# Model Tests of Cluster Separability In Relativistic Quantum Mechanics

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A relativistically invariant quantum theory first advanced by Bakamjian and Thomas has proven very useful in modeling few-body systems. For three particles or more, this approach is known formally to fail the constraint of cluster separability, whereby symmetries and conservation laws that hold for a system of particles also hold for isolated subsystems. Cluster separability can be restored by means of a recursive construction using unitary transformations, but implementation is difficult in practice, and the quantitative extent to which the Bakamjian-Thomas approach violates cluster separability has never been tested. This paper provides such a test by means of a model of a scalar probe in a three-particle system for which (1) it is simple enough that there is a straightforward solution that satisfies Poincaré invariance and cluster separability, and (2) one can also apply the Bakamjian-Thomas approach. The difference between these calculations provides a measure of the size of the corrections from the Sokolov construction that are needed to restore cluster properties. Our estimates suggest that, in models based on nucleon degrees of freedom, the corrections that restore cluster properties are too small to effect calculations of observables.

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## I. INTRODUCTION

There are two distinct requirements for describing quantum mechanical systems of particles under the requirements of special relativity. The first requirement is Poincaré invariance: probabilities, expectation values and ensemble averages for equivalent experiments performed in different inertial frames are identical. A necessary and sufficient condition for a quantum theory to be Poincaré invariant is that the dynamics is described by a unitary representation of the Poincaré group [1]. The second requirement is cluster separability: isolated subsystems must have the same observable properties as they would in a framework in which the other "spectator" particles are absent entirely. This requirement justifies tests of special relativity on isolated subsystems. It applies both to systems of particles interacting among themselves (e.g. via the strong interaction) and to the current operators that allow them to interact with external fields.

There is more than one way to implement the requirements of Poincaré invariance in a quantum system, and cluster separability is not necessarily an automatic consequence of the implementation. In local quantum field theory, Poincaré invariance and cluster separability are satisfied formally as a consequence of the covariance, spectral properties and the local commutation relations of the field operators. For systems amenable to perturbation methods, these properties can still hold. For non-perturbative systems (such as those based on the strong interaction) controlled approximations are replaced by a truncation or an expansion scheme, which replaces an infinite hierarchy of coupled non-linear field equations with a more tractable finite subset, and which may be valid only in restricted parameter domains. These schemes may still exhibit Lorentz covariance, but cluster separability does not automatically follow, and must be demonstrated on a case-by-case basis that depends on the scheme.

An alternative implementation of Poincaré invariance in a few-body system involves direct construction of the dynamical representations of the Poincaré group in the presence interactions with a finite number of particle (rather than field) degrees of freedom. Such a construction was originally studied by Dirac [2], and Bakamjian and Thomas [3] provided an explicit construction of the generators for a system of two interacting particles.

Many realistic, Poincaré invariant, quantum mechanical models of strongly interacting few-body systems are based on a generalization of the Bakamjian-Thomas construction [4]. This construction can be used to construct models of systems of arbitrary numbers of particles and systems that do not conserve particle number. A representative sample of few-body applications of Poincaré invariant quantum mechanics based on the Bakamjian-Thomas construction includes relativistic constituent quark models [5–9], relativistic few-nucleon models [10–14], relativistic models involving electromagnetic probes [15–17], and relativistic models with particle production [18, 19].

The virtue of the Bakamjian-Thomas construction is that it provides a means for constructing Poincaré generators for systems of interacting particles, and where the problems have a finite number of degrees of freedom and are in principle solvable. The limitation of this construction is that for systems of more than two particles, the dynamical representation of the Poincaré group does not become a tensor product on states representing asymptotically separated

subsystems. While this limitation does not lead to observable consequences in the three-body $S$-matrix or bound-state observables, there are observable consequences when the three-body system is embedded in the four-particle Hilbert space, as it is in four-body problems.

The most ambitious applications of the Bakamjian-Thomas construction are relativistic Faddeev calculations [10–14] of three-nucleon scattering, which have been performed using realistic nucleon-nucleon forces and three-nucleon forces. The next step in developing a relativistic few-body theory would be to model four-nucleon systems or electron scattering from a three-nucleon system. In both of these applications a three-nucleon subsystem is embedded in a four-particle Hilbert space, and, for the first time, there is the possibility of observable consequences of violations of cluster properties.

The restoration of cluster properties can be achieved through a recursive construction due to Sokolov [20, 21]. The Sokolov construction generates a dynamical representation of the Poincaré group satisfying cluster properties using of a hierarchy of unitary transformations, each of which preserves the $S$ matrix while transforming tensor products of subsystem representations of the Poincaré group to representations where the interactions can be added in a manner that preserves the underlying Poincaré symmetry. The Bakamjian-Thomas representation is retrieved by setting the relevant unitary transformations to the identity operator. The Sokolov construction is non-trivial and the unitary transformations depend upon the interactions. An isolated three-body system is a special case where the cluster property can be achieved by means of a single overall unitary transformation that preserves the $S$ matrix. For systems of more than three particles there are observable differences between the $S$ matrices in the two representations. The interested reader can find a complete discussion in [21].

The construction of the Sokolov hierarchy of unitary transformations is sufficiently complicated that the technique has never been used in realistic calculations. Before undertaking computationally intensive four-body calculations, one would like to know the importance of the corrections required by cluster properties. While Ref. [21] argued that these corrections should be small in nuclear physics applications, this was never quantified in any model calculations. An additional investigation is needed to determine if these corrections can be ignored, can be treated perturbatively, or must be treated exactly. To address this question we construct a model involving matrix elements of a scalar probe for a system of three particles. The model is simple enough that both Poincaré invariance and cluster separability are easily satisfied; this unusual pair of features is due to the simplicity of the test model. At the same time the model is sufficiently rich that it permits an equivalent Bakamjian-Thomas treatment of the three-body system that illustrates the quantitative impact of the breakdown of cluster separability in this four-body problem. The conclusion of our preliminary analysis suggests that the corrections required by cluster properties are too small to be observable in nuclear physics applications. The corrections are more important for models based on sub-nucleon degrees of freedom.

## II. CLUSTER PROPERTIES

In a few-body quantum mechanical model the requirement of cluster separability means that the Poincaré generators $\{H, \mathbf{P}, \mathbf{J}, \mathbf{K}\}$ that generate time translations, translations, rotations, and rotationless Lorentz transformations and current operators $I^\mu(x)$ have cluster expansions of the form

$$H = \sum_i H_i + \sum_{ij} H_{ij} + \sum_{jk} H_{ijk} + \cdots \tag{2.1}$$

$$\mathbf{P} = \sum_i \mathbf{P}_i + \sum_{ij} \mathbf{P}_{ij} + \sum_{ijk} \mathbf{P}_{ijk} + \cdots \tag{2.2}$$

$$\mathbf{J} = \sum_i \mathbf{J}_i + \sum_{ij} \mathbf{J}_{ij} + \sum_{ijk} \mathbf{J}_{ijk} + \cdots \tag{2.3}$$

$$\mathbf{K} = \sum_i \mathbf{K}_i + \sum_{ij} \mathbf{K}_{ij} + \sum_{ijk} \mathbf{K}_{ijk} + \cdots \tag{2.4}$$

$$I^\mu(x) = \sum_i I^\mu_i(x) + \sum_{ij} I^\mu_{ij}(x) + \sum_{jk} I^\mu_{ijk}(x) + \cdots \tag{2.5}$$

where the terms $X_{ij}, X_{ijk}, \cdots$ are short-range $2, 3 \cdots$-body operators (which only depend on the degrees of freedom of the particles in a cluster and vanish when one particle in a cluster is separated from the remaining sub-cluster), and the sums of the generators and currents over any subset of particles satisfies the Poincaré commutation relations, current covariance, and current conservation.

These constraints are realized in the Sokolov construction, but they are not generally satisfied by the Bakamjian-Thomas construction. The Sokolov construction generates all of the many-body operators that appear in these cluster expansions as functions of input two-body interactions. These interactions are necessary to preserve the Poincaré commutation relations in the presence of interactions.

## III. TEST MODEL

Poincaré invariance and the Dirac forms of dynamics for Bakamian-Thomas ($BT$) constructions are discussed extensively in Ref. [4]. The framework provided here makes use of that discussion. Conceptually, the Bakamjian-Thomas construction defines the dynamics in the rest frame of the non-interacting system and then uses simultaneous eigenstates of the mass and spin (Casimir operators for the Poincaré group) to construct unitary representations of the Poincaré group. We approach the model in a heuristic way in order to illustrate the issue of cluster separability combined with Poincaré invariance with a minimum of formal development.

A simple four-body model where we can observe the breakdown of cluster separability consists of a system of three spinless particles interacting with an external probe that represents the presence of a fourth particle. The relevant dynamical quantities are matrix elements of a one-body scalar density between eigenstates of the three-body four-momentum operator. We will see that, even for this simple model, there are differences in matrix elements of the scalar density calculated using three-body eigenstates of a $BT$ four-momentum operator compared to matrix elements calculated using three-body eigenstates of a four-momentum operator that clusters. Our simplifying assumptions show that the violation of cluster properties has nothing to do with particle spins, or properties of more realistic probes like four-current densities.

The scalar density operator, $j(x)$, is assumed to be a one-body operator. The three-particle system consists of a bound pair (which we label the (12) subsystem) and a third particle (3 which we call the struck particle) that does not interact with particles 1 or 2. Only particle 3 interacts with the scalar density $j(x)$. Therefore, the (12) subsystem acts as a spectator with respect to the action of the probe. This can be illustrated by disconnected graph shown in Fig. 1.

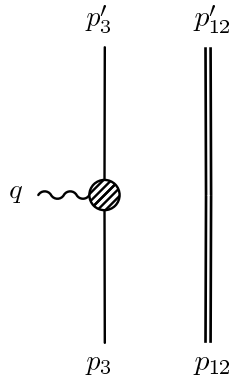


FIG. 1: Graph of one-body density interacting with a particle plus a bound two-body spectator.

In the following sections we will calculate matrix elements of the density operator in this simple system using the Bakamjian-Thomas unitary representation of the Poincaré group, and compare them with a result that exhibits manifest cluster separability. The latter result is possible only because of the simplicity of the model.

### A. Tensor-Product ($TP$) Representation

For the simple system introduced above, the unitary representation of the Poincaré group is a tensor product of a two-body representation on a two-particle Hilbert space and a one-particle representation acting on a one-body Hilbert space. In this section we consider a three-body eigenstate that is a tensor product of bound (12) eigenstate

and a one-particle eigenstate

$$|\lambda, m; \mathbf{p}_{12}, \mathbf{p}_3\rangle_{TP} = |\lambda; \mathbf{p}_{12}\rangle \otimes |m, \mathbf{p}_3\rangle. \tag{3.1}$$

where $\lambda$ labels the mass eigenvalue $M_\lambda$ of the bound state of the interacting two-body mass operator, $M_{12}$, $\mathbf{p}_{12}$ is its three-momentum, and $m$ and $\mathbf{p}_3$ are the mass and momentum of particle 3. Since the scalar density $j(x)$ acts only in the space of particle 3, the matrix element has the form

$${}_{TP}\langle \lambda, m; \mathbf{p}'_{12}, \mathbf{p}'_3|j(0)|\lambda, m; \mathbf{p}_{12}, \mathbf{p}_3\rangle_{TP} = \delta(\mathbf{p}'_{12} - \mathbf{p}_{12})\langle m; \mathbf{p}'_3|j(0)|m; \mathbf{p}_3\rangle, \tag{3.2}$$

where

$$\sqrt{\omega_m(\mathbf{p}'_3)}\langle m, \mathbf{p}'_3|j(0)|m, \mathbf{p}_3\rangle\sqrt{\omega_m(\mathbf{p}_3)} = mG(q^2); \quad q = p'_3 - p_3, \tag{3.3}$$

and $q$ is the four-momentum transferred to particle 3 and $\omega_m(\mathbf{p}_3) = \sqrt{m^2 + \mathbf{p}_3^2}$. The square roots in Eq. (3.3) ensure the Lorentz invariance of the form factor $G(q^2)$ when the states of particle 3 have a delta-function normalization:

$$\langle m, \mathbf{p}'_3|m, \mathbf{p}_3\rangle = \delta(\mathbf{p}'_3 - \mathbf{p}_3). \tag{3.4}$$

If we specify initial momenta, $\mathbf{p}_3$ and $\mathbf{p}_{12}$, and the momentum transfer, $q$, and integrate the matrix element over the final spectator momentum, $\mathbf{p}'_{12}$, the integral collapses due to the spectator momentum delta function in Eq. (3.2), and we have

$$\mathcal{F}_{TP} := \int d\mathbf{p}'_{12}\sqrt{\omega_m(\mathbf{p}'_3)}{}_{TP}\langle \lambda, m; \mathbf{p}'_{12}, \mathbf{p}'_3|j(0)|\lambda, m; \mathbf{p}_{12}, \mathbf{p}_3\rangle_{TP}\sqrt{\omega_m(\mathbf{p}_3)} = mG(q^2). \tag{3.5}$$

Equation (3.5) is the Tensor-Product ($TP$) result. It has no dependence upon the momentum $\mathbf{p}_{12}$ of the bound-state spectator, as expected from the physical requirement of cluster separability, and does not depend upon the specific values of $\mathbf{p}_3$ and $\mathbf{p}'_3$, as long as $p'_3 - p_3 = q$. The result is just the invariant form factor of particle 3 multiplied by its mass.

The representation for a three-particle system becomes a tensor product of a one-body and two-body representation of the Poincaré whenever the interactions involving one of the particles is set to zero. This implies that the generators have the form (2.5). Ideally one would simply perform all calculations using the $TP$ representation. However, for examples beyond the simple model presented here, it is a highly non-trivial task to obtain a tensor-product representation that preserves Poincaré invariance. The only known method for systems with a finite number of degrees of freedom is to obtain first the Bakamjian-Thomas representation (which has interacting generators that satisfy the Poincaré commutation relations), and then to restore cluster separability by a recursive construction involving a hierarchy Sokolov's unitary transformations. In the special case of the three-body system the two representations are related by a single unitary transformation, $A$:

$$|\psi\rangle_{TP} = A|\psi\rangle_{BT}. \tag{3.6}$$

The operator $A$ is interaction dependent and contains the corrections that restore cluster properties to the $BT$ representation. The three-body $TP$ and $BT$ representations related by $A$ have the same 3-body $S$ matrix, but off-shell differences lead to different predictions in the four-body system. The operator $A$ converts a $BT$ representation to a $TP$ representation satisfying the constraints imposed by the Poincaré commutation relations. However, for realistic three-body systems, $A$ is a complicated interaction-dependent operator.

The size of the difference

$${}_{TP}\langle \psi'|j(0)|\psi\rangle_{TP} - {}_{BT}\langle \psi'|j(0)|\psi\rangle_{BT}, \tag{3.7}$$

where the $BT$ and $TP$ states are related by Eq. (3.6), provides a measure of the importance of the corrections needed to restore cluster separability.

While in general the Sokolov operators $A$ need to be computed, the test model presented here is simple enough that the $TP$ and $BT$ states related by $A$ can be computed directly, and then used in (3.7) to estimate the size of the corrections needed to restore cluster separability.

### B. $BT$ Representations

In the following sections, we evaluate matrix elements of the scalar density (3.2) using $BT$ representations of the 2+1-particle system corresponding to different forms of dynamics [2] and compare these results to the tensor product result. These matrix elements will each depend upon the momentum dependence and mass eigenvalue of the two-body bound-state wave function, in violation of cluster separability, and our goal is to examine the magnitude of that violation. In all cases, we specify the four momentum $q$ transferred to the struck particle and bound-state momentum $\mathbf{p}_{12}$ for a given calculation and then vary these momenta for sensitivity tests. Dependence of this matrix element on $\mathbf{p}_{12}$ indicates a failure of cluster properties.

In addition to the external momenta $\mathbf{p}_3$ and $\mathbf{p}_{12}$, the calculations depend upon the momenta of the constituent particles, which we label $\mathbf{b}_1$ , $\mathbf{b}_2$ and $\mathbf{b}_3$. The constituent momenta are not experimentally accessible, and their relation to the external momenta depends in turn upon the form of dynamics, as is discussed in detail below. For purposes of comparison we vary the external momenta keeping $\mathbf{b}'_3 - b_3 = q$. This requires different kinematic conditions for each form of dynamics.

### C. $BT$ Representation: Instant Form

We now evaluate the scalar density matrix element in Eq. (3.2) using an instant-form $BT$ model. Instant-form models have no interactions in the generators of rotations and space translations.

We provide first an observation about frames that will be relevant to each of the $BT$ representations. Calculations of current-matrix observables typically use a Breit frame (in which the energy transfer is zero), a lab frame (initial target three-momentum is zero), or an anti-lab frame (final target momentum is zero). There is no frame in which both initial and final target three-momenta can be zero, so the calculation will require Lorentz transformations relating frames with different target momentum. These are provided explicitly where needed below.

First, we change variables, replacing the bound state and particle 3 momenta by the total momentum, $\mathbf{P}$, of the system and the momentum of particle 3 transformed to the frame in which $\mathbf{P} = 0$ by a rotationless Lorentz transformation $\Lambda^{-1}(\mathbf{P}/M)$:

$$(\mathbf{p}_{12}, \mathbf{p}_3) \to (\mathbf{P}, \mathbf{p}), \tag{3.8}$$

where

$$\mathbf{P} := \mathbf{p}_{12} + \mathbf{p}_3; \quad p := \Lambda^{-1}(\mathbf{P}/M)p_3; \quad \mathbf{p} = \mathbf{p}_3 + \Phi_-(\mathbf{p}_3, \mathbf{P}, M)\mathbf{P}, \tag{3.9}$$

and

$$\begin{aligned} \Phi_-(\mathbf{p}_3, \mathbf{P}, M) &:= \frac{1}{M}\left[\frac{\mathbf{P}\cdot\mathbf{p}_3}{E+M} - \omega_m(\mathbf{p}_3)\right]; \\ M &= \sqrt{E^2 - \mathbf{P}^2}; \quad E = \sqrt{m^2 + \mathbf{p}_3^2} + \sqrt{M_\lambda^2 + \mathbf{p}_{12}^2}; \\ \omega_m(\mathbf{p}_3) &= \sqrt{m^2 + \mathbf{p}_3^2}. \end{aligned} \tag{3.10}$$

With this variable change the relation between the (12) bound state and particle 3 in these bases is:

$$|\lambda, m; \mathbf{p}_{12}, \mathbf{p}_3\rangle = \left|\frac{\partial(\mathbf{P}, \mathbf{p})}{\partial(\mathbf{p}_{12}, \mathbf{p}_3)}\right|_\lambda^{\frac{1}{2}} |\lambda, m; \mathbf{P}, \mathbf{p}\rangle, \tag{3.11}$$

where the suffix $\lambda$ in the Jacobian indicates that the interacting two-body mass eigenvalue $M_\lambda$ was used in the transformation.

Equations (3.8-3.11) connect the momentum variables via a Lorentz transformation that depends upon the *interacting* mass $M_\lambda$. The $BT$ construction also uses a transformation that uses a *non-interacting* mass. To make this connection, we introduce constituent-particle momenta $\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3$. We use a different notation for these constituent momenta because they are not necessarily the same as $\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3$; for example, we will see that $\mathbf{b}_3 \neq \mathbf{p}_3$. We define variables analogous to Eqs. (3.8-3.10) for the non-interacting system

$$(\mathbf{b}_1, \mathbf{b}_3, \mathbf{p}_3) \to (\mathbf{B}, \mathbf{b}, \mathbf{k}). \tag{3.12}$$

The variables $\mathbf{B}$ and $\mathbf{b}$ are defined analogously to the definitions of $\mathbf{P}$ and $\mathbf{p}$:

$$\mathbf{B} := \mathbf{b}_{12} + \mathbf{b}_3 \qquad \mathbf{b}_{12} := \mathbf{b}_1 + \mathbf{b}_2 \tag{3.13}$$

$$b := \Lambda^{-1}(\mathbf{B}/M_0)b_3; \quad \mathbf{b} = \mathbf{b}_3 + \Phi_-(\mathbf{b}_3, \mathbf{B}, M_0)\mathbf{B}, \tag{3.14}$$

$$E_0 := \sum_i \sqrt{\mathbf{b}_i^2 + m^2} \qquad M_0 := \sqrt{E_0^2 - \mathbf{B}^2}. \tag{3.15}$$

The additional variable $\mathbf{k}$ is obtained by (1) first boosting all three $b_i$ to the rest frame of the non-interacting three-body system, followed by a boost of particle 1 to the rest frame to the (12) pair. The vector $\mathbf{k}$ is the three-momentum of particle 1 after applying these two transformations. While the expression for $\mathbf{k}$ as a function of the momenta $\mathbf{b}_i$ can be written down explicitly, only the magnitude of $k$ plays a non-trivial role in our calculations. It is related to the invariant mass of the non-interacting two and three-body systems by

$$M_k := 2\sqrt{m^2 + \mathbf{k}^2} \qquad M_0 = \sqrt{M_k^2 + \mathbf{b}^2} + \sqrt{m^2 + \mathbf{b}^2}. \tag{3.16}$$

In the instant-form Bakamjian-Thomas construction, we identify

$$\mathbf{B} = \mathbf{P}; \quad \mathbf{b} = \mathbf{p}. \tag{3.17}$$

The first of these identifications is a general property of an instant-form dynamics, while the second equation is a requirement of the Bakamjian-Thomas construction.

These identifications and the above definitions imply relations between the observable momenta $\mathbf{p}_3, \mathbf{p}_{12}$ and the constituent-particle momenta, $\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3$. It is because $\mathbf{p}$ and $\mathbf{b}$ are related to the momentum of particle 3 by Lorentz boosts involving different masses that $\mathbf{p}_3 \neq \mathbf{b}_3$.

The BT four-momentum eigenstate with particles 1 and 2 bound in the three-particle basis is

$$\langle \mathbf{B}, \mathbf{b}, \mathbf{k} | \lambda; \mathbf{P}', \mathbf{p}' \rangle_{BT} = \delta(\mathbf{P}' - \mathbf{B})\delta(\mathbf{p}' - \mathbf{b})\phi_\lambda(\mathbf{k}), \tag{3.18}$$

where $\phi_\lambda(\mathbf{k})$ is the bound-state wave function. Note that $\mathbf{b}$ depends upon the free two-particle invariant mass $M_{120}$ via Eq. (3.14-3.16), while $\mathbf{p}'$ depends upon the bound state mass eigenvalue via $\lambda$ Eq. (3.9-3.10). The association of $\mathbf{p}$ and $\mathbf{b}$ in the delta function of Eq. (3.18) is only consistent when $\mathbf{P} = 0$. This is because in the instant form Bakamjian-Thomas construction the dynamical model is solved in the rest frame and the resulting mass eigenvalue determines how the system behaves under Lorentz boosts. The identification of $\mathbf{p}$ with $\mathbf{b}$ becomes inconsistent in frames where $\mathbf{P} \neq 0$ because these quantities are defined using different boosts. Since the initial and final states in a scalar density matrix element are in different frames, this inconsistency cannot be avoided and it results in the violation of cluster properties.

We now make use of Eq. (3.18) to compute the $BT$ counterpart to $\mathcal{F}_{TP}$ that was defined by Eq. (3.5):

$$\begin{aligned}
\mathcal{F}_{BT} &:= \int d\mathbf{p}'_{12} \sqrt{\omega_m(\mathbf{p}'_3)}_{BT} \langle \lambda, m; \mathbf{p}'_{12}, \mathbf{p}'_3 | j(0) | \lambda, m; \mathbf{p}_{12}, \mathbf{p}_3 \rangle_{BT} \sqrt{\omega_m(\mathbf{p}_3)} \\
&:= \int d\mathbf{p}'_{12} d\mathbf{k}' d\mathbf{k} \sqrt{\omega_m(\mathbf{p}'_3)} \\
&\quad \times \left| \frac{\partial(\mathbf{P}', \mathbf{p}')}{\partial(\mathbf{p}'_{12}, \mathbf{p}'_3)} \right|_\lambda^{\frac{1}{2}} \left| \frac{\partial(\mathbf{P}, \mathbf{p})}{\partial(\mathbf{p}_{12}, \mathbf{p}_3)} \right|_\lambda^{\frac{1}{2}} \left| \frac{\partial(\mathbf{b}'_{12}, \mathbf{b}'_3)}{\partial(\mathbf{P}', \mathbf{p}')} \right|_k^{\frac{1}{2}} \left| \frac{\partial(\mathbf{b}_{12}, \mathbf{b}_3)}{\partial(\mathbf{P}, \mathbf{p})} \right|_k^{\frac{1}{2}} \\
&\quad \times \phi_\lambda^*(\mathbf{k}') \langle \mathbf{b}'_{12}, \mathbf{b}'_3, \mathbf{k}' | j(0) | \mathbf{b}_{12}, \mathbf{b}_3, \mathbf{k} \rangle \phi_\lambda(\mathbf{k}) \sqrt{\omega_m(\mathbf{p}_3)}.
\end{aligned} \tag{3.19}$$

The suffix $k$ in the Jacobian indicates that the non-interacting two-body invariant mass, $M_k$(Eq. 3.16), was used in the transformation of Eq. (3.14), while the suffix $\lambda$ indicates that the mass of the bound pair, $M_\lambda$, was used to compute the variable change. The $\mathbf{k}$-dependence of Jacobians with the subscript $k$ is limited to the magnitude of $\mathbf{k}$.

Since the scalar density operator $j(x)$ operates only in the space of particle 3, we have

$$\sqrt{\omega_m(\mathbf{b}'_3)} \langle \mathbf{b}'_{12}, \mathbf{b}'_3, \mathbf{k}' | j(0) | \mathbf{b}_{12}, \mathbf{b}_3, \mathbf{k} \rangle \sqrt{\omega_m(\mathbf{b}'_3)} = m\delta(\mathbf{k}' - \mathbf{k})\delta(\mathbf{b}'_{12} - \mathbf{b}_{12}) G[(b'_3 - b_3)^2]. \tag{3.20}$$

The integral over $\mathbf{p}'_{12}$ can be converted to an integral over $\mathbf{b}'_{12}$. The Jacobian of the variable change is

$$\int d\mathbf{p}'_{12} = \int d\mathbf{b}'_{12} \left| \frac{\partial(\mathbf{P}', \mathbf{p}')}{\partial(\mathbf{b}_{12}, ' \mathbf{b}'_3)} \right|_k \left| \frac{\partial(\mathbf{p}'_{12}, \mathbf{p}'_3)}{\partial(\mathbf{P}', \mathbf{p}')} \right|_\lambda . \tag{3.21}$$

The final result is

$$\mathcal{F}_{BT}^{\rm instant} = \frac{\sqrt{\omega_m(\mathbf{p}'_3)}}{\sqrt{\omega_m(\mathbf{b}'_3)}} \int d\mathbf{k} \left|\frac{\partial(\mathbf{p}'_{12},\mathbf{p}'_3)}{\partial(\mathbf{P}',\mathbf{p}')}\right|_\lambda^{\frac{1}{2}} \left|\frac{\partial(\mathbf{P},\mathbf{p})}{\partial(\mathbf{p}_{12},\mathbf{p}_3)}\right|_\lambda^{\frac{1}{2}} \left|\frac{\partial(\mathbf{P}',\mathbf{p}')}{\partial(\mathbf{b}'_{12},\mathbf{b}'_3)}\right|_k^{\frac{1}{2}} \left|\frac{\partial(\mathbf{b}_{12},\mathbf{b}_3)}{\partial(\mathbf{P},\mathbf{p})}\right|_k^{\frac{1}{2}} |\phi_\lambda(\mathbf{k})|^2 \frac{\sqrt{\omega_m(\mathbf{p}_3)}}{\sqrt{\omega_m(\mathbf{b}_3)}} mG[(b'_3 - b_3)^2]. \tag{3.22}$$

The relevant observation is that this integral has a non-trivial dependence on the momentum, $\mathbf{p}_{12}$, of the bound state, in contrast to the $\mathbf{p}_{12}$ independence of the tensor product result. In this model the $BT$ and tensor product representations of the three-particle system are related by one of Sokolov's unitary transformations, $A$, that preserve the three-body $S$-matrix:

$$|\lambda, m; \mathbf{p}_{12}, \mathbf{p}_3\rangle_{TP} = A|\lambda, m; \mathbf{p}_{12}, \mathbf{p}_3\rangle_{BT}. \tag{3.23}$$

The scale of the $\mathbf{p}_{12}$ dependence in Eq. (3.22) provides a measure of the size of the violation of cluster properties that results from ignoring $A$ by replacing it with the identity. Since we know both states in Eq. ( 3.23) we do not have to calculate $A$ explicitly to determine its impact.

For this calculation, we vary the three-momentum transfer $\mathbf{q}$ and the spectator momentum $\mathbf{p}_{12}$. The initial momentum of particle 3 and the final momentum of the system are fixed in terms of these variables:

- $\mathbf{p}_3 = -\frac{1}{2}\mathbf{q}$;
- $\mathbf{P}' = \mathbf{P} + \mathbf{q}$.

Since in the instant form,

$$\mathbf{P} = \mathbf{B}; \qquad \mathbf{P}' = \mathbf{B}', \tag{3.24}$$

and the scalar density matrix element constrains $\mathbf{b}'_{12} = \mathbf{b}_{12}$, we find that $\mathbf{b}'_3 = \mathbf{b}_3 + \mathbf{q}$, in which case the argument of the form factor is $\mathbf{q}^2$ as it is in the $TP$ case. However, the final momenta $\mathbf{p}'_{12}$ and $\mathbf{p}'_3$ are not constrained, and in general there are non-vanishing contributions to this matrix element for $\mathbf{p}'_3 \neq \mathbf{p}_3 + \mathbf{q}$ and $\mathbf{p}'_{12} \neq \mathbf{p}_{12}$.

In the nonrelativistic limit, where $\mathbf{k}$, $\mathbf{q}$ and $\mathbf{p}_{12}$ are all small with respect to the relevant masses, the Jacobians are approximately unity and can be factored out of the integral, leaving a unit wave function normalization and a result identical to the $TP$ case. The quantitative level of disagreement with the $TP$ result is therefore linked to the extent to which the model goes beyond the nonrelativistic limit.

For the other forms of dynamics we preform similar calculations. However, rather than using the natural variables for each given form of dynamics (four-velocity, light-front components of the four-momentum), we use the same variables employed in the instant-form case to facilitate a comparison of the size of the violations of cluster properties. These variables are also the quantities that are most readily measured using detectors. Calculations using front and point-form dynamics lead to results that have the same general form as Eq. (3.22) involving an integral over Jacobians and their inverses involving different sets of variables and the bound state wave functions. The Jacobians all cancel in the limit that two-body mass eigenvalue becomes the two-body invariant mass.

### D. $BT$ Representation: Front Form

Dirac's front-form dynamics is described in detail in Ref. [4]. In a front-form dynamics the generators of transformations that leave a plane tangent to the light cone invariant are free of interactions. We provide a summary here.

Basis states in the front form are eigenstates of the light-front components of the momenta

$$\tilde{\mathbf{p}} = (\mathbf{p}_\perp, p^+); \quad p^+ = p^0 + p^3; \quad \mathbf{p}_\perp = (p^1, p^2). \tag{3.25}$$

These operators generate translations in a hyperplane $x^- := x^0 - x^3 = 0$ tangent to the light cone. In what follows we use a tilde to indicate the light-front components of four-vectors.

In the front form the Lorentz transformations Eq. (3.9) and Eq. (3.14) used to define a particle momentum in the rest frame of the three-particle system in the instant form are replaced by boosts that leave the light front invariant. While these boosts are interaction independent, there is an interaction dependence in the boost parameters.

For the front-form Bakamjian-Thomas construction Eq. (3.17) is replaced by

$$\tilde{\mathbf{B}} = \tilde{\mathbf{P}}; \quad \mathbf{b} = \mathbf{p}. \tag{3.26}$$

As in the instant-form case the first equation is a general property of a light-front dynamics while the second equation is an additional requirement of the Bakamjian-Thomas construction. The second equation identifies the momentum of particle 3 in the rest frame of the non-interacting three-particle system, $\tilde{\mathbf{p}}$, with the momentum, $\tilde{\mathbf{b}}$ of particle 3 in the rest frame of the $(12)+3$ system. As in the instant-form case this identification has no consequences for the two-body $S$ matrix, but will lead to a violation of cluster separability at the four-body level. With this modification the front-form BT expression for $\mathcal{F}_{BT}^{\rm front}$ is similar to the instant-form result:

$$\mathcal{F}_{BT}^{\rm front} := \int d\mathbf{k}\sqrt{\frac{p_3^{+\prime}}{b_3^{+\prime}}}\left|\frac{\partial(\tilde{\mathbf{p}}'_{12},\tilde{\mathbf{p}}'_3)}{\partial(\tilde{\mathbf{P}}',\tilde{\mathbf{p}}')}\right|_\lambda^{\frac{1}{2}}\left|\frac{\partial(\tilde{\mathbf{P}},\tilde{\mathbf{p}})}{\partial(\tilde{\mathbf{p}}_{12},\tilde{\mathbf{p}}_3)}\right|_\lambda^{\frac{1}{2}}\left|\frac{\partial(\tilde{\mathbf{P}}',\tilde{\mathbf{p}}')}{\partial(\tilde{\mathbf{b}}'_{12},\tilde{\mathbf{b}}'_3)}\right|_k^{\frac{1}{2}}\left|\frac{\partial(\tilde{\mathbf{b}}_{12},\tilde{\mathbf{b}}_3)}{\partial(\tilde{\mathbf{P}},\tilde{\mathbf{p}})}\right|_k^{\frac{1}{2}}|\phi_\lambda(\mathbf{k})|^2\sqrt{\frac{p_3^+}{b_3^+}}mG[(b'_3-b_3)^2]. \quad (3.27)$$

This quantity has an unphysical dependence on $\tilde{\mathbf{b}}_{12}$ that does not occur in the tensor product.

The instant- and front-form calculations differ in both the constraints (3.26) vs (3.26) and the different boosts use to construct $\mathbf{p},\tilde{\mathbf{p}},\mathbf{b}$ and $\tilde{\mathbf{p}}$. For the purpose of the comparison with the instant-form calculations we make a final variable change replacing the light-front components of the momenta by the same variables that were used in the instant-form calculation.

For this calculation, we vary $\tilde{\mathbf{q}}$ and $\tilde{\mathbf{p}}_{12}$, with coordinate axes chosen such that $q^+=0$ and $p_{12}^+=0$. We define $\mathbf{p}_\perp$ and $\tilde{\mathbf{P}}'$ in terms of these quantities:

- $\mathbf{p}_\perp = -\frac{1}{2}\mathbf{q}_\perp$;
- $\tilde{\mathbf{P}}' = \tilde{\mathbf{P}} + \tilde{\mathbf{q}}$.

Analogous to the discussion following Eq. (3.24), we have that

$$\tilde{\mathbf{P}} = \tilde{\mathbf{B}} \qquad \tilde{\mathbf{P}}' = \tilde{\mathbf{B}}', \quad (3.28)$$

The scalar density matrix element constrains $\tilde{\mathbf{b}}'_{12} = \tilde{\mathbf{b}}_{12}$, and therefore $\tilde{\mathbf{b}}'_3 = \tilde{\mathbf{b}}_3 + \tilde{\mathbf{q}}$, in which case the argument of the form factor is $\mathbf{q}^2$ as it is in the $TP$ case. However, the final momenta $\tilde{\mathbf{p}}'_{12}$ and $\tilde{\mathbf{p}}'_3$ are not constrained, and in general the integral has non-zero contributions from $\tilde{\mathbf{p}}'_3 \neq \tilde{\mathbf{p}}_3 + \tilde{\mathbf{q}}$ and $\tilde{\mathbf{p}}'_{12} \neq \tilde{\mathbf{p}}_{12}$.

The constraints $\mathbf{p}_\perp = -\frac{1}{2}\mathbf{q}_\perp$; and $\tilde{\mathbf{P}}' = \tilde{\mathbf{P}} + \tilde{\mathbf{q}}$ ensure that the momentum transfer to the one-body system is q, as in the instant form example.

### E. $BT$ Representation: Point Form

Dirac's point-form dynamics are also described in detail in Ref. [4]. In this case the Lorentz group is non-interacting. We provide a summary here.

Basis states in the point form are described by four-velocity vectors $\mathbf{v}$. Momenta are obtained by multiplying the four-velocities by (interacting or non-interacting) masses. Thus, we seek to evaluate matrix elements of the scalar density operator $j(x)$ between states of particle 3 and the (12) bound state with initial four-velocity $\mathbf{V}$ and final four-velocity $\mathbf{V}'$.

In the point-form Bakamjian-Thomas construction, equations (3.17) and 3.26 are replaced by

$$\mathbf{V} = \mathbf{B}/M_0; \quad \mathbf{b} = \mathbf{p}. \quad (3.29)$$

The first of these equations corresponds to total four-velocity conservation, which is a property of all point-form representations. The boosts in the point-form are the same as those used in the instant form, but they are parameterized by the conserved four-velocities. The second equation, as in the instant and from form, is a requirement of the Bakamjian-Thomas construction.

The derivations proceed in a fashion similar to the instant form.

With these conventions, the point-form $\mathcal{F}_{BT}$ has the structure

$$\mathcal{F}_{BT}^{\rm point} := \int d\mathbf{k}\sqrt{\frac{\omega_m(\mathbf{p}'_3)}{\omega_m(\mathbf{b}'_3)}}\left|\frac{\partial(\mathbf{p}'_{12},\mathbf{p}'_3)}{\partial(\mathbf{V}',\mathbf{p}')}\right|_\lambda^{\frac{1}{2}}\left|\frac{\partial(\mathbf{V},\mathbf{p})}{\partial(\mathbf{p}_{12},\mathbf{p}_3)}\right|_\lambda^{\frac{1}{2}}\left|\frac{\partial(\mathbf{V}',\mathbf{p}')}{\partial(\mathbf{b}'_{12},\mathbf{b}'_3)}\right|_k^{\frac{1}{2}}\left|\frac{\partial(\mathbf{b}_{12},\mathbf{b}_3)}{\partial(\mathbf{V},\mathbf{p})}\right|_k^{\frac{1}{2}}|\phi_\lambda(\mathbf{k})|^2\sqrt{\frac{\omega_m(\mathbf{p}_3)}{\omega_m(\mathbf{b}_3)}}mG[(b'_3-b_3)^2]. \quad (3.30)$$

Note in this case that the velocities $\mathbf{V}$ and $\mathbf{V}'$ are the same for the kinematics using the interacting mass $M$ or the free mass $M_0$. Since the boosts depend on $\mathbf{V}$ it follows that

$$\mathbf{b}_3 = \mathbf{p}_3; \quad \mathbf{b}'_3 = \mathbf{p}'_3, \quad (3.31)$$

and therefore that $\mathbf{b}'_3 - \mathbf{b}_3 = \mathbf{p}'_3 - \mathbf{p}_3 = \mathbf{q}$. One consequence is that in general, $M'\mathbf{V}' - M\mathbf{V} \neq \mathbf{q}$.

We constrain the momentum transfer to particle 3:

- $\mathbf{b}_3 = -\frac{1}{2}\mathbf{q}$;
- $\mathbf{b}'_3 = \mathbf{b}_3 + \mathbf{q}$.

which is the same constraint on $\mathbf{b}_3$ that was used in the instant and front-form calculations.

These calculations share the property that $\mathbf{b}'_3 - \mathbf{b}_3 = \mathbf{q}$ in the instant and point forms, and $\tilde{\mathbf{b}}'_3 - \tilde{\mathbf{b}}_3 = \tilde{\mathbf{q}}$ in the front form, i.e. the constituent-particle momenta are related to each other at the three-vector level by the momentum transfer to the overall three-body system. One could choose different sets of particle momenta in calculating $BT$ matrix elements to compare to the $TP$ result. For example, we could constrain the three calculations such that $P' - P = q$ in some frame. Different choices explore different kinematic regions of the difference between $A$ and the identity. Our investigations of other choices indicate numerical effects of comparable size.

## IV. RESULTS AND DISCUSSION

In this section we discuss results of calculations of the form factor $\mathcal{F}_{BT}$ of a scalar density for instant-, front- and point-form kinematic choices. Initially we consider scales that are relevant for systems of nucleons interacting with two-body interactions. Then we turn to examples more appropriate to hadronic models with subnucleon degrees of freedom.

In all figures we display the figure of merit:

$$\frac{(\mathcal{F}_{BT} - \mathcal{F}_{TP})}{\mathcal{F}_{TP}}, \tag{4.1}$$

which represents the relative error induced by ignoring the unitary transformation Eq. (3.23) that restores cluster separability.

### A. Malfliet-Tjon Deuteron Wave Function

To model realistic conditions for nuclear physics, we use a bound-state spectator with deuteron properties constructed from Malfliet-Tjon [22] potential IV, which contains both attractive and repulsive forces. Ref. [4] discusses how to make a phase-equivalent relativistic model that has the same wave functions as the non-relativistic model, and our numerical results follow that approach. The invariant form factor $G(q^2)$ is taken as a dipole form factor.

We first consider the figure of merit as a function of the momentum transfer and momentum of the bound state in each of Dirac's forms of dynamics and for the bound state momentum perpendicular and parallel to the momentum transfer. The results are shown in the six figures below. In all cases the expected results are given by the flat planes.

The fractional deviation of this $BT$ model calculation from the $TP$ result that satisfies cluster separability is very small for all three forms of dynamics, typically of order $10^{-3}$ or smaller at the highest values of the three-momenta $\mathbf{q}$ and $\mathbf{p}_{12}$.

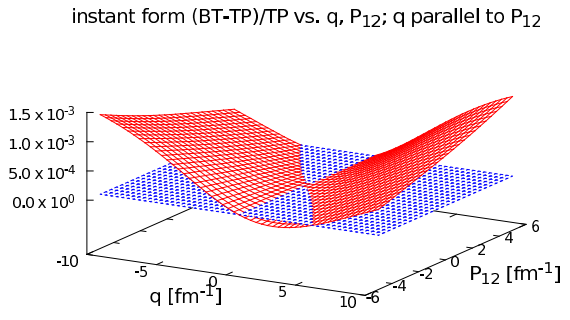


FIG. 2: (Color online) Model differences for instant-form $BT$ calculation, $\mathbf{q} \| \mathbf{p}_{12}$.

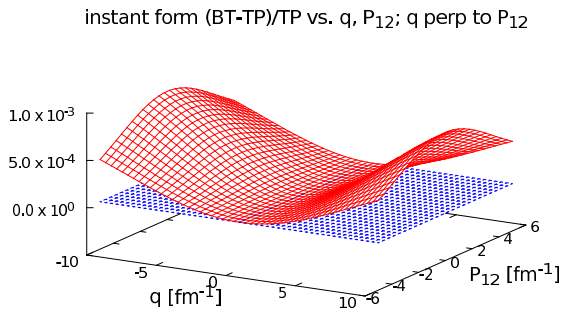


FIG. 3: (Color online) Model differences for instant-form $BT$ calculation, $\mathbf{q} \perp \mathbf{p}_{12}$.

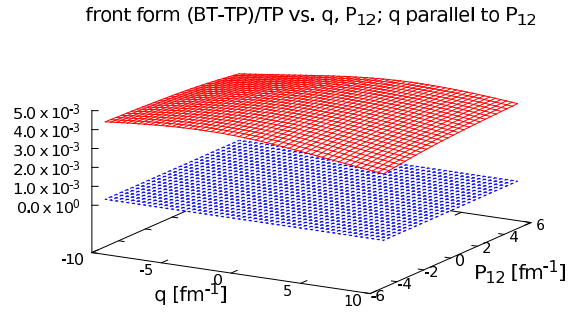


FIG. 4: (Color online) Model differences for front-form $BT$ calculation, $\mathbf{q}\|\mathbf{p}_{12}$.

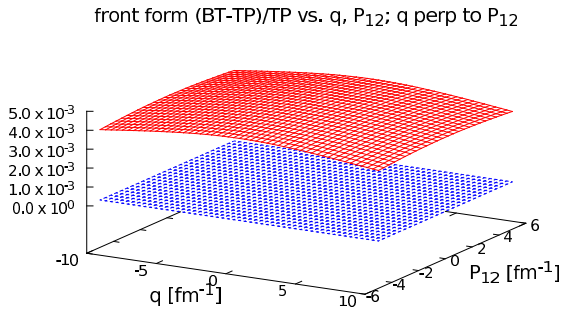


FIG. 5: (Color online) Model differences for front-form $BT$ calculation, $\mathbf{q} \perp \mathbf{p}_{12}$.

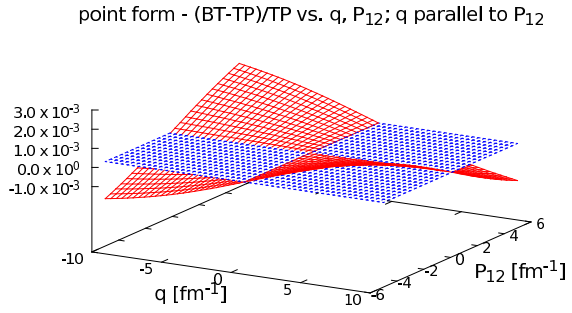


FIG. 6: (Color online) Model differences for point-form $BT$ calculation A, $\mathbf{q}\|\mathbf{p}_{12}$.

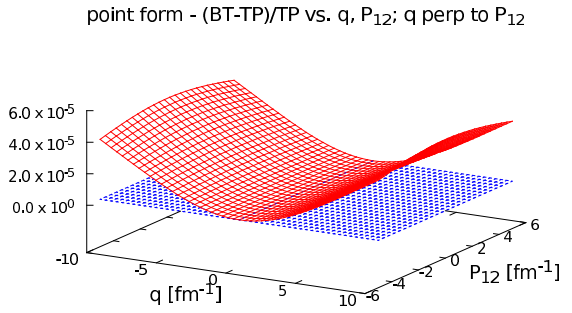


FIG. 7: (Color online) Model differences for point-form $BT$ calculation A, $\mathbf{q} \perp \mathbf{p}_{12}$.

### B. Binding Energy Variation

The calculations above assumed a bound state with a wave function having a typical dependence on the relative momentum of the constituent nucleons. The next set of curves illustrates the figure of merit for fixed values of $\mathbf{q}$ and $\mathbf{p}_{12}$ as we vary the binding energy and momentum scale of the wave function. The variations that we consider are still scales that are relevant to nuclear systems. Figures 8 and 9 show the results of calculations for varying binding energy with a Malfliet-Tjon wave function.

As with the earlier cases that employed the bound-state binding energy, the fractional deviation of the $BT$ results from the $TP$ benchmark is quite small, of order $10^{-3}$ or less.

### C. Wave Function Scale Variation

We also examined sensitivity to the momentum scale, $k_0$, of the wave function by replacing the Malfliet-Tjon wave function with a Gaussian form:

$$\phi(\mathbf{k}) = \frac{1}{\sqrt{N}} e^{-(k/k_0)^2}. \tag{4.2}$$

Figures 10 and 11 show the results for the spectator momentum perpendicular and parallel to the momentum transfer in all three forms of dynamics.

These results mirror those discussed above: the fractional deviation of the $BT$ results from the $TP$ benchmark is quite small, of order $10^{-3}$ or less.

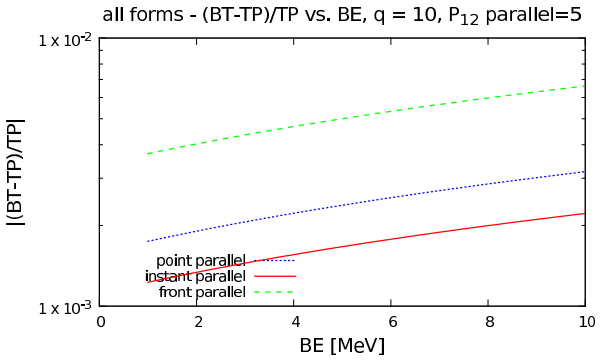


FIG. 8: (Color online) Model differences for $BT$ calculations as a function of two-body spectator binding energy, $\mathbf{q} = 10$ fm$^{-1}||\mathbf{p}_{12}$.

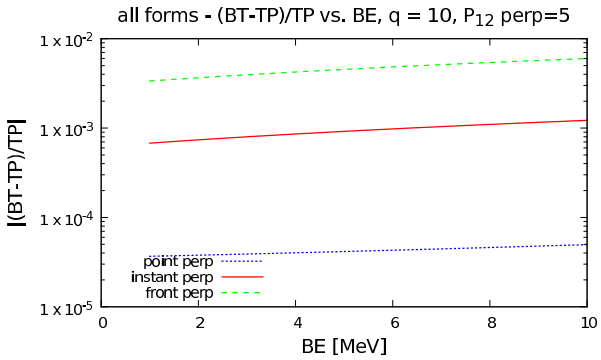


FIG. 9: (Color online) Model differences for $BT$ calculations as a function of two-body spectator binding energy, $\mathbf{q} = 10$ fm$^{-1} \perp \mathbf{p}_{12}$.

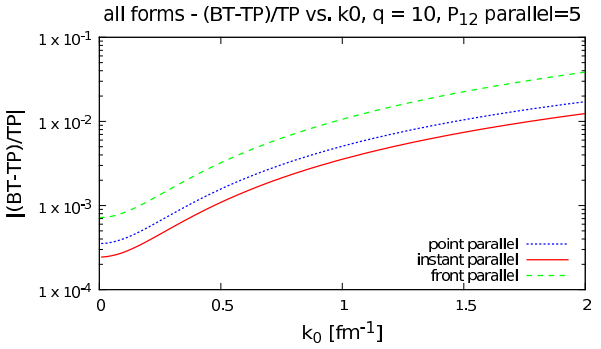


FIG. 10: (Color online) Model differences for $BT$ calculations as a function of two-body Gaussian wave function scale, $\mathbf{q} = 10$ fm$^{-1}||\mathbf{p}_{12}$.

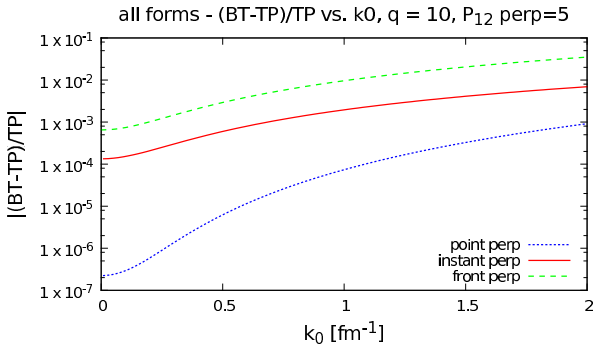


FIG. 11: (Color online) Model differences for $BT$ calculations as a function of two-body Gaussian wave function scale, $\mathbf{q} = 10$ fm$^{-1} \perp \mathbf{p}_{12}$.

### D. Implications for Nuclear Theory

Formally dynamical relativistic models should be Poincaré invariant and satisfy cluster properties; in addition operators associated with short ranged physical phenomena should have well defined cluster expansions.

While our model is considerably simpler than a realistic model of a three-nucleon system interacting with an electron, the difference between the Bakamjian-Thomas formulation of the model and the model that clusters properly is due entirely to the same unitary transformations discussed in our simplified model. This difference vanishes in the limit that these transformations become the identity. The analysis in this paper established that these operators, which implicitly depend on the interactions, are close to the identity for interactions with typical nuclear physics scales for binding energy and Fermi momentum. This provides a strong justification for applying Bakamjian-Thomas construction of dynamical representations Poincaré group to system of more than three nucleons.

Our results indicate that Bakamjian-Thomas models, which explicitly satisfy the requirements of Poincaré invariance, can be utilized in typical nuclear physics problems with minimal quantitative error due to the lack of cluster separability using any of Dirac's front-form dynamics.

### E. Implications for Hadron Models

The final set of figures show the results of calculations with scales that are more appropriate models of hadrons based on sub-nucleonic degrees of freedom.

We note here that confinement precludes separating arbitrary subsystems by large distance scales, so the requirement of cluster separability is irrelevant for models of individual hadrons. The issue may be relevant, however, for systems of hadrons described by sub-nucleonic degrees of freedom.

In these cases, in order to understand the relevant scales, we replace the particle 3 mass in the above calculations by a "constituent quark" mass of 220 MeV, and consider two-body ("diquark") masses ranging from 200 to 600 MeV for Gaussian wave functions with a 1 $\text{fm}^{-1}$ scale, and Gaussian wave functions with scale ranging from 0.5 $\text{fm}^{-1}$ to 10 $\text{fm}^{-1}$, with a diquark mass of 600 MeV.

The results are similar to those appropriate to nuclear physics discussed above, except that the scale of deviation from the $TP$ benchmark is on the scale of 10% for the instant, front and point-form $A$s. This not surprising given that the mass/momentum scale variation for these calculations is much higher than for typical cases in nuclear physics with nucleons. While these corrections may be relevant, multi-hadron models based on sub-nucleon degrees of freedom have not been developed to a precision where these effects could be identified.

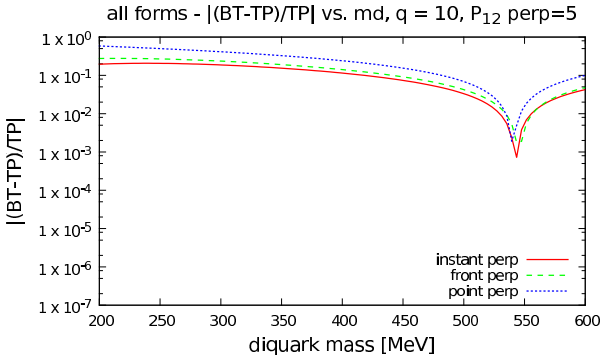


FIG. 12: (Color online) Model differences for $BT$ calculations as a function of two-body spectator mass, $\mathbf{q} = 10\ \text{fm}^{-1} \| \mathbf{p}_{12}$.

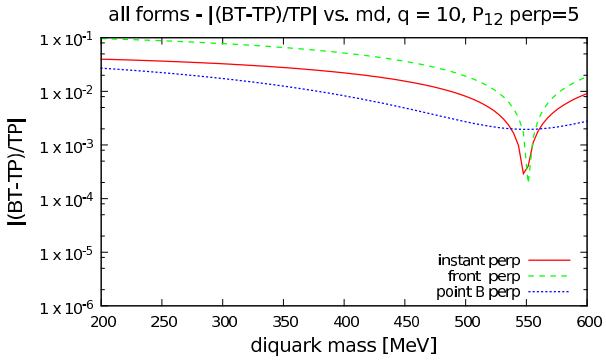


FIG. 13: (Color online) Model differences for $BT$ calculations as a function of two-body spectator mass, $\mathbf{q} = 10\ \text{fm}^{-1} \perp \mathbf{p}_{12}$.

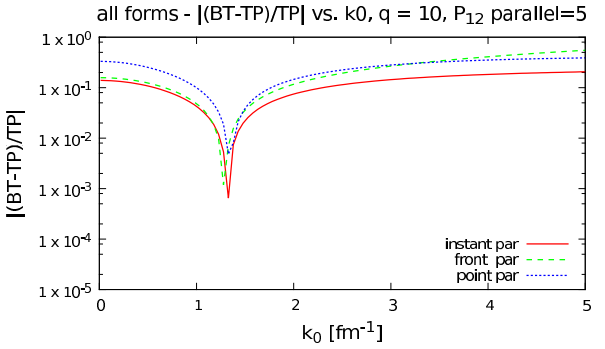


FIG. 14: (Color online) Model differences for $BT$ calculations as a function of Gaussian wave function scale, $\mathbf{q} = 10\ \text{fm}^{-1} \| \mathbf{p}_{12}$.

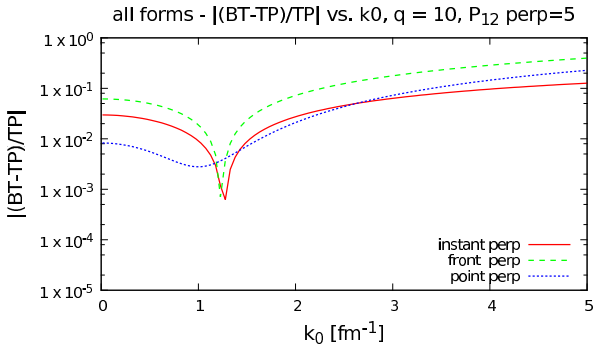


FIG. 15: (Color online) Model differences for $BT$ calculations as a function of Gaussian wave function scale, $\mathbf{q} = 10\ \text{fm}^{-1} \perp \mathbf{p}_{12}$.

## V. SUMMARY

Bakamjian-Thomas formulations, which explicitly satisfy the requirements of Poincaré invariance, do not satisfy cluster separability above the three-particle level, i.e. in systems that involve three-body subsystems whose total momentum must vary. The cluster properties can be restored via a hierarchy of unitary transformations. These transformations depend upon the full solution of a three-body problem, and are difficult to implement in practice.

Rather than attempt to calculate directly the size of these unitary transformations (e.g. the difference of matrix elements from those of the unit operator), we have developed a simple model in which the exact result consistent with cluster separability is known, and then compare to it the results of Bakamjian-Thomas calculations. The quantities that we investigate are sensitive to the part of these unitary transformations that restore cluster properties.

We conclude from these model studies that Bakamjian-Thomas models, which explicitly satisfy the requirements of Poincaré invariance, can be utilized in typical nuclear physics problems with minimal quantitative error due to the lack of cluster separability.

We also examined models utilizing mass/momentum scales appropriate for quark models. Confinement precludes separating arbitrary subsystems by large distance scales, so the general requirement of cluster separability is irrelevant for models of individual hadrons. The issue may be relevant, however, for systems of hadrons described by subnucleonic degrees of freedom. In such cases, the deviations from the model benchmark are larger (about 10%) than those for typical nuclear physics calculations with nucleons, though they are still small compared to model uncertainties.


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