1.) Consider the space of continuous complex-valued functions on the interval \([a, b]\) with inner product

\[
\int_a^b f^*(x)g(x)dx
\]

a. Show that the sum of two Cauchy sequences is a Cauchy sequence.

b. Show that if two different Cauchy sequences converge to the same function that the difference of these sequences converges to zero.

2.) Consider the functions \(\{f_n(x)\}\) defined by

\[
f(x) = \begin{cases} 
0 & : x < -1 - 1/n \\
n + 1 + nx & : -1 - 1/n < x < -1 \\
1 & : -1 \leq x \leq 1 \\
n + 1 - nx & : 1 < x < 1 + 1/n \\
0 & : x \geq 1 + 1/n
\end{cases}
\]

Show that this sequence is a Cauchy sequence. Show that it converges to the discontinuous block function

\[
b(x) = \begin{cases} 
0 & : |x| > 1 \\
1 & : |x| \leq 1
\end{cases}
\]

in the \(L_2(\mathbb{R})\) norm.

3.) Estimate the Lebesgue integral of

\[
\int_0^1 x^2 dx
\]

by dividing the range of this function into 10 equally spaced intervals between 0 and 1. Find upper and lower bounds for the integral by using the largest or smallest value of the function on each interval. Compare this to the exact value.

4.) Show that the set of rational numbers (numbers that can be expressed as ratios of integers \(m/n\)) between 0 and 1 are a set of Lebesgue measure zero.
5.) Consider the set of rational numbers between zero and one. Show that it is possible to place each rational in the interior of an open interval (i.e. $a < (m/n) < b$) in such a way that if we discard all rationals and all of the open intervals containing these rationals that what remains in the interval $[0, 1]$ has a measure as close to 1 as desired. [This exercise was a critical element in Kolomogorov, Arnold, and Moser’s solution to the classical three-body problem which asks whether the solar system is stable]

6.) Let $S_1$ and $S_2$ be subsets of a larger set $S$. For a subset $S'$ of $S$ let $S^{ec}$ the the complement of $S'$ in $S$. This means the set of points in $S$ that are not in $S'$.

a. Show

$$S_1 \cap S_2 = (S_1^{ec} \cup S_2^{ec})^c$$

b. Show how does this show that the intersection of two Lebesgue measurable sets are is measurable?

c. Show that every closed interval, $[a, b]$ of the real line is Lebesgue measurable. Find the Lebesgue measure of this set?