Assignment 5 - due October 5

1. Consider the Lagrangian

\[ L = \frac{m}{2}(\dot{x}^2 + \dot{y}^2) - V(x - y) \]

a. Show that the transformation

\[ x \rightarrow x' = x + \epsilon; \quad y \rightarrow y' = y + \epsilon; \quad t \rightarrow t' = t \]

Leaves the corresponding action invariant.

b. Use Noether’s theorem to find the associated conserved quantity.

2. A system consists of two identical point masses connected by a massless spring of force constant \( k \) and equilibrium length \( l_0 \). The system lies on an ice skating rink and can move freely on the ice without friction. Find the action. Use Noether’s theorem to identify symmetries of the action. Find the associated conserved quantities.

3. What is the conserved quantity for an action in ordinary Cartesian coordinates that is invariant under the infinitesimal transformation

\[ t \rightarrow t' = t \quad r_i(t) \rightarrow r'_i(t') = r_i(t) + \epsilon v_t \]

4. Consider a system of \( N \) point particles of the same mass. They interact via pairwise interactions \( V = \sum_{i<j} V(|r_i - r_j|) \). This interaction is invariant under rotations of both coordinates.

a. Find the form of \( (\delta r_i) \) for an infinitesimal rotation of all coordinates about the \( z \) axis.

b. Show that this transformation leaves the action for this problem invariant.

c. Find the associated conserved quantity.

5. Consider a set of generalized coordinates \( (q_1, q_2) \) in a plane where the infinitesimal distance between two points is given by

\[ ds = \sqrt{\sum_{ij=1}^{2} g_{ij}(q_1, q_2) dq_i dq_j} \]

Find the equations that give a path of the shortest distance between 2 points in the plane in terms of these coordinates.

6. Consider the Lagrangian density for a scalar field

\[ L = \frac{1}{2}[(\partial \phi(x, t)/\partial t)^2 - \sum_{i=1}^{3} (\partial \phi(x, t)/\partial x_i)^2] - m^2 \phi(x, t)^2 \]

Use the principle of stationary action to find Lagrange’s equations for this field.