Lecture 10

Second variation

\[ S^2 A[\tilde{s}_0, \tilde{s}_q] = S[S\tilde{s}_q] > 0 \] for all \( \tilde{s}_q \)

Treat as a functional

Look for \( \tilde{s}_q \) that makes \( S[S\tilde{s}_q] \) stationary

\[ S S[\tilde{s}_q, \delta \tilde{s}_q] = \frac{d}{d\lambda} S[S\tilde{s}_q + \lambda \delta \tilde{s}_q] = 0 \]

\[ S[S\tilde{s}_q] = \int_{t_i}^{t_f} \sum_{i,j} \left( C_{ij} \dot{s}_q^i s_q^j + 2 B_{ij} s_q^i \dot{s}_q^j + A_{ij} s_q^i s_q^j \right) \]

\[ -\eta \sum_{i} \int_{t_i}^{t_f} s_q^i \, dt = 0 \]

\( \eta \) is a Lagrange multiplier for the constraint

\[ \int_{t_i}^{t_f} \delta s_q^i \, dt = 1 \]

Setting the first variation of \( S[S\tilde{s}_q] \) to 0 gives the differential equation

\[
\sum_{i} \left( -\frac{d}{dt} \left( C_{ij} \dot{s}_q^i \right) + B_{ji} s_q^i \dot{s}_q^j - \frac{d}{dt} \left( B_{ij} s_q^i \right) + A_{ij} s_q^i s_q^j \right) = \eta s_q^i
\]
This is an eigenvalue equation.

Last time we showed

1. If \( \tilde{\Phi}_n(t) \) is the solution of this equation with eigenvalue \( \lambda_n \), then
   \[
   S[\tilde{\Phi}_n] = \lambda_n
   \]
   so the \( \lambda_n \) are the values of the functional acting on the \( n \)th eigenvector.

2. If \( D \) is the differential operator
   \[
   \sum_j \left( \frac{d}{dt} (C_{ij} \frac{d}{dt}) + B_{ij} \frac{d}{dt} - \frac{d}{dt} \right) \tilde{\Phi}_n = D \tilde{\Phi}_n
   \]
   then
   \[
   \int \tilde{\Phi}_m(D \tilde{\Phi}_n) \, dt = \int (D \tilde{\Phi}_m) \cdot \tilde{\Phi}_n
   \]
   this is the differential equation version of a Hermitian matrix.
many of the properties follow from

\[ D \delta \eta_n = \eta_n \delta \eta_n \]
\[ D \delta \eta_m = \eta_m \delta \eta_m \]
\[ \int \delta \eta_m \cdot D \delta \eta_n = \eta_n \int \delta \eta_m \cdot \delta \eta_n \]
\[ \int D \delta \eta_n \cdot \delta \eta_m = \eta_m \int \delta \eta_m \cdot \delta \eta_n \]

which means

\[ (\eta_n - \eta_m) \int \delta \eta_m \cdot \delta \eta_n = 0 \]

either \( \eta_n = \eta_m \) or \( \int \delta \eta_m \cdot \delta \eta_n \, dt = 0 \)

since \( D \) is real we can show

\[ \int \delta \eta_n \cdot D \delta \eta_m = \int (D \delta \eta_n) \cdot \delta \eta_m = \eta_n = \eta_m^* \]

which means the eigenvalues are real.

there are exactly the same properties satisfied by Hermitian matrices.

Linear differential equations of this form are called Strum-Liouville equations. For one variable they have the form

\[ \left[ -\frac{d}{dt} \left( C \frac{d}{dt} + B \right) + \beta \frac{d}{dt} + A \right] f(t) = \eta f(t) \]
it is easy to check that this equation has the property
\[ \int g(t) \frac{df}{dt} dt = \int (Dg) f dt. \]

These equations have the following properties:

1. they have an infinite number of eigenvalues that accumulate at \( \infty \).

   This means that for an \( \lambda > 0 \), there are only a finite number of \( \mu_n \) satisfying \( \mu_n < \lambda \).

2. the eigenvalues are real and discrete.

3. the eigenvectors are complete and can be chosen to be orthonormal, which means any \( g \) satisfying \( g(t_1) = g(t_2) = 0 \) can be expressed as

\[ g(t) = \sum_{n=1}^{\infty} c_n \phi_n(t) \]
If we choose $\mathbf{\bar{a}}_n \cdot \mathbf{\bar{a}}_n = 1$ and $\mathbf{\bar{a}} \cdot \mathbf{\bar{a}} = 1$ then

$$\sum |c_n|^2 = 1$$

and

$$\int \mathbf{\bar{a}}_n \cdot D \mathbf{\bar{a}} = \sum c_n c_m \eta_m \int \mathbf{\bar{a}}_n \cdot \mathbf{\bar{a}}_m = \sum |c_n|^2 \eta_n$$

since the $|c_n|^2$ are non-negative and not all 0 (otherwise $\sum |c_n|^2 \neq 1$) this will be positive for every $\mathbf{\bar{a}}$ satisfying the boundary conditions if and only if all of the eigenvalues are positive (just like in the finite dimensional case).

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Case of small times - 1 particle 1 dimension for simplicity

$$L = \frac{1}{2} m \dot{x}^2 - V(x)$$

The equations of motion are

$$m \ddot{x} + \frac{dV}{dx}(x) = 0$$
The solution has the form

\[ x(t) = x(0) + v(0) t - \frac{1}{2} m \frac{dv}{dx}(x) t^2 + \ldots \]

For sufficiently small time

\[ x(t) \approx x(0) + v(0) t \]

while this is expressed in terms of initial coordinate and velocity, we can also write it as

\[ x(t) = x(0) + \frac{x(t_f) - x(0)}{t_f - t} t \]

which expressed the solution in terms of the elapsed time and initial and final position.

Clearly these solutions do not depend on the potential, any on the boundary or initial conditions. The approximate Lagrangian is

\[ L = T - V \approx T = \frac{1}{2} m \dot{x}^2 \]

\[ C = m \quad A = B = 0 \]

\[ \frac{d}{dt} (m \frac{d}{dt} sX) = \eta sX \]

diffusion of the eigenvalue equation
\[ \ddot{x} = -\frac{n}{m} \dot{x} \]

The solutions satisfying \( \dot{x}(0) = \dot{x}(t_F) = 0 \) are:

\[ \dot{x} = C \sin \left( \sqrt{\frac{n}{m}} t \right) \quad \sqrt{\frac{n}{m}} t_F = n \pi \]

Let:

\[ \eta \equiv \frac{n^2 \pi^2 m}{t_F} > 0 \]

This shows that times that are short enough where the impact of the potential has not been felt, the action is a local minimum. As \( t_F \) increases, the eigenvalues can move, i.e., \( \eta \rightarrow \eta(t) \). It can happen that one of them becomes 0. Then

\[-\frac{d}{dt} \left( m \frac{d}{dt} \right) \dot{x} + \left( -\frac{d}{dx} \left( \chi_{0}(t) \right) \right) \dot{x} = 0\]

has a solution satisfying the boundary condition. (In general it is a second order equation - it will have solutions, but they may not satisfy the boundary condition.)
The equation

$$\sum_i \left( -\frac{d}{dt} C_{ij}(t) \frac{d}{dt} Q_j + B_{ij} \frac{d}{dt} - \frac{d}{dt} B_{ij}(t) + A_{ij}(t) \right) q_j q_i = 0$$

is called the Jacobi equation — in one variable

$$\left( -\frac{d}{dt} C(t) \frac{d}{dt} + B \frac{d}{dt} - \frac{d}{dt} B + A \right) q = 0$$

Returning to the case $L = \frac{1}{2} m \dot{x}^2 - V(x)$ we know from the principle of

newtonian determinacy that solutions to

$$\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{x}} \right) - \frac{\partial L}{\partial x} = 0$$

are functions of $t$ and the initial time, position and velocity,

$$x(t) = x(t, t_0, x(t_0), v(t_0))$$

$$v = \frac{\partial x}{\partial t}$$
Consider the function

\[ \frac{\partial x}{\partial v_0}(t, t_0, x, v_0) = J(t) \]

(derivative with respect to the initial velocity)

Note

\[ J(t_0) = \frac{\partial x}{\partial v_0}(t, t_0, x_0, v_0) \]

Since all curves with \( x(t_0) \) have the same starting point \( (x=x_0) \) for all initial velocities, this must vanish

\[ J(t_0) = \frac{\partial x}{\partial v_0}(t, t_0, x_0, v_0) \]

\[ \frac{d}{dt} \left( \frac{\partial x}{\partial x} \right) - \frac{\partial x}{\partial x} = 0 \quad \text{(for all \( x_0, v_0 \))} \]

\[ \frac{\partial}{\partial v_0} \left( \frac{d}{dt} \left( \frac{\partial x}{\partial x} \right) - \frac{\partial x}{\partial x} \right) = \]

\[ \frac{d}{dt} \left( \frac{\partial^2 x}{\partial x^2} \frac{2}{\partial x} + \frac{\partial^2 x}{\partial x \partial v} \frac{\partial x}{\partial v} \right) - \frac{d^2 x}{\partial x \partial t} \frac{\partial x}{\partial v} \]

\[ \left( \frac{d}{dt} C \frac{d}{dt} + \frac{d}{dt} B - B \frac{d}{dt} A \right) \frac{\partial x}{\partial v} = 0 \]
This shows that \[ \frac{\partial X(t_1, t_0, x(t_0))}{\partial V(t_0)} = J(t) \] satisfies the Jacobi equation. While it satisfies \( J(t_0) = 0 \), it does not necessarily satisfy \( J(t_1) = 0 \), so it is not a solution of the boundary value problem with eigenvalue 0.

If for some \( t_1 > t_0 \), \( J(t_1) = 0 \), then it is an eigenfunction of the boundary value problem with eigenvalue 0. This means

\[ \frac{\partial X(t_1, t_0, x(t_0))}{\partial V(t_0)} = 0 \]

or

\[ X(t_1, t_0, x(t_0)) \]

is independent of the initial velocity. This means that all solutions of the initial value problem are at the same point at time \( t_1 \), independent of \( V(t_0) \).
Points where the solution of the Jacobi equation vanishes are called conjugate points – these are the points where one of the eigenvalues of the differential operator $D$ goes through 0. If time is continued past that point that eigenvalue could become negative. In that case the solution of Lagrange's equation do not only minimize the action, they only make it stationary.

One of the nice properties of the principle of stationary action (also called Hamilton's principle) is that any continuous transformation that leaves the action invariant results in a conservation quantity. The result is called Noether's theorem.
In general the action depends on a curve $\gamma(t)$ and initial and final times, we write this as

$$A[\gamma, t_i, t_f] = \int_{t_i}^{t_f} L(\gamma(t), \dot{\gamma}(t), t) \, dt$$

corresponds to a continuous transformation that leaves the action unchanged

$$\gamma \rightarrow \gamma', \ t \rightarrow t'$$

$$A[\gamma, t_i, t_f] = A[\gamma', t_i, t_f']$$

we consider transformations that depend on a small parameter $\epsilon$.

$$\frac{d}{d\epsilon} A[\gamma', t_i, t_f'] = 0$$

This means that we only need to look at the first order term in $\epsilon$. We assume

1. $\gamma(t) \rightarrow \gamma'(t') = \gamma(t) + \epsilon \delta \gamma(t)$
2. $t \rightarrow t' = t + \epsilon \delta t(t)$
3. $\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{\gamma}} (\gamma, t) \right) - \frac{\partial L}{\partial \gamma} = 0$
The theorem is only valid for solutions of Lagrange's equations.

First note

\[ \tilde{y}'(t') = \tilde{y}'(t + \epsilon \delta t(t')) = \tilde{y}'(t) + \frac{dy'}{dt}(t) \epsilon \delta t(t) + \frac{1}{2} \frac{d^2y'}{dt^2}(t) \epsilon^2 \delta^2 t(t) + O(\epsilon^3) \]

\[ \tilde{y}^{-1}(t') = \tilde{y}^{-1}(t) + \epsilon \delta \tilde{y}(t) \]

\[ \tilde{y}'(t) + \frac{dy}{dt}(t) \epsilon \delta t(t) + O(\epsilon^2) = \tilde{y}(t) + \epsilon \delta \tilde{y}(t) \]

\[ \tilde{y}'(t) = \tilde{y}(t) + \epsilon \delta \tilde{y}(t) - \frac{d^2y}{dt^2}(t) \epsilon \delta t(t) + O(\epsilon^3) \]

\[ \frac{dy}{dt}(t) \epsilon \delta t(t) = \]

\[ \frac{dy}{dt}(t) \epsilon \delta t(t) + \epsilon \frac{d\delta t}{dt} \epsilon \delta t(t) - O(\epsilon^3) \]

This gives:

\[ \dot{y}'(t) = \dot{y}(t) + \epsilon \delta \tilde{y}(t) - \epsilon \delta t(t) \frac{d\delta y}{dt}(t) + O(\epsilon^3) \]

\[ t - t' = t + \epsilon \delta t(t) \]

For \( A[y't', t', t'] \) is independent of \( \epsilon = 0 \)

\[ \frac{d}{d\epsilon} A[y't', t', t'] = 0 \]

\( \epsilon = 0 \)
\[
\frac{d}{de} \int_{t_1 + \varepsilon \mathcal{S}(t_1)}^{t_2 + \varepsilon \mathcal{S}(t_2)} L (y'(t), \dot{y}'(t), t) \, dt = \frac{1}{\varepsilon}.
\]

\[
\frac{d}{de} \left[ \int_{t_1 + \varepsilon \mathcal{S}(t_1)}^{t_2} L (y'(t), \dot{y}'(t), t) \, dt + \int_{t_1}^{t_2} L (y'(t), \dot{y}'(t), t) \, dt \right]
\]

The first term has the form
\[
\int_{t_1 + \varepsilon \mathcal{S}(t_1)}^{t_1} \left[ L (y'(t), \dot{y}'(t), t) + L'(t) t + \frac{1}{2} L''(t) t^2 + \cdots \right]
\]

\[- L (y'(t), \dot{y}'(t), t_1) \in \varepsilon \mathcal{S}(t_1) + \frac{1}{2} L'(t_1) \varepsilon^2 \mathcal{S}^2(t_1) + O(\varepsilon^3)
\]

\[- \left[ L (y'(t), \dot{y}'(t), t_1) + O(\varepsilon) \right] \in \varepsilon \mathcal{S}(t_1)
\]

We can ignore these terms when we do \( \frac{d}{de} L \) as \( \varepsilon \to 0 \).

\[
= -L (y(t), \dot{y}(t), t_1) \varepsilon \mathcal{S}(t_1), \varepsilon + O(\varepsilon^2)
\]
Similarly
\[ \int_{t_2}^{t_3} L (\dot{\gamma}'(t) \dot{\gamma}'(t)) dt = \]
\[ \in L(\gamma(t_1), \gamma(t_1), t_1) \Delta t(t_1) + O(\varepsilon^2) \]

These two terms can be combined
\[ \in \left( L(\gamma(t_1), \gamma(t_1), t_1) \Delta t(t_1) - L(\gamma(t_1), \gamma(t_1), t_1) \Delta t(t_1) \right) + O(\varepsilon^2) \]

\[ \in \int_{t_1}^{t_2} \frac{d}{dt} \left( L(\gamma(t), \gamma(t), t) \Delta t(t) \right) dt + O(\varepsilon^1) \]

The middle term is
\[ \int_{t_1}^{t_2} L (\dot{\gamma}'(t), \dot{\gamma}'(t)) dt = \]
\[ \int_{t_1}^{t_2} \left[ L(\dot{\gamma}(t), \dot{\gamma}(t)) + \sum \frac{\partial L}{\partial \dot{\gamma}_i} \Delta \gamma_i(t) + \sum \frac{\partial L}{\partial \dot{\gamma}_i} \frac{d}{dt} \Delta \gamma_i(t) \right] dt + O(\varepsilon^1) \]

where \[ \Delta \gamma_i(t) = \delta \gamma_i(t) - \Delta t(t) \frac{d \gamma_i(t)}{dt} \]

Next we use the Lagrange's equation
\[ \sum \frac{\partial L}{\partial \gamma_i} \Delta \gamma_i(t) = \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{\gamma}_i} \right) \Delta \dot{\gamma}_i(t) \]

Then the above becomes
\[ \int_{t_1}^{t_2} L(\dot{\gamma}(t), \dot{\gamma}(t)) dt + \in \int_{t_1}^{t_2} \frac{d}{dt} \left( \sum \frac{\partial L}{\partial \dot{\gamma}_i} \Delta \gamma_i(t) \right) dt + O(\varepsilon^2) \]
Differentiating the entire expression with respect to \( \xi \), setting \( \xi \) to 0 gives

\[
0 = \int_{t_i}^{t_f} \frac{d}{dt} \left( L(\ddot{\gamma}(t)) \dot{\gamma}(t) + \gamma_i \left( \frac{\partial L}{\partial \dot{\gamma}} \right)(\gamma_i(t)) - \dot{\gamma}(t) \ddot{\gamma}(t) \right)
\]

\[
0 = \int_{t_i}^{t_f} \frac{d}{dt} \delta(t) \ dt = \delta(t_f) - \delta(t_i)
\]

which means that the integrand is independent of time (omitted)

\[
(L(\ddot{\gamma}(t)) - \gamma_i \left( \frac{\partial L}{\partial \dot{\gamma}} \right)(\gamma_i(t)) + \gamma_i \left( \frac{\partial L}{\partial \dot{\gamma}} \right)(\gamma_i(t)) = \text{constant}
\]

Noether's theorem implies that the above quantity is conserved when \( \gamma(t) \rightarrow \gamma(t') = \gamma(t) + \epsilon \delta(t) \), \( t \rightarrow t' = t + \delta t \) leave the action unchanged.

* It is always important to check this.
example assume that \( L \) has no explicit time dependence

\[
t \to t' = t + \epsilon
\]
\[
\gamma(t) \to \gamma(t') = \gamma(t) + \epsilon \quad \gamma'(t) = \gamma'(t-\epsilon)
\]
\[
\int_{t, t+\epsilon}^{t+\epsilon} L \left( \gamma(t-\epsilon), \dot{\gamma}(t-\epsilon) \right) dt
\]

Let \( t'' = t - \epsilon \)

the action

\[
L = \sum_i \frac{\dot{\gamma}_i^2}{2} = \text{conserved}
\]

If

\[
L = \sum \frac{1}{2} m_i \dot{r}_i^2 - V(r)
\]

\[
\frac{\partial L}{\partial \dot{r}_i} = m_i \dot{r}_i
\]

\[
L - m_i \ddot{r}_i \dot{r}_i = T - V - 2T = -(T + V)
\]

so in this case the conserved quantity is \((T + V)\) the total energy.

It is easy to see if \( V \) depended on time we would not have \( A = A' \)