1.) Implicit function theorem. Consider the function \( w = e^{-(x^2 + y^2)} \). We would like to solve this for \( x \) near \( x = 2 \) by constructing

\[
x = g(w, y).
\]

While this can be done analytically, write

\[
w = e^{-(2^2 + y^2)} + \frac{\partial e^{-(x^2 + y^2)}}{\partial x} |_{x=2}(x - 2) + R(x, y)
\]

Write this as

\[
x = 2 + \frac{1}{\frac{\partial e^{-(x^2 + y^2)}}{\partial x} |_{x=2}}[w - e^{-(2^2 + y^2)} - R(x, y)]
\]

Try to approximate this by iteration

\[
x_0 := 2
\]

\[
x_n := 2 + \frac{1}{\frac{\partial e^{-(x^2 + y^2)}}{\partial x} |_{x=2}}[w - e^{-(2^2 + y^2)} - R(x_{n-1}, y)]
\]

for \( x \) near 2. Compare the result of this for a few iterations to the exact result for a selected value of \( y \). The purpose of this exercise is to illustrate how the implicit function theorem works.

2. Consider Newton’s equation for a linear harmonic oscillator

\[
m \frac{d^2x}{dt^2} = -kx
\]

a. Convert this to a system of two coupled first order differential equations.

b. Convert the system of first order linear differential equations to a pair of coupled integral equations.

c. Using the initial conditions, \( x(0) = a \) and \( \frac{dx}{dt}(0) = 0 \), solve the coupled integral equations by successive approximations, using the initial conditions as the first approximation (method used in class to prove the existence of solutions of differential equations). While there are an infinite number of iterations needed to obtain the full solution, this problem is simple enough that you should be able to obtain the exact solution.

3. Convert Legendre's differential equation to an approximate finite difference equation on the interval \([-1, 1]\). Discuss the proper boundary conditions that are needed to get an approximation to the \( n \)-th Legendre polynomial.
4. Consider Legendre’s differential equation. If we do not specify boundary conditions then we know that it has two independent solutions. One of the two independent solutions is the Legendre polynomial. Find the second independent solution for the case $n = 1$.

5. Find the solution of the first order differential equation

$$\cosh(x) \frac{df}{dx} + \sinh(x)f(x) = 0$$

6. Consider the Wronskian of the differential equation

$$a(x) \frac{d^2 f}{dx^2} + b(x) \frac{df}{dx} + c(x)f(x) = 0$$

where $a(x) > 0$. Show that the Wronskian of this equation is non-zero for all $x$. Hint: Find a differential equation for the Wronskian.