1.) Let $L$ be a Hermitian differential operator with some specified homogeneous boundary conditions. Consider the resolvent operator

$$R(z) = (z - L)^{-1}$$

which is defined when

$$L[f] = z[f]$$

has no solutions satisfying the homogeneous boundary conditions. Let $\gamma(t), t \in [0, 1]$ be a parameterized circle (counter clockwise) in the complex plane with radius $r$ and center at $z = 0$ with the property that $R(z)$ is defined for all $z$ on the curve.

a. Show that

$$P = \frac{1}{2\pi i} \int_0^1 R(\gamma(t)) \frac{d\gamma}{dt} dt$$

is an orthogonal projection operator.

b. If $L[f] = \eta[f]$, what can you say about $P[f]$?

2.) The Lagrangian for a free particle in a one dimensional box is

$$L = \frac{1}{2} \left( \frac{dx}{dt} \right)^2$$

The solution that is a stationary point of the action functional

$$A[x] = \int_0^T L(\dot{x}(t)) dt$$

for $x(0) = a$ and $x(T) = b$ is known to be

$$x_0(t) = a + \frac{b - a}{T} t$$

a. Use the method used in the example in class to determine whether this solution is a minimum (among all curves satisfying $x(0) = a$ and $x(T) = b$) of the action functional or not.

3.) Use the series method to solve the second order differential equation with constant coefficients,

$$L[f] = 0$$

$$\frac{d^2}{dx^2} + a \frac{d}{dx} + b$$

with boundary conditions

$$\langle 0 | f \rangle = 1$$

$$\frac{d}{dx} \langle x | f \rangle |_{x=0} = 1$$
b. What is the domain of analyticity of your solution?

c. Put this equation in the form (14.6) (see K&D) and verify equation (14.12)
for this example.

4.) Given a Strum Liouville operator of the form

\[ L = \frac{d}{dx} g(x) \frac{d}{dx} h(x) \]

with weight \( w(x) = 1 \), \( x \in [a, b] \), and \( g(x) \) and \( h(x) \) real. Show explicitly
that the eigenvalues of \( L|f_n\rangle = \lambda_n |f_n\rangle \) are real and that eigenvectors
corresponding different eigenvalues are orthogonal on \( [a, b] \).