The relation between instant and light-front formulations of quantum field theory

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The scattering equivalence of quantum field theories formulated with light-front and instant-form kinematic subgroups is established using non-perturbative methods. The difficulty with field theoretic formulations of Dirac's forms of dynamics is that the free and interacting unitary representations of the Poincaré group are defined on inequivalent representations of the Hilbert space, which means that the concept of kinematic transformations must be modified on the Hilbert space of the field theory. This work addresses this problem by assuming the existence of a field theory with the expected properties and constructs equivalent representations with instant and front form kinematic subgroups. In this construction both the light-front and instant-form formulations share the same vacuum and one-particle states. The free field Fock space plays no role. There is no "quantization" of a classical theory. The property that survives from the perturbative approach is the notion of a kinematic subgroup, which means kinematic Poincaré transformations can be trivially implemented by acting on suitable basis vectors. This non-perturbative approach avoids dealing with issues that arise in perturbative treatments where is it necessary to have a consistent treatment of renormalization, rotational covariance, and the structure of the light-front vacuum. While addressing these issues in a computational framework is important for applications, this work suggests the nature of the expected resolution and identifies some differences between the perturbative and non-perturbative approaches.

I. INTRODUCTION

This paper discusses the relation between light-front and instant-form formulations of quantum field theory. The identification of different forms of dynamics is due to Dirac [1]. In a relativistic quantum theory the invariance of quantum probabilities in different inertial coordinate systems requires that equivalent states in different inertial coordinate systems are related by a unitary ray representation of the subgroup of the Poincaré group continuously connected to the identity [2]. The Poincaré Lie algebra has 3 independent commutators involving rotationless boost and translation generators that have the Hamiltonian on the right

$$[K^i, P^j] = i\delta_{ij}H. \tag{1}$$

If $H = H_0 + V$ for some interaction V then the operators on the left side of each commutator must also know about the interaction. Dirac identified three representations of the Poincaré Lie algebra with the minimum number (3-4) of interaction-dependent Poincaré generators. He called these the instant, point and front-forms of the dynamics. In the instant form the generators of space translations and rotations are free of interactions, in the point form, the generators of rotationless Lorentz transformations are free of interactions and in the front form the generators of transformations that leave a hyperplane tangent to the light cone invariant are free of interactions.

The equivalence of these different representations of relativistic quantum mechanics was settled by Sokolov and Shatnyi [3][4]. In quantum field theory the problem is more complicated because the free and interacting dynamics are formulated on different inequivalent representations of the Hilbert space [5], so the decomposition of the Hamiltonian into the sum of a free Hamiltonian plus interaction is not defined on the Hilbert space of the field theory. Such a decomposition makes sense in perturbative quantum field theory with cutoffs, so the notion of instant- and light-front formulations [6][7][8][9][10][11] of quantum field theory make sense perturbatively. The price paid is that as the cutoffs are removed the theory has infinities that have to be renormalized. The relation between the different forms of dynamics depends on a consistent treatment of the renormalization. The light-front formulation of quantum field theory is of particular interest for applications. The most appealing property of the theory is the apparent triviality of the light-front vacuum, which reduces the solution of the field theory to linear algebra on a Hilbert space, like non-relativistic quantum mechanics. In addition to the computational challenges of implementing this program in a

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theory with an infinite number of degrees of freedom, there are a number of puzzles that appear in comparing the two approaches. These include:

- Is the light-front vacuum the same as the Fock vacuum?
- How to understand $P^+ = 0$ (zero mode) singularities?
- How to formulate spontaneous symmetry breaking in a light-front dynamics?
- How to renormalize the theory consistent with rotational covariance.
- What is the relation between light-front and canonical quantization of a quantum field theory?
- Are both approaches equivalent?

While there is a general consensus that the two formulations are the equivalent, the answers to the above questions are not as clean as desired. These issues have been discussed extensively in the literature [6] [7] [12] [13] [8] [9] [10] [11] [14] [15] [16] [17] [18] [19] [20] [21] [22] [23] [24] [25] [26] [27] [28] [29] [30] [31] [32] [33] [34] [35] [36] [37] [38] [39] [40]. Most of these discussions are based on perturbation theory; in particular the assumption that the light-front dynamics can be formulated on the Fock space of a free field theory. The continued interest in these questions is because compelling resolutions of these questions are obscured by the need to perform a non-perturbative renormalization of the theory. In this work these complications are avoided by assuming the existence of the theory with the expected properties. The existence of an asymptotically complete scattering theory is also assumed, based on the Haag-Ruelle formulation of scattering theory [41][42][43][44]. Haag-Ruelle scattering has the advantage that it can be formulated in terms of wave operators defined by strong limits. The construction in this paper makes use of the strong limits. A two-Hilbert space representation is used, which does not require the existence of a free dynamics on the Hilbert space of the theory. The second Hilbert space is the direct sum of tensor products of single-particle Hilbert spaces with physical masses. In the simplest case it is a Fock space with physical particle masses. The mapping from the second Hilbert space to the Hilbert space of the field theory puts in the internal structure of the physical particles when they are asymptotically separated. There is a natural unitary representation of the Poincaré group on the second Hilbert space, which treats the particles as free particles. The second Hilbert space is called the asymptotic Hilbert space. There is sufficient freedom to choose the mappings from the asymptotic Hilbert space to the Hilbert space of the field theory so the unitary representation of the instant or light-front kinematic subgroup of the Poincaré group on the asymptotic space maps on to the dynamical unitary representation of the kinematic subgroup on the field-theory Hilbert space without changing the scattering operator.

A theorem due to Ekstein [45] gives necessary and sufficient conditions for unitary transformations on a Hilbert space to preserve the scattering operator. This result assumes the scattering is formulated in terms of wave operators defined as strong limits. In the two-Hilbert space formulation Ekstein's condition is the requirement that if unitary transformation is applied to the mapping from the asymptotic Hilbert space to the field theory Hilbert space, it is asymptotically "equivalent" to the original mapping (see eq. (34)). This freedom can be exploited to construct the most general class of S-matrix preserving unitary transformations that are invariant under a kinematic subgroup of the dynamical representation of the Poincaré group. The proof uses the unitary of the wave operators which only holds if the vacuum and one-body channels are included in the asymptotic Hilbert space. Ekstein's condition then implies that acceptable unitary transformations must leave the vacuum and one-particle states unchanged. When these unitary transformations are applied to the unitary representation of the Poincaré group on the physical Hilbert space, they result in two new scattering equivalent representations; with different kinematic subgroups that intertwine with the kinematic subgroups on the asymptotic Hilbert space.

The only thing that is missing is the absence of a non-interacting representation of the Poincaré group on the Hilbert space of field theory. In the event that the mapping from the asymptotic space to the field theory Hilbert space has dense range, and the unitarized mapping gives the same asymptotic condition, then it can be used to map the unitary representation of the Poincaré group on the asymptotic Hilbert space to a unitary representation of the Poincaré group on the field theory Hilbert that plays the role of a free-particle dynamics, however this is not a true free-particle representation because the vacuum is unchanged and the particle masses are physical.

The results can be summarized as follows. The construction assumes a unitary representation of the Poincaré group on the Hilbert space of the field theory and constructs scattering equivalent representations with light -front and instant-form kinematic symmetries. One consequence of the construction is that it requires that the transformations relating these representations leave the vacuum and one-particle states unchanged. Because the one-body states are eigenstates of the mass operator, there is no mass renormalization. This condition is not compatible with a $1 \leftrightarrow 2$ vertex interaction in either of these representations.

This paper has three sections and two appendices. Section two summarizes the assumptions used in this work and discusses a two-Hilbert space formulation of Haag-Ruelle scattering that has kinematically invariant injection

operators that map an asymptotic many-particle Hilbert space to the Hilbert space of the field theory. Haag-Ruelle scattering is the natural generalization of the usual formulation of time-dependent scattering theory. The construction of equivalent instant and light-front representations of the field theory is given in section 3. Section 4 has a summary of the results and a discussion of the implications. There are two appendices. The first one contains a discussion of Ekstein's treatment of scattering equivalences [45] that is used in section 3. The second appendix discusses the formulation of Haag-Ruelle injection operators with the kinematic symmetry properties that are used in sections two and three.

II. GENERAL CONSIDERATIONS

This paper examines the relation between light-front and instant-form formulations of quantum field theory from a more abstract perspective that does not assume that the Hamiltonian can be decomposed as the sum of free and interacting operators acting on one representation of the Hilbert space. The starting assumptions are typical assumptions about abstract properties of a quantum field theory. These include:

- The Hilbert space of the field theory is generated by applying bounded functions, F, of smeared field operators to a unique vacuum vector, denoted by $|0\rangle$. The smearing is assumed to be over Schwartz test functions in four space-time dimensions. It is not assumed that fields restricted to a light-front or fixed-time manifold make sense. A dense set of vectors, $|\psi\rangle$, can be taken to have the form:

$$|\psi\rangle = F|0\rangle \qquad F = \sum_{n=1}^{N} c_n e^{i\phi_n(f_n)} \tag{2}$$

  where the c_n are complex coefficients, the f_n are Schwartz functions, ϕ_n are field operators and N is finite.

- There is a unitary representation of the Poincaré group, $U(\Lambda, a)$, on this representation of the Hilbert space which is given by covariance. In the spinless case the action of $U(\Lambda, a)$ on states of the form (2) is

$$U(\Lambda, a)|\psi\rangle = \sum_{c} c_n e^{i\phi(f'_n)}|0\rangle \qquad f'_n(x) = f_n(\Lambda^{-1}(x-a)). \tag{3}$$

- The vacuum, $|0\rangle$, is Poincaré invariant and normalized to unity:

$$U(\Lambda, a)|0\rangle = |0\rangle \qquad \langle 0|0\rangle = 1. \tag{4}$$

- The unitary representation of the Poincaré group acts like the identity on the vacuum subspace and can be decomposed into a direct integral of positive-mass positive-energy irreducible representations of the Poincaré group [2][46] on the orthogonal complement of the vacuum subspace. The mass Casimir operator is assumed to have point-spectrum eigenstates representing particles of the theory.

- There is a complete set of asymptotically complete Haag-Ruelle 1[41][42][43][44] scattering states.

A characteristic element of Dirac's forms of dynamics is the notion of a *kinematic subgroup*. Kinematic subgroups leave a manifold in Minkowski space, that classically intersects every (massive) particle's world line once, invariant. In an instant-form dynamics the invariant manifold is a fixed-time plane, which is preserved under the group generated by rotations and space translations. In a light-front dynamics the manifold is a hyperplane that is tangent to the light cone. This is invariant under a seven parameter subgroup of the Poincaré group. These manifolds are prominent in Dirac's work; in this work the focus will be on the subgroups of the Poincaré group that leave these manifolds invariant. These subgroups will be referred to as kinematic subgroups.

For quantum mechanical systems of a finite number of degrees of freedom kinematical and dynamical unitary representations of the Poincaré group exist in the same representation of the Hilbert space. In the field theoretic case the representation of the Hilbert space depends on the dynamics, so *there is no free dynamics* that acts on the Hilbert space of the theory. This is one of the distinctions with the perturbative approach, which attempts to define the interacting theory as a perturbation of a free field theory. The incompatibility of these formulations leads to the infinities that complicate the analysis of the relation between light-front and instant-form treatments of field theory.

The first step in the abstract formulation is to define what is meant by the kinematic subgroup in the absence of a non-interacting representation of the Poincaré group. The unitary representation of the Poincaré group has a set

of 10 infinitesimal generators that are self-adjoint operators on the Hilbert space of the field theory satisfying the Poincaré Lie algebra. This follows because each one is the generator of a unitary one-parameter group. The unitary representation of the Poincaré group can be decomposed into a direct integral of irreducible representations. In this work a typical spectral condition is assumed, where the representations that appear in the direct integral are assumed to be positive-mass positive-energy representations, and the identity on the vacuum subspace.

The direct integral of irreducible representations of the Poincaré group results in a direct integral decomposition of the Hilbert space into irreducible subspaces or slices. The next step is to choose a basis on each irreducible subspace. Two choices will be considered in this work. The first basis consists of simultaneous eigenstates of the mass, spin, linear momentum, and the projection of the canonical spin on the z axis. These are all functions of the infinitesimal generators of the unitary representation of the Poincaré group. There will also be Poincaré invariant degeneracy parameters, which can be taken as discrete quantum numbers, since any continuous parameters can be replaced by a basis of square integrable functions of the continuous degeneracy parameters. The other basis on the same irreducible subspace consists of simultaneous eigenstates of the mass, spin, the three light-front components of the four momentum, and the projection of the light-front spin on the z axis. These are also functions of the infinitesimal generators with the same degeneracy parameters.

These bases can be formally expressed as

$$|0\rangle \cup \{|(m,s,d_n)\mathbf{p},\mu_c\rangle\} \qquad \text{and} \qquad |0\rangle \cup \{|(m,s,d_n)\tilde{\mathbf{p}},\mu_f\rangle\} \tag{5}$$

where the light-front components of the four momentum are defined by

$$\tilde{\mathbf{p}} = (\mathbf{p}\cdot\hat{\mathbf{x}}, \mathbf{p}\cdot\hat{\mathbf{y}}, p^0 + \mathbf{p}\cdot\hat{\mathbf{z}}) := (\mathbf{p}_\perp, p^+). \tag{6}$$

The variables (6) are eigenvalues of the generators of translations tangent to the light-front hyperplane. The first basis in (5) will be referred to as the "instant-form" basis and the second basis will be referred to as the "light-front" basis.

These basis states are related by a unitary change of basis. The change of basis is block diagonal in the direct integral. The different basis vectors are related by

$$|(m,s,d_n)\mathbf{p},\mu_c\rangle = \sum_{\mu_f=-s}^{s} |(m,s,d_n)\tilde{\mathbf{p}}(p),\mu_f\rangle \sqrt{\frac{p^+}{\omega_m(\mathbf{p})}} D^s_{\mu_f\mu_c}[B_f^{-1}(p/m)B_c(p/m)] \tag{7}$$

which assumes that both bases have delta function normalizations:

$$\langle(m,s,d_n)\mathbf{p},\mu_c|(m',s',d_k)\mathbf{p}',\mu_c'\rangle = \delta(\mathbf{p}-\mathbf{p}')\delta_{ss'}\delta_{\mu_c\mu_c'}\delta_{nk}\delta[m-m'] \tag{8}$$

and

$$\langle(m,s,d_n)\tilde{\mathbf{p}},\mu_f|(m',s',d_k)\tilde{\mathbf{p}}',\mu_f'\rangle = \delta(p^+-p^{+\prime})\delta(\mathbf{p}_\perp-\mathbf{p}'_\perp)\delta_{ss'}\delta_{\mu_f\mu_f'}\delta_{nk}\delta[m-m'] \tag{9}$$

where $\delta[m-m']$ is either a Dirac or Kronecker delta function depending on whether m is in the point or continuous spectrum of the mass operator. $B_c(p/m)$ and $B_f(p/m)$ are $SL(2,C)$ matrices representing a rotationless boost and a light-front preserving boost from $(1,0,0,0)$ to p/m. The combination $B_f^{-1}(p/m)B_c(p/m)$ is a $SU(2)$ representation of a Melosh rotation [47] that changes the light-front spin to the canonical spin.

There are similar transformations relating light-front bases corresponding to different orientations of the light front (i.e. replacing $\hat{\mathbf{z}}$ by some other unit vector $\hat{\mathbf{n}}$). Like (7) they involve a variable change, the square root of a Jacobian and a momentum-dependent rotation matrix.

The action of $U(\Lambda,a)$ on each of these irreducible basis states is a consequence of the transformation properties of the Poincaré generators and the basis choice. For the "instant-form" basis

$$U(\Lambda,a)|0\rangle = |0\rangle, \tag{10}$$

$$U(\Lambda,a)|(m,s,d_n)\mathbf{p},\mu\rangle = e^{ia\cdot\Lambda p}\sum_{\mu'=-s}^{s} |(m,s,d_n)\mathbf{\Lambda}p,\mu'\rangle \sqrt{\frac{\omega_m(\mathbf{\Lambda}p)}{\omega_m(\mathbf{p})}} D^s_{\mu'\mu}[B_c^{-1}(\Lambda(p/m))\Lambda B_c(p/m)] \tag{11}$$

and for the "light-front" basis.

$$U(\Lambda,a)|0\rangle = |0\rangle, \tag{12}$$

$$U(\Lambda,a)|(m,s,d_n)\tilde{\mathbf{p}},\mu\rangle = e^{ia\cdot\Lambda p}\sum_{\mu'=-s}^{s}|(m,s,d_n)\tilde{\mathbf{\Lambda}}p,\mu'\rangle\sqrt{\frac{(\Lambda p)^+}{p^+}}D^s_{\mu'\mu}[B_f^{-1}(\Lambda(p/m))\Lambda B_f(p/m)]. \tag{13}$$

These equations define the dynamical unitary representation of the Poincaré group on these two irreducible bases.

The connection of these bases with kinematic subgroups is that the coefficients of the basis functions on the right hand side of equations (11) and (13) do not depend on the mass eigenvalue m when (Λ,a) is an element of the kinematic subgroup. For the basis (11) the kinematic subgroup is generated by rotations and spatial translations while for the basis (13) the kinematic subgroup is the subgroup that leaves the light front hyperplane $x^+ = x^0 + x = 0$ invariant.

The next step is to discuss the formulation of scattering theory. A two-Hilbert space representation [48][49], will be used for this purpose. The starting point is the assumption that the mass operator, $M = \sqrt{-p^2}$ has one-particle eigenstates. In this non-perturbative context one-particle means that the mass operator, $M = \sqrt{-p^2}$, has a non-empty point spectrum. The discrete mass eigenvalues are assumed to be strictly positive. There is no distinction between elementary and composite particles.

Normalizable one-particle states in the "instant form" basis can be constructed by smearing the irreducible basis functions with a wave packet:

$$|\psi_{cg}\rangle := \int d\mathbf{p}\sum_{\mu=-s}^{s}|(m,s,d_n)\mathbf{p},\mu\rangle g(\mathbf{p},\mu) \tag{14}$$

where m is a discrete mass eigenvalue and $g(\mathbf{p},\mu)$ is a square integrable wave packet. The corresponding construction in the "light-front" basis has the form

$$|\psi_{fg}\rangle := \int d^2\mathbf{p}_\perp\int_0^\infty dp^+\sum_{\mu=-s}^{s}|(m,s,d_n)\tilde{\mathbf{p}},\mu\rangle\tilde{g}(\tilde{\mathbf{p}},\mu). \tag{15}$$

These normalizable vectors are linear in the smearing functions and can be represented as elements of the field algebra applied to the vacuum. This means that they can be expressed as

$$|\psi_{cg}\rangle = \sum_{\mu=-s}^{s}\int A^\dagger_{m,s,d_n}(p,\mu)|0\rangle d\mathbf{p}g(\mathbf{p},\mu) = A^\dagger(g)|0\rangle \tag{16}$$

and

$$|\psi_{fg}\rangle = \sum_{\mu=-s}^{s}\int \tilde{A}^\dagger_{m,s,d_n}(\tilde{p},\mu)|0\rangle dp^+d^2\mathbf{p}_\perp\tilde{g}(\tilde{\mathbf{p}},\mu) = \tilde{A}^\dagger(\tilde{g})|0\rangle \tag{17}$$

These states will be equal if $g(\mathbf{p},\mu) = \langle(m,s)\mathbf{p},\mu|g\rangle$ and $\tilde{g}(\tilde{\mathbf{p}},\mu) = \langle(m,s)\tilde{\mathbf{p}},\mu|g\rangle$ are related by the change of basis (7).

The operators $A^\dagger_{m,s,d_n}(p,\mu)$ and $\tilde{A}^\dagger_{m,s,d_n}(\tilde{p},\mu)$ are functions of the fields that create particles of mass m, spin s, momentum $\mathbf{p}$ and magnetic quantum number μ *out of the vacuum*. The construction of these operators starting from operators that couple the vacuum to the one-particle states of the theory is discussed in appendix B.

Since $A^\dagger_{m,s,d_n}(p,\mu)$ and $\tilde{A}^\dagger_{m,s,d_n}(\tilde{p},\mu)$ are in the field algebra (after smearing), they can be multiplied. While the field theory has one-particle states, *it does not have free many-particle states*, so N repeated application of these operators, while defined, cannot be interpreted as creating N-particle states out of the vacuum.

The following quantity

$$A^\dagger_{m_1,s_1,d_{n_1}}(\mathbf{p_1},\mu_1)\cdots A^\dagger_{m_k,s_k,d_{n_k}}(\mathbf{p_k},\mu_k)|0\rangle \tag{18}$$

can be considered as a mapping from a space of square integrable functions of $\mathbf{p}_1,\mu_1\cdots\mathbf{p}_k,\mu_k$ to the Hilbert space of the field theory. Note that while $A^\dagger(g)|0\rangle = \tilde{A}^\dagger(\tilde{g})|0\rangle$ for g and $\tilde{g}$ related by (7), this is not true for products of these operators applied to the vacuum. From equation (B17) in appendix B it follows that this mapping has the following properties

$$\mathbf{P}A^\dagger_{m_1,s_1,d_{n_1}}(\mathbf{p}_1,\mu_1)\cdots A^\dagger_{m_N,s_N,d_{n_N}}(\mathbf{p}_N,\mu_N)|0\rangle =$$

$$\sum_{n=1}^{N} \mathbf{p}_n A^\dagger_{m_1,s_1,d_{n_1}}(\mathbf{p}_1,\mu_1)\cdots A^\dagger_{m_N,s_N,d_{n_N}}(\mathbf{p}_N,\mu_N)|0\rangle \tag{19}$$

and

$$U(R,0)A^\dagger_{m_1,s_1,d_{n_1}}(\mathbf{p}_1,\mu_1)\cdots A^\dagger_{m_N,s_N,d_{n_N}}(\mathbf{p}_1,\mu_N)|0\rangle =$$

$$\sum A^\dagger_{m_1,s_1,d_{n_1}}(R\mathbf{p}_1,\nu_1)\cdots A^\dagger_{m_N,s_N,d_{n_N}}(R\mathbf{p}_N,\nu_N)|0\rangle|0\rangle \prod_{n=1}^{N} D^{s_n}_{\nu_n\mu_n}(R). \tag{20}$$

For the mapping in the "light-front" basis the corresponding relations are

$$\tilde{\mathbf{P}}\tilde{A}^\dagger_{m_1,s_1,d_{n_1}}(\tilde{\mathbf{p}}_1,\mu_1)\cdots \tilde{A}^\dagger_{m_N,s_N,d_{n_N}}(\tilde{\mathbf{p}}_N,\mu_N)|0\rangle =$$

$$\sum_{n=1}^{N} \tilde{\mathbf{p}}_n \tilde{A}^\dagger_{m_1,s_1,d_{n_1}}(\tilde{\mathbf{p}}_1,\mu_1)\cdots \tilde{A}^\dagger_{m_N,s_N,d_{n_N}}(\tilde{\mathbf{p}}_N,\mu_N)|0\rangle \tag{21}$$

$$U(B_f,0)\tilde{A}^\dagger_{m_1,s_1,d_{n_1}}(\tilde{\mathbf{p}}_1,\mu_1)\cdots \tilde{A}^\dagger_{m_N,s_N,d_{n_N}}(\tilde{\mathbf{p}}_N,\mu_N)|0\rangle =$$

$$\tilde{A}^\dagger_{m_1,s_1,d_{n_1}}(\tilde{\mathbf{B}}_f p_1,\mu_1)\cdots \tilde{A}^\dagger_{m_N,s_N,d_{n_N}}(\tilde{\mathbf{B}}_f p_N,\mu_N)|0\rangle \prod_{n=1}^{N}\sqrt{\frac{B_f(p_n)^+}{p_n^+}}. \tag{22}$$

$$U(R_z(\phi),0)\tilde{A}^\dagger_{m_1,s_1,d_{n_1}}(\tilde{\mathbf{p}}_1,\mu_1)\cdots \tilde{A}^\dagger_{m_N,s_N,d_{n_N}}(\tilde{\mathbf{p}}_N,\mu_N)|0\rangle =$$

$$\tilde{A}^\dagger_{m_1,s_1,d_{n_1}}(R_z(\phi)\tilde{\mathbf{p}}_1,\mu_1)\cdots \tilde{A}^\dagger_{m_N,s_N,d_{n_N}}(R(\phi)\tilde{\mathbf{p}}_N,\mu_N)|0\rangle \prod_{n=1}^{N} e^{i\mu_n\phi}. \tag{23}$$

If the Fourier transform of these states is integrated against wave packets that are localized in regions separated by large space-like separations and the field theory satisfies cluster properties (normally a consequence of uniqueness of the vacuum) then the expectation is that the resulting vectors look like states of N asymptotically separated particles when they are used in inner products [44][50]. The interpretation of the operators $A^\dagger_{m,s,d_n}$ and $\tilde{A}^\dagger_{m,s,d_n}$ is that they *asymptotically* behave like creation operators .

A scattering channel α is associated with a finite collection of particles asymptotically. It could correspond to the state of a target and incoming projectile or a collection of particles that are detected in a scattering experiment. The set of all scattering channels of the theory is denoted by $\mathcal{A}$. In what follows both the vacuum and one-particle states are included in the collection of channels, $\mathcal{A}$.

Channel injection operators are defined as products of the operators (16) or (17), applied to the vacuum, one for each particle in a scattering channel α. In the canonical basis they are

$$\Phi_\alpha(\mathbf{p}_1,\mu_1\cdots\mathbf{p}_N,\mu_N) := \prod_{k\in\alpha} A^\dagger_{m_k,s_k,d_{n_k}}(\mathbf{p}_k,\mu_k)|0\rangle \tag{24}$$

and in the light-front basis they are

$$\tilde{\Phi}_\alpha(\tilde{\mathbf{p}}_1,\mu_1\cdots\tilde{\mathbf{p}}_N,\mu_N) := \prod_{k\in\alpha} \tilde{A}^\dagger_{m_k,s_k,d_{n_k}}(\tilde{\mathbf{p}}_k,\mu_k)|0\rangle. \tag{25}$$

The order of the product does not matter in the asymptotic region. The asymptotic channel Hilbert space $\mathcal{H}_\alpha$ is the space of square integrable functions of the variables $\mathbf{p}_k,\mu_k$ or $\tilde{\mathbf{p}}_k,\mu_k$ for $k\in\alpha$. It is interpreted as a space of N particles of mass m_k and spin s_k. If any of the particles in the channel are identical then the functions representing identical Bosons should be symmetrized and those representing identical Fermions should anti-symmetrized. The

channel injection operators (24) and (25) are interpreted as mappings from the k-particle channel Hilbert space $\mathcal{H}_\alpha$ to the Hilbert space $\mathcal{H}$ of the field theory.

The asymptotic unitary representation of the Poincaré group on $\mathcal{H}_\alpha$ is defined by treating $\mathcal{H}_\alpha$ as a space of N mutually non-interacting particles of mass m_k and spin s_k for $k \in \alpha$. This representation is defined by the tensor product of one-particle irreducible representations in terms of "instant-form" variables

$$U_\alpha(\Lambda,a)|\mathbf{p}_1,\mu_1,\cdots\mathbf{p}_k,\mu_k\rangle := e^{ia\cdot\Lambda(\sum_n p_n)}|\mathbf{\Lambda}p_1,\nu_1\cdots\mathbf{\Lambda}p_k,\nu_k\rangle\prod_{n=1}^{k}\sqrt{\frac{\omega_{m_n}(\mathbf{\Lambda}p_n)}{\omega_{m_n}(\mathbf{p}_n)}}D^s_{\nu_n\mu_n}[B_c^{-1}(\Lambda p_n/m_n)\Lambda B_c(p_n/m_n)]. \tag{26}$$

The corresponding expression in terms of "light-front" variables is

$$U_\alpha(\Lambda,a)|\tilde{\mathbf{p}}_1,\mu_1,\cdots\tilde{\mathbf{p}}_k,\mu_k\rangle := e^{ia\cdot\Lambda(\sum_n p_n)}|\tilde{\mathbf{\Lambda}}p_1,\nu_1\cdots\tilde{\mathbf{\Lambda}}p_k,\nu_k\rangle\prod_{n=1}^{k}\sqrt{\frac{(\tilde{\Lambda}p_n)^+}{p_n^+}}D^s_{\nu_n\mu_n}[B_f^{-1}(\Lambda p_n/m_n)\Lambda B_f(p_n/m_n)]. \tag{27}$$

The states in (26) and (27) are basis vectors in $\mathcal{H}_\alpha$, not $\mathcal{H}$. Since the Hilbert space vectors in the range of Φ_α or $\tilde{\Phi}_\alpha$ are not N-particle states it follows that

$$U(\Lambda,a)\Phi_\alpha \neq \Phi_\alpha U_\alpha(\Lambda,a). \qquad U(\Lambda,a)\tilde{\Phi}_\alpha \neq \tilde{\Phi}_\alpha U_\alpha(\Lambda,a). \tag{28}$$

This is because (B6) does not hold for P^0 due to the p^0 integrals in (B9) and (B6) does not hold for P^- due to the p^- integrals on (B9) (see appendix B).

When Λ_K and a_K are elements of the instant-form or light-front kinematic subgroup, it follows from (19-23) that

$$U(\Lambda_K,a_K)\Phi_\alpha = \Phi_\alpha U_\alpha(\Lambda_K,a_K) \qquad U(\Lambda_K,a_K)\tilde{\Phi}_\alpha = \tilde{\Phi}_\alpha U_\alpha(\Lambda_K,a_K) \tag{29}$$

because the coefficients of the kinematic transformations are mass independent. The operators Φ_α and $\tilde{\Phi}_\alpha$ that map the channel α Hilbert space $\mathcal{H}_\alpha$ into the field theory Hilbert space $\mathcal{H}$ are Haag-Ruelle [41][42][43] injection operators. The only difference with the standard choices is that Φ_α and $\tilde{\Phi}_\alpha$ are designed to have the intertwining properties (29).

In the two-Hilbert space formulation [51][50] channel wave operators $\Omega_{\alpha\pm} = \Omega_{\alpha\pm}(H,\Phi_\alpha,H_\alpha)$ are defined by the strong limits

$$\lim_{t\to\pm\infty}\|(\Omega_{\alpha\pm}(H,\Phi_\alpha,H_\alpha) - e^{iHt}\Phi_\alpha e^{-iH_\alpha t})|\psi\rangle\| = 0 \tag{30}$$

or

$$\lim_{t\to\pm\infty}\|(\Omega_{\alpha\pm}(H,\tilde{\Phi}_\alpha,H_\alpha) - e^{iHt}\tilde{\Phi}_\alpha e^{-iH_\alpha t})|\psi\rangle\| = 0. \tag{31}$$

The wave operators satisfy the intertwining property

$$U(\Lambda,a)\Omega_{\alpha\pm}(H,\Phi_\alpha,H_\alpha) = \Omega_{\alpha\pm}(H,\Phi_\alpha,H_\alpha)U_\alpha(\Lambda,a) \qquad U(\Lambda,a)\Omega_{\alpha\pm}(H,\tilde{\Phi}_\alpha,H_\alpha) = \Omega_{\alpha\pm}(H,\tilde{\Phi}_\alpha,H_\alpha)U_\alpha(\Lambda,a) \tag{32}$$

where, unlike (28), (Λ,a) are not restricted to the kinematic subgroup.

The purpose of the injection operators in Haag-Ruelle scattering is to define the asymptotic boundary conditions. They are constructed so they behave like creation operators in asymptotically separated regions. They replace the free-particle asymptotic states of ordinary scattering theory by states in the Hilbert spaces of the field theory that only behave like a system of non-interacting particle in the asymptotic region. The two injection operators defined in appendix B both have this property. They differ in which variable enforces the one-particle mass shell delta function when applied the the vacuum. This means that as operators

$$\Omega_{\alpha\pm}(H,\Phi_\alpha,H_\alpha) = \Omega_{\alpha\pm}(H,\tilde{\Phi}_\alpha,H_\alpha) \tag{33}$$

if they are evaluated in the same one-particle basis. This identity is equivalent to the requirement

$$0 = \lim_{t\to\pm\infty}\|e^{iHt}(\Phi_\alpha - \tilde{\Phi}_\alpha)e^{-iH_\alpha t}|\psi_\alpha\rangle\| = \lim_{t\to\pm\infty}\|(\Phi_\alpha - \tilde{\Phi}_\alpha)e^{-iH_\alpha t}|\psi_\alpha\rangle\| \tag{34}$$

which is precisely the condition that the injection operators agree asymptotically.

The scattering operator for scattering from channel α to channel β is

$$S_{\beta,\alpha} : \mathcal{H}_\alpha \to \mathcal{H}_\beta := \Omega^\dagger_{\beta+}(H,\Phi_\beta,H_\beta)\Omega_{\alpha-}(H,\Phi_\alpha,H_\alpha). \tag{35}$$

It follows from the identity (34) that $S_{\alpha\beta}$ is independent of the choice of injection operator. If follows from (32) that

$$U_\beta(\Lambda,a)S_{\beta,\alpha} = S_{\beta,\alpha}U_\alpha(\Lambda,a). \tag{36}$$

This can be extended to all channels, $\mathcal{A}$, by defining the asymptotic Hilbert space as the direct sum of all channel Hilbert spaces $\mathcal{H}_\alpha$ for $\alpha \in \mathcal{A}$, where $\mathcal{A}$ is defined to include the vacuum and one-particle channels as well as the scattering channels:

$$\mathcal{H}_\mathcal{A} := \oplus_{\alpha\in\mathcal{A}}\mathcal{H}_\alpha. \tag{37}$$

The asymptotic unitary representation of the Poincaré group is defined on $\mathcal{H}_\mathcal{A}$ by

$$U_\mathcal{A}(\Lambda,a) = \oplus_{\alpha\in\mathcal{A}}U_\alpha(\Lambda,a). \tag{38}$$

Multi-channel injection operators are defined as the sum of all channel injection operators

$$\Phi_\mathcal{A} := \sum_{\alpha\in\mathcal{A}}\Phi_\alpha \qquad \text{or} \qquad \tilde{\Phi}_\mathcal{A} := \sum_{\alpha\in\mathcal{A}}\tilde{\Phi}_\alpha \tag{39}$$

where $\Phi_\alpha : \mathcal{H}_\alpha \to \mathcal{H}$. These can be used to define multi-channel wave operators

$$\Omega_{\mathcal{A}\pm}(H,\Phi_\mathcal{A},H_\mathcal{A}) = \tilde{\Omega}_{\mathcal{A}\pm}(H,\tilde{\Phi}_\mathcal{A},H_\mathcal{A}) \tag{40}$$

by the strong limits

$$\lim_{t\to\pm\infty}\|(\Omega_{\mathcal{A}\pm}(H,\Phi_\mathcal{A},H_\mathcal{A}) - e^{iHt}\Phi_\mathcal{A}e^{-iH_\mathcal{A}t})|\psi\rangle\| = 0 \tag{41}$$

or

$$\lim_{t\to\pm\infty}\|(\Omega_{\mathcal{A}\pm}(H,\tilde{\Phi}_\mathcal{A},H_\mathcal{A}) - e^{iHt}\tilde{\Phi}_\mathcal{A}e^{-iH_\mathcal{A}t})|\psi\rangle\| = 0 \tag{42}$$

where $H_\mathcal{A} := \sum_{\alpha\in\mathcal{A}}H_\alpha\Pi_\alpha$ and Π_α is the projection on the subspace $\mathcal{H}_\alpha$ of $\mathcal{H}_\mathcal{A}$. The assumed asymptotic completeness of the scattering theory means that the wave operators (including the vacuum and one-body channels) are unitary mappings from the asymptotic Hilbert space, $\mathcal{H}_\mathcal{A}$, to the Hilbert, $\mathcal{H}$, space of the quantum field theory.

With this definition (32) becomes

$$U(\Lambda,a)\Omega_{\mathcal{A}\pm}(H,\Phi_\mathcal{A},H_\mathcal{A}) = \Omega_{\mathcal{A}\pm}(H,\Phi_\mathcal{A},H_\mathcal{A})U_\mathcal{A}(\Lambda,a) \tag{43}$$

which also holds with $\Phi_\mathcal{A}$ replaced by $\tilde{\Phi}_\mathcal{A}$.

The multi-channel scattering operator is

$$S(H,\Phi_\mathcal{A},\mathcal{H}_\mathcal{A}) = \Omega_+^\dagger(H,\Phi_\mathcal{A},\mathcal{H}_\mathcal{A})\Omega_-(H,\Phi_\mathcal{A},\mathcal{H}_\mathcal{A}) \tag{44}$$

where it is defined to be the identity on the vacuum and one-particle subspaces.

Poincaré invariance of the multi-channel scattering operator on $\mathcal{H}_\mathcal{A}$ follows from the intertwining property of the wave operators (43) [51]

$$U_\mathcal{A}(\Lambda,a)S(H,\Phi_\mathcal{A},\mathcal{H}_\mathcal{A}) = U_\mathcal{A}(\Lambda,a)\Omega_+^\dagger(H,\Phi_\mathcal{A},\mathcal{H}_\mathcal{A})\Omega_-(H,\Phi_\mathcal{A},\mathcal{H}_\mathcal{A}) =$$

$$\Omega_+^\dagger(H,\Phi_\mathcal{A},\mathcal{H}_\mathcal{A})U(\Lambda,a)\Omega_-(H,\Phi_\mathcal{A},\mathcal{H}_\mathcal{A}) =$$

$$\Omega_+^\dagger(H,\Phi_\mathcal{A},\mathcal{H}_\mathcal{A})\Omega_-(H,\Phi_\mathcal{A},\mathcal{H}_\mathcal{A})U_\mathcal{A}(\Lambda,a) = S(H,\Phi_\mathcal{A},\mathcal{H}_\mathcal{A})U_\mathcal{A}(\Lambda,a). \tag{45}$$

At this point all that has been demonstrated is that the same scattering theory is obtained by using injection operators that agree asymptotically but intertwine different kinematic subgroups. This will be used in the next section to construct unitarily equivalent light-front and instant-form representations of the field theory that preserve the scattering matrix.

III. CONSTRUCTION

The next step is to use the kinematic symmetries of the injection operators to show that the Hamiltonian H in the expressions for the wave operators can be replaced by the light front Hamiltonian, P^-. The relations

$$H = P^- + P^3 \qquad H = \frac{1}{2}(P^+ + P^-) \tag{46}$$

will be used in what follows. In the first case note because P^3 intertwines with $\Phi_{\mathcal{A}}$ it follows that

$$e^{iHt}\Phi_{\mathcal{A}}e^{-iH_{\mathcal{A}}} = e^{i(P^-+P^3)t}\Phi_{\mathcal{A}}e^{-i(P^-_{\mathcal{A}}+P^3_{\mathcal{A}})t} = e^{iP^-t}\Phi_{\mathcal{A}}e^{-iP^-_{\mathcal{A}}t}. \tag{47}$$

Similarly since P^+ intertwines with $\tilde{\Phi}_{\mathcal{A}}$

$$e^{iHt}\tilde{\Phi}_{\mathcal{A}}e^{-iH_{\mathcal{A}}} = e^{i(P^-+P^+)t/2}\tilde{\Phi}_{\mathcal{A}}e^{-i(P^-_{\mathcal{A}}+P^+_{\mathcal{A}})t/2} = e^{iP^-t/2}\tilde{\Phi}_{\mathcal{A}}e^{-iP^-_{\mathcal{A}}t/2}. \tag{48}$$

This means the wave operators constructed using H are identical to the ones using P^- with both injection operators. Combining these results with (40) gives the following identifications

$$\Omega_\pm(H,\Phi_{\mathcal{A}},H_{\mathcal{A}}) = \Omega_\pm(H,\tilde{\Phi}_{\mathcal{A}},H_{\mathcal{A}}) = \Omega_\pm(P^-,\Phi_{\mathcal{A}},P^-_{\mathcal{A}}) = \Omega_\pm(P^-,\tilde{\Phi}_{\mathcal{A}},P^-_{\mathcal{A}}). \tag{49}$$

The final step in the construction is to introduce two unitary operators V_F and V_I that act on field theory Hilbert space $\mathcal{H}$ with the following properties:

$$V_F|0\rangle = V_I|0\rangle = |0\rangle \tag{50}$$

$$[U(\Lambda_{K_I},a_{K_I}),V_I = 0] \qquad [U(\Lambda_{K_F},a_{K_F}),V_F] = 0 \tag{51}$$

$$\lim_{t\to\pm\infty}\|(\Phi_{\mathcal{A}} - V_I\Phi_{\mathcal{A}})e^{-iH_{\mathcal{A}}t}|\psi_\alpha\rangle\| = 0 \tag{52}$$

$$\lim_{t\to\pm\infty}\|(\tilde{\Phi}_{\mathcal{A}} - V_F\tilde{\Phi}_{\mathcal{A}})e^{-iH_{\mathcal{A}}t}|\psi_\alpha\rangle\| = 0 \tag{53}$$

where (Λ_{K_I},a_{K_I}) and (Λ_{K_F},a_{K_F}) are in the instant-form and light-front kinematic subgroups respectively. Note that for equations (52 and 53) to hold the asymptotic Hamiltonian must have a non-trivial absolutely continuous spectrum. While this is true on the scattering channel subspaces, it is not true for the vacuum or one-particle states. This implies that in addition to (50) that the unitary operators V_I and V_f also act like the identity on the one-particle subspaces.

The next step is to define two new unitary representations of the Poincaré group on the Hilbert space $\mathcal{H}$ of the quantum field theory by:

$$U_F(\Lambda,a) := V_FU(\Lambda,a)V_F^\dagger \qquad U_I(\Lambda,a) := V_IU(\Lambda,a)V_I^\dagger \tag{54}$$

with the corresponding dynamical generators:

$$H_F := V_FHV_F^\dagger \qquad H_I := V_IHV_I^\dagger \tag{55}$$

$$P^-_F := V_FP^-V_F^\dagger \qquad P^-_I := V_IP^-V_I^\dagger. \tag{56}$$

It follows from (51) that

$$U_I(\Lambda_{K_I},a_{K_I}) = U(\Lambda_{K_I},a_{K_I}) \qquad U_F(\Lambda_{K_F},a_{K_F}) := U(\Lambda_{K_F},a_{K_F}) \tag{57}$$

where (Λ_{K_I},a_{K_I}) is an instant-form kinematic Poincaré transformation and (Λ_{K_F},a_{K_F}) is a front-form kinematic transformation.

In Appendix A it is shown [45] that as a consequence of (52) and (53) that

$$\Omega_\pm(H_I,\Phi_{\mathcal{A}},H_{\mathcal{A}}) = V_I\Omega_\pm(H,\Phi_{\mathcal{A}},H_{\mathcal{A}}) \tag{58}$$

and

$$\Omega_{\pm}(H_F,\tilde{\Phi}_{\mathcal{A}},H_{\mathcal{A}})=V_F\Omega_{\pm}(H_F,\tilde{\Phi}_{\mathcal{A}},H_{\mathcal{A}}). \tag{59}$$

Taken together with (49) gives:

$$\Omega_{\pm}(H_I,\Phi_{\mathcal{A}},H_{\mathcal{A}})=V_I\Omega_{\pm}(H,\Phi_{\mathcal{A}},H_{\mathcal{A}})=V_I\Omega_{\pm}(H,\tilde{\Phi}_{\mathcal{A}},H_{\mathcal{A}})=V_I\Omega_{\pm}(P^-,\tilde{\Phi}_{\mathcal{A}},P^-_{\mathcal{A}})=V_IV_F^{\dagger}\Omega_{\pm}(P^-_F,\tilde{\Phi}_{\mathcal{A}},P^-_{\mathcal{A}}). \tag{60}$$

Since (60) holds for the same $V_IV_f^{\dagger}$ for both time limits it follows that

$$S(H_I,\Phi\mathcal{A},H_{\alpha})=\Omega_+^{\dagger}(H_I,\Phi_{\mathcal{A}},H_{\mathcal{A}})\Omega_-(H_I,\Phi_{\mathcal{A}},H_{\mathcal{A}})=$$

$$\Omega_+^{\dagger}(H_F,\tilde{\Phi}_{\mathcal{A}},H_{\mathcal{A}})V_FV_I^{\dagger}V_IV_F^{\dagger}\Omega_-(H_F,\tilde{\Phi}_{\mathcal{A}},H_{\mathcal{A}})=$$

$$\Omega_+^{\dagger}(P^-_F,\tilde{\Phi}_{\mathcal{A}},P^-_{\mathcal{A}})\Omega_-(P^-_F,\tilde{\Phi}_{\mathcal{A}},P^-_{\mathcal{A}})=$$

$$S(P^-_F,\tilde{\Phi}_{\mathcal{A}},P^-_{\mathcal{A}}). \tag{61}$$

This means both representations of the dynamics result in the same scattering operators on the asymptotic Hilbert space. Next consider the two unitary representations of the Poincaré group, $U_I(\Lambda,a)$ and $U_F(\Lambda,a)$ defined above. For the first one the operators $\{\mathbf{P},s,s_z\}$ are mutually commuting self-adjoint functions of the instant-form kinematic generators of the representation $U_I(\Lambda,a)$. For the second one the operators $\{\tilde{\mathbf{P}},s_{z_f}\}$ are mutually commuting self-adjoint functions of the front-form kinematic generators of the representation $U_F(\Lambda,a)$.

It is possible to construct a basis for $U_I(\Lambda,a)$ consisting of the eigenvalues of $\{\mathbf{P},s,s_z\}$ and some additional kinematically invariant commuting observables x_I. Similarly it is possible to construct a basis for $U_F(\Lambda,a)$ consisting of the eigenvalues of $\{\tilde{\mathbf{P}},s_{zF}\}$ and some additional kinematically invariant commuting observables x_F.

It follows that in these bases wave functions have the form

$$\langle\mathbf{p},s,\mu,x_I|\psi\rangle \qquad \text{or} \qquad \langle\tilde{\mathbf{p}},\mu,x_F|\psi\rangle. \tag{62}$$

Matrix elements of the dynamical operators are non-trivial matrices in the $''x''$ variables:

$$\langle\mathbf{p},s,\mu,x_I|H_I|\mathbf{p}',s',\mu',x_I'\rangle=\delta(\mathbf{p}-\mathbf{p}')\delta_{ss'}\delta_{\mu\mu'}\langle x_I\|H_I(\mathbf{P},s)\|x_I'\rangle \tag{63}$$

and

$$\langle\tilde{\mathbf{p}},\mu,x_F|P^-_F|\tilde{\mathbf{p}}',\mu',x_F'\rangle=\delta(\tilde{\mathbf{p}}-\tilde{\mathbf{p}}')\delta_{\mu\mu'}\langle x_F\|P^-_F(\tilde{\mathbf{P}},\mu)\|x_F'\rangle. \tag{64}$$

The eigenvalue problems for the dynamical operators have the forms

$$\sum\int\langle x_I\|H_I(\mathbf{P},s)\|x_I'\rangle dx_I'\langle\mathbf{p},s,\mu,x_I|\psi\rangle=E(\mathbf{P},s)\langle\mathbf{p},s,\mu,x_I|\psi\rangle \tag{65}$$

and

$$\sum\int\langle x_F\|P^-_F(\tilde{\mathbf{P}},\mu)\|x_F'\rangle dx_F'\langle\tilde{\mathbf{p}},\mu,x_F'|\psi\rangle=P^-(\tilde{\mathbf{P}},\mu)\langle\tilde{\mathbf{p}},\mu,x_F|\psi\rangle. \tag{66}$$

The matrices $\langle x_I\|H_I(\mathbf{P},s)\|x_I'\rangle$ or $\langle x_F\|P^-_F(\tilde{\mathbf{P}},\mu)\|x_F'\rangle$ must be diagonalized in this basis in order to compute dynamical Poincaré transformations. On the other hand kinematic transformations can be computed by applying the inverse transformation to basis vectors:

$$\langle\mathbf{p},s,\mu,x_I|U_I(R,\mathbf{a})|\psi\rangle=\langle\psi|U_I^{\dagger}(R,\mathbf{a})|\mathbf{p},s,\mu,x_I\rangle^*=$$

$$e^{i\mathbf{a}\cdot\mathbf{p}}\langle R^{-1}\mathbf{p},s,\nu,x_I|\psi\rangle D^s_{\mu\nu}(R). \tag{67}$$

Similarly in the light-front case for light-front preserving boosts and translations:

$$\langle\tilde{\mathbf{p}},\mu,x_F|U_F(\Lambda_{KF},a_{KF})|\psi\rangle=\langle\psi|U_F^{\dagger}(\Lambda_{KF},a_{KF})|\tilde{\mathbf{p}},\mu,x_F\rangle^*=$$

$$e^{i\tilde{\mathbf{a}}\cdot\tilde{\mathbf{p}}}\langle\tilde{\mathbf{\Lambda}}_k^{-1}p,\mu,x_F|\psi\rangle\sqrt{\frac{(\Lambda_k^{-1}p)^+}{p^+}} \tag{68}$$

and for rotations about the z axis

$$\langle\tilde{\mathbf{p}},\mu,x_F|U_F(R_z(\phi),0)\psi\rangle = \langle\psi|U_F^\dagger(R_z(\phi))|\tilde{\mathbf{p}},\mu,x_F\rangle^* =$$

$$\langle\tilde{\mathbf{R}}_z(\phi)^{-1}p,\mu,x_F|\psi\rangle s^{i\mu\phi}. \tag{69}$$

Equation (67) shows that $U_I(\Lambda,a)$ has an instant form kinematic subgroup while equations (68) and (69) show that $U_F(\Lambda,a)$ has a light-front kinematic subgroup. In addition the two representation are related by a unitary transformation that preserves the vacuum and one particle states:

$$U_F(\Lambda,a) = V_F V_I^\dagger U_I(\Lambda,a) V_I V_f^\dagger \tag{70}$$

$$|0\rangle_F = V_F V_I^\dagger |0\rangle_I = |0\rangle_I. \tag{71}$$

If follows from (61) that both representations give the same unitary scattering operators on the asymptotic Hilbert space $\mathcal{H}_\mathcal{A}$. In addition because V_I and V_f are kinematically invariant (51), the injection operators satisfy

$$U_I(\Lambda_{KI},a_{KI})\Phi_\mathcal{A} = \Phi_\mathcal{A} U_\mathcal{A}(\Lambda_{KI},a_{KI}) \tag{72}$$

for the instant form kinematic subgroup and

$$U_F(\Lambda_{KF},a_{KF})\tilde{\Phi}_\mathcal{A} = \tilde{\Phi}_\mathcal{A} U_\mathcal{A}(\Lambda_{KF},a_{KF}) \tag{73}$$

for the light-front kinematic subgroup.

The result can be summarized by noting that it is possible to choose bases in the field theory Hilbert space that transforms covariantly under either kinematic subgroup. In both cases a free dynamics is not assumed. The two representations are related by S-matrix preserving unitary transformations. Both representation have the same vacuum and one particle subspaces. There are no bare particles or production vertices in these representations.

Finally note that if the range of $\tilde{\Phi}$ is all of $\mathcal{H}$ and

$$W := (\tilde{\Phi}_\mathcal{A}\tilde{\Phi}_\mathcal{A}^\dagger)^{-1/2}\tilde{\Phi}_\mathcal{A} \tag{74}$$

satisfies

$$\lim_{t\to\pm\infty}\|(W-\tilde{\Phi}_\mathcal{A})e^{-iH_\mathcal{A}t}|\psi_\alpha\rangle\| = 0 \tag{75}$$

then it is a kinematically invariant unitary mapping from $\mathcal{H}_\mathcal{A}$ to $\mathcal{H}$ and $U_0(\Lambda,a) = WU_\mathcal{A}(\Lambda,a)W^\dagger$ defines a consistent "free dynamics" on $\mathcal{H}$ that satisfies

$$U_I(\Lambda_{KI},a_{KI}) = U_0(\Lambda_{KI},a_{KI}) \tag{76}$$

with a similar relation in the light-front case.

IV. ANALYSIS AND CONCLUSIONS

This work demonstrated the equivalence of formulations of quantum field theories with light-front and instant-form kinematic symmetries. It differs from most approaches to this problem because it is completely non-perturbative. This results in a number of differences with approaches that assume the existence of both a free and interacting unitary representation of the Poincaré group on the Hilbert space representation of the field theory. The problem with this assumption is that in a local field theory the free and dynamical unitary representations of the Poincaré group act on inequivalent representations of the Hilbert space. This requires re-thinking about the meaning of instant and front-form dynamics. In this approach, kinematic Poincaré transformations can be performed without diagonalizing the mass operator. Both the light front and instant representations are unitarily equivalent and give the same multichannel scattering operators. Some of the differences are (1) there are no true multi-particle states in the field theory; that

notion only makes sense asymptotically. This also means that there is no Fock space and no "quantization" (2) the fields are not restricted to an instant or light front hyperplane (3) there is no free-field dynamics (4) there are no zero modes (5) there is one vacuum, and all of the particles are physical. The advantage of starting with the Hilbert space representation of the interacting theory is that all of the issues with renormalization are defined away.

This work provides one way to understand Dirac's forms of dynamics in a non-perturbative setting. This approach assumed the existence of local quantum field theory with all of the expected properties. Using this assumption unitarily equivalent representations with the same scattering operators were constructed. Bases that transformed covariantly with respect to light-front and instant-form kinematic subgroups where constructed for each of these equivalent representations of the dynamics. Kinematic Poincaré transformations on these wave functions could be computed without diagonalizing the dynamical operators. One aspect of the these representations of the dynamics is that they both act irreducibly on the vacuum and *one-particle* states. This means that the 0 and 1 body solutions are basis vectors in both bases. Both representations give the same Poincaré invariant scattering operator. The Haag-Ruelle injection operators that are used to formulate the scattering asymptotic conditions in both cases intertwine with corresponding kinematic subgroups. If the injection operators have full range on the Hilbert space of the field theory there is the analog of a free dynamics on the Hilbert of the field theory, it has the same vacuum as the full theory and the particles all have physical masses (no mass renormalization). The construction of equivalent instant-form and light-front unitary representations of the Poincaré group is essentially identical to the corresponding construction in the finite number of degree of freedom case [3][4][52][53]. In that case, when there is production, the particles all have their physical masses and there are no vertex interactions that can change masses [54]. The simplest interactions that change particle number are $2 \leftrightarrow 3$ interactions.

While the representations discussed in this analysis are not useful for applications, the results support the conclusions of most of the quoted references that the two formulations of the theory are equivalent. The vacuum is the same in both representations. While the vacuum is kinematically invariant, it is also rotationally invariant, which is an additional constraint on the vacuum in the perturbative case.

Appendix A: Scattering equivalences

The two-Hilbert space multi-channel wave operators are defined by the strong limits where the $\Phi_{\mathcal{A}}$ are two-Hilbert space injection operators that map the asymptotic Hilbert space $\mathcal{H}_{\mathcal{A}}$ to the Hilbert space $\mathcal{H}$ of the quantum field theory:

$$\Omega_{\pm}(H, \Phi_{\mathcal{A}}, H_{\mathcal{A}}) = s - \lim_{t \to \pm\infty} e^{iHt} \Phi_{\mathcal{A}} e^{-iH_{\mathcal{A}}t}. \tag{A1}$$

The wave operators are assumed to exist and be Poincaré invariant in the sense

$$U(\Lambda, a)\Omega^{\dagger}_{\pm}(H, \Phi_{\mathcal{A}}, H_{\mathcal{A}}) = \Omega^{\dagger}_{\pm}(H, \Phi_{\mathcal{A}}, H_{\mathcal{A}})U_{\mathcal{A}}(\Lambda, a). \tag{A2}$$

where $U_{\mathcal{A}}(\Lambda, a)$ is the natural representation of the Poincaré group on $\mathcal{H}_{\mathcal{A}}$ that has the form of a direct sum of tensor products of irreducible representations. The set of channels $\mathcal{A}$ is assumed to include the vacuum channel, and one-particle channels in addition to multi-particle scattering channels. The wave operators are assumed to be asymptotically complete unitary mappings from asymptotic Hilbert space $\mathcal{H}_{\mathcal{A}}$ to the Hilbert space $\mathcal{H}$ of the field theory.

The scattering operator is defined by

$$S(H, \Phi, H_{\mathcal{A}}) = \Omega^{\dagger}_{+}(H, \Phi_{\mathcal{A}}, H_{\mathcal{A}})\Omega_{-}(H, \Phi_{\mathcal{A}}, H_{\mathcal{A}}). \tag{A3}$$

Consider two identical scattering operators based on different Hamiltonian's

$$S(H, \Phi_{\mathcal{A}}, H_{\mathcal{A}}) = S(H', \Phi'_{\mathcal{A}}, H_{\mathcal{A}}) \tag{A4}$$

It follows from the definitions and unitarity of the wave operators that

$$W := \Omega_{+}(H', \Phi'_{\mathcal{A}}, H_{\mathcal{A}})\Omega^{\dagger}_{+}(H, \Phi_{\mathcal{A}}, H_{\mathcal{A}}) = \Omega_{-}(H', \Phi'_{\mathcal{A}}, H_{\mathcal{A}})\Omega^{\dagger}_{-}(H, \Phi_{\mathcal{A}}, H_{\mathcal{A}}) \tag{A5}$$

is a unitary operator on $\mathcal{H}$. The identification of the scattering operators (A4) means that W is the same for the incoming ($t \to +\infty$) or outgoing ($t \to -\infty$) multi-channel wave operator. It follows that

$$W\Omega_{\pm}(H, \Phi_{\mathcal{A}}, H_{\mathcal{A}}) = \Omega_{\pm}(H', \Phi'_{\mathcal{A}}, H_{\mathcal{A}}). \tag{A6}$$

In addition, the intertwining property of the wave operators [51],

$$WH' = \Omega_\pm(H,\Phi_{\mathcal{A}},H_{\mathcal{A}})\Omega_\pm^\dagger(H',\Phi'_{\mathcal{A}},H_{\mathcal{A}})H' = \Omega_\pm(H,\Phi_{\mathcal{A}},H_{\mathcal{A}})H_{\mathcal{A}}\Omega_\pm^\dagger(H',\Phi'_{\mathcal{A}},H_{\mathcal{A}})$$

$$= H\Omega_\pm(H,\Phi_{\mathcal{A}},H_{\mathcal{A}})\Omega_\pm^\dagger(H',\Phi'_{\mathcal{A}},H_{\mathcal{A}}) = HW \tag{A7}$$

means that the two Hamiltonians are related by the unitary transformation W.

It follows that

$$\Omega_-(H',\Phi'_{\mathcal{A}},H_{\mathcal{A}}) = W\Omega_-(H,\Phi_{\mathcal{A}},H_{\mathcal{A}}) =$$

$$\Omega_-(WHW^\dagger,W\Phi_{\mathcal{A}},H_{\mathcal{A}}) = \Omega_-(H',W\Phi_{\mathcal{A}},H_{\mathcal{A}}). \tag{A8}$$

Taking the difference of the right and left side of equation (A8) gives

$$0 = \lim_{t\to\pm\infty}\|e^{iH't}(\Phi'_{\mathcal{A}} - W\Phi_{\mathcal{A}})e^{-iH_{\mathcal{A}}t}|\psi\rangle\|. \tag{A9}$$

The unitarity of gives $e^{iH't}$

$$0 = \lim_{t\to\pm\infty}\|(\Phi'_{\mathcal{A}} - W\Phi_{\mathcal{A}})e^{-iH_{\mathcal{A}}t}|\psi\rangle\|. \tag{A10}$$

That this condition holds for both time limits is important.

Next consider the converse. Assume that W is a unitary operator satisfying $H' = WHW^\dagger$, and the scattering operators

$$S(H,\Phi,H_{\mathcal{A}}) = \Omega_+^\dagger(H,\Phi_{\mathcal{A}},H_{\mathcal{A}})\Omega_-(H,\Phi_{\mathcal{A}},H_{\mathcal{A}}) \tag{A11}$$

$$S(H',\Phi'_{\mathcal{A}},H_{\mathcal{A}}) = \Omega_+^\dagger(H',\Phi'_{\mathcal{A}},H_{\mathcal{A}})\Omega_-(H',\Phi'_{\mathcal{A}},H_{\mathcal{A}}) \tag{A12}$$

both exist. If W satisfies (A10) for both time limits then the S matrices are identical

$$S(H',\Phi'_{\mathcal{A}},H_{\mathcal{A}}) = S(H,\Phi_{\mathcal{A}},H_{\mathcal{A}}). \tag{A13}$$

The proof follows from

$$\Omega_\pm(H',\Phi'_{\mathcal{A}},H_{\mathcal{A}}) = \Omega_\pm(WHW^\dagger,\Phi'_{\mathcal{A}},H_{\mathcal{A}}) = W\Omega_\pm(H,W^\dagger\Phi'_{\mathcal{A}},H_{\mathcal{A}}) =$$

$$W\Omega_\pm(H,W^\dagger(\underbrace{\Phi'_{\mathcal{A}} - W\Phi_{\mathcal{A}}}_{\to 0} + W\Phi_{\mathcal{A}}),H_{\mathcal{A}}) = W\Omega_\pm(H,\Phi_{\mathcal{A}}),H_{\mathcal{A}}). \tag{A14}$$

It then follows that

$$S(H',\Phi'_{\mathcal{A}},H_{\mathcal{A}}) = \Omega_+^\dagger(H',\Phi'_{\mathcal{A}},H_{\mathcal{A}})\Omega_-(H',\Phi'_{\mathcal{A}},H_{\mathcal{A}}) =$$

$$\Omega_+^\dagger(H,\Phi_{\mathcal{A}},H_{\mathcal{A}})W^\dagger W\Omega_-(H,\Phi_{\mathcal{A}},H_{\mathcal{A}}) = S(H,\Phi_{\mathcal{A}},H_{\mathcal{A}}). \tag{A15}$$

Note that unitary equivalence is not a sufficient condition for S-matrix equivalence. This is not hard to understand in the case of ordinary quantum mechanics, where two Hamiltonian's with short-range repulsive interactions have the same spectrum (so they are unitarily equivalent) but generally have different S-matrix elements or phase shifts.

The conclusion of this section is that unitary operators W that satisfy the asymptotic condition (A10) can be used to relate Hamiltonians that have the same scattering matrix. This asymptotic condition means that W does not disturb the asymptotic structure of the states that define the asymptotic condition.

The above condition applies to both quantum mechanics and quantum field theory assuming that the have asymptotically complete scattering operators. Equation (A10) requires that W leaves the one-particle states unchanged. The field theory generalization of multi-particle scattering is the Haag-Ruelle formulation of scattering which involves strong limits used in this appendix.

Appendix B: Two Hilbert space injection operators

This appendix discusses the construction of injection operators from operators in the field algebra. The starting point is to let B be a function of the fields that creates a one-body state out of the vacuum. This means that the completeness sum

$$\langle 0|B^\dagger B|0\rangle = \sum_n \langle 0|B^\dagger|n\rangle\langle n|B|0\rangle \tag{B1}$$

has one-body intermediate states (here one-body means states with discrete positive mass eigenvalues - there is no distinction between elementary and composite one-body states). For simplicity it is assumed that the one-body spectrum is non-degenerate, which means each one-particle state has a different mass, and each one-body mass eigenstate has a given spin.

To isolate the operators that create the one-body states use space-time translations to define the operator valued distributions

$$B(x) := e^{-iP\cdot x}Be^{-iP\cdot x} = U^\dagger(I,x)BU(I,x) \tag{B2}$$

where P^μ is the four momentum operator. It follows from (B2) that

$$\frac{\partial B(x)}{\partial x_\mu} = -i[P^\mu, B(x)]. \tag{B3}$$

The next step is to compute the Fourier transform of the operator density $B(x)$:

$$\hat{B}(q) := \int \frac{d^4x}{(2\pi)^2} e^{iq\cdot x}B(x). \tag{B4}$$

Multiplying both sides of (B3) by $\frac{e^{iq\cdot x}}{(2\pi)^2}$ and integrating over d^4x by parts, assuming that $B(x)$ is an operator valued distribution, gives

$$-i[P^\mu, \hat{B}(q)] = -iq^\mu \hat{B}(q) \tag{B5}$$

or

$$[P^\mu, \hat{B}(q)] = q^\mu \hat{B}(q). \tag{B6}$$

It follows from (B6) that $\hat{B}(q)|0\rangle$ is either 0 or an eigenstate of the four momentum with eigenvalue q^μ:

$$P^\mu \hat{B}(q)|0\rangle = \hat{B}(q)P^\mu|0\rangle + q^\mu \hat{B}(q)|0\rangle = q^\mu \hat{B}(q)|0\rangle. \tag{B7}$$

Let m be a discrete mass eigenvalue of $M = \sqrt{-P^2}$ and let s denote the spin of the particle of mass m. Next let $h(q)$ be a smooth Lorentz invariant function of compact support in q^2 that is 1 when $q^2 = -m^2$ and $m > 0$, and identically 0 on the rest of the spectrum of intermediate states in (B1) and define

$$A^\dagger(q) := \hat{B}(q)h(q). \tag{B8}$$

Also define the operators:

$$A^\dagger(\mathbf{q}) := \int A^\dagger(q)dq^0 \qquad \text{and} \qquad \tilde{A}^\dagger(\tilde{\mathbf{q}}) := \int A(q)dq^-/2. \tag{B9}$$

Because $h(q)$ is localized these operators are distributions in the three momentum or light-front components of the four momentum. When these are applied to the vacuum they create one-particle states of mass m. Since by assumption, there is a unique spin s associated with each discrete mass eigenvalue, the particle created out of the vacuum also has spin s.

Because of the integrals over q^0 or q^- it follows that (B6) must be replaced by

$$[\mathbf{P}, A^\dagger(\mathbf{q})] = \mathbf{q}A^\dagger(\mathbf{q}). \qquad [\tilde{\mathbf{P}}, \tilde{A}^\dagger(\tilde{\mathbf{q}})] = \tilde{\mathbf{q}}\tilde{A}^\dagger(\tilde{\mathbf{q}}) \tag{B10}$$

which when applied to the vacuum becomes

$$\mathbf{P}A^\dagger(\mathbf{q})|0\rangle = \mathbf{q}A^\dagger(\mathbf{q})|0\rangle \qquad \tilde{\mathbf{P}}\tilde{A}^\dagger(\tilde{\mathbf{q}})|0\rangle = \tilde{\mathbf{q}}\tilde{A}^\dagger(\tilde{\mathbf{q}})|0\rangle. \tag{B11}$$

This means that $A^\dagger(\mathbf{q})$ creates a one-particle state with momentum $\mathbf{q}$ and mass m and spin s out of the vacuum. Similarly

$$\tilde{\mathbf{P}}\tilde{A}^\dagger(\tilde{\mathbf{q}})|0\rangle = \tilde{\mathbf{q}}A^\dagger(\tilde{\mathbf{q}})|0\rangle. \tag{B12}$$

creates a one-particle state with light front momentum $\mathbf{q}$ and mass m and spin s out of the vacuum.

The next step is to construct operators that create simultaneous eigenstates of mass and spin and magnetic quantum numbers.

Define the operator

$$A^\dagger_{m,s}(\mathbf{p},\mu) := \int_{SU(2)} U(R,0)A^\dagger(R\mathbf{p})U^\dagger(R,0)D^{s*}_{\mu s}(R)dR \tag{B13}$$

where the integral is over the $SU(2)$ Haar measure and $D^{s*}_{\mu s}(R)$ is the spin-s $SU(2)$ Wigner function

$$D^s_{\mu s}(R) = \langle s,\mu|U(R)|s,s\rangle. \tag{B14}$$

It follows from (B13) that

$$U(R',0)A^\dagger_{m,s}(\mathbf{p},\mu)U^\dagger(R',0) =$$

$$\int_{SU(2)} U(R'R,0)A^\dagger(R^{-1}\mathbf{p})U^\dagger(R'R,0)D^{s*}_{\mu s}(R)dR. \tag{B15}$$

Changing variables, $R'' = R'R$ using the invariance of the Haar measure $dR = dR''$ for fixed R' gives

$$(B15) = \int_{SU(2)} U(R'',0)A^\dagger(R''^{-1}R'\mathbf{p})U^\dagger(R'',0)dR''D^{s*}_{\mu s}(R'^{-1}R'') =$$

$$\int_{SU(2)} \sum_{\nu=-s}^{s} U(R'',0)A^\dagger(R''^{-1}R'\mathbf{p})U^\dagger(R'',0)dR''D^{s*}_{\mu\nu}(R'^{-1})D^{s*}_{\nu s}(R'') =$$

$$\int_{SU(2)} \sum_{\nu=-s}^{s} U(R'',0)A^\dagger(R''^{-1}R'\mathbf{p})U^\dagger(R'',0)dR''D^{s*}_{\nu s}(R'^{-1}R'')D^{s}_{\nu\mu}(R') =$$

$$\sum_{\nu=-s}^{s} A^\dagger_{(m,j)}(R'\mathbf{p},\nu)D^s_{\nu\mu}(R'). \tag{B16}$$

When applied to the vacuum (B16) gives

$$U(R,0)A^\dagger_{(m,s)}(\mathbf{p},\mu)|0\rangle = \sum_{\nu=-s}^{s} A^\dagger_{m,s}(R\mathbf{p},\nu)|0\rangle D^s_{\nu\mu}(R) \tag{B17}$$

which means that either this vanishes or it transforms like a particle of mass m, spin s, momentum $\mathbf{p}$ and magnetic quantum number μ. The spin in these states created out of the vacuum is the canonical spin. This will vanish if there are no one-particle intermediate states with mass m and spin s in (B1). While the notation is purposely suggestive, $A^\dagger_{m,s}(\mathbf{p},\mu)$ is *not a creation operator*. In addition, p^0 is only equal to $\sqrt{\mathbf{p}^2+m^2}$ when $A^\dagger_{m,s}(\mathbf{p},\mu)$ is applied to the vacuum.

Because rotations do not change the time component, these operators also satisfy

$$[\mathbf{P}, A^\dagger_{(m,s)}(\mathbf{q},\mu)] = \int_{SU(2)} dR[\mathbf{P}, U(R,0)A^\dagger(R^{-1}\mathbf{q})U^\dagger(R,0)]D^{s*}_{\mu s}(R) =$$

$$\int_{SU(2)} dRU(R,0)[R\mathbf{P}, A^\dagger(R^{-1}\mathbf{q})]U^\dagger(R,0)D^{s*}_{\mu s}(R) =$$

$$RR^{-1}\mathbf{q}\int_{SU(2)} dRU(R,0)A^\dagger(R^{-1}\mathbf{q})U^\dagger(R,0)D^{s*}_{\mu s}(R) = \mathbf{q}A^\dagger_{(m,s)}(\mathbf{q},\mu). \tag{B18}$$

The normalization can be chosen so the states created out of the vacuum have the normalization (9).

The construction above cannot be used in the "light-front" case due to the integral over p^- in (B9). However in that case it is enough to project out the magnetic quantum number using a rotation $R_z(\phi)$ which leaves p^- in (B9) unchanged. The spin is identified with the highest non-zero weight, $s = \mu_{max}$. In this case equation(B13) is replaced by

$$\tilde{A}^\dagger_{m,s}(\tilde{\mathbf{p}},\mu) := \int_0^{2\pi} U(R_z(\phi),0)\tilde{A}^\dagger(R_z(\phi)^{-1}\mathbf{p})U^\dagger(R_z(\phi),0)e^{-i\mu\phi}\frac{d\phi}{2\pi}, \tag{B19}$$

(B17) and (B18) are replaced by

$$[\tilde{\mathbf{P}}, \tilde{A}^\dagger_{(m,s)}(\tilde{\mathbf{q}},\mu)] = \tilde{\mathbf{q}}\tilde{A}^\dagger_{(m,s)}(\tilde{\mathbf{q}},\mu), \tag{B20}$$

$$U(\Lambda_K,0)\tilde{A}^\dagger_{(m,s)}(\tilde{\mathbf{q}},\mu)U^\dagger(\Lambda_K,0) = \tilde{A}^\dagger_{(m,s)}(\tilde{\Lambda}_k q,\mu), \tag{B21}$$

and

$$U(R_z(\phi),0)\tilde{A}^\dagger_{(m,s)}(\tilde{\mathbf{q}},\mu)U^\dagger(R_z(\phi),0) = \tilde{A}^\dagger_{(m,s)}(R_z(\phi)\tilde{\mathbf{q}},\mu)e^{i\mu\phi}. \tag{B22}$$

The proof (B20) is essentially the same as the proof of (B18). To show (B22) note that for rotations about the z axis

$$B_f(p/m)R_z = R_z B_f(R_z^{-1}p/m) \tag{B23}$$

where $B_f(p/m)$ is a light front boost. It follows that

$$U(B_f(p/m),0)\tilde{A}^\dagger_{m,s}(\tilde{\mathbf{q}},\mu)U(B_f(p/m),0)^\dagger =$$

$$\int_0^{2\pi} U(B_f(p/m)R_z(\phi),0)\tilde{A}^\dagger(R_z(\phi)^{-1}\mathbf{q})U^\dagger(B_f(p/m)R_z(\phi),0)e^{-i\mu\phi} =$$

$$\int_0^{2\pi} U(R_z(\phi)B_f(R_z^{-1}(\phi)p/m),0)\tilde{A}^\dagger(R_z(\phi)^{-1}\mathbf{q})U^\dagger(R_z(\phi)B_f(R_z^{-1}(\phi)p/m),0)e^{-i\mu\phi} =$$

$$\int_0^{2\pi} U(R_z(\phi),0)U(B_f(R_z^{-1}(\phi)p/m),0)\tilde{A}^\dagger(R_z(\phi)^{-1}\mathbf{q})U(B_f(R_z^{-1}(\phi)p/m),0))^\dagger U^\dagger(R_z(\phi),0)^\dagger e^{-i\mu\phi} =$$

$$\int_0^{2\pi} U(R_z(\phi),0)U(,0)\tilde{A}^\dagger(B_f(R_z^{-1}(\phi)p/m)R_z(\phi)^{-1}\mathbf{q})U^\dagger(R_z(\phi),0)^\dagger e^{-i\mu\phi} =$$

$$\int_0^{2\pi} U(R_z(\phi),0)U(,0)\tilde{A}^\dagger(R_z(\phi)^{-1}B_f(p/m)\mathbf{q})U^\dagger(R_z(\phi),0)^\dagger e^{-i\mu\phi} =$$

$$\tilde{A}^\dagger_{m,s}(\tilde{\mathbf{B}}_f(p/m)q,\mu) \tag{B24}$$

which proves (B21). For (B22) note that

$$U(R_z(\phi'),0)\tilde{A}^\dagger_{(m,s)}(\tilde{\mathbf{q}},\mu)U^\dagger(R_z(\phi'),0) =$$

$$\int_0^{2\pi} U(R_z(\phi+\phi'),0)\tilde{A}^\dagger(R_z(\phi)^{-1}\mathbf{p})U^\dagger(R_z(\phi+\phi'),0)e^{-i\mu\phi}\frac{d\phi}{2\pi}. \tag{B25}$$

Let $\phi''=\phi+\phi'$ so (B25)) becomes

$$\int_0^{2\pi} U(R_z(\phi''),0)\tilde{A}^\dagger(R_z(\phi''-\phi')^{-1}\mathbf{p})U^\dagger(R_z(\phi''),0)e^{-i\mu(\phi''-\phi')}\frac{d\phi''}{2\pi}$$

$$\int_0^{2\pi} U(R_z(\phi''),0)\tilde{A}^\dagger(R_z(\phi'')^{-1}R_z(\phi')\mathbf{p})U^\dagger(R_z(\phi''),0)e^{-i\mu(\phi''}e^{i\phi'\mu)} =$$

$$\tilde{A}^\dagger_{(m,s)}(R_z(\phi')\tilde{\mathbf{q}},\mu)e^{i\mu\phi'}. \tag{B26}$$

The normalization can be chosen so these states created out of the vacuum have the normalization (9).

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