Flow equations and wavelet truncations

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We investigate both theoretical and computational aspects of using wavelet bases to decouple physics on different scales in quantum field theory.

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I. INTRODUCTION

In this paper we investigate the use of wavelet methods [1][2][3][4][5][6][7][8][9][10][11][12][13][14] [15][16][17][18][19] to decouple degrees of freedom on different distance scales in local quantum field theory. Daubechies wavelets and scaling functions can be used to construct an orthonormal basis of compactly supported functions [20][21][22][23] [24][25][26][27][28][29][30][31][32][33][34]. These basis functions decompose the Hilbert space into a direct sum of orthogonal subspaces associated with different resolutions. Basis functions on the different resolution subspaces are related by unitary scale transformations.

Local fields can be expanded in this basis. This replaces the operator-valued distributions by infinite linear combinations of basis functions with operator-valued coefficients. The operator valued coefficients are defined by smearing the local fields with the basis functions. While the full expansion is exact, there are natural volume and resolution truncations that are defined by retaining only terms in the expansion that have support intersecting a given volume and with a finest resolution.

We limit our considerations to the use of Hamiltonian methods, however the representations used in this paper could also be employed in any field theory framework to provide natural volume and resolution truncations. They could also be utilized in alternative wavelet approaches [8][10] [12][13][16] [17] [18]. When the fields are replaced by these expansions in field-theory Hamiltonians, local products of field operators are replaced infinite linear combinations of products of well-defined operators. The singularities that arise from the local operator products reappear as non-convergence of sums, so the renormalization problem takes on a different form. The theory is naturally regularized by truncating the basis.

The problem of constructing a local limit involves first solving the field equations for the truncated theories with different volume and resolution cutoffs, and adjusting dimensionless parameters of each truncated field theory to preserve some common observables. These are systems with a finite number of degrees of freedom that can in principle be solved, just like lattice truncations. The problem is to identify a sequence of truncated theories and a limiting procedure that results in a well-defined infinite-volume, infinite-resolution limit that satisfies the axioms of a local field theory. The existence of such a limit is an unsolved problem, and is beyond the scope of this paper.

However, for measurements involving a fixed energy scale and finite volume, the number of relevant degrees of freedom is finite. Under these conditions both the accessible volume and resolution are limited. Truncated field theory that include degrees of freedom associated with this volume and resolution should describe physics on this scale after determining the parameters of the truncated theory by experiment. The predictions at this scale should be improvable as the volume and resolution are increased by finite amounts. This is independent of the existence of an infinite volume - infinite resolution limit that describes physical phenomena on all scales.

While it is possible to work at successively finer resolutions, there are reasons to eliminate short-distance degrees of freedom that are much smaller than the scales accessible to a given experiment. In this manner the effects of the eliminated degrees of freedom appear in a more complicated effective Hamiltonian that only involves the physically relevant degrees of freedom. This is similar in spirit to the program initiated by Glöckle and Müller to eliminate explicit pion degrees of freedom in a field theory of interacting pions and nucleons [35] using an "Okobu" transformation[36]. A feature of the wavelet representation is that the commutation relations among the field operators are all discrete, and there are irreducible canonical pairs of operators associated with each resolution and volume.

In the wavelet representation truncated Hamiltonians with different resolutions have the same form, with coefficients that are rescaled as a function of the resolution. There is a natural transformation that transforms the high-resolution truncated Hamiltonian to the sum of the corresponding low-resolution truncated Hamiltonian and corrections that involve the missing high-resolution degrees of freedom.

Block diagonalizing this Hamiltonian according to resolution gives an effective Hamiltonian entirely in the low-resolution degrees of freedom that includes the physics of the eliminated high-resolution degrees of freedom. This can be compared to the original low-resolution Hamiltonian to see how it must be modified to include the effects of the eliminated degrees of freedom. In this representation explicit high-resolution degrees of freedom are replaced by more complicated effective interactions in the low-resolution degrees of freedom.

While this process generates new effective operators, the coefficients of these operators are well-defined functions of the original parameters of the theory, so in a renormalizable theory there is no need to introduce new parameters associated with the new effective operators.

In this paper we investigate the use of flow equation methods [37][38][39][40][41][43] to perform the block diagonalization of the high-resolution Hamiltonian. In general the flow equation will generate an infinite collection of complicated effective operators. In order to separate the problem of convergence of the flow equation from an analysis of the scaling properties of the effective interactions, we consider the case of a free field. For free fields the different resolution degrees of freedom are coupled by spatial derivatives, but the structure of the operators generated by the flow equation remain quadratic functions of the fields, which restricts the structure of the operators that are generated by the flow equation to a finite number of classes. Because of this, the flow equation can be solved without addressing the problem of the generated effective interactions. This provides a first test of the proposed flow equation method to separate scales.

II. BACKGROUND - WAVELET BASIS

In this section the basis of functions that will be used to expand the field operators are defined. The basis functions on the real line are Daubechies scaling functions [20][21][29][31][32][33] on a fixed scale and Daubechies wavelets on all smaller scales.

Our preference for the Daubechies basis is because the basis functions are orthonormal and have compact support. The scale is associated with the size of the support of different basis functions. In higher dimensions the basis functions are products of the one-dimensional basis functions defined in this section. This leads to a representation of the theory in terms of local observables. The structure of the truncated theory is similar to lattice truncations which are also formulated in terms of local degrees of freedom. One advantage of wavelet truncations is that it is possible to include independent degrees on different scales, so large-scale degrees of freedom do not have to be generated by the collective dynamics of many small-scale degrees of freedom.

One useful property of the scaling-wavelet basis is that all of the basis functions can be constructed from a single function, $s(x)$, called the scaling function by integer translations and dyadic scale transformations. The scaling function, $s(x)$, is the solution of the following linear renormalization group equation

$$s(x) = \underbrace{D\,(\underbrace{\sum_{l=0}^{2K-1} h_l T^l s(x)}_{\text{block average}})}_{\text{rescale}} \tag{1}$$

The normalization of the solution of this homogeneous equation is fixed by the condition

$$\int s(x) = 1. \tag{2}$$

In equation (1) T is a unitary integer translation operator and D is a unitary scale transformation operator that shrinks the support of a function by a factor of two. These operators are

$$Ts(x) = s(x-1) \qquad Ds(x) = \sqrt{2}s(2x). \tag{3}$$

Equation (1) implies that $s(x)$ is the fixed point of the operation of taking a weighted average of a finite number of translated of copies of $s(x)$ scaled to half of the original support. The weights, h_l, are real numbers determined by the three conditions:

TABLE I: Scaling Coefficients for Daubechies K=3 Wavelets

h_0	$(1+\sqrt{10}+\sqrt{5+2\sqrt{10}})/16\sqrt{2}$
h_1	$(5+\sqrt{10}+3\sqrt{5+2\sqrt{10}})/16\sqrt{2}$
h_2	$(10-2\sqrt{10}+2\sqrt{5+2\sqrt{10}})/16\sqrt{2}$
h_3	$(10-2\sqrt{10}-2\sqrt{5+2\sqrt{10}})/16\sqrt{2}$
h_4	$(5+\sqrt{10}-3\sqrt{5+2\sqrt{10}})/16\sqrt{2}$
h_5	$(1+\sqrt{10}-\sqrt{5+2\sqrt{10}})/16\sqrt{2}$

1. Orthonormality of integer translates of $s(x)$:

$$\int s(x)s(x-n)dx = \delta_{n0}. \tag{4}$$

2. Consistency:

$$\sum h_l = \sqrt{2}. \tag{5}$$

3. Ability to locally pointwise represent low-degree polynomials:

$$x^n = \sum c_m s(x-m) \qquad 0 \leq n \leq K. \tag{6}$$

Although the sum in (6) is infinite, there are no convergence problems because only a finite number of terms in this sum are non-zero at any given point.

There are two solution of equations (4-6) for the h_l. They are related by $h'_l = h_{2K-1-l}$. The corresponding fixed points, $s(x)$, are mirror images of each other. The resulting $s(x)$ has compact support on the finite interval $[0, 2K-1]$. The values for $K=3$, which are used in this work, are given in table I. These h_l are simple algebraic numbers.

Scaling functions are defined by translating and rescaling $s(x)$:

$$s_n^k(x) := D^k T^n s(x) = 2^{k/2} s\left(2^k(x - 2^{-k}n)\right). \tag{7}$$

It follows from (4) and the unitarity of D that the functions $s_n^k(x)$ are orthonormal for each fixed k.

Subspaces $\mathcal{S}_k(\mathbb{R}) \subset L^2(\mathbb{R})$ of resolution $1/2^k$ are defined by

$$\mathcal{S}_k := \{f(x) | f(x) = \sum_{n=-\infty}^{\infty} c_n s_n^k(x), \quad \sum_{n=-\infty}^{\infty} |c_n|^2 < \infty\}.$$

It follows from (1) that these subspaces are related by

$$\mathcal{S}_k := D^k \mathcal{S}_0 \qquad \mathcal{S}_k \subset \mathcal{S}_{k+n} \qquad n \geq 0$$

or more generally they are nested

$$\mathcal{S}_0 \subset \mathcal{S}_1 \subset \mathcal{S}_2 \subset \cdots . \tag{8}$$

The inclusions in (8) are proper in the sense that they have non-empty orthogonal complements

$$\mathcal{S}_{k+1} = \mathcal{S}_k \oplus \mathcal{W}_k \qquad \mathcal{W}_k \neq \{\emptyset\}.$$

The space $\mathcal{W}_k$ is the orthogonal complement of $\mathcal{S}_k$ in $\mathcal{S}_{k+1}$. From a physical point of view $\mathcal{S}_{k+1}$ is a finer resolution subspace than $\mathcal{S}_k$, and $\mathcal{W}_k$ fills in the missing degrees of freedom that are in $\mathcal{S}_{k+1}$ but not in $\mathcal{S}_k$.

Combining these decompositions we have the following relation between the subspaces $\mathcal{S}_{k+n}$ and $\mathcal{S}_k$ of different resolutions

$$\mathcal{S}_{k+n} = \mathcal{S}_k \oplus \mathcal{W}_k \oplus \mathcal{W}_{k+1} \oplus \cdots \oplus \mathcal{W}_{k+n-1}. \tag{9}$$

The limit of this chain as $n \to \infty$ leads to an exact decomposition of $L^2(\mathbb{R})$ by resolution:

$$L^2(\mathbb{R}) = \mathcal{S}_k \oplus \mathcal{W}_k \oplus \mathcal{W}_{k+1} \oplus \mathcal{W}_{k+2} \oplus \mathcal{W}_{k+3} \oplus \cdots . \tag{10}$$

The subspaces $\mathcal{W}_k$ are called wavelet spaces. Orthonormal bases for the subspaces $\mathcal{W}_k$ are constructed from the mother wavelet, $w(x)$, which is defined by taking a different weighted average of translations of the scaling function $s(x)$ scaled to half of the support of $s(x)$:

$$w(x) := \sum_{l=0}^{2K-1} g_l D T^l s(x) \qquad g_l = (-)^l h_{2K-1-l} \tag{11}$$

The weights g_l in equation (11) are related to the weights h_l used in (1) except the signs alternate and the order of the indices is reversed.

Applying powers of the dyadic scale transformation operator, D, and integer translation operator, T, to $w(x)$ gives the following basis functions for $\mathcal{W}_k$:

$$w^k_m(x) := D^k T^m w(x) = 2^{k/2} w\left(2^k(x - 2^{-k}m)\right) . \tag{12}$$

The functions $w^k_m(x)$ are called wavelets. It can be shown that for each fixed k, $\{w^k_m(x)\}_{m=-\infty}^{\infty}$ is an orthonormal basis for the subspace $\mathcal{W}_k$. Because of (9) the $w^k_n(x)$ for different values of k are also orthogonal.

From (7),(11) and (12) it follows that both $s^k_m(x)$ and $w^k_m(x)$ can be constructed from the fixed point, $s(x)$, of the renormalization group equation (1), using elementary transformations.

The decomposition (10) implies that for any fixed starting scale 2^{-k},

$$\{s^k_n\}_{n=-\infty}^{\infty} \cup \{w^l_n\}_{n=-\infty, l=k}^{\infty,\infty}.$$

is an orthonormal basis for $L^2(\mathbb{R})$ consisting of compactly supported functions. The support of both $s^k_m(x)$ and $w^k_m(x)$ is $[2^{-k}m, 2^{-k}(m + 2K - 1)]$. For any point on the real line there are basis functions of arbitrarily small support that include that point.

The basis functions $s^k_n(x)$ are associated with degrees of freedom of scale 2^{-k} and the $w^l_n(x)$ are associated with degrees of freedom of scale $2^{-(l+1)}$ that are not of scale 2^{-l}. Thus they are identified with localized degrees of freedom with distance scales 2^{-k-l} for all integers, $l \geq 0$.

Equation (9) implies that the functions

$$\{s^{k+m}_n\}_{n=-\infty}^{\infty} \qquad \text{and} \qquad \{s^k_n\}_{n=-\infty}^{\infty} \cup \{w^l_n\}_{n=-\infty, l=k}^{\infty, k+m-1}$$

are related by an orthogonal transformation. This transformation is called the wavelet transform. It can be computed more efficiently than a fast Fourier transform, using the h_l and g_l as weights that define "low"- and "high-pass" filters

$$s^{k-1}_n(x) = \sum h_l s^k_{2n+l}(x) \qquad w^{k-1}_n(x) = \sum g_l s^k_{2n+l}(x).$$

The inverse of this orthogonal transformation is

$$s^k_n(x) = \sum_m h_{m-2n} s^{k-1}_m(x) + \sum_m g_{m-2n} w^{k-1}_m(x).$$

It is precisely these transformations (or their three-dimensional generalization) that relate a fine-resolution Hamiltonian to the sum of a coarse resolution Hamiltonian plus fine scale corrections.

The Daubechies wavelets and scaling functions are fractal functions. This is because $s(x)$ is the solution of a renormalization group equation, and all of the basis functions are obtained by applying a finite number of scale transformations, translations and sums to $s(x)$. In spite of their fractal nature, these basis functions have a finite number of derivatives that increase with increasing K. This paper uses the $K = 3$ basis. These basis functions have one continuous derivative which provides a means for computing derivatives of fields that appear in the Hamiltonian without having to use finite difference approximations. Increasing K leads to more smoothness at the expense of larger support and increasing overlap with basis functions on the same scale.

While quantum fields are generally assumed to be operator-valued tempered distributions, which suggest that field operators smeared with test functions that only have a finite number of derivatives might not be well-defined operators, direct calculations show that free field Wightman functions smeared with the Daubechies $K \geq 3$ wavelet or scaling functions are well-defined. This is analogous to the observation that a delta function, which is a distribution, is also a well-defined linear functional on the space of continuous functions. Equation (6) shows that certain linear combinations of these functions can be much smoother. If there are any issues with the fractal nature of the wavelet basis functions they must arise when one tries to establish the existence of a local limit.

For theories truncated to a finite number of degrees of freedom, the interaction picture still exists, which makes it to possible to formulate the dynamics of the truncated theory in terms of the well-defined wavelet-smeared free fields on the free-field Fock space.

Another potential issue with fractal basis functions involves their computation. This turns out to be a non-issue because they have compact support and integrals of products of these functions with polynomials of arbitrarily high-degree can be computed exactly (reduced to finite linear algebra) using the renormalization group equation (1). Since any continuous function on a compact interval can be approximated by a polynomial, it is possible to accurately compute integrals of products of these basis functions with any continuous function. The renormalization group equation can also be used to reduce the computation of arbitrary products of these basis function and their derivatives to finite linear algebra. It is even possible to use these methods to evaluate integrals of products of these basis functions with functions having logarithmic or principal-value singularities [31][32][33][34]. The computational methods relevant to this work are discussed in the appendix.

III. WAVELET DISCRETIZED FIELDS

Given a pair of scalar fields $\Phi(x)$ and $\Pi(x)$ satisfying canonical equal-time commutation relations

$$[\Pi(\mathbf{x},t),\Phi(\mathbf{y},t)] = -i\delta(\mathbf{x}-\mathbf{y})$$

$$[\Phi(\mathbf{x},t),\Phi(\mathbf{y},t)] = [\Pi(\mathbf{x},t),\Pi(\mathbf{y},t)] = 0,$$

discrete fields satisfying the discrete form of these commutation relations can be constructed by smearing the spatial coordinates of the fields with an orthonormal set of basis functions.

For the scaling-wavelet basis (we consider the $1+1$ dimensional case for notational simplicity) the discrete fields are

$$\Phi^k(s,n,t) = \int dx \Phi(x,t) s^k_n(x) \qquad \Phi^l(w,n,t) = \int dx \Phi(x,t) w^l_n(x) \qquad (l \geq k)$$

$$\Pi^k(s,n,t) = \int dx \Pi(x,t) s^k_n(x) \qquad \Pi^l(w,n,t) = \int dx \Pi(x,t) w^l_n(x) \qquad (l \geq k).$$

These fields represent degrees of freedom localized on the support of the associated basis function.

As a result of the orthonormality of the basis functions the discrete equal-time commutators are:

$$[\Phi^k(s,n,t),\Phi^k(s,m,t)] = 0 \qquad [\Pi^k(s,n,t),\Pi^k(s,m,t)] = 0 \tag{13}$$

$$[\Phi^k(s,n,t),\Pi^k(s,m,t)] = i\delta_{n,m} \tag{14}$$

$$[\Phi^r(w,n,t),\Phi^s(w,m,t)] = 0 \qquad [\Pi^r(w,n,t),\Pi^s(w,m,t)] = 0 \tag{15}$$

$$[\Phi^r(w,n,t),\Pi^s(w,m,t)] = i\delta_{rs}\delta_{n,m} \tag{16}$$

$$[\Phi^r(w,n,t),\Phi^k(s,m,t)] = 0 \qquad [\Pi^r(w,n,t),\Pi^k(s,m,t)] = 0 \tag{17}$$

$$[\Phi^r(w,n,t),\Pi^k(s,m,t)] = 0 \qquad [\Pi^r(w,n,t),\Phi^k(s,m,t)] = 0 \tag{18}$$

The field operators have the *exact* representation in terms of these discrete operators

$$\Phi(x,t) = \sum_n \Phi^k(s,n,t) s^k_n(x) + \sum_{l\geq k;n} \Phi^l(w,n,t) w^l_n(x) \tag{19}$$

$$\Pi(x,t) = \sum_n \Pi^k(s,n,t) s^k_n(x) + \sum_{l\geq k;n} \Pi^l(w,n,t) w^l_n(x). \tag{20}$$

These expansion can be inserted in the free field Hamiltonian,

$$H = \frac{1}{2}\int (\Pi(\mathbf{x},0)\Pi(\mathbf{x},0) + \boldsymbol{\nabla}\Phi(\mathbf{x},0)\cdot\boldsymbol{\nabla}\Phi(\mathbf{x},0) + \mu^2\Phi(\mathbf{x},0)\Phi(\mathbf{x},0))d\mathbf{x}, \tag{21}$$

which can also be expressed exactly in terms of the $t = 0$ discrete fields. The discrete form of the exact Hamiltonian is the sum of an operator with only scaling functions fields, H_s, one with only wavelet fields, H_w and one that has products of both types of fields, H_{sw}:

$$H = H_s + H_w + H_{sw} \tag{22}$$

where

$$H_s := \frac{1}{2}(\sum_n : \Pi^k(s,n,0)\Pi^k(s,n,0) : + \sum_{mn} : \Phi^k(s,m,0) D^k_{s;mn} \Phi^k(s,n,0) :$$

$$+\mu^2 \sum_n : \Phi^k(s,n,0)\Phi^k(s,n,0) :)$$

$$H_w := \frac{1}{2}(\sum_{n,l} : \Pi^l(w,n,0)\Pi^l(w,n,0) : + \sum_{m,l,n,j} : \Phi^l(w,m,0) D^{lj}_{w;mn} \Phi^j(w,n,0) :$$

$$+\mu^2 \sum_{l,n} : \Phi^l(w,n,0)\Phi^l(w,n,0) :).$$

$$H_{sw} := \frac{1}{2}\sum_{m,l,n} : \Phi^l(w,m,0) D^{lk}_{sw;mn} \Phi^k(s,n,0).$$

The coefficients D^k_{smn}, $D^{lj}_{w,m,n}$ and $D^{lk}_{sw;m,n}$ that couple near neighbor fields and fields with different scales are the constant matrices given by

$$D^k_{s;mn} = \int dx \frac{d}{dx} s^k_m(x) \frac{d}{dx} s^k_n(x) \tag{23}$$

$$D^{lj}_{w;mn} = \int dx \frac{d}{dx} w^l_m(x) \frac{d}{dx} w^j_n(x) \tag{24}$$

$$D^{lk}_{ws;mn} = 2\int dx \frac{d}{dx} w^l_m(x) \frac{d}{dx} s^k_n(x). \tag{25}$$

The support properties of the basis functions imply that the matrices $D^x_{y;mn}$ vanish if the support of the functions in the integrand have empty intersection, so they have a structure similar to a finite difference approximation. For a free field the matrices $D^{lk}_{ws;mn}$ and $D^{lj}_{w,m,n}$ for $l \neq j$ are responsible for coupling of physical degrees of freedom on different resolution scales. In interacting theories there are additional couplings that come from local products of more than two fields. For example

$$\int \phi^4(x,t)dx = \sum_{n_1 n_2 n_3 n_4} \Gamma^k_{s;n_1\cdots n_4} \Phi^k(s,n_1,t)\Phi^k(s,n_2,t)\Phi^k(s,n_3,t)\Phi^k(s,n_4,t) + \cdots$$

where

$$\Gamma^k_{s;n_1\cdots n_4} := \int s^k_{n_1}(x) s^k_{n_2}(x) s^k_{n_3}(x) s^k_{n_4}(x) dx \tag{26}$$

and the $\cdots$ represent additional terms in the sum that also involve the wavelet basis functions and fields. Like the $D^k_{s;mn}$, the coefficients $\Gamma^k_{s,n_1\cdots n_4}$ are almost local in the sense that they vanish unless all of the functions in the integral (26) have overlapping support.

The other important feature of the fractal nature of the scaling -wavelet basis is that these constant coefficients have simple scaling properties. For example

$$D^k_{s;mn} = 2^{2k} D^0_{s;mn} = 2^{2k} D^0_{s;0,n-m}$$

$$\Gamma^k_{s;n_1\cdots n_4} = 2^k \Gamma^0_{s;n_1\cdots n_4} = 2^k \Gamma^0_{s;0,n_2-n_1,n_3-n_1,n_4-n_1}.$$

Translational invariance can be used to express corresponding coefficients involving integrals of products of scaling functions and wavelets with different scales in terms of $D^0_{s;0,n-m}$ and $\Gamma^0_{s;0,n_2-n_1,n_3-n_1,n_4-n_1}$. In addition, these constant coefficients can all be computed exactly (i.e. reduced to finite linear algebra) using (2-3) and the scaling equation (1). This is discussed in detail in [11] and the appendix. See also [26][27][22][23] for general methods to compute integrals involving wavelets and scaling functions.

The result is that for a free field all of the coupling coefficients can be expressed in terms of the nine non-zero coefficients $D^0_{s;0m}$ with $-4 \leq m \leq 4$. These are given in the appendix.

Note that the evaluation of the above expressions are reduced to solving a small linear system of equations. Volume truncations involve limiting the n's in $\Phi^k(s,n)$, $\Pi^k(s,n)$ $\Phi^l(w,n)$, and $\Pi^l(w,n)$ while resolution truncations involve limiting the largest values of $l \geq k$.

If we let $H(k,l)$ be a truncated Hamiltonian of the form (22) with scaling functions of resolution 2^{-k} and wavelets with resolutions between 2^{-k-1} and 2^{-k-l} then equation (9) implies the identity

$$H_s(k+l,0) = H_s(k,l) + H_w(k,l) + H_{sw}(k,l) \tag{27}$$

where $H_s(k+l,0)$ and $H_s(k,l)$ only differ by the rescaled coefficients $D^{k+l}_{s;mn} = 2^{2l} D^k_{s;mn}$.

What we want to investigate is how to block diagonalize $H_s(k+l,0)$ so one of the blocks only involves the degrees of freedom of resolution 2^{-k}. This leads to an effective Hamiltonian in 2^{-k} resolution degrees of freedom that includes the effects of eliminated degrees of freedom up to the resolution 2^{-k-l}

IV. FLOW EQUATION

The wavelet basis decomposes the field into a sum of operators that are localized in different finite volumes. Each of these operators are also associated with different resolutions. For free fields the coefficients $D^{kl}_{sw;mn}$, $D^{kl}_{ws;mn}$, and $D^{rs}_{ww;mn}$ in the Hamiltonian couple degrees freedom with different resolutions.

One can think of $1/2^k$ as the physically relevant resolution scale. The canonical scaling function fields $\Phi^k(s,n,t)$ and $\Pi^k(s,n,t)$ are an irreducible set of operators for degrees of freedom on this scale. The wavelet fields also appear in the Hamiltonian; they are associated with finer-scale degrees of freedom. Finally products of wavelet and scaling function fields represent terms that couple degrees of freedom on the physical scale to those on smaller scales.

From a physics point of view, while the smaller scales may not be experimentally relevant, they may represent important contributions to the dynamics. One can imagine integrating them out in a functional integral representation to get an effective theory involving only the experimentally relevant degrees of freedom. This is a difficult calculation in the wavelet representation.

A more direct approach would be to decouple the scaling function part of the Hamiltonian from the wavelet part. This would also lead to an effective Hamiltonian involving only the physically relevant degrees of freedom $\Phi^k(s,n,t)$ and $\Pi^k(s,n,t)$ and a complementary Hamiltonian that acts only on the remaining degrees of freedom. The decoupling will necessarily generate more complicated effective interactions among the physically relevant degrees of freedom.

We also remark that the free field Hamiltonian(21) is still a many-body Hamiltonian. Decoupling at the operator level is a stronger condition than decoupling on a finite number of particle subspace.

Flow equation were introduced by Wegner [37] as a method to continuously evolve a Hamiltonian to a unitarily equivalent simpler form. Flow equations methods [38] [39] [40] [41] [42] [44] [43] are an alternative to direct diagonalization or block diagonalization methods [36]. They have been applied to problems in quantum field theory and quantum mechanics. They have the advantage that they are simpler to implement than integrating out short-distance degrees of freedom in a functional integral. Flow equations are designed to perform this diagonalization using a continuously parameterized unitary transformation, $U(\lambda)$. The transformed Hamiltonian has the form

$$H(\lambda) = U(\lambda) H U^\dagger(\lambda).$$

Here $H(0) = H$ is the original Hamiltonian; the generator of the flow equation is chosen to continuously evolve the initial Hamiltonian into the desired form as λ increases. Here λ is called the flow parameter. As λ increases from 0 the Hamiltonian evolves towards the desired form. The evolution is constructed to exponentially approach the desired form, but it is possible for the exponent to become small. Nevertheless, evaluating $H(\lambda)$ at any value of λ still yields a Hamiltonian that is unitarily equivalent to the original Hamiltonian with weaker scale coupling terms.

The preference for flow equations methods in the wavelet representation is that the simple form of the commutators of the discrete canonical fields, (13-18), reduces the integration of the flow equation to simple algebra. The problem is to find a generator of the flow that leads to the desired outcome.

In general the unitarity of $U(\lambda)$ implies that it satisfies the differential equation

$$\frac{dU(\lambda)}{d\lambda} = \frac{dU(\lambda)}{d\lambda} U^\dagger(\lambda) U(\lambda) = K(\lambda) U(\lambda)$$

where

$$K(\lambda) = \frac{dU(\lambda)}{d\lambda} U^\dagger(\lambda) = -K^\dagger(\lambda)$$

is the anti-Hermitian generator of this unitary transformation. We are free to choose a generator that leads to the desired outcome. It follows that $H(\lambda)$ satisfies the differential equation

$$\frac{dH(\lambda)}{d\lambda} = [K(\lambda), H(\lambda)].$$

For this application it is useful to choose a generator, $K(\lambda)$, that is a function of the evolved Hamiltonian:

$$K(\lambda) = [G(\lambda), H(\lambda)]$$

where $G(\lambda)$ is the part of $H(\lambda)$ with the operators that couple different scales turned off. With this choice $G(\lambda) = G^\dagger(\lambda)$ so $K(\lambda)$ is anti-Hermetian.

It follows that

$$\frac{dH(\lambda)}{d\lambda} = [K(\lambda), H(\lambda)] = [[G(\lambda), H(\lambda)], H(\lambda)] = [H(\lambda), [H(\lambda), G(\lambda)]]. \tag{28}$$

Equation (28) is the desired flow equation for our free field Hamiltonian. A fixed point of this equation occurs when

$$[H(\lambda), G(\lambda)] = 0.$$

It follows from the structure of the equation that this commutator only contains operators that couple the wavelet and scaling functions degrees of freedom. Below we discuss the argument that this non-linear equation drives this commutator to zero for the case of a free field Hamiltonian.

The following considerations are limited to the case of a free field. This is because for interacting fields integrating the flow equation generates an infinite number of new operators. The non-zero commutators of polynomials of field operators of degree n and m are polynomials of degree $n + m - 2$. Each iteration of the scaling equation increases the degree of the polynomials. The new operators represent many-body interactions in the transformed Hamiltonian. A separate analysis of the scaling properties of these many-body polynomial operators is needed to determine the relative strength of these operators, and which, if any operators can be safely discarded. This analysis is separate from considerations about the flow equation and need to be considered in applications to realistic systems.

To understand what happens in the case of the free field Hamiltonian (21) first note that for the starting Hamiltonian all of the operators are quadratic in the $\Phi^k(s/w, n, t)$ and $\Pi^k(s/w, n, t)$ operators, and commutators of these quadratic polynomials remain quadratic polynomials. The commutator of the scaling function or wavelet part of the Hamiltonian with the scale coupling part is either zero or replaces one scaling function operator with a wavelet function operator, or one wavelet function operator with a scaling function operator, resulting in another scale-coupling operator. Likewise the commutator of different scale coupling operator parts gives zero or a product of two scaling or two wavelet function operators. This allows us to separate the flow equation into separate equations for the scale coupling term $H_{sw}(\lambda)$ and uncoupled terms $G(\lambda) = H_s(\lambda) + H_w(\lambda)$. To do this consider the equations

$$H(\lambda) = G(\lambda) + H_{sw}(\lambda),$$

$$[G(\lambda), H_{sw}(\lambda)] =: H_{sw}(\lambda)',$$

Here the prime indicates a different scale coupling operator. Commuting this operator with $H(\lambda)$ gives

$$[G(\lambda)+H_{sw}(\lambda),H'_{sw}(\lambda)] =: H''_{sw}(\lambda)+[H_{sw}(\lambda),H'_{sw}(\lambda)]$$

where

$$H''_{sw}(\lambda)=[G(\lambda),H'_{sw}(\lambda)]=[G(\lambda),[G(\lambda),H_{sw}(\lambda)]].$$

is another scale coupling operator. The commutator of the scale-coupling terms gives a sum of scaling and wavelet terms

$$[H_{sw}(\lambda),H'_{m}(\lambda)]=H''_{s}(\lambda)+H''_{w}(\lambda).$$

This can be used to identify the terms on the right hand side of the scaling equation that couple degrees of freedom on different scales and those that map degrees of freedom on the same scale into themselves. Defining

$$H_A(\lambda)=G(\lambda) \qquad H_B(\lambda)=H_{sw}(\lambda)$$

The flow equations can now be separated into coupled equations for the mixed (B) and non-mixed (A) parts of the Hamiltonian

$$\frac{dH_A(\lambda)}{d\lambda}=[H_B(\lambda),[H_B(\lambda),H_A(\lambda)]]$$

$$\frac{dH_B(\lambda)}{d\lambda}=[H_A(\lambda),[H_B(\lambda),H_A(\lambda)]]=-[H_A(\lambda),[H_A(\lambda),H_B(\lambda)]]$$

These equations have a symmetric form under $H_A(\lambda)\leftrightarrow H_B(\lambda)$ except for a sign, which can be seen by changing the order in the commutator in the second equation.

To understand how these equations evolve the Hamiltonian to the desired form, we express the first equation in a basis of eigenstates of $H_B(\lambda)$ with eigenvalues $e_{bn}(\lambda)$ and the second in a basis of eigenstates of $H_A(\lambda)$ with eigenvalues $e_{an}(\lambda)$. The equations for the matrix elements in each of these bases have the form

$$\frac{dH_{Amn}(\lambda)}{d\lambda}=(e_{bm}(\lambda)-e_{bn}(\lambda))^2H_{Amn}(\lambda) \tag{29}$$

and

$$\frac{dH_{Bmn}(\lambda)}{d\lambda}=-(e_{am}(\lambda)-e_{an}(\lambda))^2H_{Bmn}(\lambda) \tag{30}$$

which shows that matrix elements of H_A increase exponentially while matrix elements of H_B decrease exponentially. Of course this evolution can stall if there are degeneracies in the eigenvalues, approximate degeneracies, or if eigenvalues parameterized by λ cross.

To solve these equation numerically the Hamiltonian needs to be truncated to finite number of degrees of freedom. This means that it is necessary to truncate both the volume and resolution. Any system with a finite energy in a finite volume is expected to be dominated by a finite number of degrees of freedom. These can be separated into degrees of freedom associated with an experimental scale and additional relevant degrees of freedom at smaller scales. We can use scaling function fields as the degrees of freedom on the experimental scale and wavelet degrees of freedom the smaller scales that are still relevant to the given volume and energy scale.

While similar remarks apply to Hamiltonians with interactions, in general a different flow generator may be needed separate the desired degrees of freedom.

V. TEST

To determine if solving the flow equation eliminates the coupling terms, we consider a truncation of the free Hamiltonian to a finite volume with two resolutions, using 32 basis functions - 16 scaling functions and 16 wavelets to expand the fields. For simplicity, we only keep wavelets on one scale.

The coefficients $D^{k}_{s;mn}$, $D^{lj}_{w;mn}$ and $D^{lk}_{sw;mn}$, in (23),(24),(25) are computed in [11].

The truncated fields have expansions of the form

$$\Phi(x) = \sum_{n=0}^{15} s_n(x)\Phi(s,n,t) + \sum_{n=0}^{15} w_n(x)\Phi(w,n,t)$$

$$\Pi(x) = \sum_{n=0}^{15} s_n(x)\Pi(s,n,t) + \sum_{n=0}^{15} w_n(x)\Pi(w,n,t)$$

The truncated Hamiltonian is constructed by inserting these truncated fields in the free field Hamiltonian. The resulting Hamiltonian is quadratic in these fields, and has the form

$$H = \sum_{mn} a_{ssmn}(\lambda)\Phi(s,m,t)\Phi(s,n,t) + \sum_{mn} b_{ssmn}(\lambda)\Pi(s,m,t)\Pi(s,n,t)+$$

$$\sum_{mn} c_{ssmn}(\lambda)\Phi(s,m,t)\Pi(s,n,t) + \sum_{mn} d_{ssmn}(\lambda)\Pi(s,m,t)\Phi(s,n,t)+$$

$$\sum_{mn} a_{wwmn}(\lambda)\Phi(w,m,t)\Phi(w,n,t) + \sum_{mn} b_{wwmn}(\lambda)\Pi(w,m,t)\Pi(w,n,t)+$$

$$\sum_{mn} c_{wwmn}(\lambda)\Phi(w,m,t)\Pi(w,n,t) + \sum_{mn} d_{wwmn}(\lambda)\Pi(w,m,t)\Phi(w,n,t)+$$

$$\sum_{mn} a_{wsmn}(\lambda)\Phi(w,m,t)\Phi(s,n,t) + \sum_{mn} b_{wsmn}(\lambda)\Pi(w,m,t)\Pi(s,n,t)+$$

$$\sum_{mn} c_{wsmn}(\lambda)\Phi(w,m,t)\Pi(s,n,t) + \sum_{mn} d_{wsmn}(\lambda)\Pi(w,m,t)\Phi(s,n,t)+$$

$$\sum_{mn} a_{swmn}(\lambda)\Phi(s,m,t)\Phi(w,n,t) + \sum_{mn} b_{swmn}(\lambda)\Pi(s,m,t)\Pi(w,n,t)+$$

$$\sum_{mn} c_{swmn}(\lambda)\Phi(s,m,t)\Pi(w,n,t) + \sum_{mn} d_{swmn}(\lambda)\Pi(s,m,t)\Phi(w,n,t) \qquad (31)$$

The initial condition ($\lambda = 0$) corresponds to the original truncated Hamiltonian, including all of the wavelet-scale coupling terms:

$$a_{ssmn}(0) = \frac{\mu}{2}\delta_{mn} + D_{ssmn} \qquad b_{ssmn}(0) = \frac{1}{2}\delta_{mn} \qquad c_{ssmn}(0) = 0 \qquad d_{ssmn}(0) = 0$$

$$a_{ssmn}(0) = \frac{\mu}{2}\delta_{mn} + D_{wwmn} \qquad b_{ssmn}(0) = \frac{1}{2}\delta_{mn} \qquad c_{ssmn}(0) = 0 \qquad d_{ssmn}(0) = 0$$

$$a_{wsmn}(0) = D_{wsmn} \qquad b_{wsmn}(0) = 0 \qquad c_{wsmn}(0) = 0 \qquad d_{wsmn}(0) = 0$$

$$a_{swmn}(0) = D_{swmn} \qquad b_{swmn}(0) = 0 \qquad c_{swmn}(0) = 0 \qquad d_{swmn}(0) = 0$$

To test the flow equation method the mass is set to $\mu = 1$. The mass sets the energy scale for the parameter λ. The equation is solved using the Euler method, which uses the differential equation to step to successive values of λ. We use a step size of .001. This is determined by examining the size and number of matrix elements, to ensure that

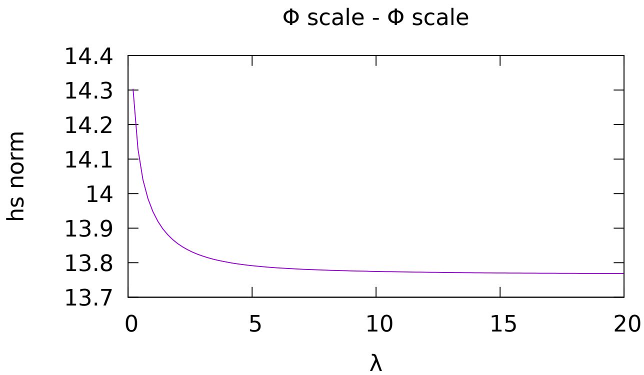

FIG. 1: Hilbert-Schmidt norm: Φ scale - Φ scale

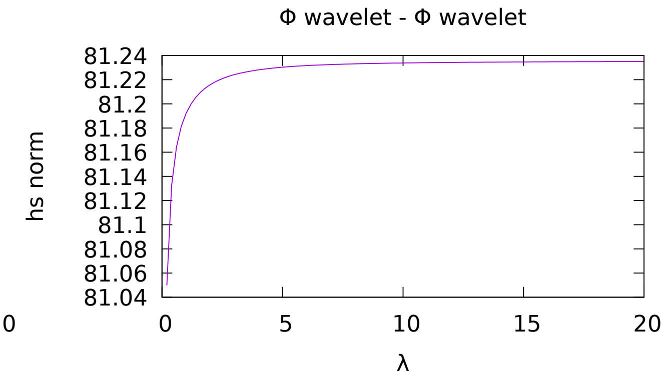

FIG. 2: Hilbert-Schmidt norm: Φ wavelet - Φ wavelet

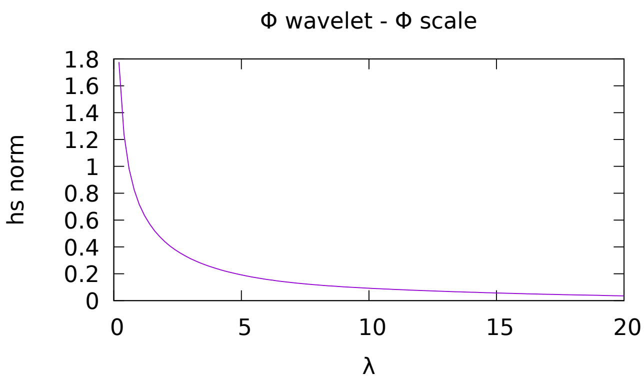

FIG. 3: Hilbert-Schmidt norm: Φ scale - Φ scale

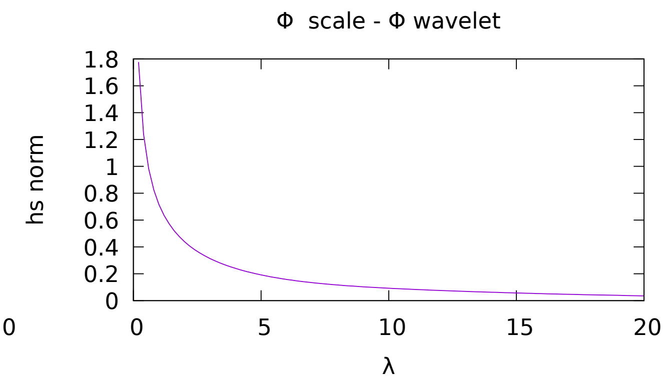

FIG. 4: Hilbert-Schmidt norm: Φ scale - Φ wavelet

the errors remain small. While the error could be improved with a higher order solution method, in this example the Euler method is sufficient to see the that flow equation drives the coupling term to zero.

To illustrate the evolution of the coefficients in the expansion (31) we plot the Hilbert-Schmidt norms of the non-zero coefficients

$$\sqrt{\sum a_{xyij}^2} \qquad \sqrt{\sum b_{xyij}^2} \qquad \sqrt{\sum c_{xyij}^2} \qquad \sqrt{\sum d_{xyij}^2}$$

as a function of λ.

There are four types of operators, $\Phi(s,n,t), \Pi(s,n,t), \Phi(w,m,t)$ and $\Pi(w,m,t)$ leading to 16 types of quadratic expressions. Figures 1-8 show the Hilbert-Schmidt norms of the coefficients of each of the non-zero quadratic expressions as a function of the flow parameter.

The norms of coefficients involving all scaling or all wavelet fields evolve to non-zero values, while the norms of the coupling matrices all evolve to zero. The plots show that initially the size of the coupling terms fall off very fast, but the rate of decrease slows significantly as λ gets larger. For $\lambda = 20$ the Hilbert Schmidt norms of the coupling coefficients are reduced by about two orders of magnitude from their original values.

The Hilbert-Schmidt norms of the coefficients are dominated by the largest matrix elements. It is also useful to understand how the individual matrix coefficients converge.

Figures 9-11 provide a graphical representation of the coefficient matrices for different values of λ. The figures should be viewed as a montage of sixteen 16×16 matrices. Indices 0-15 correspond to $\Phi(s)$, 16-31 correspond to $\Phi(w)$, 32-47 correspond to $\Pi(s)$ and 48-63 correspond to $\Pi(w)$. The grey scale shows the size of the coefficients of the quadratic expressions in the Hamiltonian as a function of λ.

The four figures correspond to different values of the flow parameter $\lambda = 0$, $\lambda = .2$, $\lambda = 2$ and $\lambda = 20$.

The two narrow diagonal bands in in the bottom right corner of figure 9 represent the coefficients of $\Pi(n) - \Pi(n)$ for the scaling functions and wavelet fields respectively. The fatter diagonal bands in the upper left hand part of this figure are associated with the scale-scale and wavelet-wavelet derivative terms. They are almost diagonal because the matrices $D_{x;mn}$ only couple neighboring degrees of freedom.

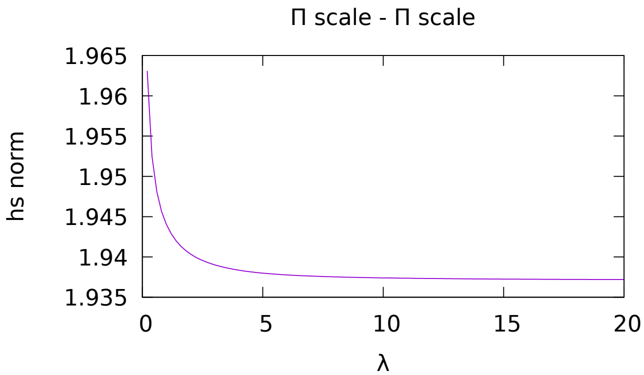

FIG. 5: Hilbert-Schmidt norm: Π scale - Π scale

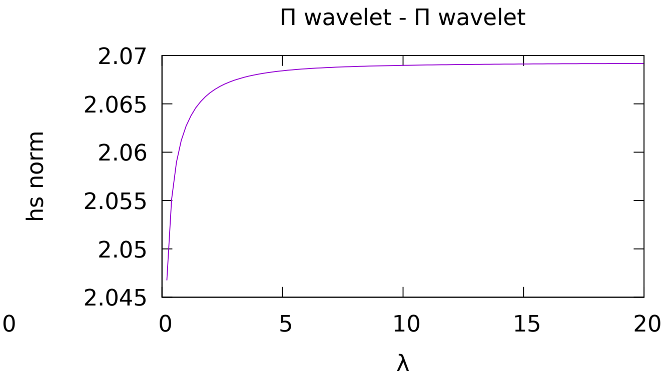

FIG. 6: Hilbert-Schmidt norm: Π wavelet - Π wavelet

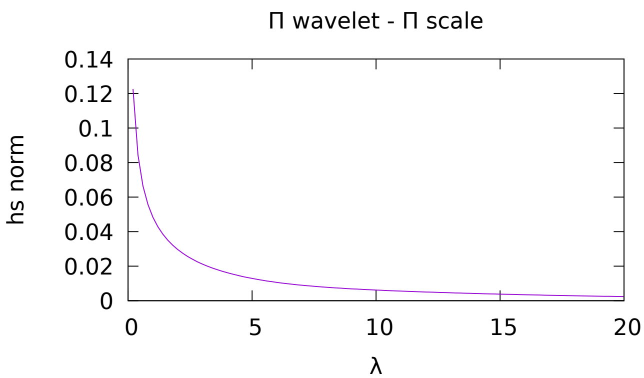

FIG. 7: Hilbert-Schmidt norm: Π wavelet - Π scale

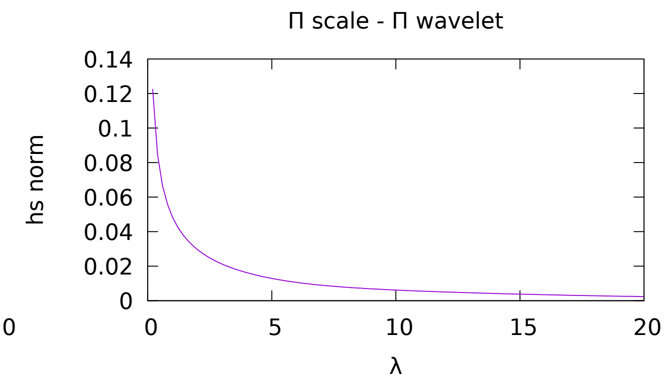

FIG. 8: Hilbert-Schmidt norm: Π scale - Π wavelet

The terms above and below the diagonal are the coefficients of the scale-wavelet and wavelet-scale derivative terms. These are responsible for the coupling of the two scales and are the terms that the flow equation is designed to suppress.

Figure 10 shows the value of these coefficients for $\lambda = .2$. For this value of λ the coupling terms have become smaller and more non-local. This is because repeated applications of the derivative matrix widens the support of the degrees of freedom that are coupled together.

Figure 11 shows that by $\lambda = 2$ the scale coupling terms have essentially disappeared.

Figure 12 show that solving the flow equation out to $\lambda = 20$ does not lead to any big changes. This is consistent with the behavior shown in figures 1-8, that the exponential suppression slows as λ is increased. It is worth noting that the width of the diagonal band in the uncoupled Hamiltonian at $\lambda = 20$ is about the same size as the width of the corresponding band in the original Hamiltonian. This shows that at least for this example the flow equation preserves the local nature of the truncated theory.

The figures support the contention that flow equation methods can be successfully applied to wavelet discretized field theory.

For any value of the flow parameter the flow equation generates a new equivalent Hamiltonian. Ignoring the small terms that couple the wavelet to the scaling function fields, the Hamiltonian becomes a sum of two commuting operators that have degrees of freedom associated with different scale degrees of freedom, but each operator includes the effects of the eliminated degrees that appear in the other operator.

It is still necessary to solve the field equations. In this case there are two independent systems of field equations using the decoupled Hamiltonians:

$$\dot{\Phi}(s,n,t) = i[H_s(\lambda), \Phi(s,n,t)] \qquad \dot{\Pi}(s,n,t) = i[H_s(\lambda), \Pi(s,n,t)]$$

and

$$\dot{\Phi}(w,n,t) = i[H_w(\lambda), \Phi(w,n,t)] \qquad \dot{\Pi}(w,n,t) = i[H_w(\lambda), \Pi(w,n,t)]$$

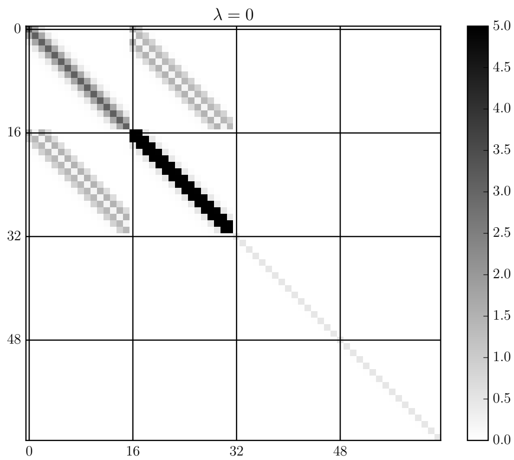

FIG. 9: Full matrix, λ=0

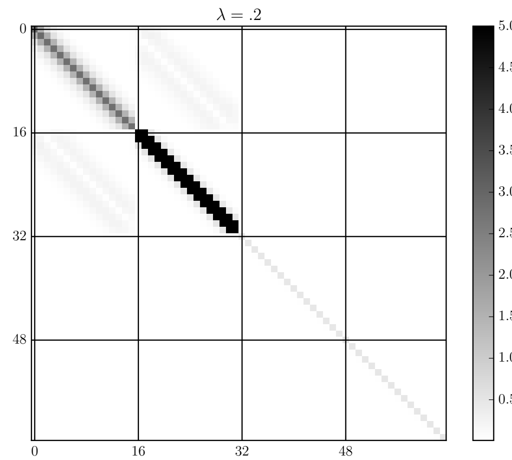

FIG. 10: Full matrix, λ=0.2

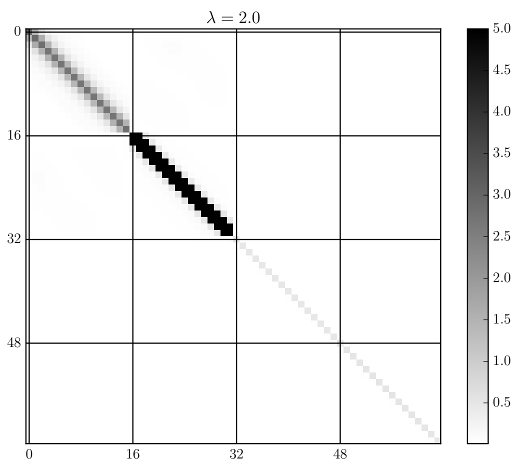

FIG. 11: Full matrix, λ=2.0

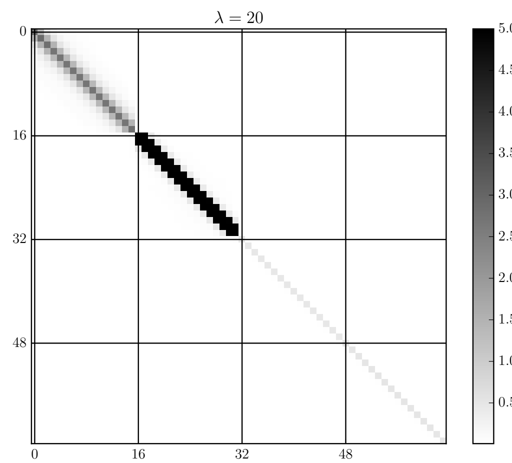

FIG. 12: Full matrix, λ=20.0

The effective Hamiltonian, $H_s(\lambda)$, is the Hamiltonian with the relevant (scaling-function) degrees of freedom. The diagonal part of $H_s(\lambda)$ is the Hamiltonian for a finite system of uncoupled oscillators. Since this system has a finite number of degrees of freedom, the ground state of this Hamiltonian can be calculated in the Fock space associated with the uncoupled oscillators. Solutions of the Heisenberg equations give the time dependence of the discrete field operators and can be used with the expansions (19-20) to compute approximate spacetime correlation functions.

VI. SUMMARY, CONCLUSIONS AND OUTLOOK

The purpose of this work is to examine the use of flow equation methods to separate the physics on different resolution scales in an exact wavelet discretization of quantum field theory. While quantum field theory couples all distance scales, there is a physically relevant scale (or resolution) and it is desirable to formulate the theory directly in terms of the degrees of freedom associated with the physically relevant degrees of freedom. This can be done by eliminating the short distance degrees of freedom.

While it may not be possible to get a well-defined local theory by eliminating all arbitrarily small distance degrees of freedom, it is possible formulate an effective theory that includes the important short-distance physics by eliminating degrees of freedom between the physically relevant scale and a chosen minimal resolution scale. The justification for this is that for a given application there is a relevant volume and energy scale. These restrictions generally imply that the dynamics is dominated by a finite number of degrees of freedom. This can be understood in a number of ways. For a free field theory restricting the energy leads to a subspace of the Fock space with an upper bound on the number of particles. If this system is put in a finite volume, the free particles in a finite volume have discrete energies and only a finite number of these states have energy less than the energy scale. These degrees of freedom are sufficient to formulate the dynamics relevant to the system.

In the scaling-wavelet representation this provides a justification for a volume/resolution truncation of the field theory. The smallest scales that influence the physics that be eliminated by block diagonalizing the Hamiltonian. In this representation the field was expressed as a linear combination of almost local operators classified by position and resolution.

These operators satisfy simple discrete canonical commutation relations. In this representation the exact Hamiltonian is a finite degree polynomial with known constant coefficients in an infinite number of these operators. Resolution and volume truncations lead to truncated Hamiltonians involving a finite number of degrees of freedom.

The approach taken in this paper differs from other wavelet approaches to quantum field theory[2][4][8][9][10][12][13][16]. The basis functions that are used in this work are orthonormal with compact support, but they have a limited amount of smoothness. In most wavelet approaches the wavelet functions are overcomplete, smooth functions that do not have compact support. The justification for smearing fields with functions that have a limited amount of smoothness is that when these basis functions are integrated against free field Wightman functions the results are well-defined. This means the resulting operators are well defined on the free field Fock space, and for theories truncated to a finite number of degrees of freedom it is not necessary to pass to an inequivalent representation of the field algebra in order to solve the field equations.

While there are a number of potential methods to eliminate the short distance degrees of freedom, the simple nature of the commutation relations of the discrete field operators suggest using flow equation methods, where the generator of the desired unitary transformation involves commutators products of canonical pairs of operators. These methods have been proposed to be used in QCD[38][39] as well as in potential theory[43]. Those applications focus on momentum scales and the equations are designed to drive the Hamiltonian to a diagonal form. In this work the goal is to formulate the problem in terms of distance scales and to block diagonalize the Hamiltonian rather than diagonalize it. In this example the transformed theory consists of two sets of canonically conjugate set of operators that operate on different distance scales. The generator of the flow equation is chosen to eliminate the terms that couple these scales in the Hamiltonian.

This method was tested using the Hamiltonian for a free scalar field. While the free field is trivial, in the wavelet representation the space derivatives in the Hamiltonian generate non-trivial terms that couple the degrees of freedom on different scales. This Hamiltonain has the advantage that the flow equation applied to this Hamiltonian does not generate an infinite number of new types of many-body operators. This allows us to focus on testing the flow equation as a method to separate scales without the complication of understanding the relative importance of the generated interactions.

The method was applied to a free field Hamiltonian with a volume and resolution truncation. The field expansions were limited to 32 basis functions that were associated with two different scales. Rather than imposing boundary conditions, the exact expansion was simply truncated. The flow was generated by the commutator of the Hamiltonian with the block diagonal part of the Hamiltonian. The flow equation was solved by stepwise integration using the Euler method.

The dimensions were fixed by the mass in the free field Hamiltonian which was taken as 1 and the Daubechies K=3 wavelets and scaling functions were used to decompose the fields. The result is that the flow equation has the desired property of driving the scale-coupling terms to 0, resulting in a transformed Hamiltonian is that approximately the sum of mutually commuting high and low resolution Hamiltonians.

In practice the coefficients of the coupling terms initially fell off quickly, but the rate of fall off slowed down significantly as the flow parameter increased. The method reduced the coupling coefficients by a factors of about 100 for a modest value of the flow parameter.

Due to the local nature of the theory, the non-zero coefficients matrices of the field operators in the initial Hamiltonian were in narrow bands. Solving the flow equation did two things. First, as expected it eliminated the band associated the coefficients that couple fields on the different scales. Second, to a good approximations it maintained the local structure of the theory. Thus, while each Euler iterations in principle generates more non-localities, and examination of figures 9-12 show that the generated non-local terms are significantly smaller than the generated local almost terms.

Initially we tried to solve these equations using perturbation theory. We found that it did not converge. We successfully solved the equation using the Euler method, which is more like Newton's method for differential equations. While higher order methods should converge faster, we found Euler's method was sufficient for this truncation.

The next step in this program is to consider models with interactions and to consider models in 3+1 dimensions.

The complication with interactions is that the flow equation generates new operators with each step of the Euler method. A different flow generator needs to be formulated to get results comparable to the results outlined above. This is because the analysis that led to (29-30) does not apply to the interacting case. On the other hand, the truncations suggest that in the interacting case only a finite dimensional subspace of the Fock space is relevant. Flow equation methods are more naturally designed to work on subspaces and diagonalize or block diagonalize operators projected on subspaces. These projections limit the types and nature of the many-body operators that are generated by solving the flow equations. Generalizations to 3+1 dimensions are straightforward. Single basis functions are replaced by products of product of three basis functions. While the bookkeeping becomes more difficult, the basic structure is essentially unchanged.

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VII. APPENDIX

Daubechies wavelets and scaling functions are fractal functions. Integrals involving these functions cannot be computed using conventional methods, however the scaling equation (1) and normalization condition (2) lead to linear constraints that reduce the exact integration of all of relevant integrals to finite linear algebra.

For example, if we define the m^{th} moment of the scaling function by

$$M^m = \int s(x) x^m dx$$

the normalization condition implies

$$M^0 = 1$$

and the identity, which is a consequence of unitarity of D and the scaling equation,

$$M^n = \int s(x) x^m dx = \int D^{-1} s(x) D^{-1} x^m dx = \sum_l h^l \int s(x-l) \frac{1}{\sqrt{2}} (x/2)^m dx$$

leads to the recursion for all moments with $m > 0$

$$M^m = \frac{1}{2^m - 1} \frac{1}{\sqrt{2}} \sum_l \sum_{k=0}^{m-1} \frac{m!}{k!(m-k)!} M^k l^{m-k}.$$

Similar remarks apply to integrals of arbitrary products of scaling functions, wavelets, derivatives of these functions and monomials. In the wavelet representation local products of fields and derivatives of fields are replaced by sums of products of discrete fields with constant coefficients. The coefficients are integrals of products of the scaling wavelet basis functions and their derivatives. These coefficients are non-zero only when the intersection of the support of the basis functions in the integrand is not empty. The coefficients are invariant with respect to discrete translations and have well-defined scaling properties. As a result all of them can be expressed in terms of a small number of elementary coefficients. The integrals needed to compute the free field Hamiltonian are

$$D^k_{s;mn} = \int s^{k\prime}_m(x) s^{k\prime}_n(x) dx = 2^{2k} \int s'(x-m) s'(x-n) = 2^k D^0_{s;mn}$$

$$D^{kl}_{sw;mn} = \int s^{k\prime}_m(x) w^{l\prime}_n(x) dx = 2^{2(l+1)} \sum_{m'n'} H^{l+1-k}_{mm'} G_{nn'} D^0_{s;m'n'}$$

$$D^{w;jl}_{mn} = \int w^{j\prime}_m(x) w^{l\prime}_n(x) dx = 2^{2(l+1)} \sum_{m'n'} (GH^{l-j})_{mm'} G_{nn'} D^0_{s;m'n'} \qquad (l \geq j).$$

where the matrices H_{mn} and G_{mn} are defined in terms of the scaling function and wavelet weights by

$$H_{mn} = h_{n-2m} \qquad G_{mn} = g_{n-2m}$$

and

$$D^0_{s;mn} = \int s'_m(x) s'_n(x) dx.$$

Translational invariance implies that all of these coefficients can be expressed in terms of

$$D^0_{s;mn} = D^0_{s;0,n-m}$$

These coefficients can be expressed in terms of another set of integrals

$$D^0_{s;mn} = \sum_k D^0_{s:mnk} \qquad D^0_{s;knm} := \int s_k(x) s'_m(x) s'_n(x) dx$$

using the partition of unity property

$$\sum_n s_n(x) = 1$$

It follows from the scaling equation that the coefficients $D^0_{s:nlm} = D^0_{s;0,l-n,m-n}$ satisfy the homogeneous equations

$$D^0_{s;0lm} = 4\sqrt{2} \sum_{n,k,j} H_{0n} H_{l,k+n} H_{m,j+n} D^0_{s;0kj}$$

as a consequence of the scaling equation. They also satisfy

$$\sum_m D^0_{s;0lm} = 0 \qquad D^0_{s;0lm} - D^0_{s;0ml} = 0$$

and the inhomogeneous equation

$$\sum_l l D^0_{s;nlm} = \int s_n(x) s'_m(x) dx = \Gamma_{s;nm} = \Gamma_{s;0,m-n}.$$

where $\Gamma_{s;0l}$ satisfies

$$\Gamma_{s;0l} = 2 \sum_{m,n} H_{0m} H_{l,n+m} \Gamma_{s;0n}$$

and

$$\sum_n n \Gamma_{s;0n} = 1$$

An independent set of these linear equations can be solved for the coefficients $\Gamma_{s;0n}$, $D^0_{s;nlm}$, $D^0_{s;lm}$. The results of these calculations are given in [11].

The result is that all of the coefficients that appear in the free field Hamiltonian can be expressed in terms of the following nine numbers

$$D^0_{s;mn} = D^0_{s;0,n-m}$$

where

$$D^0_{s;40} = \int s'(x-4) s'(x) dx = -0.005357$$

$$D^0_{s;30} = \int s'(x-3) s'(x) dx = -0.1143$$

$$D^0_{s;20} = \int s'(x-2) s'(x) dx = 0.8762$$

$$D^0_{s;10} = \int s'(x-1)s'(x)dx = -3.390$$

$$D^0_{s;00} = \int s'(x)s'(x)dx = 5.268$$

$$D^0_{s;-10} = \int s'(x+1)s'(x)dx = -3.390$$

$$D^0_{s;-20} = \int s'(x+2)s'(x)dx = 0.8762$$

$$D^0_{s;-30} = \int s'(x+3)s'(x)dx = -0.1143$$

$$D^0_{s;-40} = \int s'(x+4)s'(x)dx = -0.005357$$

and

$$D^0_{ww;mn} = \sum_{kl} g_{k-2m} g_{l-2n} D^0 0_{s;kl}$$

$$D^0_{ws;mn} = \sum_{kl} h_{k-2m} g_{l-2n} D^0_{s;kl}$$

$$D^0_{sw;mn} = \sum_{kl} g_{k-2m} h_{l-2n} D^0_{s;kl}$$

These same methods [31] can be used to compute the overlap integrals (26) that appear in interacting theories.

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