Lecture 23

Last time

\[ \vec{r}_{\text{i}} = \vec{b}(t) + Q^T(t) \vec{r}_{\text{bi}} \]

\[ Q^T = R_z(\psi) R_y(\theta) R_z(\phi) \]

polar angles of \( \hat{z}_i \) in body fixed coordinate system \((\theta, \phi)\)

\[ \vec{b}(t) = \text{origin of body fixed coordinate system in inertial CS} \]

\[ Q^T \hat{e}_b = \hat{e}_i \]

We also have

\[ \mathbf{T} = \frac{1}{2} \sum m_i \vec{r}_{\text{i}} \cdot \dot{\vec{r}}_{\text{i}} = \]

\[ \frac{1}{2} m \dot{\vec{b}} \cdot \dot{\vec{b}} + \]

\[ \dot{\vec{b}} \cdot (\sum m_i Q^T \vec{r}_{\text{bi}}) + \]

\[ \frac{1}{2} \sum m_i \left( \overline{\Omega} \times Q^T \vec{r}_{\text{bi}} \right) \cdot \left( \overline{\Omega} \times Q^T \vec{r}_{\text{bi}} \right) \]

Middle term vanishes if

a) \( \dot{\vec{b}} = \vec{0} \)

b) \( \sum m_i \vec{r}_{\text{bi}} = 0 \) \( \Rightarrow \) \( \vec{b} = \text{center of mass} \)

Note:

\[ \dot{Q}^T = \overline{\Omega} \cdot \overline{e} \]

\[ \overline{e} = \left( \begin{array}{c} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{array} \right), \overline{\Omega} = \left( \begin{array}{cccc} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{array} \right) \]
Inertia tensor (Body fixed)

\[ I_{ij}^B = \sum m_i \left( \vec{r}_{nB} \times \vec{r}_{nB} \right) \left( \vec{r}_{nB} \times \vec{r}_{nB} \right)^T \]

With this choice

\[ \frac{1}{2} \sum m_i \left( \vec{\Omega} \times \vec{r}_{nB} \right) \cdot \left( \vec{\Omega} \times \vec{r}_{nB} \right)^T = \]

\[ \frac{1}{2} \sum (\vec{\Omega} \times \vec{r}_{nB}) (\vec{\Omega} \times \vec{r}_{nB})^T I_{ij}^B = \]

\[ \frac{1}{2} \sum \vec{\omega} \times \vec{\omega} (\Omega_{ik}^T \Omega_{jk}^T I_{ij}^{kp}) \]

\[ I_{ij}^{kp} \]

The advantage of using \( I_{ij}^{kp} \) over \( I_{ij}^k \) is that the body fixed inertia tensor is time independent.

Defining \( \vec{\Omega} \) by \( \vec{\Omega} = \vec{\Omega} \cdot \vec{I} \) we found

\[ \vec{\Omega} = -\vec{\Omega} \]

Last time we proved this using

\[ \vec{\Omega} \cdot \vec{I} = (\Omega \cdot \vec{I}) \]

This gives

\[ \frac{1}{2} \sum \vec{\omega} \times \vec{\omega} I_{ij}^B \vec{\omega} \cdot \vec{\omega} \]
Check

\[
R_z(\phi) \cdot L = R_z(\phi)' = \\
\begin{pmatrix}
\cos \phi & -\sin \phi & 0 \\
\sin \phi & \cos \phi & 0 \\
0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & -1 \\
0 & 1 & 0
\end{pmatrix}
\begin{pmatrix}
\cos \phi & \sin \phi & 0 \\
-\sin \phi & \cos \phi & 0 \\
0 & 0 & 1
\end{pmatrix} = \\
= \cos \phi L_x + \sin \phi L_y
\]

\[
O^T = L_x L_y = \\
\begin{pmatrix}
\cos \phi L_x + \sin \phi L_y \\
\cos \phi L_y - \sin \phi L_y \\
L_z
\end{pmatrix}
\]

Comparing we see the claimed transformation property

Note

\[
O^T = R_z(\phi) R_y(\theta) R_z(\psi) \\
O = R_z(-\phi) R_y(-\theta) R_z(-\psi) = \\
\begin{pmatrix}
\cos \phi & \sin \phi & 0 \\
-\sin \phi & \cos \phi & 0 \\
0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
\cos \theta & 0 & -\sin \theta \\
0 & \cos \theta & 0 \\
\sin \theta & 0 & \cos \theta
\end{pmatrix} = \\
\begin{pmatrix}
\cos \phi \cos \theta & \sin \phi \cos \theta & -\sin \theta \\
-\sin \phi \cos \theta & \cos \phi \cos \theta & 0 \\
\sin \phi \sin \theta & -\cos \phi \sin \theta & \cos \theta
\end{pmatrix}
\]

To compute \( \bar{\Sigma}' \)

\[
O^T \bar{O}^T = R_z(-\phi) R_y(-\theta) R_z(-\psi) R_z(\phi) R_y(\theta) R_z(\psi) + \\
R_z(-\phi) R_y(-\theta) R_z(-\psi) R_z(\phi) R_y(\theta) R_z(\psi) + \\
R_z(-\phi) R_y(-\theta) R_z(-\psi) R_z(\phi) R_y(\theta) R_z(\psi)
\]
Recall \( R_z(-\Theta) R_z(\Theta) = \)
\[
\begin{pmatrix}
-\sin \Phi & \cos \Phi & 0 \\
-\cos \Phi & -\sin \Phi & 0 \\
0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
\cos \Phi & -\sin \Phi & 0 \\
\sin \Phi & \cos \Phi & 0 \\
0 & 0 & 1
\end{pmatrix} = 
\begin{pmatrix}
0 & 0 & 0 \\
-1 & 0 & 0 \\
0 & 0 & 1
\end{pmatrix}
\]
\( \dot{\phi} = -L_z \dot{\phi} \)

Similarly
\( R_y(-\Theta) R_y(\Theta) = -L_y \dot{\Theta} \)

Using these in the expression on the previous page
\[
\dot{\phi} \Omega = -L_z \dot{\phi} + R_z(-\Theta)(-L_y \dot{\Theta}) R_z(\Theta) + R_z(-\Theta) R_y(-\Theta) (-L_z \dot{\phi}) (R_y(\Theta) R_z(\Theta))
\]

Computation
\[
- R_z(-\Theta) L_y R_z(\Theta) = \]
\[
- \begin{pmatrix}
\cos \Phi & \sin \Phi & 0 \\
-\sin \Phi & \cos \Phi & 0 \\
0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
0 & 0 & 1 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
\cos \Phi & -\sin \Phi & 0 \\
\sin \Phi & \cos \Phi & 0 \\
0 & 0 & 1
\end{pmatrix} = 
\begin{pmatrix}
0 & 0 & 0 \\
-\cos \Phi & -\sin \Phi & 0 \\
\sin \Phi & \cos \Phi & 0
\end{pmatrix}
\]

\(- \cos \Phi L_y + \sin \Phi L_y \)
\[- R_z (-\theta) R_y (-\phi) L_z \dot{\psi} R_y (\alpha) R_z (\theta) \]

\[
(R_y (\theta) R_z (\phi)) = \begin{pmatrix}
    \cos \phi & -\cos \theta \sin \phi & \sin \phi \\
    \sin \phi & \cos \theta \cos \phi & -\sin \theta \\
    -\sin \theta & \cos \theta \sin \phi & \cos \phi
\end{pmatrix}
\]

\[
\begin{pmatrix}
    \cos \phi & -\cos \theta \sin \phi & \sin \phi \\
    \sin \phi & \cos \theta \cos \phi & -\sin \theta \\
    -\sin \theta & \cos \theta \sin \phi & \cos \phi
\end{pmatrix}
\]

about becomes

\[
- \begin{pmatrix}
    \cos \phi & \sin \phi & -\sin \theta \\
    -\sin \phi & \cos \phi & \cos \theta \\
    \sin \theta & -\cos \theta & \cos \phi
\end{pmatrix}
\begin{pmatrix}
    0 & -1 & 0 \\
    1 & 0 & 0 \\
    0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
    \cos \phi & -\cos \theta \sin \phi & \sin \phi \\
    \sin \phi & \cos \theta \cos \phi & -\sin \theta \\
    -\sin \theta & \cos \theta \sin \phi & \cos \phi
\end{pmatrix}
\]

\[
- \begin{pmatrix}
    \cos \phi & \sin \phi & -\sin \theta \\
    -\sin \phi & \cos \phi & \cos \theta \\
    \sin \theta & -\cos \theta & \cos \phi
\end{pmatrix}
\begin{pmatrix}
    -\sin \phi & -\cos \theta & 0 \\
    \cos \phi & \cos \theta & 0 \\
    0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
    \cos \phi & -\cos \theta \sin \phi & \sin \phi \\
    \sin \phi & \cos \theta \cos \phi & -\sin \theta \\
    -\sin \theta & \cos \theta \sin \phi & \cos \phi
\end{pmatrix}
\]

\[
- \begin{pmatrix}
    0 & -\cos \theta & \sin \phi \\
    \cos \theta & 0 & \sin \phi \\
    -\sin \phi & -\cos \phi & 0
\end{pmatrix}
= - \Theta L_z - \sin \phi L_y + \cos \phi L_x
\]

putting everything together

\[
\Omega = L_x (-\sin \phi \dot{\phi} + \sin \theta \cos \phi \dot{\theta}) + L_y (-\cos \phi \dot{\theta} - \sin \theta \sin \phi \dot{\phi}) + L_z (-\dot{\phi} - \cos \theta \dot{\phi})
\]
we can read off

\[ \bar{\Omega}' = ( -\sin \phi \dot{\theta} + \sin \theta \cos \phi \dot{\psi} ) \\
- \cos \phi \dot{\theta} - \sin \theta \sin \phi \dot{\psi} \]
- \dot{\theta} - \cos \theta \dot{\psi} \)

what we need is

\[ \bar{\Omega}' = 0 \bar{\Omega} = \\
- ( \sin \phi \dot{\theta} - \sin \theta \cos \phi \dot{\psi} ) \\
\cos \phi \dot{\theta} + \sin \theta \sin \phi \dot{\psi} \\
\dot{\theta} + \cos \theta \dot{\psi} \)

Note in order to calculate \[ \bar{\Omega} = \bar{\Omega}^T \]

\( \theta \) change \( \psi \rightarrow -\phi \) \( \phi \rightarrow -\psi \) \( \theta \rightarrow -\theta \)

in the expression in \( \bar{\Omega}' \)

\[ \bar{\Omega} = ( \sin \phi \dot{\theta} - \sin \theta \cos \phi \dot{\psi} \\
- \cos \phi \dot{\theta} - \sin \theta \sin \phi \dot{\psi} \\
- \dot{\theta} - \cos \theta \dot{\psi} \) \)
\[ T = \frac{1}{2} \sum_{i=1}^{3} \Omega_i^0 I_{ii}^0 \Omega_i^0 = \frac{1}{2} \sum (\Omega_i^0 \Omega_i^0) \]

where
\[
- \vec{\Omega}' = \begin{pmatrix}
\sin \phi \dot{\phi} - \sin \theta \cos \phi \dot{\psi} \\
\cos \phi \dot{\phi} + \sin \theta \sin \phi \dot{\psi} \\
\dot{\theta} + \cos \theta \dot{\psi}
\end{pmatrix}
\]

where
\[
Q^T: \hat{e}_g \rightarrow \hat{e}_3 = R_2(\phi) R_1(\theta) R_2(\psi)
\]

Note that \( T_{ii} \) is a scalar quantity - it does not matter how it is computed; \( \psi \phi \theta \) mean the same thing.

Note that \( I_{ii}^B \) is a real symmetric matrix. While the matrix depends on the choice of origin of the coordinate system, in any origin it is real and symmetric and we are free to choose the body-fixed \( x' y' z' \) axes.

We can always choose a set of coordinates that make \( I_{ii}^B \) diagonal:
\[
I_{ii}^B = \lambda_i \delta_{ii}^B
\]

then \( O_{ii} = \delta_{ii}^B \) and
\[
O^T I O = \Lambda = \begin{pmatrix}
\lambda_1 & 0 & 0 \\
0 & \lambda_2 & 0 \\
0 & 0 & \lambda_3
\end{pmatrix}
\]
\( \mathbf{r} \) define a rotation that make the inertia tensor diagonal

\[ \mathbf{I}_1 \cdot \mathbf{I}_2 = 0 \]

The \( \mathbf{I}_i \) define the principal axes of inertia; the \( \lambda_i \) are called the principal moments of inertia.

We define

\[ I_1 = \lambda_1, I_2 = \lambda_2, I_3 = \lambda_3 \]

In this basis

\[ T = \frac{1}{2} \left( I_1 \mathbf{\Omega}_1^2 + I_2 \mathbf{\Omega}_2^2 + I_3 \mathbf{\Omega}_3^2 \right) \]

This can be used to formulate Lagrange's equations.

Example - symmetric top with one point fixed.

\[ r_1 - r_3 = \]

\[ R_z (-\theta) \cdot R_y (\theta) \cdot R_z (-\theta) \]
To interpret this in terms of polar angles in the inertial coordinate system

Let
- \psi -> \phi'
- \theta -> \theta'
- \phi -> \phi'

\[ T = \frac{1}{2} I_1 \left(-\sin \theta \dot{\phi} + \sin \psi \cos \theta \dot{\phi}'\right)^2 \]
\[ I_2 \left(\cos \psi \dot{\phi}' + \sin \theta \sin \psi \dot{\phi}\right)^2 \]
\[ I_3 \left(\dot{\theta}' + \cos \psi \dot{\phi}'\right)^2 \]

for a symmetric top \( I_1 = I_2 \)

\[ T = \frac{1}{2} I_1 \left(\dot{\phi}^2 + \sin^2 \theta \dot{\phi}'^2\right) + \frac{1}{2} I_3 \left(\dot{\theta}' + \cos \psi \dot{\phi}'\right)^2 \]

\[ V = m q \dot{\theta} = m q \dot{\theta}_m \cos \theta \]

\[ L = \frac{1}{2} I_1 \left(\dot{\phi}^2 + \sin^2 \theta \dot{\phi}'^2\right) + \frac{1}{2} I_3 \left(\dot{\theta}' + \cos \psi \dot{\phi}'\right)^2 - m q \dot{\theta}_m \cos \theta \]

where \( \dot{\theta}_m \) is the displacement of the center of mass from the fixed point.

Now we can write down Lagrange's equations.
0 \quad \frac{d}{dt} \left( \frac{2L}{2\dot{\theta}} \right) = 0 \quad \frac{d}{dt} \left( \frac{2L}{2\dot{\phi}} \right) = 0 \quad \frac{d}{dt} \left( \frac{2L}{2\dot{\psi}} \right) = \frac{2L}{2\dot{\phi}}

\frac{\partial L}{\partial \dot{\theta}} = I_1 \sin^2 \theta \dot{\theta} + I_3 (\dot{\psi} + \cos \theta \dot{\phi}) \cos \theta = \text{const} = c_1,

\frac{\partial L}{\partial \dot{\phi}} = I_3 (\dot{\psi} + \cos \theta \dot{\phi}) = c_2

Using the second equation in the first,

\[ I_1 \sin \theta \dot{\phi} + c_2 \cos \theta = c_1 \]

\[ I_3 (\dot{\psi} + \cos \theta \dot{\phi}) = c_2 \]

The \( \theta \) equation is

\[ I_1 \ddot{\theta} = I_1 \sin \theta \cos \theta \dot{\phi}^2 + I_3 (\dot{\psi} + \cos \theta \dot{\phi}) (-\sin \theta \dot{\phi}) + m q R \sin \theta \]

we don't need the \( \psi \) equation because we know energy is conserved.

\[ \frac{1}{2} I_1 (\dot{\theta}^2 + \sin^2 \theta \dot{\phi}^2) + \frac{1}{2} I_3 (\dot{\theta}^2 + \cos^2 \theta \dot{\phi}^2) + m q R \cos \theta = c_3 \]

This gives an algebraic set of equations in \( \phi, \dot{\phi} \),

\[ \dot{\phi} = \frac{c_1 - c_2 \cos \theta}{I_1 \sin^2 \theta} \]

\[ \dot{\psi} = \frac{c_2^2}{I_3} - \cos \theta \left( \frac{c_1 - c_2 \cos \theta}{I_1 \sin^2 \theta} \right) \]

\[ \dot{\theta}^2 = \frac{2}{I_1} \left( c_3 - \frac{I_1}{2} \sin^2 \theta \dot{\phi}^2 - \frac{1}{2} \frac{c_2^2}{I_3} - m q R \cos \theta \right) \]
This results in first order differential equation in θ.

change variables: \( u = \cos \theta \) \( \dot{u} = -\sqrt{1-u^2} \)

\( u^2 = (1-u^2) \dot{u}^2 \)

\[
\begin{align*}
\dot{u}^2 &= (1-u^2) \left( \frac{2c_3}{I_1} - (1-u^2) \left( \frac{C_1 - C_3 u}{I_1^2 (1-u^2)} \right) - \frac{C_2}{I_3} + \frac{2m g R}{I_1} u \right) \\
&= (1-u^2) \left( \frac{2c_3 - C_2}{I_1} \right) - \left( \frac{C_1 - C_3 u}{I_1^2} \right)^2 - \frac{2m g R}{I_1} u (1-u^2)
\end{align*}
\]

The quantity on the right is a cubic polynomial

\[
\dot{u}^2 = (1-u^2) (\alpha - \beta u) - (\gamma - \delta u)^2
\]

\[
\begin{align*}
\alpha &= \frac{2c_3}{I_1} - \frac{C_2}{I_3} \\
\beta &= \frac{2m g R}{I_1} \\
\gamma &= \frac{C_1}{I_1^2} \\
\delta &= \frac{C_3}{I_1^2}
\end{align*}
\]

\[
\dot{u} = \frac{\gamma - \delta u}{1-u^2} \quad \psi = \frac{c_3}{I_3} - u \frac{r - s u}{1-u^2}
\]

This is a cubic polynomial so the right side eventually changes sign however the value of \( u \) is limited to \(-1 \leq u \leq 1\).

\( r > s \), sign of \( \dot{u} \) is fixed
\( s > r \), \( \psi \) can change sign

The \( u^2 \) of this polynomial is when \( \dot{u} \) changes sign
when $u = \pm 1$, $u^2 < 1$ must have 1 unphysical root

\[ u \]

turning points

\[ u^+ \]

\[ u^- \]

\[ \phi \text{ may } \alpha \text{ may not change sign depending on } \delta, \phi \geq \delta \text{ no sign change } \]

\[ \phi \text{ changes sign} \]

Euler angles are not the only way to parameterize rotations

Cayley–Klein parameter

Define Pauli matrices

\[ \vec{\sigma} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \]

basis for traceless Hermitian 2x2 matrices
Properties:

1. \( \sigma_i \sigma_j = S_{ij} I + i \sum_k \epsilon_{ijk} \sigma_k \)

2. \( \sigma_i = \sigma_i^+ \)

3. \( \text{Tr} (\sigma_i \sigma_j) = 2 S_{ij} \)

Note:

\[
\begin{align*}
X &= \bar{X} \cdot \bar{X} = \left( \begin{array}{cc} \bar{z} & x - iy \\ x + iy & -\bar{z} \end{array} \right) \\
\det X &= -\|X^2\| = -(x^2 + y^2 + \bar{z}^2) \\
\bar{X} &= \frac{1}{2} \text{Tr} (\sigma_i \bar{X})
\end{align*}
\]

Consider linear transformations that preserve \( \det X \), \( \text{Tr} X \)

\[
X \rightarrow X' \equiv \\
\det (\lambda I - X) = \lambda^2 - \lambda \text{Tr} X + \det X = 0
\]

\[
\lambda = \frac{1}{2} \left( \text{Tr} X \pm \sqrt{[\text{Tr} X]^2 - 4 \det X} \right) = \pm \sqrt{\det X}
\]

The transformation must preserve eigenvalues.

This means

\[
X' = A X A^{-1} = (A^{-1})^T X A^T \quad (\text{could have } X = -X \text{ because } \text{Tr} X = 0)
\]

\[
A^T AX = X A^T A \quad \therefore \\
\left[ A^T A, X \right] = 0
\]
since $\tilde{\mathcal{G}}$ is a basis for any matrix $(+I)$

$$(A^+A) = (\cos\theta I) =$$

divide by $\sqrt{c}$ $u = \frac{1}{\sqrt{c}} A$ $u^+ = \frac{1}{\sqrt{c}} A^+$ $uu^+ = 1$

$x' = \pm u x u^+$

- corresponds to a space reflection
- eliminate space reflection

$$\tilde{x'} = u x u^+$$

corresponds to a rotation

$$\tilde{x'} \tilde{c} = u \tilde{x'} \tilde{c} u^+ = \tilde{x'} \tilde{x} \tilde{u} \tilde{u}^+ = (R \tilde{x}) \cdot \tilde{c}$$

$$R \tilde{x} = \frac{1}{2} \text{Tr} (R \tilde{x} \cdot \tilde{c}) = \frac{1}{2} \text{Tr} (\tilde{c}^I u \tilde{c}^I u^+ x')$$

$$R_{ij} = \frac{1}{2} \text{Tr} (c_i u c_j u^+)$$

we can choose $\det u = 1$ because the phases cancel.