Reflection positivity in Euclidean formulations of relativistic quantum mechanics of particles.

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This paper discusses the general structure of reflection positive Euclidean covariant distributions that can be used to construct Euclidean representations of relativistic quantum mechanical models of systems of a finite number of degrees of freedom. Because quantum systems of a finite number of degrees of freedom are not local, reflection positivity is not as restrictive as it is in a local field theory. The motivation for the Euclidean approach is that it is straightforward to construct exactly Poincaré invariant quantum models of finite number of degrees of freedom systems that satisfy cluster properties and a spectral condition. In addition the quantum mechanical inner product can be computed without requiring an analytic continuation. Whether these distributions can be generated by a dynamical principle remains to be determined, but understanding the general structure of the Euclidean covariant distributions is an important first step.

I. INTRODUCTION

The purpose of this work is to explore a Euclidean formulation of relativistic quantum mechanics [1] [2],[3] [4] [5]. From a purely academic perspective this Euclidean formulation yields a Hilbert space structure with a representation of the Poincaré Lie algebra by self-adjoint operators. The Poincaré generators satisfy cluster properties and the Hamiltonian is bounded from below. These elements are sufficient to formulate any kind of quantum mechanical calculation. The interesting feature of the Euclidean representation is that it is not necessary to perform an analytic continuation to perform these calculations. The dynamical input to this representation of relativistic quantum mechanics is a collection of Euclidean covariant distributions that satisfy a reflection positivity condition. Reflection positivity is used to construct a Hilbert space representation with a non-trivial kernel. The reflection positivity requirement in the quantum mechanical case is weaker than the corresponding condition in quantum field theory. The purpose of this work is to understand the general structure of these distributions. What is still missing is a dynamical framework to generate these model distributions.

The motivation for this exploration is discussed below. Understanding the internal structure of elementary hadrons is a goal of nuclear and particle physics. Hadronic structure can be studied by scattering weak or electromagnetic probes off of strongly interacting hadronic systems. In a typical reaction the probe scatters off of an initial hadronic state and causes a transition to a final hadronic state. These reactions are complex since the target and strongly interacting reaction products are composite systems, and particle number is not generally conserved. In addition, because the probe can transfer momentum to the initial hadronic state, the final hadronic state(s) is in a moving frame relative to the initial state. Relativistic momentum transfers are needed for a resolution that is sensitive to the internal structure of hadrons. QCD is the appropriate theory to model the hadronic states. It is a challenging problem to use QCD to solve for these strongly interacting Poincaré covariant states. Most non-perturbative calculations utilize truncations, which are not mathematically controlled approximations. Lattice calculations are directly based on QCD, but they break Poincaré symmetry and the treatment of scattering problems with composite initial and final states is challenging, particularly in the Euclidean formulation.

While there are many difficulties associated with solving quantum field theories, one of them is due to the local nature of quantum field theories, which cannot be satisfied in theories of a finite number of degrees of freedom. The expectation is that for a given momentum transfer the dynamics will be approximately governed by a relativistic model of a finite number of relevant degrees of freedom, limited by the available energy and interaction volume.

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Axiomatic formulations of quantum field theory [6][7] [8][9] provide a mathematical formulation of physical properties that are expected to hold in any reasonable formulation of relativistic quantum field theory. The axioms are internally consistent since they are satisfied by free field theories. The challenge is the absence of examples of non-trivial quantum theories that satisfy the axioms in 3+1 dimensional spacetime. The property responsible for most of the difficulties is the locality axiom. While direct tests of locality on arbitrarily small distances are not possible, consequences of locality such as the PCT theorem, crossing symmetry and the need for antiparticles make it a compelling constraint.

One approach that avoids some of these difficulties is to use relativistic quantum mechanical models of a finite number of degrees of freedom to model the initial and final hadronic states [10] [11] [12] [13] [14] [15] [16] [17] [18] [19] [20] [21] [22] [23] [24] [25] [26] [27] [28] [29] [30] [31] [32] [33] [34] [35] [36] [37] [38] [39] [40] [41] [42]. These phenomenological models can be constructed so they satisfy most of the axioms of a quantum field theory, however they are not local, since they are formulated in terms of a finite number of degrees of freedom. The advantage is that model states can be calculated using the same methods that are used in non-relativistic calculations and the relativistic invariance is exact. While relativistic models have been used successfully, their relation to QCD or a more fundamental local field theory is not direct.

While relativistic quantum models do not satisfy microscopic locality, in order to be useful they need to satisfy cluster properties (macroscopic locality). This is needed to justify tests of special relativity on isolated subsystems and more importantly to provide a connection between few and many-body systems. While macroscopic locality can be realized in relativistic models, the construction of dynamical Poincaré generators satisfying macroscopic locality is based on a recursive construction that uses chains of unitary transformations that map tensor products of subsystem unitary representations of the Poincaré group and transforms them to a form where the interactions can be added in a manner that preserves the Poincaré invariance. This is followed by another overall unitary transformation that restores cluster properties [43][13] [14][21]. This is a complicated construction that has never been used in applications. Allowing for production reactions in these models presents an additional set of problems.

While relativistic quantum models can provide a formal construction of a relativistically invariant quantum theory satisfying cluster properties and a spectral condition, it is not a practical solution for more than two or three-body systems due to the complexity and non-uniqueness of the construction. In addition there is no direct relation to an underlying local relativistic quantum field theory that can be used to systematically improve the models. The complexity of this construction, due to requiring macroscopic locality, and the absence of a direct connection to a local field theory provides motivation for exploring alternative non-local formulations of relativistic quantum mechanics where cluster properties can be established in a straightforward manner.

This work contributes to an approach based on a Euclidean formulation of Hamiltonian dynamics [1] [2],[3] [4]. In the axioms of Euclidean quantum field theory [8] [9] the locality axiom is logically independent of the other axioms. This means that it should be possible to formulate relativistic quantum mechanical models satisfying all of the other Euclidean axioms. The theory is expressed in terms of a collection of Euclidean covariant distributions that satisfy a condition called reflection positivity [8] [9][44]. The motivation for considering this approach is that cluster properties are easy to satisfy. In addition, because of the reflection positivity, physical Hilbert space inner products can be computed without analytic continuation. The consequence of not requiring locality is that there is no required relation between N-point functions with different numbers of initial and final coordinates. This makes the positivity of the Hilbert space norm into a more manageable problem. It has the additional advantage that the N-point distributions of a local quantum theory satisfy all of the required conditions, so there is a clear relation to the Schwinger functions of a local field theory.

While Wightman functions of a local quantum field theory [6] also have some of the same advantages, they are not as closely tied to the dynamics. Euclidean N-point functions can be approximated in lattice calculations and they are solutions of the infinite hierarchy of Euclidean Schwinger-Dyson equations. Scattering is normally formulated in terms of time-ordered Green functions, which can be obtained from the Euclidean N-point functions by analytic continuation [45]. The relation to time-ordered distributions focuses on the scattering matrix rather than on the Hamiltonian as a Hilbert space operator [46][47] [48][49]. An alternative analytic continuation gives the Wightman distributions, which are kernels of the physical Hilbert space inner product. When the Euclidean N-point distributions are reflection positive, the quantum mechanical inner product on a dense set of states can be constructed from the Euclidean distributions without analytic continuation. There is a representation of the Poincaré Lie algebra with self-adjoint operators on this Hilbert space satisfying both a spectral condition and cluster properties. These conditions also hold in the non-local case. It is even possible to directly formulate time-dependent scattering calculations in this representation of the Hilbert space without analytic continuation [1] [2][3] [4].

While analytic continuation is needed to construct the Wightman distributions from the Euclidean distributions, it is not needed to formulate quantum mechanical calculations. The elements needed for a relativistic quantum mechanical model are a representation of the physical Hilbert space inner product, a self-adjoint representation of the Poincaré Lie algebra satisfying cluster properties on this space, and a Hamiltonian satisfying spectral condition.

Reflection positivity allows these properties to be satisfied without explicit analytic continuation.

Reflection positivity ensures both the positivity of the physical Hilbert space norm and that the spectrum of Hamiltonian is bounded from below [44]. The resulting spectral condition implies the existence of the analytic condition, however analytic continuation is not needed to calculate physical Hilbert space inner products. In the Euclidean axioms of a local field theory the N-point Schwinger functions are completely symmetric (anti-symmetric). They serve as the kernel of a quadratic form related to the Hilbert space inner product. Because of the symmetry one N-point distribution appears in the kernel of the Euclidean distributions with M initial and K final variables for any $M + K = N$ [8] [9] [44]. By relaxing the locality condition a single N-point distribution can be replaced by $N-1$ N-point distributions with M initial points and $N-M$ final points. Because these are no longer required to be related by locality, the reflection positivity condition is less restrictive.

Having a Hilbert space representation and a set of self-adjoint generators of the Poincaré group satisfying cluster properties on this space are all that is needed to perform calculations. Cluster properties can be used to formulate scattering asymptotic conditions [50][51] [52][4]. In the Euclidean case the dynamics is encoded in the collection of Euclidean covariant distributions which will be referred to as quasi-Schwinger functions. While some aspects of this approach have been previously discussed [1] [2][3] [4] [5], the purpose of this paper is to understand the general structure of the reflection positive $M + K = N$ point distributions which provide the dynamical content of this formulation of non-local relativistic quantum mechanics. This needs to be understood in order to address the problem of formulating realistic models.

While the purpose of this work is to demonstrate that it is possible to construct a large class of non-trivial examples of sets of Euclidean covariant distributions that are consistent with all of the desired properties, what is still missing is a dynamical principle that can be used to determine reflection positive quasi-Schwinger functions from some simple input. The fact that Schwinger functions can be approximated using Euclidean lattice theories or Schwinger-Dyson equations suggests that this is possible, however this is beyond the scope of this paper, and will be investigated in future work.

II. NOTATION

The following notation will be used. Euclidean four vectors are denoted with a subscript e:

$$x_e = (x_e^0, x_e^1, x_e^2, x_e^3). \tag{1}$$

Minkowski 4 vectors are denoted with a subscript m:

$$x_m = (x_m^0, x_m^1, x_m^2, x_m^3). \tag{2}$$

The signature of the Minkowski metric is $(-+++)$. The following matrices:

$$\sigma_{m\mu} = \left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right) \tag{3}$$

$$\sigma_{e\mu} = \left(\begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right) \tag{4}$$

are used to construct the 2×2 matrix representations of Euclidean or Minkowski four vectors:

$$X_e = \sum_{\mu=0}^{3} x_e^\mu \sigma_{e\mu} = \begin{pmatrix} ix_e^0 + x_e^3 & x_e^1 - ix_e^2 \\ x_e^1 + ix_e^2 & ix_e^0 - x_3^3 \end{pmatrix} \qquad x_e^\mu = \frac{1}{2}\mathrm{Tr}(\sigma_{e\mu}^\dagger X_e) \tag{5}$$

$$X_m = \sum_{\mu=0}^{3} x_m^\mu \sigma_{m\mu} = \begin{pmatrix} x_m^0 + x_m^3 & x_m^1 - ix_m^2 \\ x_m^1 + ix_m^2 & x_m^0 - x_m^3 \end{pmatrix} \qquad x_m^\mu = \frac{1}{2}\mathrm{Tr}(\sigma_{m\mu} X_). \tag{6}$$

The Euclidean time-reflection operator, θ, which changes the sign of the Euclidean time, is defined by

$$\theta x_e = (-x_e^0, x_e^1, x_e^2, x_e^3). \tag{7}$$

The Euclidean or Minkowski parity operator Π is defined by

$$\Pi x_e = (x_e^0, -x_e^1, -x_e^2, -x_e^3) \qquad \Pi x_m = (x_m^0, -x_m^1, -x_m^2, -x_m^3). \tag{8}$$

III. WIDDER'S THEOREM

The goal of this work is to understand the general structure of reflection positive distributions. The simplest prototype of a reflection positive kernel is a $k(t)$ satisfying

$$\int_0^\infty dt dt' f(t) k(t+t') f(t') \geq 0 \tag{9}$$

for continuous functions $f(t)$ with compact support on $(0, \infty)$.

The general form of continuous $k(t)$ satisfying (9) is (see of [53])

$$k(t) = \int \rho(\lambda) d\lambda e^{-t\lambda} \tag{10}$$

where $\rho(\lambda)$ is non-decreasing and the integral converges for $0 \leq t \leq 2T$ where support$(f(t)) \in [0, T]$.

For $t > 0$ this can be written as

$$k(t) = \int \tilde{\rho}(\lambda) \frac{e^{-itp}}{\lambda^2 + p^2} dp d\lambda \tag{11}$$

where $\tilde{\rho}(\lambda) = \rho(\lambda) \frac{\lambda}{\pi}$ and the restriction on the support of $f(t)$ means that the p integral can be computed using the residue theorem. The relevant observation is that (11) has a structure similar to a Lehmann representation. In what follows it is shown that all positive mass, positive energy unitary irreducible representations of the Poincaré group can be constructed using a generalization of this construction. Since any unitary representation of the Poincaré group can be decomposed into direct integrals of irreducible representations, this provides the general structure of reflection positive distributions.

IV. RELATIVISTIC COVARIANCE: $SL(2,\mathbb{C})$ AND $SU(2) \times SU(2)$

The relation between the Lorentz and orthogonal groups in four dimensions plays a central role in the connection between the Euclidean and Minkowski representation of relativistic quantum mechanics.

The covering groups of the complex Lorentz and the four dimensional complex orthogonal groups are isomorphic. Because of this there are formal expressions for the Poincaré generators as linear combinations of the generators of the Euclidean group. These operators satisfy the commutation relations of the Poincaré Lie algebra, but they are not self-adjoint on the Hilbert space where the Euclidean generators are self-adjoint. Reflection positivity leads to a Hilbert space inner product where the Poincaré generators constructed from the Euclidean generators are represented by self-adjoint operators.

The purpose of this section is to review the relation between the complex Lorentz and complex orthogonal groups. The starting point is to note that Euclidean and Minkowski 4-vectors can be represented by 2×2 matrices. Minkowski 4-vectors x_m can be represented by Hermitian matrices (5) while the Euclidean 4 vectors x_e can be represented by matrices of the form (6).

The advantage of these matrix representations is that the determinants of these matrices are related to the Minkowski and Euclidean invariant line elements respectively:

$$\det(X_m) = (x_m^0)^2 - \mathbf{x}_m \cdot \mathbf{x}_m \qquad \det(X_e) = -\left((x_e^0)^2 + \mathbf{x}_e \cdot \mathbf{x}_e\right). \tag{12}$$

In both cases the determinants are preserved under linear transformations of the form

$$X \to X' = A X B^t \tag{13}$$

where A and B are both $SL(2,\mathbb{C})$ matrices and X can be X_m or X_e. In both cases the relevant symmetry group is $SL(2,\mathbb{C}) \times SL(2,\mathbb{C})$. For arbitrary elements $(A, B) \in SL(2,\mathbb{C}) \times SL(2,\mathbb{C})$ the transformation (13) defines a complex Lorentz or complex orthogonal transformation that preserves the real invariant line elements. The corresponding complex 4×4 Lorentz and orthogonal matrices are

$$\Lambda(A,B)^\alpha{}_\beta = \frac{1}{2} \mathrm{Tr}(\sigma_\alpha A \sigma_\beta B^t) \tag{14}$$

$$O(A,B)^\alpha{}_\beta = \frac{1}{2} \mathrm{Tr}(\sigma_{e\alpha}^\dagger A \sigma_{e\beta} B^t). \tag{15}$$

For real Lorentz transformations $B = A^*$ and for real orthogonal transformations A and B are independent $SU(2)$ matrices. When the left sides of (14) and (15) are real, the 4×4 transformations can also be expressed by taking complex conjugates

$$\Lambda(A,B)^{\alpha}{}_{\beta} = \frac{1}{2}\mathrm{Tr}(\sigma_{\alpha}^* A^* \sigma_{\beta}^* B^{\dagger}) \tag{16}$$

$$O(A,B)^{\alpha}{}_{\beta} = \frac{1}{2}\mathrm{Tr}(\sigma_{e\alpha}^t A^* \sigma_{e\beta}^* B^{\dagger}). \tag{17}$$

There are two kinds of spinors associated with $SL(2,\mathbb{C}) \times SL(2,\mathbb{C})$ that are distinguished by their $SL(2,\mathbb{C})$ transformation properties

$$\xi^a \to \xi^{a\prime} = \sum_b A_{ab}\xi^b \tag{18}$$

$$\xi^{\dot a} \to \xi^{\dot a\prime} = \sum_{\dot b} B_{\dot a\dot b}\xi^{\dot b}. \tag{19}$$

For reasons that will be discussed later spinors transforming like (18) are called right-handed spinors while spinors transforming like (19) are called left-handed spinors. Dot superscripts are are used to distinguish left handed spinors from right handed spinors. For real Lorentz transformations $B_{\dot a\dot b} = A^*_{ab}$ while for real orthogonal transformations A_{ab} and $B_{\dot a\dot b}$ are independent $SU(2)$ matrices. In both cases the two representations are inequivalent; which means that a general A and B or A and A^* cannot be related by a single constant similarity transformation:

$$\nexists \qquad M \text{ satisfying } MAM^{-1} = A^* \text{ or } MAM^{-1} = B \qquad \forall\, A, B. \tag{20}$$

Because they have determinant 1, a general $SL(2,\mathbb{C})$ matrix can be expressed as the exponential of the dot product of a complex 3-vector $\mathbf{z}$ with the vector of Pauli matrices:

$$A = \pm e^{\mathbf{z}\cdot\boldsymbol{\sigma}} \qquad \det(A) = e^{\mathbf{z}\cdot\mathrm{tr}(\boldsymbol{\sigma})} = 1. \tag{21}$$

It follows from the representation (21) and properties of the Pauli matrices that

$$\sigma_2 A^t \sigma_2 = A^{-1} \qquad \sigma_2 B^t \sigma_2 = B^{-1}. \tag{22}$$

This means that σ_2 behaves like a metric tensor with respect to these spinors. Dual right- and left-handed spinors are identified by lower indices:

$$\xi_a := \sum_b (\sigma_2)_{ab}\xi^b \qquad \xi_{\dot a} := \sum_{\dot b} (\sigma_2)_{ab}\xi^{\dot b}. \tag{23}$$

The transformation properties of the dual spinors follow from the transformation properties of the corresponding upper index spinors

$$\xi_a \to \xi_a' = \sum_{bc} (\sigma_2)_{ab} A_{bc}\xi^c = \sum_{bc} (\sigma_2)_{ab} A_{bc} (\sigma_2)_{cd}\xi_d = \sum_b (A^t)^{-1}_{ab}\xi_b \tag{24}$$

$$\xi_{\dot a} \to \xi_{\dot a}' = \sum_{\dot b\dot c} (\sigma_2)_{\dot a\dot b} B_{\dot b\dot c}\xi^{\dot c} = \sum_{\dot b\dot c} B_{\dot a\dot b}(\sigma_2)_{\dot b\dot c}\xi_{\cdot c} = \sum_{\dot b} (B^t)^{-1}_{\dot a\dot b}\xi_{\dot b}. \tag{25}$$

The dual spinors define invariant linear functionals on the corresponding spinors. This follows since the contraction of a spinor and a dual spinor of the same type is invariant under $SL(2,\mathbb{C})$:

$$\sum_a \xi^{a\prime}\chi_a' = \sum_{abc} A_{ab}\xi^b (A^t)^{-1}_{ac}\chi_c = \sum_{abc} x^a (A^t)_{ab}((A^t)^{-1})_{bc}\chi_c = \sum_a \xi^a\chi_a \tag{26}$$

and replacing A by B

$$\sum_{\dot a} \xi^{\dot a\prime}\chi_{\dot a}' = \sum_{\dot a} \xi^{\dot a}\chi_{\dot a}. \tag{27}$$

For Lorentz transformations when $\mathbf{z} = -i\frac{\boldsymbol{\phi}}{2}$, $A = \pm e^{\mathbf{z}\cdot\boldsymbol{\sigma}}$ represents a rotation through an angle $|\boldsymbol{\phi}|$ about the $\hat{\boldsymbol{\phi}}$ axis, while for $\mathbf{z} = \frac{\boldsymbol{\rho}}{2}$, $A = \pm e^{\mathbf{z}\cdot\boldsymbol{\sigma}}$ represents a rotationless Lorentz boost with rapidity $\boldsymbol{\rho}$. For rotations A is unitary while for rotationless boosts A is a positive Hermitian matrix. Both (A, B) and $(-A, -B)$ correspond to the same 4×4 complex Lorentz or orthogonal transformation.

The 4×4 matrix representation of the complex orthogonal group can be expressed in a number of different ways in terms of the $SL(2, \mathbb{C})$ matrices A and B using properties of the Pauli matrices

$$\sum_\mu (O(A, B)x_e)^\mu \cdot \sigma_{e\mu} = \sum_\mu x_e^\mu \cdot A\sigma_{e\mu}B^t. \tag{28}$$

Taking transposes gives

$$\sum_\mu (O(A, B)x_e)^\mu \cdot \sigma_{e\mu}^t = \sum_\mu x_e^\mu \cdot B\sigma_{e\mu}^t A^t. \tag{29}$$

Multiplying by σ_2 on the right and left using $\sigma_2 A\sigma_2 = (A^t)^{-1}$ on (28) and (29) gives

$$\sum_\mu (O(A, B)x_e)^\mu \cdot \sigma_2\sigma_{e\mu}\sigma_2 = \sum_\mu x_e^\mu \cdot (A^t)^{-1}\sigma_2\sigma_{e\mu}\sigma_2 B^{-1} \tag{30}$$

and

$$\sum_\mu (O(A, B)x_e)^\mu \cdot \sigma_2\sigma_{e\mu}^t\sigma_2 = \sum_\mu x_e^\mu \cdot (B^t)^{-1}\sigma_2\sigma_{e\mu}^t\sigma_2 A^{-1}. \tag{31}$$

In all four cases the complex orthogonal matrix, $O(A, B)^\mu{}_\nu$, remains unchanged.

A general $SL(2, \mathbb{C})$ matrix has a polar decomposition of the form

$$A = RP = P'R' \tag{32}$$

where

$$P := (A^\dagger A)^{1/2} \qquad R := A(A^\dagger A)^{-1/2}$$

$$P' := (AA^\dagger)^{1/2} \qquad R' := (A^\dagger A^\dagger)^{-1/2}A \tag{33}$$

and P and P' are positive Hermitian matrices and R and R' are $SU(2)$ matrices. This implies that any Lorentz transformation can be expressed as the product of a rotation and a "rotationless" boost in either order. Since rotations leave a rest four momentum invariant, both A and P' in (32) transform the rest four momentum to the same final four momentum, p, but the "rotationless" boost, P', is distinguished by being a positive matrix. The rotationaless boost is a function of the 4-velocity, $v := p/m$. The notation $P(p/m)$ is used for the positive $SL(2, \mathbb{C})$ matrix corresponding to a boost that transforms $p_0 := (m, \mathbf{0})$ to p.

V. IRREDUCIBLE REPRESENTATIONS

The elementary building blocks of a relativistic theory of particles are unitary irreducible representations of the Poincaré group [54]. In this section transformation properties of single-particle states of mass m and spin s are used to construct equivalent states in the Euclidean representation. This work is limited to positive-mass positive-energy irreducible representations. These representations are also the building blocks of general unitary representations of the Poincaré group, which can be decomposed into direct integrals of irreducible representations.

In this section two-point quasi-Schwinger functions that describe particles of mass m and spin s are constructed. This construction motivates the structure of multi-point reflection positive quasi-Schwinger functions, which are discussed in the following section.

The starting point is the standard quantum mechanical description of a particle of mass m and spin s. The state of the particle is determined by a complete set of compatible measurements. Compatible quantities that can be measured are a particle's linear momentum, $\mathbf{p}$, and the projection, μ, of its spin on a fixed axis in a fixed reference frame. Lorentz boosts can change the momentum to any value and, because the spin satisfies $SU(2)$ commutation relations, the spin projection can take on $2s + 1$ values in integer steps between $-s$ and s. These considerations determine the spectrum of the momentum and spin projection operators. A Hilbert space representation for such a particle is the space of square integrable functions of the eigenvalues of the linear momentum and a spin projection operator,

$$\langle (m,s)\mathbf{p},\mu|\psi\rangle, \tag{34}$$

satisfying

$$\sum_{\mu=-s}^{s}\int_{\mathbb{R}^3} d\mathbf{p}|\langle (m,s)\mathbf{p},\mu|\psi\rangle|^2 < \infty. \tag{35}$$

Poincaré transformation properties of these wave functions follow from the transformation properties of the basis of simultaneous eigenstates of mass, spin, linear momentum and spin projection. For particles, m and s are fixed. In the general case the mass spectrum is no longer discrete and there are additional invariant degeneracy quantum numbers. The group representation property implies that a general Poincaré transformation can be expressed as a Lorentz transformation followed by a spacetime translation. It follows that the unitary representation of the Poincaré group, $U(\Lambda, a)$, can be expressed as

$$U(\Lambda,a) = U(I,a)U(\Lambda,0) \tag{36}$$

where $U(I,a)$ is a four-parameter unitary representation of the group of spacetime translations and $U(\Lambda,0)$ is a unitary representation of $SL(2,\mathbb{C})$. The notation Λ is used to represent the $SL(2,\mathbb{C})$ matrix or the corresponding 4×4 Lorentz transformation. The interpretation should be clear from the context.

The spacetime translation operator $U(I,a)$ is a multiplication operator acting on the basis states

$$U(I,a)|(m,s)\mathbf{p},\mu\rangle = |(m,s)\mathbf{p},\mu\rangle e^{ip\cdot a}. \tag{37}$$

For rotations, $\Lambda = R$, on a $\mathbf{0}$ linear momentum eigenstate, the rotation does not change the momentum; it only transforms the magnetic quantum number. This means that the transformed state is a linear combination of zero momentum spin states:

$$U(R,0)|(m,s)\mathbf{0},\mu\rangle =$$

$$\sum_{\nu=-s}^{s}|(m,s)\mathbf{0},\nu\rangle\langle s,\nu|U(R,0)|s,\mu\rangle = \sum_{\nu=-s}^{s}|(m,s)\mathbf{0},\nu\rangle D^s_{\nu\mu}[R] \tag{38}$$

where $D^s_{\nu\mu}[R]$ is the $2s+1$ dimensional unitary representation of $SU(2)$ in the basis of eigenstates of the spin projection $\hat{\mathbf{z}}\cdot\mathbf{s}$, $D^s_{\nu\mu}[R] = \langle s,\mu|U(R,0)|s,\nu\rangle$.

The matrix, $D^s_{\nu\mu}[R]$, can be computed explicitly [45]:

$$D^s_{\mu,\nu}[R] = \langle s,\mu|U(R,0)|s,\nu\rangle =$$

$$\sum_{k=0}^{s+\mu}\frac{\sqrt{(s+\mu)!(s+\nu)!(s-\mu)!(s-\nu)!}}{k!(s+\nu-k)!(s+\mu-k)!(k-\mu-\nu)!}R^k_{++}R^{s+\nu-k}_{+-}R^{s+\mu-k}_{-+}R^{k-\mu-\nu}_{--} \tag{39}$$

where

$$R = \begin{pmatrix} R_{++} & R_{+-} \\ R_{-+} & R_{--}\end{pmatrix} = e^{-\frac{i}{2}\boldsymbol{\theta}\cdot\boldsymbol{\sigma}} = \sigma_0\cos(\frac{\theta}{2}) - i\hat{\boldsymbol{\theta}}\cdot\boldsymbol{\sigma}\sin(\frac{\theta}{2}) \tag{40}$$

is a $SU(2)$ matrix. $D^s_{\mu\nu}[R]$ in (39) is a homogeneous polynomial of degree $2s$ in the $SU(2)$ matrix elements, R_{ij}, with real coefficients, while the matrix elements R_{ij} are entire functions of angles, $\boldsymbol{\theta}$. This means that $D^s_{\mu\nu}[R]$ is an entire function of the rotation angles.

The positive matrix, $P(p/m) = (AA^\dagger)^{1/2}$, in the polar decomposition of the $SL(2,\mathbb{C})$ matrix, A, represents a rotationless Lorentz boost. It can be used to define states $|(m,s)\mathbf{p},\mu\rangle$ with non-zero linear momentum in terms of zero momentum states,

$$|(m,s)\mathbf{p},\mu\rangle := U(P(p/m),0)|(m,s)\mathbf{0},\mu\rangle N(\mathbf{p}), \tag{41}$$

where $N(\mathbf{p})$ is a normalization factor that is chosen to ensure that this transformation is unitary. This definition means that the eigenvalue of the magnetic quantum number in the state (41) is the value that would be measured in the particle's rest frame if it was boosted to its rest frame by the inverse of a rotationless boost. The spin defined by (41) is called the canonical spin. (Different spins, like helicity or light-front spin, can be defined by replacing $P(p/m)$ in (41) with a different boost, $A(p/m) =: P(p/m)R(p/m)$, where $R(p/m)$ is a momentum dependent rotation).

For basis states with a Dirac delta-function normalization

$$\langle (m,s)\mathbf{p}',\mu' | (m,s)\mathbf{p},\mu\rangle = \delta(\mathbf{p}'-\mathbf{p})\delta_{\mu'\mu} \tag{42}$$

the transformation (41) is unitary for

$$N(\mathbf{p}) = \sqrt{\frac{m}{\omega_m(\mathbf{p})}}, \qquad \omega_m(\mathbf{p}) := \sqrt{m^2+\mathbf{p}^2}. \tag{43}$$

The unitary transformations, (37), (38) and (41) can be combined to construct a unitary representation of the Poincaré group on the single-particle Hilbert space

$$U(\Lambda,a)|(m,s)\mathbf{p},\mu\rangle = U(I,a)U(\Lambda,0)|(m,s)\mathbf{p},\mu\rangle =$$

$$U(I,a)U(P(\Lambda p/m),0)U(P^{-1}(\Lambda p/m),0)U(\Lambda,0)U(P(p/m),0)|(m,s)\mathbf{0},\mu\rangle \sqrt{\frac{m}{\omega_m(\mathbf{p})}} =$$

$$\sum_{\nu=-s}^{s} U(I,a)U(P(\Lambda p/m),0)|(m,s)\mathbf{0},\nu\rangle \sqrt{\frac{m}{\omega_m(\mathbf{p})}} D^s_{\nu\mu}[(P^{-1}(\Lambda p/m))\Lambda(P(p/m)] =$$

$$\sum_{\nu=-s}^{s} U(I,a)|(m,s)\mathbf{\Lambda}p,\nu\rangle \sqrt{\frac{\omega_m(\mathbf{\Lambda}p)}{\omega_m(\mathbf{p})}} D^s_{\nu\mu}[(P^{-1}(\Lambda p/m))\Lambda(P(p/m)] =$$

$$\sum_{\nu=-s}^{s} e^{i\Lambda p\cdot a}|(m,s)\mathbf{\Lambda}p,\nu\rangle \sqrt{\frac{\omega_m(\mathbf{\Lambda}p)}{\omega_m(\mathbf{p})}} D^s_{\nu\mu}[(P^{-1}(\Lambda p/m))\Lambda(P(p/m)] \tag{44}$$

where $R_w(\Lambda,p) := P^{-1}(\Lambda p/m))\Lambda(P(p/m))$ is a canonical-spin Wigner rotation. The representation (44) is referred to as a Poincaré covariant unitary representation of the Poincaré group.

The next step is to transform (44) into a Lorentz covariant representation of the Poincaré group. Because the Wigner functions (39) are entire functions of angles, both the group representation property and equations for adding angular momenta with real Clebsch-Gordan coefficients also hold for complex angles - or equivalently for all $\Lambda \in SL(2,\mathbb{C})$.

Specifically since

$$\sum_{\alpha=-s}^{s} D^s_{\mu\alpha}[R_2]D^s_{\alpha\nu}[R_1] - D^s_{\mu\nu}[R_2R_1] = 0 \tag{45}$$

and

$$D^{s_1}_{\mu_1\nu_1}[R]D^{s_2}_{\mu_2\nu_2}[R] -$$

$$\sum_{s,\mu,\nu} < s_1,\mu_1,s_2,\mu_2|s,\mu > D^s_{\mu\nu}[R] < s_1,\nu_1,s_2,\nu_2|s,\mu >= 0 \tag{46}$$

are entire and vanish for all real angles, they vanish for complex angles by analyticity (i.e. for $R = e^{-i\frac{1}{2}\boldsymbol{\theta}\cdot\boldsymbol{\sigma}} \to e^{\frac{\mathbf{z}}{2}\cdot\boldsymbol{\sigma}}$). The coefficients $< s_1,\mu_1,s_2,\mu_2|s,\mu >$ in (46) are real $SU(2)$ Clebsch-Gordan coefficients. This means that the $2s+1$ dimensional representation of the Wigner rotations can be factored into a matrix product of three $2s+1$ dimensional Wigner functions of $SL(2,\mathbb{C})$ matrices (see 39):

$$D^s_{\nu\mu}[R_w(\Lambda,p)] = \sum_{\nu'\nu''} D^s_{\nu\nu'}[P^{-1}(\Lambda p/m)]D^s_{\nu'\nu''}[\Lambda]D^s_{\nu''\mu}[P(p/m)]. \tag{47}$$

Using this decomposition equation (44) can be rewritten as

$$U(\Lambda,0)\sum_{\mu'=-s}^{s}|(m,s)\mathbf{p},\mu'\rangle D^s_{\mu'\mu}[(P^{-1}(p/m)]\sqrt{\omega_m(\mathbf{p})} =$$

$$\sum_{\mu',\mu''=-s}^{s}|(m,s)\mathbf{\Lambda}p,\mu'\rangle D^s_{\mu'\mu''}[P^{-1}(\Lambda p/m)]\sqrt{\omega_m(\mathbf{\Lambda}p)}D^s_{\mu''\mu}[\Lambda]. \tag{48}$$

This leads to the definition of "Lorentz covariant" basis states

$$|(m,s)\mathbf{p},\mu\rangle_{cov} := \sum_{\mu'=-s}^{s}|(m,s)\mathbf{p},\mu'\rangle D^s_{\mu'\mu}[P^{-1}(p/m)]\sqrt{\omega_m(\mathbf{p})} \tag{49}$$

where the spins transform under a $2s+1$ dimensional representation of $SL(2,\mathbb{C})$:

$$U(\Lambda)|(m,s)\mathbf{p},\mu\rangle_{cov} = \sum_{\mu'=-s}^{s}|(m,s)\mathbf{\Lambda}p,\mu'\rangle_{cov}D^s_{\mu'\mu}[\Lambda]. \tag{50}$$

Since for $SU(2)$ matrices, $R=(R^\dagger)^{-1}$, it also follows that

$$|(m,s)\mathbf{p},\mu\rangle_{\overline{cov}} := \sum_{\mu'=-s}^{s}|(m,s)\mathbf{p},\mu'\rangle D^s_{\mu'\mu}[P(p/m)]\sqrt{\omega_m(\mathbf{p})} \tag{51}$$

which transforms like

$$U(\Lambda)|(m,s)\mathbf{p},\mu\rangle_{\overline{cov}} = \sum_{\mu'=-s}^{s}|(m,s)\mathbf{\Lambda}p,\mu'\rangle_{\overline{cov}}D^s_{\mu'\mu}[(\Lambda^\dagger)^{-1}]. \tag{52}$$

Equations (50) and (52) define inequivalent (see (20)) unitary representations of the Lorentz group.

In terms of the notation for the transformation properties of spinors, the spins in (50) transform like right handed spinors, ξ^μ, while the spins in (52) transform like dual left handed spinors, $\xi_{\dot\mu}$.

Since $D^s_{\mu'\mu}[P(p/m)]$ is invertible, both Lorentz covariant representations are isomorphic to the original Poincaré covariant representation. While either Lorentz covariant representation can be used, these inequivalent representations get transformed into each other under the discrete transformation of space reflection. This is because complex conjugation changes the sign of σ_2 which results in X_m being reflected about the $x-z$ plane. This means that in order to treat space reflections and Lorentz transformations consistently in the Lorentz covariant representations, both representations (49) and (51) must appear either as tensor products, as they do for four vectors, or direct sums, as they do for Dirac spinors.

These covariant basis vectors can be used to construct Lorentz covariant wave functions:

$${}_{cov}\langle(m,s)\mathbf{p},\mu|\psi\rangle \tag{53}$$

or

$${}_{\overline{cov}}\langle(m,s)\mathbf{p},\mu|\psi\rangle. \tag{54}$$

The Poincaré covariant wave functions are related to the Lorentz covariant wave functions by

$$\langle(m,s)\mathbf{p},\mu|\psi\rangle = \sum_{\mu'=-s}^{s}D^s_{\mu\mu'}[(P)(p/m)]_{cov}\langle(m,s)\mathbf{p},\mu'|\psi\rangle\sqrt{\frac{1}{\omega_m(\mathbf{p})}} \tag{55}$$

or

$$\langle(m,s)\mathbf{p},\mu|\psi\rangle = \sum_{\mu'=-s}^{s}D^s_{\mu\mu'}[(P^{-1})(p/m)]_{\overline{cov}}\langle(m,s)\mathbf{p},\mu'|\psi\rangle\sqrt{\frac{1}{\omega_m(\mathbf{p})}}. \tag{56}$$

The Hilbert space inner product in the Lorentz covariant representation follows from relations (55) and (56) and the inner product in the Poincaré covariant irreducible representations (44):

$$\langle \phi|\psi\rangle = \sum_{\mu=-s}^{s} \int_{\mathbb{R}^3} d\mathbf{p} \langle \phi|(m,s)|\mathbf{p},\mu\rangle \langle (m,s)\mathbf{p},\mu|\psi\rangle =$$

$$\sum_{\mu,\nu=-s}^{s} \int_{\mathbb{R}^3} d\mathbf{p} \langle \phi|(m,s)\mathbf{p},\mu\rangle_{cov} \frac{D^s_{\mu\nu}[P(p/m)P^\dagger(p/m)]}{\omega_m(\mathbf{p})}{}_{cov}\langle (m,s)\mathbf{p},\nu|\psi\rangle =$$

$$\sum_{\mu,\nu=-s}^{s} \int_{\mathbb{R}^4} d^4p \langle \phi(m,s)\mathbf{p},\mu\rangle_{cov} 2\delta(m^2+p^2)\theta(p^0)\times$$

$$D^s_{\mu\nu}[P(p/m)P(p/m))]_{cov}\langle (m,s)\mathbf{p},\nu|\psi\rangle \tag{57}$$

with a similar expression for the inner product in the representation (52):

$$\langle \phi|\psi\rangle =$$

$$\sum_{\mu,\nu=-s}^{s} \int_{\mathbb{R}^3} d\mathbf{p} \langle \phi|(m,s)\mathbf{p},\mu\rangle_{\overline{cov}} \frac{D^s_{\mu\nu}[P^{-1}(p/m)(P^\dagger(p/m))^{-1}]}{\omega_m(\mathbf{p})}{}_{\overline{cov}}\langle (m,s)\mathbf{p},\nu|\psi\rangle =$$

$$\sum_{\mu,\nu=-s}^{s} \int_{\mathbb{R}^4} d^4p \langle \phi(m,s)\mathbf{p},\mu\rangle_{\overline{cov}} 2m\delta(m^2+p^2)\theta(p^0)\times$$

$$D^s_{\mu\nu}[P^{-1}(p/m))P^{-1}(p/m)]_{\overline{cov}}\langle (m,s)\mathbf{p},\nu|\psi\rangle \tag{58}$$

where (33), $P = P^\dagger$, was used in (57)-(58). Because $P(p/m) = e^{\frac{1}{2}\boldsymbol{\rho}\cdot\boldsymbol{\sigma}}$, where $\boldsymbol{\rho}$ is the rapidity of the boost, it follows that

$$P(p/m)P(p/m) = e^{\boldsymbol{\rho}\cdot\boldsymbol{\sigma}} = \frac{\sigma_m \cdot p}{m} \tag{59}$$

and

$$P^{-1}(p/m)P^{-1}(p/m) = \frac{(\sigma_2\sigma_m^t\sigma_2)\cdot p}{m} = \frac{\sigma_m \cdot \Pi p}{m} \tag{60}$$

where Π is the parity operator. These are called "right" and "left-handed" representations because they differ by a space reflection. The resulting kernel is independent of the type of boost used to define the type of spin (41) since for a general $SL(2,\mathbb{C})$ boost, $A(p/m)$, the polar decomposition (32) gives

$$A(p/m)A^\dagger(p/m) = P(p/m)R(p/m)R^\dagger(p/m)P^\dagger(p/m) =$$

$$P(p/m)P^\dagger(p/m) = P(p/m)P(p/m) = \frac{\sigma_m \cdot p}{m}. \tag{61}$$

Using (59) and (60) in (57) and (58) gives the following expression for the inner product of right- or left-handed Lorentz covariant wave functions:

$$\langle \phi|\psi\rangle =$$

$$\sum_{\mu,\nu=-s}^{s} \int_{\mathbb{R}^3} d^3p \langle \phi|(m,s)\mathbf{p},\mu\rangle_{cov} \frac{D^s_{\mu\nu}[\sigma_m \cdot p/m]}{\omega_m(\mathbf{p})}{}_{cov}\langle (m,s)\mathbf{p},\nu|\psi\rangle =$$

$$\sum_{\mu,\nu=-s}^{s} \int_{\mathbb{R}^3} d^4p \langle \phi | (m,s)\mathbf{p},\mu\rangle_{cov} 2\delta(m^2+p^2)\theta(p^0) D^s_{\mu\nu}[\sigma_m \cdot p/m]_{cov} \langle (m,s)\mathbf{p},\nu|\psi\rangle \tag{62}$$

and

$$\sum_{\mu,\nu=-s}^{s} \int_{\mathbb{R}^3} d^3p \langle \phi | (m,s)\mathbf{p},\mu\rangle_{\overline{cov}} \frac{D^s_{\mu\nu}[\sigma_m \cdot \Pi p/m]}{\omega_m(\mathbf{p})}_{\overline{cov}} \langle (m,s)\mathbf{p},\nu|\psi\rangle =$$

$$\sum_{\mu,\nu=-s}^{s} \int_{\mathbb{R}^3} d^4p \langle \phi | (m,s)\mathbf{p},\mu\rangle_{\overline{cov}} 2\delta(m^2+p^2)\theta(p^0) D^s_{\mu\nu}[\sigma_m \cdot \Pi p/m]_{\overline{cov}} \langle (m,s)\mathbf{p},\nu|\psi\rangle. \tag{63}$$

Note that both $D^s_{\mu\nu}[\sigma_m \cdot p/m]$ and $D^s_{\mu\nu}[\sigma_m \cdot \Pi p/m]$ are positive since each one can expressed as the square of a Hermitian matrix.

The Fourier transforms of the kernels of these inner products are

$$W^s_{R\mu\nu}(x,y) = \int_{\mathbb{R}^3} \frac{d^3p}{(2\pi)^3} e^{ip\cdot(x-y)} \frac{D^s_{\mu\nu}[\sigma_m \cdot p/m]}{\omega_m(\mathbf{p})} \tag{64}$$

and

$$W^s_{L\mu\nu}(x,y) = \int_{\mathbb{R}^3} \frac{d^3p}{(2\pi)^3} e^{ip\cdot(x-y)} \frac{D^s_{\mu\nu}[\sigma_2\sigma_m^t\sigma_2 \cdot p/m]}{\omega_m(\mathbf{p})} \tag{65}$$

where $p^0 = \omega_m(\mathbf{p})$ and R and L stand for "right" and "left". The Lorentz covariance properties of these distributions are

$$W^s_{R\mu\nu}(\Lambda x, \Lambda y) = \sum_{\mu'\nu'=-s}^{s} D^s_{\mu\mu'}[\Lambda] W^s_{R\mu'\nu'}(x,y) D^s_{\nu'\nu}[\Lambda^\dagger] \tag{66}$$

and

$$W^s_{L\mu\nu}(\Lambda x, \Lambda y) = \sum_{\mu'\nu'=-s}^{s} D^s_{\mu\mu'}[\Lambda^*] W^s_{L\mu'\nu'}(x,y) D^s_{\nu'\nu}[(\Lambda)^{-1}]. \tag{67}$$

Note that in terms of the spinor transformation properties of the matrices in the kernel, $D^s_{\mu\nu}[\sigma \cdot p/m]$ transforms a left-handed spinor to a right-handed dual spinor while $D^s_{\mu\nu}[\sigma \cdot \Pi p/m]$ transforms a right-handed spinor to a left-handed dual spinor. This is because complex conjugation transforms right-handed spinors to left-handed spinors and left-handed spinors to right-handed spinors. Because of this property of the Wigner functions, the dotted index notation will not be used, instead $D^s_{\mu\nu}[\sigma \cdot p/m]$ will be referred to as the right-handed representation while $D^s_{\mu\nu}[\sigma \cdot \Pi p/m]$ will be referred to as the left-handed representation. The invariance of the inner product is due to the momentum dependence in the kernel.

The kernels of these inner products are representations of two-point Wightman distributions for right- or left-handed particles with of spin s [6]. The price paid in order to have wave functions with spins that transform under finite dimensional representations of $SL(2,\mathbb{C})$ is that the inner product has a momentum and spin-dependent kernel. In the covariant representations (49) and (51) the mass dependence has been moved from the Hamiltonian to the kernel of the inner product. This is necessary since time translations and rotationless boosts, which are dynamical, transform trivially in the Lorentz covariant representation.

A general covariant wave function can transform as a product or direct sum of right- and left-handed spinor representations. For product representations the basis states are replaced by

$$\langle \phi | (m,s)\mathbf{p},\mu\rangle_{cov} \to \langle \phi | (m,s_r,s_l)\mathbf{p},\mu_r,\mu_l\rangle_{cov} \tag{68}$$

where the inner product of Lorentz covariant wave functions has the form

$$\langle \phi|\psi\rangle =$$

$$\int_{\mathbb{R}^3} d\mathbf{p} \sum_{\mu_r,\nu_r=-s_r}^{s_r} \sum_{\mu_l,\nu_l=-s_l}^{s_l} \langle \phi | (m,s_r,s_l)\mathbf{p},\mu_r,\mu_l\rangle_{cov} \times$$

$$\frac{D^{s_r}_{\mu_r\nu_r}[\sigma_m\cdot p/m]D^{s_l}_{\mu_l\nu_l}[\sigma_m\cdot \Pi p/m]}{\omega_m(\mathbf{p})}\langle (m,s_r,s_l)\mathbf{p},\nu_r,\nu_l|\psi\rangle_{cov} =$$

$$\int_{\mathbb{R}^4} d^4p \sum_{\mu_r,\nu_r=-s_r}^{s_r}\sum_{\mu_l,\nu_l=-s_l}^{s_l}\langle\phi|(m,s_r,s_l)\mathbf{p},\mu_r,\mu_l\rangle_{cov}\times$$

$$2\delta(m^2+p^2)\theta(p^0)D^{s_r}_{\mu_r\nu_r}[\sigma\cdot p/m]D^{s_l}_{\mu_l\nu_l}[\sigma\cdot\Pi p/m]\langle (m,s_r,s_l)\mathbf{p},\nu_r,\nu_l|\psi\rangle_{cov}. \tag{69}$$

The kernel of this inner product

$$W^{s_rs_l}_{\mu_r\mu_l\nu_r\nu_l}(x,y) =$$

$$\int \frac{d^4p}{(2\pi)^4}e^{ip\cdot(x-y)}2\delta(m^2+p^2)\theta(p^0)D^{s_r}_{\mu_r\nu_r}[\sigma\cdot p/m]D^{s_l}_{\mu_l\nu_l}[\sigma\cdot\Pi p/m] \tag{70}$$

is a 2-point Wightman distribution. The notation (s_r,s_l) will be used to denote the spin representations in (70).

The next step is to relate the Lorentz covariant representation of the Hilbert space inner product given in terms of the Wightman distributions to a Hilbert space inner product given in terms of Euclidean covariant distributions. The representation (70) of the Wightman distributions can be related to a reflection positive Euclidean two-point distribution.

To show this consider the following Euclidean covariant distribution:

$$S^{s_rs_l}_{\mu_r\mu_l;\nu_r\nu_l}(x_e,y_e) := \int \frac{d^4p_e}{(2\pi)^4}\frac{D^{s_r}_{\mu_r\nu_r}[p_e\cdot\sigma_e]D^{s_l}_{\mu_l\nu_l}[\Pi p_e\cdot\sigma_e]}{p_e^2+m^2}e^{ip_e\cdot(x_e-y_e)}. \tag{71}$$

Because the Wigner functions are polynomials in the components of p_e, the p_e integral in (71) will not generally converge, however this expression represents a distribution, where it is necessary to perform the x_e and y_e integrals before computing the p_e integrals. If the test functions are Schwartz functions, their Fourier transforms are Schwartz functions. This means that the p_e integrals converge as distributions. For test functions satisfying the Euclidean time-support condition, it follows that

$$h^{s_rs_l}_{\mu_r\mu_l;\nu_r\nu_l}(\mathbf{x},\mathbf{y},p_e^0) := \int f^*(\theta x_e)e^{ip_e^0x_e^0}dx_e^0\int g(y_e)e^{-ip_e^0y_e^0}dy_e^0 D^{s_r}_{\mu_r\nu_r}[p_e\cdot\sigma_e]D^{s_l}_{\mu_l\nu_l}[\Pi p_e\cdot\sigma_e] \tag{72}$$

is analytic in the lower-half p_e^0 plane, and the p_e^0 integral can be computed using the residue theorem, closing the contour in the lower half plane. The poles in the p_e^0 integration are at $p_e^0 = \pm i\sqrt{\mathbf{p}^2+m^2}$. The integral over the contour in the lower half plane gets a contribution from the pole at $-i\sqrt{\mathbf{p}^2+m^2}$.

This distribution transforms covariantly under the complex orthogonal group

$$S^{s_rs_l}_{\mu_r\mu_l;\nu_r\nu_l}(O(A,B)x_e,O(A,B)y_e) =$$

$$\sum_{\mu_r',\nu_r'=-s_r}^{s_r}\sum_{\mu_l'\nu_l'=-s_l}^{s_l} D^{s_r}_{\mu_r\mu_r'}[A]D^{s_l}_{\mu_l\mu_l'}[(B^t)^{-1}]S^{s_rs_l}_{\mu_r'\mu_l';\nu_r'\nu_l'}(x_e,y_e)D^{s_r}_{\nu_r'\nu_r}[B^t]D^{s_l}_{\nu_l'\nu_l}[A^{-1}]. \tag{73}$$

Note that as in the $SL(2,\mathbb{C})$ case, $D^{s_r}_{\mu_r\nu_r}[p_e\cdot\sigma_e]$ maps a right-handed $SU(2)\times SU(2)$ spinor to a left-handed dual spinor while $D^{s_l}_{\mu_l\nu_l}[\Pi p_e\cdot\sigma_e]$ maps a left-handed $SU(2)\times SU(2)$ spinor to a right-handed dual spinor.

Of interest is when the test functions have support for positive Euclidean time and the Euclidean time on the final test function is reflected. Then the Euclidean time difference in the exponent $\theta x_0^e - y_e^0$ is strictly negative. In this case the result of this integration (for $\theta x_e^0 - y_e^0 < 0$) is

$$\sum_{\mu_r,\nu_r=-s_r}^{s_r}\sum_{\mu_l,\nu_l=-s_l}^{s_l}\int d^4x_ed^4y_e f^{s_rs_l*}_{\mu_r\mu_l}(\theta x_e)S^{s\dot{s}}_{\mu_r\mu_l;\nu_r\nu_l}(x_e,y_e)g^{s_rs_l}_{\nu\dot{\nu}}(y_e) =$$

$$\sum_{\mu_r,\nu_r=-s_r}^{s_r}\sum_{\mu_l,\nu_l=-s_l}^{s_l}\int d\mathbf{p} f^{s_rs_l*}_{\mu_r\mu_l}(x_e)d^4x_e\frac{e^{-\omega_m(\mathbf{p})x_e^0+i\mathbf{p}\cdot\mathbf{x}}}{(2\pi)^{3/2}}\times$$

$$\frac{D^{s_r}_{\mu_r,\nu_r}[p_m\cdot\sigma_m]D^{s_l}_{\mu_l,\nu_l}[\Pi p_m\cdot\sigma_m]}{2\omega_m(\mathbf{p})}d^4y_e g^{s_r s_l}_{\nu_r\nu_l}(y_e)\frac{e^{-\omega_m(\mathbf{p})y^0_e-i\mathbf{p}\cdot\mathbf{y}}}{(2\pi)^{3/2}} \tag{74}$$

where

$$(-i\omega_m(\mathbf{p}),\mathbf{p})\cdot(\sigma_e,\boldsymbol{\sigma}) = (-i\omega_m(\mathbf{p}),\mathbf{p})\cdot(i\sigma_0,\boldsymbol{\sigma}) =$$

$$(\omega_m(\mathbf{p}),\mathbf{p})\cdot(\sigma_0,\boldsymbol{\sigma}) = p_m\cdot\sigma_m \tag{75}$$

was used.

By defining the wave functions

$$\psi_{\nu_r\nu_l}(\mathbf{p}) := \int_{\mathbb{R}^4} d^4y_e g_{\nu_r\nu_l}(y_e)\frac{e^{-\omega_m(\mathbf{p})y^0_e-i\mathbf{p}\cdot\mathbf{y}}}{(2\pi)^{3/2}} \tag{76}$$

and

$$\phi^*_{\mu_r\mu_l}(\mathbf{p}) = \int_{\mathbb{R}^4} d^4x_e f^*_{\mu_r\mu_l}(x_e)\frac{e^{-\omega_m(\mathbf{p})x^0_e+i\mathbf{p}\cdot\mathbf{x}}}{(2\pi)^{3/2}} \tag{77}$$

(74) becomes

$$\sum_{\mu_r,\nu_r=-s_r}^{s_r}\sum_{\mu_l,\nu_l=-s_l}^{s_l}\int_{\mathbb{R}^3}(\phi^{s_r s_l}_{\mu_r\mu_l}(\mathbf{p}))^*\frac{d\mathbf{p}D^{s_r}_{\mu_r\nu_r}[p_m\cdot\sigma_m]D^{s_l}_{\mu_l\nu_l}[\Pi p_m\cdot\sigma_m]}{2\omega_m(\mathbf{p})}\psi^{s_r s_l}_{\nu_r\nu_l}(\mathbf{p}) \tag{78}$$

which is exactly the expression for the inner product in the Lorentz covariant representation with wave functions (69)

$$(\phi^{s_r s_l}_{\mu_r\mu_l}(\mathbf{p}))^* = \langle\phi|(m,s_r,s_l)\mathbf{p},\mu_r,\mu_l\rangle_{cov} \tag{79}$$

and

$$\psi^{s_r s_l}_{\mu_r\mu_l}(\mathbf{p}) = \langle(m,s_r,s_l)\mathbf{p},\nu_r,\nu_l|\psi\rangle_{cov}. \tag{80}$$

Since the Wigner functions of $p_m\cdot\sigma$ and $\Pi p_m\cdot\sigma$ that appear in these expressions are squares of Hermitian matrices, it follows that (78) is non-negative. When $B\to A^*$ the complex Euclidean covariance condition (73) becomes the Lorentz covariance condition (66-67):

$$S^{s_r s_l}_{\mu_r\mu_l;\nu_r\nu_l}(O(A,A^*)x_e, O(A,A^*)y_e)$$

$$\sum_{\mu'_r,\nu'_r=-s_r}^{s_r}\sum_{\mu'_l,\nu'_l=-s_l}^{s_l} D^{s_r}_{\mu_r\mu'_r}[A]D^{s_l}_{\mu_l\mu'_l}[(A^\dagger)^{-1}]S^{s_r s_l}_{\mu'_r\mu'_l;\nu'_r\nu'_l}(x_e,y_e)D^{s_r}_{\nu'_r\nu_r}[A^\dagger]D^{s_l}_{\nu'_l\nu_l}[A^{-1}]. \tag{81}$$

This means that this complex subgroup of the complex orthogonal transformations defines a unitary representation of the Poincaré group on the Hilbert space defined by the inner product (75).

It follows from (81) that

$$\sum_{\mu'_r,\nu'_r=-s_r}^{s_r}\sum_{\mu'_l,\nu'_l=-s_l}^{s_l}\int f^*(\theta x_e,\mu'_r,\mu'_l)S^{s_r s_l}_{\mu'_r\mu'_l;\nu'_r\nu'_l}(x_e,y_e)f(y_e,\nu_r,\nu_l)d^4x_e d^4y_e =$$

$$\int\sum_{\mu'_r,\nu'_r=-s_r}^{s_r}\sum_{\mu'_l,\nu'_l=-s_l}^{s_l} f^*(O(A,A^*)\theta x_e,\mu'_r,\mu'_l)D^{s_r}_{\mu_r\mu'_r}[A]D^{s_l}_{\mu_l\mu'_l}[(A^\dagger)^{-1}]S^{s_r s_l}_{\mu'_r\mu'_l;\nu'_r\nu'_l}(x_e,y_e)\times$$

$$D^{s_r}_{\nu'_r\nu_r}[A^\dagger]D^{s_l}_{\nu'_l\nu_l}[A^{-1}]f(O(A,A^*)y_e,\nu'_r,\nu'_l)d^4x_e d^4y_e \tag{82}$$

or that

$$f(y_e,\nu_r,\nu_l) \to \sum_{\nu'_r,\nu'_l=-s_l}^{s_l} f(O(A,A^*)y_e,\nu'_r,\nu'_l) D^{s_r}_{\nu'_r\nu_r}[A^\dagger] D^{s_l}_{\nu'_l\nu_l}[A^{-1}] \tag{83}$$

is a unitary representation of $SL(2,\mathbb{C})$.

In order to understand how the support conditions work for finite Lorentz transformations it is useful to express the Euclidean coordinates in terms of the matrices X_e. X_e can be decomposed into the sum of Hermitian and anti-Hermitian parts

$$X_e = \frac{1}{2}(X_e + X_e^\dagger) + \frac{1}{2}(X_e - X_e^\dagger) = X_h + X_a \tag{84}$$

where

$$X_h = X_h^\dagger \qquad X_a = -X_a^\dagger . \tag{85}$$

Under real Lorentz transformations

$$AX_eA^\dagger = AX_hA^\dagger + AX_aA^\dagger \tag{86}$$

the Hermitian and anti Hermitian parts transform independently

$$X'_h = AX_hA^\dagger \qquad X'_a = AX_aA^\dagger . \tag{87}$$

For real Euclidean X_e, $X_a = ix_e^0 I$ so

$$AX_aA^\dagger = AA^\dagger X_a = ix_e^0 AA^\dagger \tag{88}$$

which means $Tr(AX_aA^\dagger) = ix_e^0 Tr(AA^\dagger)$, or that the sign of the imaginary part of X_e is unchanged, while the Hermitian part, associated with real Lorentz transformations transforms independently. This means that the sign of the real part of the Euclidean time is preserved under real Lorentz transformations.

Since the integrals are over 4-dimensional Euclidean variables

$$|\det AX_eA^\dagger| = |\det X_e| = |\det X_e^\dagger| = |\det \theta X_e| \tag{89}$$

which implies $d^4x_e = d^4\theta x_e = d^4x'_e$ where x'_e is the real Lorentz transformed x_e.

The Fourier transforms of these distributions can be computed explicitly (see [46])

$$S^{s_rs_l}_{\mu_r\mu_l;\nu_r\nu_l}(x_e,y_e) =$$

$$\int \frac{d^4p_e}{(2\pi)^4} \int \frac{e^{ip_e\cdot(x_e-y_e)} D^{s_r}_{\mu_r\nu_r}[p_e\cdot\sigma_e] D^{s_l}_{\mu_l\nu_l}[p_e\cdot\sigma_2\sigma_e^*\sigma_2]}{p_e^2+m^2} =$$

$$\frac{m^2}{(2\pi)^2} D^{s_r}_{\mu_r\nu_r}[-i\sigma_e\cdot\nabla_{x_e}] D^{s_l}_{\mu_l\nu_l}[-i\sigma_2\sigma_e^*\sigma_2\cdot\nabla_{x_e}] \frac{K_1(m|x_e-y_e|)}{m|x_e-y_e|} . \tag{90}$$

The Wigner functions are polynomial in the derivatives. While this kernel is singular as $x_e \to y_e$, the Euclidean time support condition along with the Euclidean time reflection ensures that $|x_e - y_e|$ never vanishes.

VI. EUCLIDEAN COVARIANCE

The spinor transformation properties of the Schwinger functions in the previous section,(73), were determined by requiring that the Lorentz invariant inner product is recovered when the Euclidean wave functions have support for positive Euclidean time and the final Euclidean time is reflected.

The covariance condition for the Schwinger functions follow from equation (73) which can be expressed in the form

$$S^{s_rs_l}_{\mu'_r\mu'_l;\nu'_r\nu'_l}((O(A,B)x_e,(O(A,B)y_e) =$$

$$\sum_{\mu_r',\nu_r'=-s_r}^{s_r} \sum_{\mu_l',\nu_l'=-s_l}^{s_l} S^{s_r s_l}_{\mu_r'\mu_l';\nu_r'\nu_l'}(x_e,y_e) D^{s_r}_{\mu_r'\mu_r}[A^t] D^{s_r}_{\nu_r'\nu_r}[B^t] D^{s_l}_{\mu_l'\mu_l}[(B)^{-1}] D^{s_l}_{\nu_l'\nu_l}[A^{-1}]. \tag{91}$$

where A and B are independent $SU(2)$ matrices. It is straightforward to extend this condition to Euclidean covariant kernels with more initial and final Euclidean coordinates and spins. The distributions that replace the Schwinger functions of axiomatic field theory will be referred to as quasi-Schwinger functions, since they are not required to satisfy the locality (symmetry) requirement.

In the Euclidean case, when spinors are involved, this kernel is no longer positive. This is most easily seen in equation (71) where $D^{s_r}_{\mu_r\nu_r}[p_e\cdot\sigma_e]$ and $D^{s_l}_{\mu_l\nu_l}[p_e\cdot\Pi\sigma_e]$ are not Hermitian for real p_e.

VII. CONNECTED DISTRIBUTIONS

The construction of positive-mass positive-energy irreducible representations of the Poincaré group using Euclidean covariant distributions was discussed in section V. Tensor products of these distributions describe non-interacting many-particle systems.

The quasi-Schwinger functions for interacting particles have cluster expansions, which are sums of tensor products of connected quasi-Schwinger functions. Since reflection positivity is preserved under addition and tensor products, reflection positivity follows if the connected quasi-Schwinger functions are reflection positive. Connected quasi-Schwinger functions are the building blocks of general quasi-Schwinger distributions. The purpose of this section is to show that, in the absence of the symmetry requirement it is straightforward to construct connected quasi-Schwinger functions satisfying the conditions needed to for a relativistic quantum theory.

The structure of connected quasi-Schwinger functions is motivated by the construction of the quasi-Schwinger functions for positive-mass positive-energy irreducible representations derived in section (V). The general structure of a connected quasi-Schwinger function can be understood by considering the example of a connected four-point quasi-Schwinger function with two initial and two final coordinates and right handed spinors. It is assumed that it has a contribution from an "intermediate state" with mass λ and spin $(s,0)$. This exhibits the structural elements of a general connected quasi-Schwinger functions.

The following structure of the connected quasi-Schwinger function is assumed:

$$S^c_4(x_{e1},s_1,\mu_1,x_{e2},s_2,\mu_2;y_{e1},s_1,\nu_1,y_{e2},s_2,\nu_2) =$$

$$\sum_{\mu,\nu=-s}^{s} \int \mathcal{S}^{*s_1,s_2:s}_2(\frac{1}{2}(x_{1e}-x_{e2}),p_e)\langle s_1,\mu_2,s_2,\mu_2|s,\mu\rangle \times$$

$$\frac{D^s_{\mu\nu}(p_e\cdot\sigma_e)}{p_e^2+m^2} e^{ip_e\cdot(x_{e1}+x_{e2}-y_{e1}-y_{e2})} d^4p_e \times$$

$$\langle s_1,\nu_2,s_2,\nu_2|s,\nu\rangle \mathcal{S}^{s_1,s_2:s}_2(\frac{1}{2}(y_{e1}-y_{e2}),p_e). \tag{92}$$

where

$$\mathcal{S}^{s_1 s_2:s}_2(\frac{1}{2}(x_{e1}-x_{e2}),p_e) \tag{93}$$

is a connected Euclidean invariant function of $\frac{1}{2}(x_{e1}-x_{e2})$ and p_e. It is assumed to vanish as $(x_{e1}-x_{e2})^2\to\infty$, but be analytic in p_e. For identical particles the coefficient

$$\langle s_1,\nu_2,s_2,\nu_2|s,\nu\rangle \mathcal{S}^{s_1 s_2:s}_2(\frac{1}{2}(y_{e1}-y_{e2}),p_e) \tag{94}$$

is assumed to be either symmetric or antisymmetric with respect to interchange of $1\leftrightarrow 2$.

To show that this expression satisfies the Euclidean covariance condition (91) note

$$S^c_4(O(A,B)x_{e1},s_1,\mu_1,O(A,B)x_{e2},s_2,\mu_2;O(A,B)y_{e1},s_1,\nu_1,O(A,B)y_{e2},s_2,\nu_2) = \tag{95}$$

$$\sum_{\mu,\nu=-s}^{s} \mathcal{S}_2^{*s_1s_2:s}(\frac{1}{2}O(A,B)(x_{e1}-x_{e2}),p_e)\langle s_1,\mu_2,s_2,\mu_2|s,\mu\rangle\times$$

$$\frac{D^s_{\mu\nu}(p_e\cdot\sigma_e)}{p_e^2+m^2}e^{ip_e\cdot O(A,B)(x_{e1}+x_{e2}-y_{e1}-y_{e2})}d^4p_e\times$$

$$\langle s_1,\nu_2,s_2,\nu_2|s,\nu\rangle\mathcal{S}_2^{s_1s_2:s}(O(A,B)(\frac{1}{2}(y_{e1}-y_{e2}),p_e). \tag{96}$$

Euclidean invariance of the dot product in the exponent can be used to move $O(A,B)$ to p_e,

$$=\int\sum_{\mu,\nu=-s}^{s} \mathcal{S}_2^{*s_1s_2:s}(\frac{1}{2}O(A,B)(x_{e1}-x_{e2}),p_e)\langle s_1,\mu_2,s_2,\mu_2,|s,\mu\rangle\times$$

$$\frac{D^j_{\mu\nu}(p_e\cdot\sigma_e)}{p_e^2+m^2}e^{i(O(A,B)^{-1}p_e)\cdot(x_{e1}+x_{e2}-y_{e1}-y_{e2})}d^4p_e$$

$$\langle s_1,\nu_2,s_2,\nu_2|s,\nu\rangle\mathcal{S}_2^{s_1s_2:s}(\frac{1}{2}O(A,B)(y_{e1}-y_{e2}),p_e) \tag{97}$$

while changing variables $p_e'=O^{-1}(A,B)p_e$ gives

$$=\int\sum_{\mu,\nu=-s}^{s} \mathcal{S}_2^{*s_1s_2:s}(\frac{1}{2}O(A,B)(x_{e1}-x_{e2}),O(A,B)p_e)\langle s_1,\mu_2,s_2,\mu_2,|s,\mu\rangle\times$$

$$\frac{D^s_{\mu\nu}(O(A,B)p_e\cdot\sigma_e)}{p_e^2+m^2}e^{ip_e\cdot(x_{e1}+x_{e2}-y_{e1}-y_{e2})}d^4p_e\times$$

$$\langle s_1,\nu_2,s_2,\nu_2|s,\nu\rangle\mathcal{S}_2^{s_1s_2:s}(\frac{1}{2}O(A,B)(y_{e1}-y_{e2}),O(A,B)p_e). \tag{98}$$

The Euclidean invariance, $\mathcal{S}_2^{s_1s_2:s}(\frac{1}{2}(y_{e1}-y_{e2}),p_e)=\mathcal{S}_2^{s_1s_2:s}(O(A,B)\frac{1}{2}(y_{e1}-y_{e2}),O(A,B)p_e)$ means that the factors of $O(A,B)$ can be removed while (23), $(O(A,B)(p_e\cdot\sigma_e)=A(p_e\cdot\sigma_e)B^t$, gives

$$=\sum_{\mu,\nu=-s}^{s} \mathcal{S}_2^{*s_1s_2:s}(\frac{1}{2}(x_{e1}-x_{e2}),p_e)\langle s_1,\mu_1,s_2,\mu_2|s,\mu\rangle\times$$

$$D^s_{\mu\mu'}(A)\frac{D^s_{\mu'\nu'}(p_e\cdot\sigma_e)}{p_e^2+m^2}D^s_{\nu'\nu}(B^t)e^{ip_e\cdot(x_{e1}+x_{e2}-y_{e1}-y_{e2})}d^4p_e$$

$$\langle s_1,\nu_1,s_2,\nu_2|s,\nu\rangle\mathcal{S}_2^{s_1s_2:s}(\frac{1}{2}(y_{e1}-y_{e2}),p_e). \tag{99}$$

Finally the property (46) of the Clebsch-Gordan coefficients can be used to get

$$=\int\sum_{\mu,\nu=-s}^{s}\sum_{\mu_1',\nu_1'=-s_1}^{s_1}\sum_{\mu_2',\nu_2'=-s_2}^{s_2} D^{s_1}_{\mu_1\mu_1'}(A)D^{s_2}_{\mu_2\mu_2'}(A)\mathcal{S}_2^{*s_1s_2:s}(\frac{1}{2}(x_{e1}-x_{e2}),p_e)\langle s_1,\mu_1',s_2,\mu_2'|s,\mu\rangle\times$$

$$\frac{D^s_{\mu\nu}(p_e\cdot\sigma_e)}{p_e^2+m^2}e^{ip_e\cdot(x_{e1}+x_{e2}-y_{e1}-y_{e2})}d^4p_e$$

$$\langle s_1,\nu_1',s_2,\nu_2'|s,\nu\rangle \mathcal{S}_2^{s_1s_2:s}(\frac{1}{2}(y_{e1}-y_{e2}),p_e)D^{s_1}_{\nu_2'\nu_2}(B^t)D^{s_2}_{\nu_2'\nu_2}(B^t) \tag{100}$$

which shows that the two-point quasi Schwinger function (92) satisfies the Euclidean covariance condition (91):

$$= \sum_{\mu_1',\nu_1'=-s_1}^{s_1}\sum_{\mu_2',\nu_2'=-s_2}^{s_2} D^{s_1}_{\mu_1\mu_1'}(A)D^{s_2}_{\mu_2\mu_2'}(A)S_4^c(x_{e1},s_1,\mu_1',x_{e2},s_2,\mu_2';y_{e1},s_1,\nu_1',y_{e2},s_2,\nu_2')D^{s_1}_{\nu_2'\nu_2}(B^t)D^{s_2}_{\nu_2'\nu_2}(B^t) \tag{101}$$

The other requirement is reflection positivity. In this case it is sufficient to assume that the test functions have support for positive Euclidean times. It follows that they have support for positive $X_e^0 = \frac{1}{2}(x_{e1}^0 + x_{e2}^0)$.

To show reflection positivity note that $(\theta f, Sf)$ is:

$$\int \sum_{\mu_1,\nu_1=-s_1}^{s_1}\sum_{\mu_2,\nu_2=-s_2}^{s_2} f^*(\theta x_{e1},s_1,\mu_1,\theta x_{e2},s_2,\mu_2)S_4^c(x_{e1},s_1,\mu_1,x_{e2},s_2,\mu_2;y_{e1},s_1,\nu_1,y_{e2},s_2,\nu_2)f(y_{e1},s_1,\nu_1,y_{e2},s_2,\nu_2). \tag{102}$$

Moving the reflection operators from the final test functions to the quasi-Schwinger function gives

$$= \int \sum_{\mu_1,\nu_1=-s_1}^{s_1}\sum_{\mu_2,\nu_2=-s_2}^{s_2} f^*(x_{e1},s_1,\mu_1,x_{e2},s_2,\mu_2)\mathcal{S}_2^{*s_1s_2:s}(\frac{1}{2}(\theta(x_{e1}-x_{e2}),p_e)\langle s_1,\mu_1,s_2,\mu_2|s,\mu\rangle \times$$

$$\frac{D^s_{\mu\nu}(p_e\cdot\sigma_e)}{p_e^2+\lambda^2}e^{ip_e\cdot(\theta x_{e1}+\theta x_{e2}-y_{e1}-y_{e2})}\times$$

$$\langle s_1,\nu_1,s_2,\nu_2|s,\nu\rangle d^4p_e \mathcal{S}_2^{s_1s_2:s}(\frac{1}{2}(y_{e1}-y_{e2}),p_e)f(y_{e1},s_1,\nu_1,y_{e2},s_2,\nu_2). \tag{103}$$

Using the fact that $\theta(x_{e1}-x_{e2})\cdot P_e = ((x_{e1}-x_{e2})\cdot P_e)^*$ gives

$$= \sum_{\mu_1,\nu_1=-s_1}^{s_1}\sum_{\mu_2,\nu_2=-s_2}^{s_2}\sum_{\mu,\nu=-s}^{s}\int f^*(x_{e1},s_1,\mu_1,x_{e2},s_2,\mu_2)(\mathcal{S}_2^{s_1s_2:s}(\frac{1}{2}((x_{e1}-x_{e2}),p_e)\langle s_1,\mu_1,s_2,\mu_2|s,\mu\rangle)^*\times$$

$$\frac{D^s_{\mu\nu}(p_e\cdot\sigma_e)}{p_e^2+\lambda^2}e^{ip_e\cdot(\theta x_{e1}+\theta x_{e2}-y_{e1}-y_{e2})}\times$$

$$\langle s_1,\nu_1,s_2,\nu_2|s,\nu\rangle d^4p_e \mathcal{S}_2^{s_1s_2:s}(\frac{1}{2}(y_1-y_2),p_e)f(x_1,s_1,\nu_1,x_2,s_2,\nu_2). \tag{104}$$

Because the $\mathcal{S}_2^{s_1s_2:s}$ are analytic in p_e and the Wigner functions are polynomials in the components of p_e, as in the single-particle case, after integrating over test functions in X_e and Y_e satisfying the Euclidean time support condition, the integral over p_e^0 can be computed using the residue theorem. The only contributing pole is at $p^0 = -i\omega_\lambda(\mathbf{p})$:

$$= \int \sum_{\mu_1,\nu_1=-s_1}^{s_1}\sum_{\mu_2,\nu_2=-s_2}^{s_2}\sum_{\mu,\nu=-s}^{s}\Bigg(\int dx_{e1}dx_{e2}f((x_{e1},s_1,\mu_1,x_{e2},s_2,\mu_2)\times$$

$$\mathcal{S}_2^{s_1s_2:s}(\frac{1}{2}(x_{e1}-x_{e2}),p_e)\langle s_1,\mu_1,s_2,\mu_2|s,\mu\rangle)e^{-\omega_\lambda(\mathbf{p})(x_{e1}^0+x_{e2}^0)}\Bigg)^*\frac{\pi d\mathbf{p}}{\omega_\lambda(\mathbf{p})}D^s_{\mu\nu}(p_m\cdot\sigma_m)\times$$

$$\Bigg(\int dy_{e1}dy_{e2}f(y_{e1},s_1,\nu_1,y_{e2},s_2,\nu_2)\mathcal{S}_2^{s_1s_2:s}(\frac{1}{2}(y_1-y_2),p_e)\langle s_1,\nu_1,s_2,\nu_2|s,\nu,\rangle e^{-\omega_\lambda(\mathbf{p})(y_{e1}^0+y_{e2}^0)}\Bigg) \tag{105}$$

which is non-negative since it has the form $\sum_{mn}a_m^*P_{mn}a_n$ where P_{mn} is positive definite, so it is positive.

Note that unlike the local case, the Euclidean time differences $x_1^0-x_2^0$ do not have to be different from zero. This is because in the non-local case there are different N-point quasi Schwinger functions with different numbers of initial and final coordinates.

This four-point example is easy to generalize;

1. The connected four point function can be replaced by a square matrix of $m+n$-point connected distributions with the same "intermediate states".
2. The spinors of rank $(s,0)$ can be replaced by spinors of rank (s_r, s_l) or direct sums of rank $(s,0)\oplus(0,s)$.
3. A single "intermediate state" of mass, $\lambda > 0$, and spin (s_r, s_l) can be replaced by linear superpositions of states with different λ's and spins with a positive weight..
4. The initial and final distributions $\mathcal{S}_2^{s_1 s_2:s}(\frac{1}{2}(y_1 - y_2), p_e)\langle s_1, \nu_1, s_2, \nu_2 | s, \nu, \rangle$ are replaced sums of products of invariant distributions and constant coupling coefficients of the form

$$\sum_a \mathcal{S}_{n,a}^{(s_{r1},s_{l1})\cdots(s_{rn},s_{ln});(s_r,s_l)}(X_e - x_{e1}\cdots X_e - x_{en-1}; p_e^c) C^{(s_{r1},s_{l1}),\cdots(s_{rn},s_{ln});(s_r s_l)}_{(\mu_{r1},\mu_{l1})\cdots(\mu_{rn},\mu_{ln});(\mu_r,\mu_l)}(a) \tag{106}$$

where $X = \frac{1}{n}(x_{e1} + \cdots + x_{en})$, each $\mathcal{S}_{n,a}$ is a connected is a Euclidean invaraint function of the $X_e - x_{ei}$ and p_e, analytic in p_e, and the coefficients $C^{(s_{r1},s_{l1}),\cdots(s_{rn},s_{ln});(s_r s_l)}_{(\mu_{r1},\mu_{l1}\cdots(\mu_{rn},\mu_{ln});(\mu_r,\mu_l)}(a)$ decompose the tensor products of $D(A)$, $D(B)$ to direct sums with spin $(s_r s_l)$, similar to (101),(46).

It isstraightforward to construct $S_{n,a}$ with these properties. In the general case a single value of m is replaced by a linear superpostion of states with different values of m with a positive weight, $\rho(m)$:

$$m \to \int \rho(m) dm \tag{107}$$

The combinations (106) should be symmetric or antisymmetric with respect to exchange of identical particles. This can be realized by projecting the initial and final states on the symmetric or antisymmetric subspace of the Hilbert space.

With these generalizations the proof of Euclidean covariance and reflection positivity follows the proof in the four-point case.

The discussion in this section shows that it is not difficult to construct connected quasi-Schwinger functions that satisfy Euclidean covariance and reflection positivity. These can be used to construct a Hilbert space inner product where the vectors are function of Euclidean variables with support for positive Euclidean times.

VIII. CLUSTER EXPANSIONS

The motivation for exploring the Euclidean approach to relativistic quantum mechanics is the difficulty in satisfying cluster properties in relativistic direct interaction models.

In this section a generalization of the linked cluster theorem is used to construct quasi Schwinger functions that satisfy cluster properties using the connected quasi-Schwinger functions introduced in the previous section.

In this section reflection positive quasi-Schwinger functions will be expressed as linear combinations of products of connected reflection positive quasi-Schwinger functions. The connected reflection positive quasi-Schwinger functions are the elementary building blocks of reflection positive quasi-Schwinger functions that satisfy cluster properties. For systems of identical particles the sums have to include all combinations of tensor products of connected quasi-Schwinger functions that are generated by permutations.

Let

$$\{S_{mn}(x_{me}, \cdots, x_{1e} : y_{1e}, \cdots, y_{ne})\} \tag{108}$$

be a collection reflection-positive quasi-Schwinger distribution $(1 \leq m, n \leq N \leq \infty$ for a system of identical particles. In this and the following expressions the spinor indices are suppressed.

Each S_{mn} can be expanded as a sum of tensor products of connected reflection positive kernels. The tensor products contributing to this sum are products of a total of l kernels; k_1 connected kernels of type $S^c_{m_1 n_1}$, k_2 connected kernels of the type $S^c_{m_2 n_2}$, $\cdots$, k_l connected kernels of the type $S^c_{m_l n_l}$ where

$$n = \sum_{i=1}^{l} k_i n_i \qquad m = \sum_{i=1}^{l} k_i m_i. \tag{109}$$

For systems of identical particles the sums include all distributions generated by $m!$ permutations of the final coordinates and $n!$ permutations of the initial coordinates.

Assume each kernel, $S^c_{m_in_i}$, is invariant up to sign under $m_i!n_i!$ permutations that separately interchange the initial and final arguments. If a given kernel appears k_i times in the product, there are $k_i!$ additional permutations that exchange the k_i identical terms in the product. After accounting for these invariances there remain

$$N = \frac{m!n!}{k_1!\cdots k_l!(n_1!m_1!)^{k_1}(n_2!m_2!)^{k_2}\cdots(n_l!m_l!)^{k_l}} \tag{110}$$

kernels with this structure that differ by permutations, where the integers in (110) are constrained by (109). For identical particles for each product that contributes to S_{mn}, the sum must also include all N distributions that are generated by these additional permutations.

It is possible to construct a generating function for the S_{mn} in terms of the individual connected S^c_{mn}. To do this consistently for Bosons and Fermions define formal creation and annihilation operators. The $a^\dagger(y_i)$ operators create initial states and $\mathbf{b}^\dagger(x_i)$ create final states. These operators satisfy

$$[a(x_i), a^\dagger(y_i)]_\pm = \delta^4(x_i - y_i)\delta_{\mu_{ir}\mu'_{ir}}\delta_{\mu_{il}\mu'_{il}} \tag{111}$$

$$[b(x_i), b^\dagger(y_i)]_\pm = \delta^4(x_i - y_i)\delta_{\mu_{ir}\mu'_{ir}}\delta_{\mu_{il}\mu'_{il}} \tag{112}$$

$$[a(x_i), b(y_i)]_\pm = 0 \tag{113}$$

$$a(x_i)|0\rangle = b(x_i)|0\rangle = 0 \qquad \langle 0|0\rangle = 1. \tag{114}$$

In these expressions, $[A, B]_-$ is a commutator and $[A, B]_+$ is an anticommutator. The operators $a(x_i)$ and $b(y_i)$ are just formal operators that are useful for bookkeeping purposes. The same is true for the formal "vacuum", $|0\rangle$; it has nothing to do with the ground state of the theory. They are just for the purpose of constructing quasi-Schwinger functions from the connected distributions.

With this notation define the generating functional, S, as the formal sum

$$S = \sum_{mn} \frac{1}{m!n!} \int d^{4m}x d^{4n}y \int b^\dagger(x_{me})\cdots b^\dagger(x_{1e})\times$$

$$S_{mn}(x_{me}, \cdots, x_{1e} : y_{1e}, \cdots y_{ne})a^\dagger(x_{1e})\cdots a^\dagger(x_{ne}). \tag{115}$$

The individual symmetrized $S_{mn}(X : Y)$ can be extracted from S using products of annihilation operators and the formal vacuum vector

$$S_{mn}(X : Y) = \langle 0|a(y_{ne})\cdots a(y_{1e})b(x_{1e})\cdots b(x_{me})S|0\rangle. \tag{116}$$

Note that in (116) there are $n!m!$ pairings of creation and annihilation operators of the same type that are equivalent to adding all possible exchanges. If S_{mn} is already symmetric or antisymmetric each product of pairings gives the same result. This results in an overcounting by $n!m!$ which is canceled by the denominator in (115).

Each kernel S_{mn} is a sum of all allowed products of connected kernels that satisfy (109):

$$S_{mn} = \sum \prod_{i=1}^{l} (S^c_{m_in_i})^{k_i} \tag{117}$$

where the sum is over all products of connected kernels satisfying (109). It is useful to express each one of these as operators in terms of the creation and annihilation operators:

$$S^c_i = \sum \frac{1}{m_i!n_i!} \int d^{4m}x_e d^{4n}y_e \int b^\dagger(x_{me})\cdots b^\dagger(x_{1e})$$

$$S^c_{m_in_i}(x_{m_ie}\cdots x_{1_ie} : y_{1_ie}\cdots y_{n_ie})a^\dagger(y_{1_ie})\cdots a^\dagger(y_{n_ie}) \tag{118}$$

where the $S^c_{m_in_i}$ are connected factors that appear in the product.

In this notation the operator S is a sum of products of these different types of connected operators. While the factor $\frac{1}{m_i!n_i!}$ divides by the number of permutations give the same kernel when $S^c_{m_in_i}$ is symmetric, if a given S^c_i appears k_i times in the product, there are $k_i!$ permutations that exchange all of the coordinates in each factor in the product. In terms of the different types of connected operators the generating function has the form

$$S = \sum \frac{1}{k_1!}(S^c_1)^{k_1} \cdots \frac{l}{k_l!}(S^c_l)^{k_l} = e^{\sum_l S^c_l} \tag{119}$$

and the individual S_{mn} can be extracted from this expression using (116) which can be expressed as

$$S_{mn}(X:Y) = \langle 0|a(y_{en})\cdots a(y_{e1})b(x_{e1})\cdots b(x_{em})e^{\sum_l S^c_l}|0\rangle . \tag{120}$$

which is the form of the linked cluster theorem for the quasi-Schwinger distributions. The convergence of the series is irrelevant since only a finite number of terms contribute to a given $S_{mn}(X:Y)$.

Note that while this generating function involves systems of arbitrary numbers of degrees of freedom, it is possible the individual connected components can involve a finite number of degrees of freedom.

IX. CONCLUSION

The purpose of this work is to show that by relaxing the requirement of locality it is possible to construct a set of reflection positive Euclidean covariant distributions satisfying cluster properties. The new feature is that a single N-point Schwinger function is replaced by $N-1$ distributions with m final and k initial degrees of freedom with $m+k=N$. This simplifies the reflection positivity requirement. These Euclidean distributions define the kernel of a Hilbert space inner product. There is a representation of the Poincaré Lie algebra on this Hilbert space, [5]. These generators are self-adjoint operators satisfying cluster properties. The spectrum of the Hamiltonian is bounded from below. It follows that there is a unitary representation of the Poincaré group that satisfies space-like cluster properties and a spectral condition. Given these distributions it is possible to perform any kind of quantum calculation without the need for analytic continuation.

While establishing the existence of a large class of reflection positive quasi Schwinger functions is an important first step for constructing dynamical models, the problem is that this construction assumed an acceptable spectral density, which is dynamical information that should be calculated rather than assumed.

The formulation of a dynamical principle that could generate these distributions is beyond the scope of this paper, but it in important question that needs to be addressed in the future.

Some aspects of this program are discussed elsewhere. They all assume the existence of a set of distributions with the properties shown in this work. Reference [1] provides a computational justification that this formalism may be applied to compute scattering cross sections. The scattering computations used a variation of the time-dependent formulation of scattering that utilized the invariance principle [55] with narrow wave packets to approximate sharp-momentum transition matrix elements. Reference [4] provided a formulation of scattering with composite particles, using a generalization of Haag-Ruelle scattering. This required a Euclidean construction to isolate composite one-body states using functions of the mass operator, which is represented by the Euclidean Laplacian, with compact support. This was done by establishing the completeness of polynomials in the Euclidean Laplacian using the Carleman condition [56]. Reference [5] provides explicit expressions for the Poincaré generators with any spin and proved their self-adjointness .

One of the interesting observations about this approach is that given reflection positive Euclidean distributions, it is possible to perform quantum mechanical calculations directly in a Euclidean representation, without analytic continuation.

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