1. Show that if $R$ is a real $3 \times 3$ matrix that preserves the length of all vectors,

$$ r' = Rr \quad r' \cdot r' = r \cdot r, $$

that $R$ must satisfy $R^T R = I$ where $R^T$ is the transpose of $R$. Matrices with this property are called orthogonal matrices.

2. Show that if $R_1$ and $R_2$ are orthogonal matrices that the matrix product $R_3 = R_2 R_1$ is also an orthogonal matrix.

3. Show that the product of 2 Galilean transformations

$$ G = \begin{pmatrix} R & v_0 & r_0 \\ 0 & 1 & t_0 \\ 0 & 0 & 1 \end{pmatrix} $$

is another Galilean transformation. This means that multiple applications of these transformations do not lead to a new type of transformation.

4. Show that every Galilean transformation has an inverse that is also a Galilean transformation. Write down the inverse transformation.

5. An isolated mechanical system consists of three point particles that are initially at rest in an inertial coordinate system. Show that they remain in the initial plane for all time (assume for this problem that inertial coordinate systems are related by Galilean transformations where $R$ can also include reflections).

6. The difference between Galilean and special relativity is the relation between different inertial coordinate systems. In special relativity the transformations that relate different inertial coordinate systems preserve

$$ c^2(t_1 - t_2)^2 - (r_1 - r_2) \cdot (r_1 - r_2) = c^2(t'_1 - t'_2)^2 - (r'_1 - r'_2) \cdot (r'_1 - r'_2) \quad (1) $$

where $c$ is the speed of light in a vacuum. These transformations leave the form of Maxwell’s equations invariant, but do not leave Newton’s second law invariant, while the Galilean transformations do the opposite. Define

$$ x^\mu = (ct, x, y, z) = (x^0, x^1, x^2, x^3) \quad x'^\mu = f^\mu(x^0, x^1, x^2, x^3) $$

Show that the general form of a transformation that preserves (1) is

$$ x'^\mu = \sum_{\nu=0}^3 M_{\mu \nu} x^\nu + a^\mu $$

where $a^\mu$ is a constant (4) vector and $M_{\mu \nu}$ is a constant matrix satisfying

$$ M^T \eta M = \eta $$

1
where $\eta$ is the matrix

$$
\eta = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & -1 \\
\end{pmatrix}
$$
Homework 1 - Solutions

1. \[ r_i' = \sum_{j=1}^{3} R_{ij} r_j \]
   \[ \sum_{i=1}^{3} r_i' r_i' = \sum_{i=1}^{3} R_{ij} r_i r_i = \sum_{i=1}^{3} r_i r_i^T r_i r_i \]

   If this is equal to \( \sum r_i r_i \)
   we must have

   \[ 0 = \sum_{i=1}^{3} \sum_{j=1}^{3} R_{ij} \left( \sum_{k=1}^{3} R_{ki} r_i - s_{kj} \right) r_j \]

   for any choice of \( \bar{r} = (r_1, r_2, r_3) \)

   This equation can be written as

   \[ \bar{r} \cdot (R^T R - I) \bar{r} = 0 \]  \hspace{1cm} (1)

   Let \( \bar{r} = \bar{r}_A - \bar{r}_B = 0 \)

   \[ \left( \bar{r}_A + \bar{r}_B \right) (R^T R - I) \left( \bar{r}_A + \bar{r}_B \right) = \]

   \[ \bar{r}_A \cdot (R^T R - I) \bar{r}_A + \bar{r}_B \cdot (R^T R - I) \bar{r}_B \]

   \[ \pm \left( \bar{r}_A \cdot (R^T R - I) \bar{r}_B - \bar{r}_B \cdot (R^T R - I) \bar{r}_A \right) \]  \hspace{1cm} (2)

   Line 2 vanishes by line (1) - note also

   \[ \sum R_{AR} R_{ki}^T r_i r_j = \sum R_{AR} R_{ki}^T r_i r_j = \sum R_{BR} R_{ki}^T r_i r_j \]

   \[ \sum r_{BR} R_{ki}^T r_i r_j = \sum r_{BR} R_{ki}^T r_i r_j \]

   \[ \sum r_{BR} R_{ki}^T r_i r_j = \sum r_{BR} R_{ki}^T r_i r_j \]
so equation (3) becomes

\[ 2 \vec{\alpha}_i \cdot (R^T R - I) \cdot \vec{\beta}_j = 0 \]

This must be true for any \( \vec{\alpha}_i, \vec{\beta}_j \)
which gives

\[ R^T R = I \]

2. If \( R^T R = I \)

\[ (R_1 R_2)^T_{ij} = \sum_k (R_1)_{ik} (R_2)_{kj} \]

\[ \sum_k (R_1)_{ik} (R_2)_{kj} = \sum_h (R_2)_{hi} (R_1)_{jh} \]

or

\[ \sum_n (R_2^T)_{in} (R_1^T)_{nj} \]

\[ (R_1 R_2)^T = R_2^T R_1^T \]

\[ (R_1 R_2)^T (R_1 R_2) = \]

\[ R_2^T (R_1^T R_1) R_2 = \]

\[ R_2^T I R_2 = \]

\[ R_2^T R_2 = \]

\[ I \]
\[ G_2 G_1 = \begin{pmatrix} R_2 & \bar{V}_{20} & \bar{R}_{20} \\ 0 & 1 & C_2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} R_1 & \bar{V}_{10} & \bar{R}_{10} \\ 0 & 1 & C_1 \\ 0 & 0 & 1 \end{pmatrix} = \]

using matrix multiplication

\[ = \begin{pmatrix} R_2 R_1 & \bar{R}_2 \bar{V}_{10} + \bar{V}_{20} & \bar{R}_2 \bar{R}_{10} + \bar{V}_{20} C_1 + \bar{R}_{20} \\ 0 & 1 & C_1 + C_2 \\ 0 & 0 & 1 \end{pmatrix} \]

Let

\[ R_{21} = R_2 R_1, \]

which satisfies \( R_{21}^T R_{21} = I \) by problem 2

\[ \bar{V}_{21} = R_{21} \bar{V}_{10} + \bar{V}_{20}, \]

\[ \bar{R}_{21} = R_{21} \bar{R}_{10} + \bar{V}_{20} C_1 + \bar{R}_{20}, \]

\[ C_{21} = C_1 + C_2 \]

\[ G_2 G_1 = G_{21} = \begin{pmatrix} R_{21} & V_{210} & R_{210} \\ 0 & 1 & C_{21} \\ 0 & 0 & 1 \end{pmatrix} \]

which has the same form as \( G_1 \) and \( G_2 \)
For $\mathbf{G}_2 \mathbf{G}_1 = I \Rightarrow$

\[ \mathbf{R}_{21} = \mathbf{R}_2 \mathbf{R}_1 = I \]
\[ \mathbf{R}_2 \mathbf{R}_1 \mathbf{R}_1^T = \mathbf{R}_1^T = \mathbf{R}_2 \mathbf{I} = \mathbf{R}_2 \]
\[ \Rightarrow \mathbf{R}_2 = \mathbf{R}_1^T \]
\[ \mathbf{V}_{210} = \mathbf{0} = \mathbf{R}_2 \mathbf{V}_{10} + \mathbf{V}_{20} = \mathbf{R}_1^T \mathbf{V}_{10} + \mathbf{V}_{20} \]
\[ \Rightarrow \mathbf{V}_{20} = -\mathbf{R}_1^T \mathbf{V}_{10} \]
\[ \mathbf{\bar{r}}_{210} = \mathbf{0} = \mathbf{R}_2 \mathbf{r}_{10} + \mathbf{V}_{20} \mathbf{c}_1 + \mathbf{r}_{20} \]
\[ = \mathbf{R}_1^T \mathbf{r}_{10} - \mathbf{R}_1^T \mathbf{V}_{10} \mathbf{c}_1 + \mathbf{r}_{20} \]
\[ \Rightarrow \mathbf{r}_{20} = -\mathbf{R}_1^T \mathbf{r}_{10} + \mathbf{R}_1^T \mathbf{V}_{10} \mathbf{c}_1 \]
\[ \mathbf{0} = \mathbf{c}_{12} = \mathbf{c}_1 + \mathbf{c}_2 \]
\[ \mathbf{c}_2 = -\mathbf{c}_1 \]
\[ \mathbf{G}_1^{-1} = \begin{pmatrix} \mathbf{R}_1^T - \mathbf{R}_1^T \mathbf{V}_{10} & -\mathbf{R}_1^T \mathbf{r}_{10} + \mathbf{R}_1^T \mathbf{V}_{10} \mathbf{c}_1 \\
\mathbf{0} & 1 & -\mathbf{c}_1 \\
\mathbf{0} & 0 & 1 \end{pmatrix} \]

which is also a Galilean transformation since $\mathbf{R}_1^T \mathbf{R}_1 = \mathbf{R}_1 \mathbf{R}_1^T = I$
(1) From the Newtonian principle of determinacy

$$\vec{F}_i(t) = \vec{g}_i(t_j, t_0, \vec{r}_{i0}, \vec{r}_{20}, \vec{r}_{30}, \vec{V}_{i0}, \vec{V}_{20}, \vec{V}_{30})$$

In this case all of the velocities are $\vec{0}$ so

$$\vec{F}_i(t) = \vec{g}_i(t, t_0, \vec{r}_{i0}, \vec{r}_{20}, \vec{r}_{30})$$

By the principle of Galilean relativity $\vec{g}_i$ must be a function of coordinate differences satisfying

$$\vec{F}_i(t) = R^T \vec{g}_i(t, t_0, R(\vec{r}_{20} - \vec{r}_{i0}), R(\vec{r}_{30} - \vec{r}_{i0}))$$

This will be true if $\vec{g}$ is a vector constructed out of

$$\vec{g}_i = A_i(t)(\vec{r}_{20} - \vec{r}_{i0}) +$$

$$B_i(t)(\vec{r}_{30} - \vec{r}_{i0}) +$$

$$C_i(t)(\vec{r}_{20} - \vec{r}_{i0}) \times (\vec{r}_{30} - \vec{r}_{i0})$$

If $R = -I$ is a space reflection $\vec{g}$ should change sign - but the cross product term does not so $C_i(t) = 0 \Rightarrow \vec{F}_i(t)$ stays in the initial plane.
\( \sum \eta_{\mu \nu} (f^\nu(x) - f^\nu(y)) (f^\nu(x) - f^\nu(y)) = \\
\sum \eta_{\mu \nu} (x^\mu - y^\mu)(x^\nu - y^\nu) \)

(0) Differentiate with respect to \( y^\mu \); set \( y = 0 \)

\[ \sum \eta_{\mu \nu} \left( \frac{\partial f^\nu}{\partial y^\mu} \right) (f^\nu(x) - f^\nu(y)) + \\
\sum \eta_{\mu \nu} (f^\nu(x) - f^\nu(y)) \left( - \frac{\partial f^\nu}{\partial y^\mu} \right) \]

\[ \sum \eta_{\mu \nu} (- s^\nu)(s^\nu) + \\
\sum \eta_{\mu \nu} (x^\nu)(- s^\nu) \]

(2) Noting that \( \eta_{\mu \nu} = \eta_{\nu \mu} \) and replacing \( \mu \) with \( \nu \) in the second and fourth term gives

\[ -2 \sum \eta_{\mu \nu} \frac{\partial f^\mu}{\partial y^\nu}(c) (f^\nu(x) - f^\nu(c)) = \\
-2 \sum \eta_{\mu \nu} x^\mu \]

or

\[ \eta_{\mu \nu} x^\mu = \eta_{\mu \nu} \frac{\partial f^\mu}{\partial y^\nu}(c) (f^\nu(x) - f^\nu(c)) \] (1)
Let $x 	o y$ and differentiate with respect to $y^B$ and set $\gamma = \nu$

$$\sum M_{\nu a} y^\nu = \sum M_{\nu \nu} \frac{\partial f^\nu}{\partial y_a} (\nu - f^\nu (c))$$  \hspace{0.5cm} (1)$$

$$\sum M_{\nu a} \delta^a_B = \sum M_{\nu \nu} \frac{\partial f^\nu}{\partial y^\nu_a} \frac{\partial f^\nu}{\partial y^\nu_B}$$

$$M_{\nu B} = \sum \frac{\partial f^\nu}{\partial y^\nu_a} (\nu - f^\nu (c)) \frac{\partial f^\nu}{\partial y^\nu_B}$$  \hspace{0.5cm} (2)$$

set

$$M^T_{\nu \nu} \equiv \frac{\partial f^\nu}{\partial y^\nu_B} (\nu - f^\nu (c)) = M_{\nu a}$$

then equation (2) reads

$$\begin{bmatrix} M &= M^T \kappa \kappa M \end{bmatrix} \quad I = \frac{\kappa M^T \kappa M}{M^{-1}} \quad (\kappa^2 = I)$$

equation (1) has the matrix form

$$\kappa M \kappa. X = M^T \kappa (\kappa f(x) \kappa - \kappa f(c) \kappa)$$

multiply this by $\kappa M \kappa \kappa$ gives

$$\kappa M \kappa \kappa . X = \kappa M \kappa \kappa M^T \kappa (\kappa f(x) \kappa - \kappa f(c) \kappa) \frac{1}{M^{-1}}$$

$$\kappa M \kappa \kappa \kappa \kappa X = \kappa (\kappa f(x) \kappa - \kappa f(c) \kappa)$$
\begin{align*}
\text{since } \eta^2 &= I \\
M \cdot x &= \eta^2 (f(x) - f(\zeta)) = f(x) - f(\zeta) \\
\Rightarrow f(x) &= f(\zeta) + M \eta x \\
\frac{\partial f^\mu}{\partial x^\nu}(\zeta) \times \nu &= \sum_{\nu} \frac{\partial f^\mu}{\partial x^\nu}(\zeta) \times \nu + f^\mu(\zeta) \\
M_{\nu \nu} &\quad a^\nu
\end{align*}