \tag{238}$$

The factor p is a parameter. The functions $\psi_{t,s}(x)$ are the wavelets associated with the mother function $\psi(x)$. The wavelet $\psi_{t,s}(x)$ has two continuous parameters. We investigate conditions on the mother function that allow one to expand any function in terms of wavelets.

To choose the parameter p note that

$$\int_{-\infty}^{\infty}\left||s|^{-p}\psi(\frac{x-t}{s})\right|^{q}dx =$$

$$|s|^{1-qp}\int_{-\infty}^{\infty}|\psi(u)|^{q}du. \tag{239}$$

It follows that if $p = 1/q$ the L^q-norm of ψ

$$\|\psi\|_q := \left(\int_{-\infty}^{\infty} |\psi(u)|^q du \right)^{1/q} \tag{240}$$

is preserved under scale transformations. Thus for $p = 1/q$:

$$\|\psi\|_q = \|\psi_{s,t}\|_q \qquad \text{for all} \qquad s, t. \tag{241}$$

The **continuous wavelet transform** of f is defined by taking the scalar product of f with the wavelet $\psi_{s,t}$:

$$\hat{f}(s,t) := \int_{-\infty}^{\infty} \psi_{s,t}^*(x) f(x) dx = (\psi_{s,t}, f) \tag{242}$$

where asterik $'*'$ indicates the complex conjugate for a complex mother function. In what follows a $\hat{f}$ is used to indicate the wavelet transform of the function f.

Parseval's identity for the Fourier transform implies that the wavelet transform can be expressed in terms of the original function and the mother function or alternatively in terms of their Fourier transforms:

$$\hat{f}(s,t) = (\psi_{s,t}, f) = (\tilde{\psi}_{s,t}, \tilde{f}) \tag{243}$$

where the $\sim$ indicates the Fourier transform defined by:

$$\tilde{\psi}_{s,t}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx} \psi_{s,t}(x) dx \tag{244}$$

$$\tilde{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx} f(x) dx. \tag{245}$$

Note that Parseval's identity states $(f, f) = (\tilde{f}, \tilde{f})$. Using this with $f = g + h$ and $f = g + ih$ gives

$$(g,g) + (h,h) + (g,h) + (h,g) = (\tilde{g},\tilde{g}) + (\tilde{h},\tilde{h}) + (\tilde{g},\tilde{h}) + (\tilde{h},\tilde{g}) \tag{246}$$

and

$$(g,g) + (h,h) + i(g,h) - i(h,g) = (\tilde{g},\tilde{g}) + (\tilde{h},\tilde{h}) + i(\tilde{g},\tilde{h}) - i(\tilde{h},\tilde{g}) \tag{247}$$

which, using the identities $(g,g) = (\tilde{g},\tilde{g})$ and $(h,h) = (\tilde{h},\tilde{h})$, gives the solution to (246) and (247):

$$(g,h) = (\tilde{g},\tilde{h}) \tag{248}$$

which is the form of Parseval's identity used in (243).

The Fourier transform of the wavelet $\psi_{s,t}(x)$ can be expressed in terms of the Fourier transform of the mother function:

$$\tilde{\psi}_{s,t}(k) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx} |s|^{-p} \psi(\frac{x-t}{s}) dx =$$

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-iksu} e^{-ikt} |s|^{-p+1} \psi(u) du =$$

$$|s|^{1-p} e^{-ikt} \tilde{\psi}(sk). \tag{249}$$

The inner product of the Fourier transforms gives

$$\hat{f}(s,t) = (\tilde{\psi}_{s,t}, \tilde{f}) =$$

$$\int_{-\infty}^{\infty} \tilde{\psi}^*_{s,t}(k) \tilde{f}(k) dk$$

$$\int_{-\infty}^{\infty} |s|^{1-p} e^{ikt} \tilde{\psi}^*(sk) \tilde{f}(k) dk. \tag{250}$$

Multiplying both sides of (250) by $e^{-ik't}$ and integrating over t gives

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ik't} (\tilde{\psi}_{s,t}, \tilde{f}) dt =$$

$$|s|^{1-p} \tilde{\psi}^*(sk') \tilde{f}(k'), \tag{251}$$

where the representation of the delta function:

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i(k'-k)t} dt = \delta(k'-k). \tag{252}$$

was used to get (251).

The right-hand side of (251) is a product of the Fourier transform of the original function with another function. We can't divide by the function $\tilde{\psi}^*(sk')$ because it might be zero for some values of k'. Instead, the trick is to eliminate it using the variable s.

Multiply both sides of this equation by $\tilde{\psi}(sk')$ and a yet to be determined weight function $w(s)$ and integrate over s. This gives

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} w(s)ds \int_{-\infty}^{\infty} dte^{-ik't}\tilde{\psi}(sk')\hat{f}(s,t) =$$

$$\tilde{f}(k')\int_{-\infty}^{\infty} w(s)ds|s|^{1-p}\tilde{\psi}^*(sk')\tilde{\psi}(sk') = \tilde{f}(k')Y(k') \tag{253}$$

where

$$Y(k') = \int_{-\infty}^{\infty} dsw(s)|s|^{1-p}|\tilde{\psi}(sk')|^2. \tag{254}$$

In order to be able to extract the Fourier transform of the original function, it is sufficient that $Y(k')$ satisfies $0 < A \leq Y(k') \leq B < \infty$ for some numbers A and B. In this case

$$\tilde{f}(k) = \frac{1}{2\pi Y(k)}\int_{0}^{\infty} w(s)ds \int_{-\infty}^{\infty} dte^{-ikt}\tilde{\psi}(sk)\hat{f}(s,t). \tag{255}$$

It is convenient to rewrite this in terms of the wavelet basis:

$$\tilde{f}(k) = \frac{1}{2\pi Y(k)}\int_{-\infty}^{\infty} w(s)|s|^{p-1}ds \int_{-\infty}^{\infty} dt\tilde{\psi}_{s,t}(k)\hat{f}(s,t). \tag{256}$$

We define the **dual wavelet** by

$$\tilde{\psi}^{s,t}(k) = \frac{1}{2\pi Y(k)}\tilde{\psi}_{s,t}(k). \tag{257}$$

The dual wavelet is distinguished from the ordinary wavelet by having the parameters s, t appearing as superscripts rather than subscripts.

The inversion formula can be expressed in terms of the dual wavelet by

$$\tilde{f}(k) = \int_{-\infty}^{\infty} w(s)|s|^{p-1}ds \int_{-\infty}^{\infty} dt\tilde{\psi}^{s,t}(k)\hat{f}(s,t). \tag{258}$$

In order to recover the original function, take the inverse Fourier transform of this expressions:

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dke^{ikx}\tilde{f}(k) =$$

$$\int_{-\infty}^{\infty} w(s)|s|^{p-1} ds \int_{-\infty}^{\infty} dt \psi^{s,t}(x) \hat{f}(s,t) \tag{259}$$

where

$$\psi^{s,t}(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk e^{ikx} \tilde{\psi}^{s,t}(k). \tag{260}$$

In general this is a tedious procedure because the dual wavelet $\psi^{s,t}(x)$ must be computed using (257) and (260) for each value of s and t. If the dual wavelet also had a mother function, then it would only be necessary to Fourier transform the "dual mother" and then all of the other Fourier transforms could be expressed in terms of the transform of the "dual mother".

The first step in constructing a "dual mother" is to investigate the structure of the dual wavelets in x-space:

$$\psi^{s,t}(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk e^{ikx} \tilde{\psi}^{s,t}(k) =$$

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk e^{ikx} \frac{1}{2\pi Y(k)} |s|^{1-p} e^{-ikt} \tilde{\psi}(sk) =$$

$$\psi^{s,0}(x-t)$$

where

$$\psi^{s,0}(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk e^{ikx} \frac{1}{2\pi Y(k)} |s|^{1-p} \tilde{\psi}(sk).$$

This shows for a single scale the dual wavelet and its translation can be expressed in terms of a single function. This is not necessarily true for the dual wavelet and the scaled quantity.

$$\psi^{s,0}(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk e^{ikx} \frac{1}{2\pi Y(k)} |s|^{1-p} \tilde{\psi}(sk)$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} du e^{iu\frac{x}{s}} \frac{1}{2\pi Y(u/s)} |s|^{-p} \tilde{\psi}(u).$$

This fails to be a rescaling of a single function because of the s dependence in the quantity $Y(u)$. It follows that if *a weight function* $w(s)$ is chosen so $Y(u/s) = Y$ is constant, the dual wavelet will satisfy

$$\psi^{s,0}(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} du e^{iu\frac{x}{s}} \frac{1}{2\pi Y} |s|^{-p} \tilde{\psi}(u) = |s|^{-p} \psi^{1,0}(x/s). \tag{261}$$

In this case $Y(u)$ is a constant which we denote by Y. The function $\psi^{1,0}(x)$ serves as the dual mother wavelet.

To determine $w(s)$ note that

$$Y(sk) = \int_{-\infty}^{\infty} dt w(t)|t|^{1-p}|\tilde{\psi}(tsk)|^2.$$

Let $t' = st$ to get

$$Y(sk) = \int_{-\infty}^{\infty} dt w(t)|t|^{1-p}|\tilde{\psi}(tsk)|^2 =$$

$$|s|^{p-2}\int_{-\infty}^{\infty} dt' w(t'/s)|t|'^{1-p}|\tilde{\psi}(t'k)|^2.$$

This will equal $Y(k)$ provided

$$w(t') = |s|^{p-2}w(t'/s) \qquad \text{or} \qquad w(s) = |s|^{p-2}w(1).$$

With this choice

$$Y(k) = Y = w(1)\int_{-\infty}^{\infty} \frac{dt}{|t|}|\tilde{\psi}(t)|^2.$$

Assuming this choice of weight the admissibility condition becomes

$$0 < A \leq Y \leq B < \infty.$$

Having computed the constant Y it is now possible to write down an explicit expression for the dual mother wavelet:

$$\psi^{s,0}(x-t) =$$

$$\frac{|s|^{-p}}{\sqrt{2\pi}}\int_{-\infty}^{\infty} du e^{iu\frac{(x-t)}{s}}\frac{1}{2\pi Y}\tilde{\psi}(u)$$

Letting $k = u/s$

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dk \frac{1}{2\pi Y}|s|^{1-p}e^{ik(x-t)}\tilde{\psi}(ks)$$

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dk \frac{1}{2\pi Y}e^{ikx}\tilde{\psi}_{s,t}(k).$$

This has the form

$$\psi^{s,t}(x) = \frac{1}{2\pi}\frac{1}{Y}\psi_{s,t}(x). \tag{262}$$

Thus the inversion procedure can be summarized by the formulas:

$$f(x) = \int_{-\infty}^{\infty} |s|^{2p-3} ds \int_{-\infty}^{\infty} dt \psi^{s,t}(x) \hat{f}(s,t) \tag{263}$$

$$Y = \int_{-\infty}^{\infty} \frac{dt}{|t|} |\tilde{\psi}(t)|^2 \tag{264}$$

$$\psi^{s,t}(x) = \frac{\psi_{s,t}(x)}{2\pi Y} \tag{265}$$

$$\psi_{s,t} = |s|^{-p} \psi(\frac{x-t}{s}). \tag{266}$$

The mother function must satisfy $0 < Y < \infty$. This requires that the Fourier transform of the mother function vanishes at the origin. This is equivalent to saying that the integral of the mother function is zero.

Using the representation for the wavelet transform gives a representation of a delta function:

$$\delta(x-y) =$$

$$\int_{-\infty}^{\infty} |s|^{2p-3} ds \int_{-\infty}^{\infty} dt \psi^{s,t}(x) \psi^*_{s,t}(y) =$$

$$\frac{1}{2\pi Y} \int_{-\infty}^{\infty} |s|^{2p-3} ds \int_{-\infty}^{\infty} dt \psi_{s,t}(x) \psi^*_{s,t}(y).$$

We can also use this representation of the delta function to formulate a Parseval's identity for continuous wavelets

$$(f,f) = \frac{1}{2\pi Y} \int_{-\infty}^{\infty} |s|^{2p-3} ds \int_{-\infty}^{\infty} dt |\hat{f}(s,t)|^2. \tag{267}$$

Consider the example of the **Mexican hat** wavelet. The mother function is

$$\psi(x) = \frac{1}{\sqrt{2\pi}}(x^2-1)e^{-x^2/2}.$$

To work with the Mexican hat mother function it is useful to use properties of Gaussian integrals:

$$\int_{-\infty}^{\infty} e^{-ax^2+bx+c}dx =$$

$$\int_{-\infty}^{\infty} e^{-a(x-\frac{b}{2a})^2+\frac{b^2}{4a}+c}dx.$$

Change variables to $y = \sqrt{a}(x - \frac{b}{2a})$ to obtain:

$$\frac{e^{\frac{b^2}{4a}+c}}{\sqrt{a}} \int_{-\infty}^{\infty} e^{-y^2}dy =$$

$$\sqrt{\frac{\pi}{a}} e^{\frac{b^2}{4a}+c}.$$

This can be used to compute the Fourier transform of the Mexican hat mother function:

$$\tilde{\psi}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx}\psi(x)dx =$$

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} (x^2 - 1)e^{-x^2/2-ikx}dx.$$

To do the integral insert a parameter a which will be set to 1 at the end of the calculation:

$$(-2\frac{d}{da} - 1)\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-x^2a/2-ikx}dx =$$

$$(-2\frac{d}{da} - 1)\frac{1}{2\pi}\sqrt{\frac{2\pi}{a}}e^{-\frac{k^2}{2a}} =$$

$$(\frac{1}{a} - \frac{k^2}{a^2} - 1)\sqrt{\frac{1}{2\pi a}}e^{-\frac{k^2}{2a}}.$$

In the limit that $a \to 1$ this becomes

$$-\sqrt{\frac{1}{2\pi}}k^2e^{-\frac{k^2}{2}}.$$

Using this expression it is possible to calculate the coefficient Y

$$Y = \int_{-\infty}^{\infty} \frac{dk}{|k|}|\tilde{\psi}(k)|^2 =$$

$$\int_{-\infty}^{\infty} \frac{dk}{|k|} |\tilde{\psi}(k)|^2 =$$

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} |k|^3 dk e^{-k^2} =$$

$$\frac{1}{\pi} \int_{0}^{\infty} k^3 dk e^{-k^2}.$$

Inserting a parameter a which will eventually be set to 1 gives

$$Y = \frac{1}{2\pi}(-\frac{d}{da}) \int_{0}^{\infty} 2k dk e^{-ak^2} =$$

$$\frac{1}{2\pi}(-\frac{d}{da})\frac{1}{a} \int_{0}^{\infty} dv e^{-v} =$$

$$\frac{1}{2\pi}.$$

This satisfies the essential inequality $0 < Y < \infty$ which ensures the admissibility of the Mexican hat mother function.

The expression for the wavelet transform and its inverse can be written as:

$$\hat{f}(s,t) = |s|^{-p} \int_{-\infty}^{\infty} dx \frac{1}{\sqrt{2\pi}}((\frac{x-t}{s})^2 - 1) e^{-(\frac{x-t}{s})^2/2} f(x) =$$

$$|s|^{1-p} \int_{-\infty}^{\infty} du \frac{1}{\sqrt{2\pi}}(u^2 - 1) e^{-u^2/2} f(su + t).$$

where $x = su + t$

The inverse is formally given by

$$f(x) = \int_{-\infty}^{\infty} |s|^{2p-3} ds \int_{-\infty}^{\infty} dt \frac{\psi_{st}(x)}{2\pi Y} \hat{f}(s,t) =$$

$$\int_{-\infty}^{\infty} |s|^{2p-3} ds \int_{-\infty}^{\infty} dt \frac{1}{\sqrt{2\pi}} |s|^{-p}((\frac{x-t}{s})^2 - 1) e^{-(\frac{x-t}{s})^2/2} \hat{f}(s,t) =$$

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} |s|^{p-3} ds \int_{-\infty}^{\infty} dt((\frac{x-t}{s})^2 - 1) e^{-(\frac{x-t}{s})^2/2} \hat{f}(s,t) =$$

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} |s|^{p-3} ds \int_{-\infty}^{\infty} du(u^2 - 1) e^{-u^2/2} \hat{f}(s, su + x)$$

where $t = su + x$.

Initially we were concerned because we were representing an arbitrary function by a linear superposition of functions that all have zero integral. We could not understand how wavelets could be used to represent a function with non-zero integral.

We tested this by computing the wavelet transform and its inverse for a Gaussian function with the Mexican hat wavelet. The original Gaussian function was recovered.

The resolution of this paradox has to do with the difference between L^1 and L^2 convergence. The wavelet transform has a vanishing L^1 norm, but the L^2 norm is non-zero.

11 Appendix II - Spline Wavelets

We use the convention which defines the Fourier transform of a function $f(x)$ as

$$F(k) = \int_{-\infty}^{\infty} e^{-ikx} f(x)\, dx\,, \tag{268}$$

and the inverse transform by

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(k) e^{ikx}\, dk\,. \tag{269}$$

For this convention, Parseval relation is

$$\int_{-\infty}^{\infty} f^*(x) g(x)\, dx = \frac{1}{2\pi} \int_{-\infty}^{\infty} F^*(k) G(k)\, dk\,. \tag{270}$$

The cardinal B-splines, $N_m(x)$, are defined by first defining

$$N_1(x) = \begin{cases} 0, & \text{if } x < 0 \\ 1, & \text{if } 0 < x < 1 \\ 0, & \text{otherwise.} \end{cases} \tag{271}$$

Then for $m \geq 2$, $N_m(x)$ is defined recursively by the convolution integral

$$\begin{aligned} N_m(x) &= \int_{-\infty}^{\infty} N_{m-1}(x-t) N_1(t)\, dt \\ &= \int_0^1 N_{m-1}(x-t)\, dt\,. \end{aligned} \tag{272}$$

Since $N_m(x)$ is defined by a convolution integral, the Fourier transform will be defined by a product. To show this, we evaluate the Fourier transform

$$\begin{aligned}
\tilde{N}_m(k) &= \int_{-\infty}^{\infty} e^{-ikx} N_m(x)\, dx \\
&= \int_{-\infty}^{\infty} dx\, e^{-ikx} \int_{-\infty}^{\infty} N_{m-1}(x-t) N_1(t)\, dt \\
&= \int_{-\infty}^{\infty} dt\, e^{-ikt} N_1(t) \int_{-\infty}^{\infty} e^{-ik(x-t)} N_{m-1}(x-t)\, dx\,. \qquad (273)
\end{aligned}$$

Now setting $x - t = y$ in the second integral, we find

$$\begin{aligned}
\tilde{N}_m(k) &= \int_{-\infty}^{\infty} dt\, e^{-ikt} N_1(t) \int_{-\infty}^{\infty} e^{-iky} N_{m-1}(y)\, dy \\
&= \tilde{N}_{m-1}(k) \int_{-\infty}^{\infty} e^{-ikt} N_1(t)\, dt \\
&= \tilde{N}_{m-1}(k) \tilde{N}_1(k) \\
&= \left[\tilde{N}_1(k)\right]^m\,, \qquad (274)
\end{aligned}$$

where

$$\begin{aligned}
\tilde{N}_1(k) &= \int_{-\infty}^{\infty} e^{-ikx} N_1(x)\, dx \\
&= \int_0^1 e^{-ikx}\, dx \\
&= e^{-ik/2} \frac{\sin(k/2)}{k/2}\,. \qquad (275)
\end{aligned}$$

For this example we use the quadratic spline shifted to the left by one unit

$$\begin{aligned}
\tilde{B}(k) &= e^{ik} \tilde{N}_3(k) \\
&= e^{-ik/2} \left[\frac{\sin(k/2)}{k/2}\right]^3\,. \qquad (276)
\end{aligned}$$

Now evaluating the inverse transform using Maple, we get

$$B(x) = \begin{cases} 0, & \text{if } x < -1 \\ \frac{1}{2}(x+1)^2, & \text{if } -1 \le x < 0 \\ \frac{3}{4} - (x - \frac{1}{2})^2, & \text{if } 0 \le x < 1 \\ \frac{1}{2}(x-2)^2, & \text{if } 1 \le x < 2 \\ 0, & \text{otherwise.} \end{cases} \tag{277}$$

The splines are not orthogonal; however, we can use them to construct a scaling function $\phi(x)$ which has the orthonormality property

$$\int_{-\infty}^{\infty} \phi^*(x-l)\phi(x-m)\,dx = \delta_{lm}\,. \tag{278}$$

To do this we follow the procedure given in the books by Chui[1] and Daubechies[2]. Note that this a general procedure; we are using the spline as a convenient example. The method gives an expression for the Fourier transform, $\tilde{\phi}(k)$, of $\phi(x)$. The Fourier transform of $\phi(x-l)$ is given by

$$\begin{aligned} \tilde{\phi}_l(k) &= \int_{-\infty}^{\infty} e^{-ikx}\phi(x-l)\,dx \\ &= e^{-ikl}\int_{-\infty}^{\infty} e^{-ik(x-l)}\phi(x-l)\,dx \\ &= e^{-ikl}\tilde{\phi}(k)\,. \end{aligned} \tag{279}$$

Now we show that if

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikm}\left|\tilde{\phi}(k)\right|^2\,dk = \delta_{m,0}\,, \tag{280}$$

then the functions are orthogonal. To show this, we use the Parseval relation

$$\begin{aligned} \int_{-\infty}^{\infty} \phi^*(x-l)\phi(x-m)\,dx &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{\phi_l}^*(k)\tilde{\phi_m}(k)\,dk \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikl}\tilde{\phi}^*(k)e^{-ikm}\tilde{\phi}(k)\,dk \end{aligned}$$

[1]Charles K. Chui, *An Introduction to Wavelets*, Academic Press, 1992
[2]Ingrid Daubechies, *Ten Lectures on Wavelets*, SIAM, 1992

$$
\begin{aligned}
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ik(l-m)}\left|\tilde{\phi}(k)\right|^2 dk \\
&= \delta_{l-m,0} \\
&= \delta_{lm}\,. \qquad (281)
\end{aligned}
$$

Finally, we show that if we can find a $\tilde{\phi}(k)$ such that

$$
\sum_{l=-\infty}^{\infty}\left|\tilde{\phi}(k+2\pi l)\right|^2 = 1\,, \qquad (282)
$$

then the functions are orthonormal. The infinite sum in Equation (282) is periodic in k with a period of 2π; thus it has the Fourier series expansion

$$
\sum_{l=-\infty}^{\infty}\left|\tilde{\phi}(k+2\pi l)\right|^2 = \sum_{j=-\infty}^{\infty} c_j e^{ikj} \qquad (283)
$$

where the expansion coefficients are give by

$$
\begin{aligned}
c_j &= \frac{1}{2\pi}\int_0^{2\pi} e^{-ijk}\sum_{l=-\infty}^{\infty}\left|\tilde{\phi}(k+2\pi l)\right|^2 dk \\
&= \frac{1}{2\pi}\sum_{l=-\infty}^{\infty}\int_0^{2\pi} e^{-ikj}\left|\tilde{\phi}(k+2\pi l)\right|^2 dk \\
&= \frac{1}{2\pi}\sum_{l=-\infty}^{\infty}\int_{2\pi l}^{2\pi(l+1)} e^{-i(k-2\pi l)j}\left|\tilde{\phi}(k)\right|^2 dk \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ikj}\left|\tilde{\phi}(k)\right|^2 dk\,. \qquad (284)
\end{aligned}
$$

Since the sum in Equation (282) is equal to one, $c_j = \delta_{j,0}$, and one finds

$$
\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ikj}\left|\tilde{\phi}(k)\right|^2 dk = \delta_{j,0}\,. \qquad (285)
$$

Thus, the functions are orthonormal.

Now given a function, $B(x)$, we construct a scaling function by taking its Fourier transform and defining

$$
\tilde{\phi}(k) = \frac{\tilde{B}(k)}{\left[\sum_{l=-\infty}^{\infty}\left|\tilde{B}(k+2\pi l)\right|^2\right]^{\frac{1}{2}}}\,. \qquad (286)
$$

This function satisfies Equation (282), and the $\phi(x)$ will have the orthonormality property given in Equation (278). To evaluate the infinite sum in Equation (286), we use the finite Fourier series expansion of the function

$$g(k) = \sum_{l=-\infty}^{\infty} \left| \tilde{B}(k + 2\pi l) \right|^2 . \tag{287}$$

This function has period 2π, and the Fourier expansion has the form

$$g(k) = \sum_{j=-\infty}^{\infty} c_j e^{ijk} . \tag{288}$$

Following the derivation in Equation (284), the expansion coefficients are given by

$$\begin{aligned} c_j &= \frac{1}{2\pi} \int_0^{2\pi} e^{-ijk} g(k)\, dk \\ &= \frac{1}{2\pi} \int_0^{2\pi} e^{-ijk} \sum_{l=-\infty}^{\infty} \left| \tilde{B}(k + 2\pi l) \right|^2 dk \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ijk} \left| \tilde{B}(k) \right|^2 dk \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{B}^*(k) e^{-ijk} \tilde{B}(k)\, dk \\ &= \int_{-\infty}^{\infty} B^*(x) B(x - j)\, dx , \end{aligned} \tag{289}$$

where the Parseval relation was used for the last step. The integral in Equation (289) is easy to evaluate for the B-splines, and we find

$$\int_{-\infty}^{\infty} B^*(x) B(x - j)\, dx = \begin{cases} \frac{1}{120}, & \text{if } j = -2 \\ \frac{13}{60}, & \text{if } j = -1 \\ \frac{11}{20}, & \text{if } j = 0 \\ \frac{13}{60}, & \text{if } j = 1 \\ \frac{1}{120}, & \text{if } j = 2 \\ 0, & \text{otherwise.} \end{cases} \tag{290}$$

Using these coefficients for the expansion given in Equation (288), we find

$$g(k) = \frac{11}{20} + \frac{13}{30}\cos(k) + \frac{1}{60}\cos(2k)\,. \tag{291}$$

To find $\phi(x)$ the inverse Fourier transform of $\tilde{\phi}(k)$ must be done numerically; however, there is a nice method which gives an efficient algorithm. Since the function $g(k)$ has period 2π, we can use the expansion

$$\frac{1}{\sqrt{g(k)}} = \sum_{n=-\infty}^{\infty} c_n e^{-ink}\,, \tag{292}$$

where the coefficients

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{e^{ink}}{\sqrt{g(k)}}\,dk \tag{293}$$

must be computed numerically. For the B-spline, the $g(k)$ is an even function of k, and one finds

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{\cos(nk)}{\sqrt{g(k)}}\,dk\,. \tag{294}$$

In addition, from Equation (294) we see that $c_{-n} = c_n$. Now, using Equation (292) we get

$$\begin{aligned}
\phi(x) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\tilde{B}(k)}{\sqrt{g(k)}}\,e^{ikx}\,dk \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{B}(k)\left[\sum_{n=-\infty}^{\infty} c_n e^{-ink}\right] e^{ikx}\,dk \\
&= \sum_{n=-\infty}^{\infty} c_n \left[\frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{B}(k) e^{ik(x-n)}\,dk\right] \\
&= \sum_{n=-\infty}^{\infty} c_n B(x-n)\,.
\end{aligned} \tag{295}$$

Now we need to find the wavelet $\psi(x)$ with the properties

$$\int_{-\infty}^{\infty} \psi^*(x-l)\phi(x-m)\,dx = 0\,, \tag{296}$$

and

$$\int_{-\infty}^{\infty} \psi^*(x-l)\psi(x-m)\,dx = \delta_{lm}\,. \tag{297}$$

To do this we introduce the functions

$$\phi_{-1,l}(x) = \sqrt{2}\phi(2x-l)\,. \tag{298}$$

The Fourier transform of these functions is given by

$$\begin{aligned} \tilde{\phi}_{-1,l}(k) &= \int_{-\infty}^{\infty} e^{-ikx}\phi_{-1,l}(x)\,dx \\ &= \sqrt{2}\int_{-\infty}^{\infty} e^{-ikx}\phi(2x-l)\,dx \end{aligned} \tag{299}$$

and setting $2x - l = y$ yields

$$\begin{aligned} \tilde{\phi}_{-1,l}(k) &= \frac{1}{\sqrt{2}}e^{-ikl/2}\int_{-\infty}^{\infty} e^{-iky/2}\phi(y)\,dy \\ &= \frac{1}{\sqrt{2}}e^{-ikl/2}\tilde{\phi}(k/2)\,. \end{aligned} \tag{300}$$

Using the $\phi_{-1,n}(x)$ as an orthonormal basis set, we can write

$$\phi(x) = \sum_{n=-\infty}^{\infty} h_n\phi_{-1,n}(x)\,, \tag{301}$$

where

$$h_n = \int_{-\infty}^{\infty} \phi^*_{-1,,n}(x)\phi(x)\,dx\,.. \tag{302}$$

This is the scaling equation for this system. In this case there are an infinite number of non-zero scaling coefficients. Since the $\phi(x-n)$ are orthonormal, the h_n must have the property

$$\sum_{n=-\infty}^{\infty} |h_n|^2 = 1\,. \tag{303}$$

Taking the Fourier transform of Equation (301) gives

$$\tilde{\phi}(k) = \frac{1}{\sqrt{2}}\sum_{n=-\infty}^{\infty} h_n e^{-ikn/2}\tilde{\phi}(k/2)\,, \tag{304}$$

which can be written as

$$\tilde{\phi}(k) = m_0(k/2)\tilde{\phi}(k/2)\,, \tag{305}$$

where

$$m_0(k) = \frac{1}{\sqrt{2}} \sum_{n=-\infty}^{\infty} h_n e^{-ikn}\,. \tag{306}$$

Using Equations (282) and (305) we see that

$$\begin{aligned}
\sum_{l=-\infty}^{\infty} \left|\tilde{\phi}(2k+2\pi l)\right|^2 &= \sum_{l=-\infty}^{\infty} \left|m_0(k+\pi l)\right|^2 \left|\tilde{\phi}(k+\pi l)\right|^2 \\
&= \sum_{l,even} \left|m_0(k+\pi l)\right|^2 \left|\tilde{\phi}(k+\pi l)\right|^2 \\
&\qquad + \sum_{l,odd} \left|m_0(k+\pi l)\right|^2 \left|\tilde{\phi}(k+\pi l)\right|^2 \\
&= \sum_{l=-\infty}^{\infty} \left|m_0(k+2\pi l)\right|^2 \left|\tilde{\phi}(k+2\pi l)\right|^2 \\
&\qquad + \sum_{l=-\infty}^{\infty} \left|m_0(k+2\pi l+\pi)\right|^2 \left|\tilde{\phi}(k+2\pi l+\pi)\right|^2 \\
&= |m_0(k)|^2 \sum_{l=-\infty}^{\infty} \left|\tilde{\phi}(k+2\pi l)\right|^2 \\
&\qquad + |m_0(k+\pi)|^2 \sum_{l=-\infty}^{\infty} \left|\tilde{\phi}(k+\pi+2\pi l)\right|^2 \\
&= |m_0(k)|^2 + |m_0(k+\pi)|^2
\end{aligned} \tag{307}$$

where we have used the periodicity of $m_0(k)$. The sum on the left-hand side of Equation (307) is equal to unity; thus, we have shown that

$$|m_0(k)|^2 + |m_0(k+\pi)|^2 = 1 \tag{308}$$

Now we use a similar procedure to find $\psi(x)$. Using the $\phi_{-1,n}(x)$ as an orthonormal basis set, we write

$$\psi(x) = \sum_{n=-\infty}^{\infty} f_n \phi_{-1,n}(x)\,, \tag{309}$$

with

$$f_n = \int_{-\infty}^{\infty} \phi^*_{-1,n}(x)\psi(x)\,dx\,. \tag{310}$$

Taking the Fourier transform of Equation (309) gives

$$\begin{aligned}\tilde{\psi}(k) &= \frac{1}{\sqrt{2}}\sum_{n=-\infty}^{\infty} f_n e^{-ikn/2}\tilde{\phi}(k/2) \\ &= m_1(k/2)\tilde{\phi}(k/2)\,, \end{aligned} \tag{311}$$

where

$$m_1(k) = \frac{1}{\sqrt{2}}\sum_{n=-\infty}^{\infty} f_n e^{-ikn}\,. \tag{312}$$

If $m_1(k)$ has the same property as that given in Equation (308) for $m_0(k)$, then the functions $\psi(x-m)$ will be orthonormal. In addition, we want $\psi(x-n)$ to be orthogonal to $\phi(x-m)$. Thus, we want to find a $m_1(k)$ such that

$$\begin{aligned}\int_{-\infty}^{\infty} \psi^*(x-n)\phi(x-m) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikn}\tilde{\psi}^*(k)e^{-ikm}\tilde{\phi}(k)\,dk \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i(n-m)k}\tilde{\psi}^*(k)\tilde{\phi}(k)\,dk \\ &= \frac{1}{2\pi}\int_{0}^{2\pi} dk\, e^{i(n-m)k}\sum_{l=-\infty}^{\infty}\tilde{\psi}^*(k+2\pi l)\tilde{\phi}(k+2\pi l) \\ &= 0. \end{aligned} \tag{313}$$

This condition is satisfied if

$$\sum_{l=-\infty}^{\infty}\tilde{\psi}^*(k+2\pi l)\tilde{\phi}(k+2\pi l) = 0\,. \tag{314}$$

Substituting Equations (305) and (311) into Equation (314) and replacing k by $2k$ gives

$$\sum_{l=-\infty}^{\infty} m_1^*(k)\tilde{\phi}^*(k+\pi l)m_0(k)\tilde{\phi}(k+\pi l) = 0\,. \tag{315}$$

Regrouping the sums for odd and even l, and following the procedure used in Equation (307) gives

$$m_1^*(k)m_0(k) + m_1^*(k+\pi)m_0(k+\pi) = 0\,. \tag{316}$$

This condition will be satisfied if we choose

$$m_1(k) = e^{-ik} m_0^*(k+\pi)\,. \tag{317}$$

Note this choice for $m_1(k)$ is not unique; we can multiply $m_1(k)$ by any function $\rho(k)$ which has period π and $|\rho(k)| = 1$, and still satisfy the constraints on $m_1(k)$. Substituting this result into Equation (311) gives

$$\begin{aligned}
\tilde{\psi}(k) &= e^{-ik/2} m_0^*(k/2+\pi)\tilde{\phi}(k/2) \\
&= e^{-ik/2}\frac{1}{\sqrt{2}}\sum_{n=-\infty}^{\infty} h_n^* e^{i(k/2+\pi)n}\tilde{\phi}(k/2) \\
&= \frac{1}{\sqrt{2}}\sum_{n=-\infty}^{\infty} (-1)^n h_n^* e^{-i(-n+1)k/2}\tilde{\phi}(k/2) \\
&= \sum_{n=-\infty}^{\infty} (-1)^n h_n^* \tilde{\phi}_{-1,-n+1}(k)\,.
\end{aligned} \tag{318}$$

Replacing n by $-n+1$ in Equation (318) gives

$$\tilde{\psi}(k) = -\sum_{n=-\infty}^{\infty} (-1)^n h_{-n+1}^* \tilde{\phi}_{-1,n}(k)\,. \tag{319}$$

For convenience, we drop the minus sign in front of the sum, and write

$$\tilde{\psi}(k) = \sum_{n=-\infty}^{\infty} g_n \tilde{\phi}_{-1,n}(k)\,, \tag{320}$$

where $g_n = (-1)^n h_{-n+1}$ for h_n a real number. Taking the Fourier transform of Equation (320) gives the result

$$\begin{aligned}
\psi(x) &= \sum_{n=-\infty}^{\infty} g_n \phi_{-1,n}(x) \\
&= \sqrt{2}\sum_{n=-\infty}^{\infty} g_n \phi(2x-n)\,.
\end{aligned} \tag{321}$$

To evaluate $\psi(x)$ we use the expansion given in Equation (295) in Equation (321). This gives

$$\psi(x) = \sqrt{2}\sum_{n=-\infty}^{\infty} g_n \sum_{m=-\infty}^{\infty} c_m B(2x-n-m)\,. \tag{322}$$

Now replace m by $l-n$, this gives

$$\begin{aligned}\psi(x) &= \sum_{l=-\infty}^{\infty}\left[\sqrt{2}\sum_{n=-\infty}^{\infty} g_n c_{l-n}\right] B(2x-l) \\ &= \sum_{l=-\infty}^{\infty} d_l B(2x-l)\,. \end{aligned} \tag{323}$$

Numerical Methods

To determine the h_n we need to evaluate the overlap integral given in Equation (302). From Equations (298) and (295), we find

$$\phi_{-1,n}(x) = \sqrt{2}\sum_{m=-\infty}^{\infty} c_m B(2x-m-n)\,. \tag{324}$$

Then using

$$B(x) = \sum_{j=-\infty}^{\infty} b_j B(2x-j)\,, \tag{325}$$

where, the b_j for the quadratic B-splines are given by

$$b_j = \begin{cases} \frac{1}{4}, & \text{if } j=-1 \\ \frac{3}{4}, & \text{if } j=0 \\ \frac{3}{4}, & \text{if } j=1 \\ \frac{1}{4}, & \text{if } j=2 \\ 0, & \text{otherwise.} \end{cases}\,, \tag{326}$$

we can write

$$\begin{aligned}\phi(x) &= \sum_{n=-\infty}^{\infty} c_n \sum_{j=-\infty}^{\infty} b_j B(2x-2n-j) \\ &= \sum_{l=-\infty}^{\infty}\left[\sum_{n=-\infty}^{\infty} c_n b_{l-2n}\right] B(2x-l) \\ &= \sum_{l=-\infty}^{\infty} s_l B(2x-l)\,. \end{aligned} \tag{327}$$

Using these expansions, the overlap integral is given by

$$\begin{aligned} h_n &= \sqrt{2} \sum_{m=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} c_m s_l \int_{-\infty}^{\infty} B(2x-m-n)B(2x-l)\,dx \\ &= \sqrt{2} \sum_{m=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} c_m s_l \frac{1}{2} \int_{-\infty}^{\infty} B(x-m-n+l)B(x)\,dx \\ &= \frac{1}{\sqrt{2}} \sum_{m=-\infty}^{\infty} \sum_{j=-2}^{2} c_m s_{m+n-j} \int_{-\infty}^{\infty} B(x)B(x-j)\,dx\,, \end{aligned} \tag{328}$$

where, we have set $l = m + n - j$ in the second summation. The values for the integrals of the quadratic B-splines are given in Equation (289).

To derive Equation (325), we use $\sin(2\theta) = 2\cos(\theta)\sin(\theta)$ to write Equation (276) as

$$\begin{aligned} \tilde{B}(k) &= e^{-ik/2} \left[\frac{\cos(k/4)\sin(k/4)}{k/4} \right]^3 \\ &= e^{-ik/4} \left(\frac{e^{ik/4} + e^{-ik/4}}{2} \right)^3 e^{-ik/2} \left[\frac{\sin(k/4)}{k/4} \right]^3 \\ &= e^{-ik/4} \left(\frac{e^{ik/4} + e^{-ik/4}}{2} \right)^3 \tilde{B}(k/2) \\ &= \left(\frac{e^{ik/2} + 3 + 3\,e^{-ik/2} + e^{-ik}}{8} \right) \tilde{B}(k/2)\,. \end{aligned} \tag{329}$$

Now taking the Fourier transform of (329) and using

$$\begin{aligned} \int_{-\infty}^{\infty} e^{-ikx} B(2x-j)\,dx &= \frac{1}{2}\, e^{ikj/2} \int_{-\infty}^{\infty} e^{-ikx/2} B(x)\,dx \\ &= \frac{1}{2}\, e^{ikj/2} B(k/2)\,, \end{aligned} \tag{330}$$

we find

$$B(x) = \frac{1}{4}B(2x-1) + \frac{3}{4}B(2x) + \frac{3}{4}B(2x+1) + \frac{1}{4}B(2x-2)\,. \tag{331}$$

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