The light-front vacuum

W. N. Polyzou
The University of Iowa
Marc Herrmann
The University of Iowa

abstract: We discuss the relation between the trivial light-front vacuum and the non-trivial Heisenberg vacuum.

The light-front representation [1] of quantum field theory [2] has the advantage that the vacuum of the interacting theory is identical to the Fock vacuum of the corresponding free field theory. This is in contrast to the canonical representation of field theory where the interacting vacuum has infinite norm in the free field Hilbert space. On the other hand the light-front and canonical representations of the same field theory should be equivalent. The purpose of this work is to understand the relation between the vacuum vectors in these two representations of the same field theory.

The light front is a hyperplane tangent to the light cone defined by $x^+ = x^0 + \hat{\mathbf{n}} \cdot \mathbf{x} = 0$. Translations tangent to the light-front hyperplane are independent of interactions and are generated by the following components of the four momentum

$$P^+ := P^0 + \hat{\mathbf{n}} \cdot \mathbf{P} \geq 0 \qquad \mathbf{P}_\perp := \mathbf{P} - \hat{\mathbf{n}}(\hat{\mathbf{n}} \cdot \mathbf{P}). \tag{1}$$

Interactions, V, appear in the generator of translations normal to the light-front hyperplane

$$P^- = P_0^- + V. \tag{2}$$

The translation generators P^+ and $\mathbf{P}_\perp$ are sums of single-particle operators

$$P^+ = \sum_n p_n^+; \qquad \mathbf{P}_\perp = \sum_n \mathbf{p}_{n\perp}, \tag{3}$$

where each $p_n^+ \geq 0$.

The origin of the triviality of the light-front vacuum follows because both P^- and P_0^- commute with P^+. This means that $V|0\rangle$, where $|0\rangle$ is the free Fock vacuum, is an eigenstate of P^+ with eigenvalue zero. If V is expressed in normal ordered form, the part of V with all creation operators can only increase the value of p_n^+

$$V = \int \mathcal{V}(\tilde{\mathbf{p}}_1, \cdots, \tilde{\mathbf{p}}_n) a^\dagger(\tilde{\mathbf{p}}_1) \cdots a^\dagger(\tilde{\mathbf{p}}_n) d\tilde{\mathbf{p}}_1 \cdots d\tilde{\mathbf{p}}_n + \cdots , \tag{4}$$

however the sum of all of the non-negative p_n^+ must vanish, so it follows that either $\mathcal{V}|0\rangle = 0$ or the pure creation operator coefficient is proportional to $\delta(p_n^+)$ or derivatives of $\delta(p_n^+)$. We refer to these contributions, if they appear, as zero modes. These contributions cannot appear if the kernel $\mathcal{V}$ is smooth. In this case the interaction does not change the Fock vacuum. This does not happen in the canonical case because although the interaction is translationally invariant, the single-particle momenta can sum to zero without being all zero.

The simplest way to understand the relation between the light-front and canonical vacuua is to first consider the case of free fields, where the precise relations between the Heisenberg field, fields restricted to a fixed time surface and fields restricted to a light front are known. For a free scalar field the fixed time and light front representations of the field are given by the well-known expressions:

$$\phi(x) = \frac{1}{(2\pi)^{3/2}} \int \frac{d\mathbf{p}}{\sqrt{2\omega_m(\mathbf{p})}} (e^{ip\cdot x} a(\mathbf{p}) + e^{-ip\cdot x} a^\dagger(\mathbf{p})) \tag{5}$$

$$= \frac{1}{(2\pi)^{3/2}} \int \frac{d\tilde{\mathbf{p}}}{\sqrt{2p^+}} (e^{ip\cdot x} a(\tilde{\mathbf{p}}) + e^{-ip\cdot x} a^\dagger(\tilde{\mathbf{p}})), \tag{6}$$

where $\omega_m(\mathbf{p})$ is the energy of a particle of mass m and momentum $\mathbf{p}$ and $\tilde{\mathbf{p}} = (p^+, \mathbf{p}_\perp)$.

In either representation the vacuum is normally defined by the condition that it is annihilated by the annihilation operator. Identifying these two representations of the field and changing variables from the three momenta to the light-front components of the four momentum gives the relation between the annihilation operators

$$a_i(\mathbf{p}) = a_i(\tilde{\mathbf{p}}) \sqrt{\frac{p^+}{\omega_{m_i}(\mathbf{p})}}. \tag{7}$$

This suggest that the vacuum defined by the annihilation operator in these two representations is the same. When the field is restricted to the light front all of the mass dependent (dynamical) terms in (6) disappear. Furthermore, the field restricted to the light front is irreducible (i.e. it is possible to extract both the creation and annihilation operator directly from the field restricted to the light front - there is no need for derivatives off of the light front). It follows that all inner products and all expectation values of operators are identical, which means that free fields with different masses restricted to the light front are unitarily equivalent. This suggests that there is one vacuum vector, up to unitary equivalence, for free scalar fields of any mass. On the other hand, the creation and annihilation operators for two free scalar fields with different masses are related by the canonical transformation [4]

$$a_2(\mathbf{p}) = \cosh(\eta(\mathbf{p})) a_1(\mathbf{p}) + \sinh(\eta(\mathbf{p})) a_1^\dagger(\mathbf{p}) \tag{8}$$

where

$$\cosh(\eta(\mathbf{p})) = \frac{1}{2} \left(\sqrt{\frac{\omega_{m_2}}{\omega_{m_1}}} + \sqrt{\frac{\omega_{m_1}}{\omega_{m_2}}} \right). \tag{9}$$

The problem is that the generator of the associated unitary transformation

$$G = -i \int \frac{d\mathbf{p}\eta(\mathbf{p})}{2} \left(a_1(\mathbf{p}) a_1(\mathbf{p}) - a_1^\dagger(\mathbf{p}) a_1^\dagger(\mathbf{p}) \right) \tag{10}$$

has infinite norm when applied to the vacuum of particle 1, which means the canonical transformation (8) cannot be realized as a unitary transformation on the Hilbert space associated with particle 1.

These results have the appearance of being inconsistent - specifically identifying the vacuum of the canonical and light-front fields of the same mass, and the vacuum of the light-front fields with different masses, suggest that all of the vacuua are identical or related by a unitary transformation. On the other hand a direct comparison on the vacuua of canonical field theories with different masses show that they are not even in the same Hilbert space .

The resolution of this apparent inconsistency is that the annihilation operator does not completely characterize the vacuum[5]. Another characterization of the vacuum is as an invariant linear functional on an algebra of operators. The relevant algebras are the algebra of free fields integrated against test functions of four space-time variables, the algebra of free fields restricted to a light front, integrated against test functions in coordinates of the light-front hypersurface, and the algebra of free fields and their time derivatives restricted to a fixed time surface, integrated against test functions in three space variables. We call these algebras the local algebra, the light-front algebra, and the canonical algebra respectively. Each algebra is invariant with respect to a different symmetry group.

The formal expressions (5-6) of the fields in terms of creation and annihilation operators can still be used to construct elements of each of these algebras by making the appropriate restrictions and integrating against the appropriate test functions.

From the algebraic perspective it is clear that both the light-front and canonical algebras do not contain enough operators to formulate Lorentz invariance or locality. Schlieder and Seiler [3] give an example that illustrates the importance of the underlying algebra. They consider two free fields of different mass, but they restrict the algebra by limiting the test functions to functions with Fourier transforms that are related on the two different mass shells by

$$\frac{f(\sqrt{m_1^2+\mathbf{p}^2},\mathbf{p})}{(m_1^2+\mathbf{p}^2)^{1/4}} = \frac{f(\sqrt{m_2^2+\mathbf{p}^2},\mathbf{p})}{(m_2^2+\mathbf{p}^2)^{1/4}}. \tag{11}$$

For this class of test functions

$$\langle 0_1|\phi_1(f_1)\cdots\phi_1(f_n)|0_1\rangle = \langle 0_2|\phi_2(f_1)\cdots\phi_2(f_n)|0_2\rangle. \tag{12}$$

In this case the inner products are identical, and the two fields restricted to this algebra are related by a unitary transformation. In addition, because there is no restriction on the test functions on one mass shell, it follows that both of these algebras are irreducible. On the other hand, these properties are not preserved when the restricted class of test functions are enlarged to include the full set of test functions. This illustrates the importance of the role of the algebra.

In the case of the two representations of the free fields, the structure of the field (5-6) plays a key role in extending the algebra from an algebra of operators restricted to a hypersurface to the local algebra. In the case of the light-front representation the free Heisenberg field of mass m can be expressed as a linear operator acting on a field restricted to a light front

$$\phi(x) = \int F_m(x,\tilde{\mathbf{y}})\phi(\tilde{\mathbf{y}})d\tilde{\mathbf{y}} \tag{13}$$

where

$$F_m(x,\tilde{\mathbf{y}}) = \frac{1}{(2\pi)^2}\int \frac{d\tilde{\mathbf{p}}}{2} e^{-i\frac{p_\perp^2+m^2}{p^+}x^+ + i\tilde{\mathbf{p}}\cdot(\tilde{\mathbf{x}}-\tilde{\mathbf{y}})}. \tag{14}$$

Here the kernel $F_m(x,\tilde{\mathbf{y}})$ that defines this extension has the mass dependence, which is the dynamical information for a free field. We can also alternatively interpret this kernel as a map of test functions in four variables to test functions restricted to a light front. The Fourier transforms of these test functions are related by

$$\tilde{f}(\tilde{\mathbf{p}}) = \tilde{f}\left(\frac{\mathbf{p}_\perp^2+m^2}{p^+}, \tilde{\mathbf{p}}\right). \tag{15}$$

The important observation is that because the Fourier transform of a Schwartz function in four variables is a Schwartz function, the function $\tilde{f}(\tilde{\mathbf{p}})$ vanishes faster than $(p^+)^n$ for any n at the origin. We will see that this observation has important implications when we discuss properties of the light-front Fock algebra.

We can interpret $F_m(x,\tilde{\mathbf{y}})$ as a mapping from the local Heisenberg algebra to a sub-algebra of the light-front Fock algebra that preserves all Wightman functions, and is hence unitary. Under this mapping the vacuum becomes the free field Fock vacuum, and the dynamics is contained in the mapping.

In order to consider the potential role of zero modes it is useful to summarize the key properties of the light-front Fock algebra [6][3][7], that is that algebra generated by free fields restricted to a light front. Operators in this algebra have the form

$$\phi(\tilde{f}) = \int d\tilde{\mathbf{x}}\phi(\tilde{\mathbf{x}}, x^+=0)f(\tilde{\mathbf{x}}). \tag{16}$$

This algebra is closed under operator multiplication, which can be summarized by

$$e^{i\phi(\tilde{f})}e^{i\phi(\tilde{g})} = e^{i\phi(\tilde{f}+\tilde{g})}e^{-\frac{1}{2}((\tilde{f},\tilde{g})-(\tilde{g},\tilde{f}))} \tag{17}$$

where the light-front scalar product is given by

$$(\tilde{f},\tilde{g}) = \int \frac{d\tilde{\mathbf{p}}\theta(p^+)}{p^+} f(-\tilde{\mathbf{p}})g(\tilde{\mathbf{p}}). \tag{18}$$

One of the important properties of this algebra is that it is irreducible. The simplest way to understand this is to note that it is possible to extract the creation and annihilation parts of $\phi(\tilde{f})$ without extending the algebra. Specifically the creation and annihilation operators are related to the Fourier transform of the restricted field by

$$\phi_+(\tilde{\mathbf{p}}) := \theta(p^+)\phi(\tilde{\mathbf{p}}) = \frac{\theta(p^+)}{\sqrt{p^+}}a(\tilde{\mathbf{p}}) \tag{19}$$

$$\phi_-(\tilde{\mathbf{p}}) := \theta(-p^+)\phi(\tilde{\mathbf{p}}) = \frac{\theta(-p^+)}{\sqrt{-p^+}}a^\dagger(-\tilde{\mathbf{p}}) \tag{20}$$

$$\phi(\tilde{\mathbf{p}}) = \phi_+(\tilde{\mathbf{p}}) + \phi_-(\tilde{\mathbf{p}}). \tag{21}$$

One consequence of this is that there is a purely algebraic normal ordering, which can be summarized by

$$: e^{i\phi(\tilde{f})} := e^{i\phi_-(\tilde{f})} e^{i\phi_+(\tilde{f})}. \tag{22}$$

It follows, using the Campbell-Baker-Hausdorff formula that

$$e^{i\phi(\tilde{f})} =: e^{i\phi(\tilde{f})} : e^{\frac{1}{2}(\tilde{f},\tilde{f})}. \tag{23}$$

We note that independent of the choice of vacuum, if the test function vanishes for $p^+ = 0$, the light-front inner product in (23) is well behaved and the vacuum expectation value of the normal product must be one, since for $p^+ \neq 0$ the annihilation operator necessarily reduces the value of p^+, which means that it must annihilate the vacuum. In this case the vacuum is uniquely determined by this equation. Since the operator F_m maps four variable test functions into functions on the light front with Fourier transforms that vanish for $p^+ = 0$, the vacuum is always the free field Fock vacuum in this case and there can be no zero mode contributions.

It is well known that the equivalence of calculations of some observables based on light-front field theory and canonical field theory require 0-mode contributions in the light-front expressions. So far we have only considered free fields, so there is a question concerning whether zero modes are a consequence of a more complex dynamics or not.

To answer this first note that the classical Noether's theorem gives formal expressions for the Poincaré generators as local products of operators at the same point on the light front. These products are not elements of the light-front algebra. One can see the problem in the following expression for the two point Wightman function in light-front variables:

$$\langle 0|\phi(x)\phi(y)|0\rangle = \tag{24}$$

$$\frac{1}{(2\pi)^3}\int \frac{d\tilde{\mathbf{p}}}{p^+} e^{-i\frac{\mathbf{p}_\perp^2+m^2}{p^+}(x^+-y^+)+i\tilde{\mathbf{p}}(\tilde{\mathbf{x}}-\tilde{\mathbf{y}})}. \tag{25}$$

Although there is an apparent logarithmic singularity due to the p^+ integral, if $(x^+ - y^+)$ is not zero the $1/p^+$ singularity in the above expression is regulated by oscillations in the exponent. This can be seen by considering the integral of the same form

$$\int_0^a \frac{e^{ic/p^+}}{p^+} dp^+ = \int_{c/a}^\infty \frac{e^{iu}}{u} du = \frac{\pi}{2} - (Ci(c/a) + iSi(c/a)). \tag{26}$$

If both fields are restricted to the light front, this mechanism that regulates the singularity at $p^+ = 0$ is turned off leading an infrared singularity. There will also be ultraviolet singularities associated with the other variables. In this case there is no reason for the mapped test functions to vanish at $p^+ = 0$. In this case if we try to make sense out of (25) when the test function does not vanish at $p^+ = 0$ two things happen. First the logarithmic singularity in the light front scalar product needs to be regularized. Typically this regularization breaks longitudinal boost invariance. A second thing that can happen is that because it is possible to have non-zero contributions from $p^+ = 0$ the algebraic normal product can be extended to include additional singular contributions that have support on $p^+ = 0$. These can have a non-trivial dependence on transverse momentum and may be needed to restore full rotational covariance and longitudinal boost invariance.

It follows that zero modes are a consequence of renormalizing local operator products. They are available tools that may be needed to make the theory finite and consistent with the corresponding canonical theory.

We note that Lorentz invariance of the S-matrix in a light-front quantum theory is equivalent to invariance with respect to a change of orientation of the light front [8]. Changing the orientation of the light front can exchange infrared and ultraviolet singularities. This suggest that the problem of zero modes and renormalization is more complicated in 3+1 dimensional theories than it is in 1+1 dimensional theories.

While the above analysis is limited to free fields, in an asymptotically complete field theory an interacting Heisenberg field can be expanded in normal ordered products of an irreducible set of asymptotic (in or out) fields[4][9]:

$$\phi(x) = \sum \int L(x; z_1, \cdots, z_n) \colon \phi_{in}(z_1) \cdots \phi_{in}(z_n) \colon dz_1 \cdots dz_n. \tag{27}$$

The asymptotic fields are free fields, and following the above analysis, can be expressed as products of free fields on the light front using the mapping (14):

$$\phi(x) = \sum \int \mathcal{L}(x; \tilde{\mathbf{y}}_1, \cdots, \tilde{\mathbf{y}}_n) \colon \phi_0(\tilde{\mathbf{y}}_1) \cdots \phi_0(\tilde{\mathbf{y}}_n) \colon d\tilde{\mathbf{y}}_1 \cdots d\tilde{\mathbf{y}}_n \tag{28}$$

$$\mathcal{L}(x; \tilde{\mathbf{y}}_1, \cdots, \tilde{\mathbf{y}}_n) = \int L(x; z_1, \cdots, z_n) \prod_i F_m(z_i, \tilde{\mathbf{y}}_i) d^4 z_1 \cdots d^4 z_n. \tag{29}$$

Furthermore, if the Heisenberg field $\phi(x)$ is an operator valued distribution we expect that the smeared coefficient will behave like (15):

$$\int f(x) \tilde{\mathcal{L}}(x; \tilde{\mathbf{p}}_1, \cdots, \tilde{\mathbf{p}}_n) d^4 x \to 0 \quad \text{any} \quad p_i^+ \to 0. \tag{30}$$

If this is the case then $\{\mathcal{L}(x; \tilde{\mathbf{y}}_1, \cdots, \tilde{\mathbf{y}}_n)\}$ is the kernel of a unitary mapping from the Heisenberg algebra of the local field field theory to a sub-algebra of the light-front Fock algebra, where the unitary transformation maps the Heisenberg vacuum to the trivial Fock vacuum:

$$\phi(x) = \sum \int \mathcal{L}(x; \tilde{\mathbf{y}}_1, \cdots, \tilde{\mathbf{y}}_n) : \phi_0(\tilde{\mathbf{y}}_1) \cdots \phi_0(\tilde{\mathbf{y}}_n) : d\tilde{\mathbf{y}}_1 \cdots d\tilde{\mathbf{y}}_n. \tag{31}$$

At this level zero modes cannot occur, however they can occur as intermediate steps in the construction of the coefficient functions $\{\mathcal{L}(x; \tilde{\mathbf{y}}_1, \cdots, \tilde{\mathbf{y}}_n)\}$.

The conclusion of this work is that there is a unitarily equivalent representation of the dynamics with the free-field vacuum. The Hilbert space is generated from this vacuum by a sub-algebra of the light-front Fock algebra. While the vacuum functional has no zero mode contributions on this sub algebra, zero modes can appear in intermediate steps in the construction of the operators that map the local Heisenberg algebra to this sub-algebra.

References

[1] Dirac, P. A. M. (1949), Forms of Relativistic Dynamics, Rev. Mod. Phys. **21**, 392

[2] Chang, S. J., Root, R. G., Yan, T. M. (1973), Quantum field theories in the infinite momentum frame. 1. Quantization of scalar and Dirac fields, Phys. Rev. **D**7, 1133

[3] Schlieder, S., Seiler, E., (1972), Remarks on the null plane development of a relativistic quantum field theory, Comm. Math. Phys. **25**,62

[4] Haag, R. (1955), On quantum field theories, Mat-Fys. Medd. K. Danske Vidensk. Selsk., **29**,1

[5] Herrmann, M. Polyzou, W. N., (2015), The light front vacuum, Phys. Rev. **D** 91,085043

[6] Leutwyler, H, Klauder, J. R., Streit, L., (1970), Il. Nuovo Cimento, Quantum field theory on lightlike slabs, **A66**,536

[7] Coester, F. and Polyzou, W. N., (1994), Vacuum Structures in Hamiltonian Light Front Dynamics, Foundations of Physics, **24**, 387

[8] Polyzou, W. N.,(1999), Left Coset Invariance and Relativistic Invariance, Few-Body Systems **27**,57

[9] Glaser, V., Lehmann, H., Zimmermann, W., (1957), Field Operators and retarded functions, Il Nuovo Cimento **6**,1122