Hamiltonian Mechanics

1. generalized momentum

\[ p_i = \frac{\partial L}{\partial \dot{q}_i} \]

for

\[ L = \sum \frac{1}{2} m_i \dot{q}_i^2 - V \]

\[ \dot{p}_i = \frac{\partial L}{\partial \dot{q}_i} = m_i \ddot{q}_i \]

which is the linear momentum. This explains why \( \frac{\partial L}{\partial \dot{q}_i} \) is called the generalized momentum.

2. a function of several variables is convex if

\[ \frac{\partial^2 f}{\partial x_i \partial x_j} > 0 \]

meaning that the matrix has positive eigenvalues.

3. If \( L(q, \dot{q}) \) is a convex function of \( \dot{q} \), then

\[ \frac{\partial^2 L}{\partial q_i \partial \dot{q}_i} - \frac{\partial p_i}{\partial q_i} > 0 \]
If \( L \) is convex then

\[
\bar{p}_i = \frac{\partial L}{\partial \bar{q}_i} (\bar{q})
\]

has a unique solution for \( \bar{q} (\bar{p}, q) \).

**Proof:** Assume that there are two \( \bar{q}_1, \bar{q}_2 \) satisfying

\[
\bar{\bar{p}}_i = \frac{\partial L}{\partial \bar{q}_i} (\bar{q}_1) = \frac{\partial L}{\partial \bar{q}_i} (\bar{q}_2)
\]

Assume this is true, let

\[
\bar{y}(\lambda) = \lambda \bar{q}_2 + (1-\lambda) \bar{q}_1,
\]

\[
\bar{y}(0) = \bar{q}_1, \quad \bar{y}(1) = \bar{q}_2.
\]

Consider

\[
\sum (\bar{q}_2 - \bar{q}_1)^i \int \frac{\partial L}{\partial \bar{q}_i} (\bar{y}, q) \frac{d\bar{x}_j}{d\lambda} d\lambda
\]

\[
\sum (\bar{q}_2 - \bar{q}_1) \int \frac{\partial L}{\partial \bar{q}_i} (\bar{y}, q) (\bar{q}_2 - \bar{q}_1) d\lambda
\]

If \( \frac{\partial L}{\partial \bar{q}_i} (\bar{y}, q) \) is convex then this is strictly positive. On the other hand if we do the integral

\[
\sum (\bar{q}_2 - \bar{q}_1) ^i \left( \frac{\partial L}{\partial \bar{q}_i} (\bar{q}_2) - \frac{\partial L}{\partial \bar{q}_i} (\bar{q}_1) \right)
\]

\[
\sum (\bar{q}_2 - \bar{q}_1) (\bar{p} - \bar{p}) = 0
\]

which is a contradiction.
while this means that we can in principle we can always \( \varphi \) solve \( \varphi \) for \( \varphi_i(p) \)

geometry

\[
\begin{align*}
p'q - L &= \dot{p} = \frac{\partial L}{\partial \dot{q}} \\
\dot{q} &= \text{maximum distance} \dot{q}
\end{align*}
\]

\[
L = \frac{1}{2} \sum m_i \dot{r}_i \cdot \dot{r}_i - V
\]

\[
\frac{\partial^2 L}{\partial \dot{q} \partial \dot{q}} = \delta_{ij} m_i > 0
\]

This happens when the kinetic energy is positive

\[
H = \sum p_i \dot{q}_i - L(\dot{q}, \ddot{q}, t)
\]

is called the Legendre transformation

Note - treating \( p_i, q_i, \dot{q}_i, t \) as independent variables

\[
dH = \sum p_i dq_i + \sum q_i dp_i - \sum \frac{\partial L}{\partial q_i} dq_i - \sum \frac{\partial L}{\partial p_i} dp_i - \frac{\partial L}{\partial t}
\]

These cancel

which means that \( H = H(p, \dot{q}, t) \)

\[
= \sum \frac{\partial H}{\partial q_i} dq_i + \sum \frac{\partial H}{\partial p_i} dp_i + \frac{\partial H}{\partial t} dt
\]
equating coefficients gives

\[
\frac{\partial H}{\partial \dot{q}_i} = -\frac{\partial L}{\partial q_i}, \quad \frac{\partial H}{\partial p_i} = \dot{q}_i, \quad \frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t}
\]

if we assume Lagrange's equation

\[
\frac{\partial L}{\partial q_i} = \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_i} \right) = p_i
\]

This gives

\[
\dot{q}_i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial q_i}
\]

These equations are called Hamilton's equations. If \( \dot{q}_i, \dot{p}_i \) satisfy Lagrange's equations, then \( p(q_i, t), \dot{q}_i \) satisfy Hamilton's equations.

Note

\[
\frac{\partial H}{\partial p_i} = \dot{q}_i, \quad \frac{\partial^2 H}{\partial p_i \partial q_j} = \frac{\partial^2 H}{\partial q_j \partial p_i} = \left( \frac{\partial p_i}{\partial q_j} \right)^T > 0
\]

If all of the eigenvalues of \( (\frac{\partial p_i}{\partial q_j}) \) - \( \lambda \) is the eigenvalues of the inverse one \( \frac{1}{\lambda} > 0 \), so if \( L \) is a convex function of \( \dot{q}_i \), \( H \) is a convex function of \( \dot{p}_i \). Consider the Legendre transformation of \( H \).
\[ L' = \sum p_i \dot{q}_i - H(p_i, q_i) \]
\[ \delta L' = \sum p_i \delta \dot{q}_i + \sum \delta q_i \delta p_i - \frac{\partial H}{\partial \dot{q}_i} \delta p_i - \frac{\partial H}{\partial q_i} \delta q_i - \frac{\partial H}{\partial t} \delta t \]

No cancel

This means \( L' \) is a function of \( \dot{q}_i, \ddot{q}_i \)
\[ \delta L' = \sum \frac{\partial L'}{\partial \dot{q}_i} \delta \dot{q}_i + \sum \frac{\partial L'}{\partial q_i} \delta q_i + \frac{\partial L'}{\partial t} \delta t \]

Equating coefficients of independent variables
\[ p_i = \frac{\partial L'}{\partial \dot{q}_i} = -\frac{\partial H}{\partial q_i} = \frac{\partial L}{\partial \dot{q}_i} = \frac{\partial L'}{\partial \dot{q}_i} \]

Assuming \( \dot{q} \ddot{q} \) are solutions of Hamilton's equations
\[ \dot{p}_i = \frac{d}{dt} \left( \frac{\partial L'}{\partial \ddot{q}_i} \right) = -\frac{\partial H}{\partial q_i} = \frac{\partial L'}{\partial q_i} \]

Which means
\[ \frac{d}{dt} \left( \frac{\partial L'}{\partial \dot{q}_i} \right) - \frac{\partial L'}{\partial q_i} = 0 \]

Which shows that \( \dot{q}(\ddot{q}) \ddot{q} \) are solutions of Lagrange's equations.
This means that whenever the Lagrangian is a convex function of the generalized velocities Lagrange equations and Hamiltonian equations are equivalent.

Convexity means that the kinetic energy is an increasing function of the generalized velocities.

To translate from $\dot{q} \to p$ it is necessary to solve $\dot{p}_i = \frac{\partial L}{\partial \dot{q}_i}$ to get $\dot{q}(\dot{q},\ddot{q})$, while this solution must exist and be unique - it may not be easy to find.

Application - simple pendulum

![Diagram of a simple pendulum with a length L and angle $\theta$]

generalized coordinate $= \theta$

$L = \frac{1}{2} mL^2 \dot{\theta}^2 + mgL \cos \theta$

generalized momentum

$\frac{\partial L}{\partial \dot{\theta}} = p_\theta = mL^2 \dot{\theta}$
calculate $\dot{\omega} (p, \theta)$

$$\dot{\omega} = \frac{p_0}{mL^2}$$

check convexity $\frac{\partial^2 L}{\partial \dot{\theta}^2} = mL^2 > 0$

construct $H$

$$H = p \dot{q} - \frac{1}{2} mL^2 \dot{\theta}^2 - mgL \cos \theta$$

substitute $\dot{q} = \dot{q} (p, \theta)$

$$= p_0 \left( \frac{p_0}{mL^2} \right) - \frac{1}{2} mL^2 \left( \frac{p_0}{mL^2} \right)^2 - mgL \cos \theta$$

$$+ \frac{1}{2} \frac{p_0^2}{mL^2} - mgL \cos \theta$$

Hamilton equation,

$$\dot{\theta} = \frac{\partial H}{\partial p_0} = \frac{p_0}{mL^2}$$

$$p_0 = -\frac{\partial H}{\partial \theta} = -mgL \sin \theta$$

we can replace this pair of equations by a single second order equation

$$\ddot{\theta} = \frac{p_0}{mL^2} = -\frac{mgL \sin \theta}{mL^2} = -\frac{g}{L} \sin \theta$$

which is exactly the same equation resulting from Lagrange's equation or Newton's second law
Hamilton's equations can also be derived directly using Hamilton's principle:

\[ A = \int_{t_1}^{t_2} \left( \sum p_i \dot{q}_i - H(p_i, q_i) \right) \, dt \]

In this case, \( \bar{p}, \bar{q} \) are treated as independent variables:

\[ \bar{p} = p + \lambda \bar{p} \quad \bar{q} = q + \lambda \bar{q} \]

\[ SA = 0 = \frac{d}{dt} A(\bar{q}, \bar{p} + \lambda \bar{p}, p + \lambda \bar{p}) = \]

\[ = \int_{t_1}^{t_2} \left( \sum \left( \frac{\partial \mathcal{L}}{\partial \dot{q}_i} \dot{q}_i + \frac{\partial \mathcal{L}}{\partial p_i} \dot{p}_i - \frac{2\mathcal{H}}{\dot{q}_i} \partial_{\dot{q}_i} \mathcal{L} \right) + \bar{p}_i \dot{q}_i \right) \, dt \]

For the boundary term to vanish we demand \( \dot{q}_i(t_2) = \dot{q}_i(t_1) \), but \( \dot{p}_i \) is unconstrained. For this to vanish for all \( \dot{q}_i, \dot{p}_i \) satisfying the boundary conditions we must have:

\[ \dot{p}_i = -\frac{\partial \mathcal{H}}{\partial \dot{q}_i} \quad \dot{q}_i = \frac{\partial \mathcal{H}}{\partial \dot{p}_i} \]

which are Hamilton's equations.
Assume that \( \bar{q}(t) \), \( \bar{p}(t) \) are solutions of Hamilton's equations.

Let \( F(\bar{q}, \bar{p}, t) \) be a function of these variables and time. It follows that

\[
\frac{dF}{dt} = \sum_{i=1}^{n} \left( \frac{\partial F}{\partial \bar{q}_i} \frac{d\bar{q}_i}{dt} + \frac{\partial F}{\partial \bar{p}_i} \frac{d\bar{p}_i}{dt} \right) + \frac{\partial F}{\partial t}
\]

Using Hamilton's equations, give

\[
\frac{dF}{dt} = \sum_{i=1}^{n} \left( \frac{\partial F}{\partial \bar{q}_i} \frac{d\bar{q}_i}{dt} + \frac{\partial F}{\partial \bar{p}_i} \frac{d\bar{p}_i}{dt} \right) + \frac{\partial F}{\partial t}
\]

If we have 2 functions, \( A(\bar{q}, \bar{p}) \), \( B(\bar{q}, \bar{p}) \), the Poisson bracket of these functions is defined by

\[
\{A, B\} = \sum_{i=1}^{n} \left( \frac{\partial A}{\partial \bar{q}_i} \frac{\partial B}{\partial \bar{p}_i} - \frac{\partial A}{\partial \bar{p}_i} \frac{\partial B}{\partial \bar{q}_i} \right)
\]
The Poisson bracket satisfies the following conditions (CH0)

1. \( \{ A, B \} = -\{ B, A \} \)
2. \( \{ A, xB + yC \} = x\{ A, B \} + y\{ A, C \} \)
   (where \( x, y \) are constants)
3. \( \{ A, B, C \} = \{ A, B \} C + B \{ A, C \} \)
4. \( \{ A \{ B, C \} \} + \{ B \{ C, A \} \} + \{ C \{ A, B \} \} \)

The third equation is called the Jacobi identity. Using the first identity, it can be written as:

\[ \{ A \{ B, C \} \} = -\{ B \{ C, A \} \} - \{ C \{ A, B \} \} \]

(4') \( \{ A \{ B, C \} \} = \{ A, B \} C + B \{ A, C \} \)

(2) means \( D_AC = \{ A, C \} \) is linear:
\[
D_A(\alpha B + \beta C) = \alpha D_AB + \beta D_AC
\]

(3) means \( D_A \) satisfies the chain rule:
\[
D_A(BC) = (D_AB)C + B(D_AC)
\]

(4') means \( D_A \) satisfies another kind of chain rule:
\[
D_A \{ B, C \} = \{ D_A, B \} C + \{ B, D_A \} C
\]
Application

\[ \frac{dF}{dt} = \{ F, H \} + \frac{\partial F}{\partial t} \]

\[ \dot{q}_i = \{ q_i, H \} = \frac{\partial H}{\partial p_i} \]

\[ \dot{p}_i = \{ p_i, H \} = -\frac{\partial H}{\partial q_i} \]

\{ \text{recovery Hamiltonian equations} \}

Note that if

\[ \{ F, H \} = 0 \text{ then } \frac{df}{dt} = -\frac{\partial F}{\partial t} \text{ and} \]

if F has no explicit time dependence

then F is conserved

\[ \{ F, H \} = 0, \quad \frac{\partial F}{\partial t} = 0 \Rightarrow F \text{ conserved} \]

We can write

\[ \frac{dF}{dt} = \{ F, H \} + \frac{\partial F}{\partial t} = -D_H F + \frac{\partial F}{\partial t} \]

If F has no explicit time dependence

\[ \frac{d^n F}{dt^n} = (-D_H)^n F \]

If F has a convergent Taylor series

\[ F(Q(t), \theta(t)) = \sum_{n=0}^{\infty} \frac{t^n}{n!} (-D_H)^n F(Q, \theta) \bigg|_{t=0} \]

\[ = e^{-tD_H} F(Q, \theta) \]
\( e^{-\beta H} \) is called the Liouville operator.

**Example**

\[
L = \frac{1}{2} m \dot{q}^2 - \frac{1}{2} k q^2
\]

\[
\dot{q} = m \ddot{q}
\]

\[
H = \frac{D_0^2}{2m} + \frac{1}{2} k q^2
\]

\[
D_0^2 \dot{p} = \{ \dot{p}, H \} = -\frac{\partial H}{\partial q} = -k q
\]

\[
D_0^2 q = \{ q, H \} = -\frac{\partial H}{\partial p} = \frac{p}{m}
\]

\[
D_0^2 \dot{p} = -k \{ \dot{q}, H \} = -\frac{k}{m} p
\]

\[
D_0^2 q = \frac{1}{m} \{ p, H \} = -\frac{k}{m} q
\]

\[
D_4^2 \dot{p} = (-)^n \left( \frac{k}{m} \right)^n p
\]

\[
D_4^2 \dot{q} = (-)^n \left( \frac{k}{m} \right)^n q
\]

\[
D_4^2 \dot{p} = -k (-)^n \left( \frac{k}{m} \right)^n q
\]

\[
D_4^2 q = (-)^n \left( \frac{k}{m} \right)^n q
\]

\[
D_4^2 \dot{p} = \frac{1}{m} \{ p, H \} = \frac{k}{m} \left( \frac{k}{m} \right)^n p
\]

\[
q(t) = \sum_{n=0}^{\infty} \frac{t^{2n}}{2^n} \cos \left( \frac{k}{m} \right)^n \cos \left( \frac{k}{m} \right)^n q(0) + \frac{1}{\sqrt{k m}} \sum_{n=1}^{\infty} \frac{t^{2n+1}}{2^n} \sin \left( \frac{k}{m} \right)^n \sin \left( \frac{k}{m} \right)^n q(0)
\]

\[
p(t) = p(0) \cos \left( \frac{k}{m} t \right) - \sqrt{k m} q(0) \sin \left( \frac{k}{m} t \right)
\]

\[
q(t) = \sum_{n=0}^{\infty} \frac{t^{2n}}{2^n} \cos \left( \frac{k}{m} \right)^n q(0) + \frac{1}{\sqrt{k m}} \sum_{n=1}^{\infty} \frac{t^{2n+1}}{2^n} \sin \left( \frac{k}{m} \right)^n \sin \left( \frac{k}{m} \right)^n q(0)
\]

\[
p(t) = p(0) \cos \left( \frac{k}{m} t \right) + \frac{1}{\sqrt{k m}} \sum_{n=1}^{\infty} \frac{t^{2n+1}}{2^n} \sin \left( \frac{k}{m} \right)^n \sin \left( \frac{k}{m} \right)^n q(0)
\]
Checking
\[
\dot{\mathbf{p}} = -\sqrt{\frac{k}{m}} \mathbf{p} \cos \left( \sqrt{\frac{k}{m}} t \right) - k \mathbf{q} \mathbf{r} \cos \left( \sqrt{\frac{k}{m}} t \right)
\]
\[= -k \left( \mathbf{q} \mathbf{r} \cos \left( \sqrt{\frac{k}{m}} t \right) + \frac{i}{\sqrt{2m}} \mathbf{p} \mathbf{r} \sin \left( \sqrt{\frac{k}{m}} t \right) \right)
\]
\[= -k \mathbf{q}
\]

We can also check the other equation:

In general, if \( F(q, p) = F(q) \)

Then
\[\mathbf{F}(q(t), p(t)) = e^{-\frac{i}{\hbar} \mathbf{H} t} \mathbf{F}(q(0), p(0)) \]

Connection with quantum mechanics

\[\{ q_i, p_j \} = \delta_{ij} \]
\[\{ q_i, q_j \} = 0 \]
\[\{ p_i, p_j \} = 0 \]

\[\frac{dF}{dt} = \{ F, \mathbf{H} \} \]
\[F(t) = e^{-\frac{i}{\hbar} \mathbf{H} t} F(0) \]

\[\{ q_i, p_j \} = i \hbar \delta_{ij} \]
\[\{ q_i, q_j \} = 0 \]
\[\{ p_i, p_j \} = 0 \]

\[\frac{dF}{dt} = -\frac{i}{\hbar} \mathbf{H} \]
\[F(t) = e^{\frac{i}{\hbar} \mathbf{H} t} e^{-\frac{i}{\hbar} \mathbf{H} t} F(0) \]
Comparing we get the perscript:

1. Replace $q_i, p_i$ by operators
2. Replace $\{ \}$ by $-\frac{i}{\hbar}$

This perscriptum is called canonical quantization.

The theories are not equivalent because

$q_i p_j = p_j q_i$ in classical mechanics

$\hat{q}_i \hat{p}_j = \hat{p}_j \hat{q}_i + i\hbar$ in quantum mechanics

**Canonical Transformations**

Define

$Z = (q_1, q_2, p_1, p_2)$

$Z_i = q_i \quad i = 1, \ldots, N$

$Z_{i+N} = p_i \quad i = 1, \ldots, N$

$$\{Z_i, Z_j\} = \begin{pmatrix} [q q] & [q p] \\ [p q] & [p p] \end{pmatrix} = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} = J_{i j}$$

$J_{i,i+N} = 1 \quad J_{i+N,i} = -1 \quad$ all other components vanish

$$\{Z_i, Z_j\} = J_{i j}$$
Next we write the general expression for the Poisson Bracket in this notation

\[ \{ F, G \} = \sum_{i=1}^{\nu} \left( \frac{\partial F}{\partial z_i} \frac{\partial G}{\partial z_{i,nu}} - \frac{\partial F}{\partial z_{i,nu}} \frac{\partial G}{\partial z_i} \right) = \sum_{i=1}^{\nu} \left( \frac{\partial F}{\partial z_i} \frac{\partial G}{\partial z_{i,nu}} + \frac{\partial F}{\partial z_{i,nu}} \frac{\partial G}{\partial z_i} \right) \]

since these are the only non zero \( \partial_i J \),
we can replace this with the double sum

\[ \{ F, G \} = \sum_{ij=1}^{\nu} \frac{\partial F}{\partial z_i} \frac{\partial G}{\partial z_j} \]

For \( F = z_m \) \( G = z_n \) we get

\[ \{ z_m, z_n \} = \sum_{i} \frac{\partial z_m}{\partial z_i} J_{ij} \frac{\partial z_n}{\partial z_i} = J_{mn} \]

Finally,

\[ \{ \Sigma z_i H^j \} = \sum_{i} \frac{\partial z_i}{\partial z_j} J_{IR} \frac{\partial H}{\partial z_R} = \sum_{i} J_{IR} \frac{\partial H}{\partial z_R} \]

\[ \frac{\partial \Sigma z_i H^j}{\partial z_R} = \sum_{i} \frac{\partial z_i H^j}{\partial z_R} \]

\[ \{ \Sigma z_i, H^j \} = \sum_{R} \frac{\partial z_i}{\partial z_R} J_{IR} \frac{\partial H}{\partial z_R} \]
Definition. A transformation of the coordinates and momenta

\[ z_i \rightarrow z_i'(\bar{z}) \]

is called a canonical transformation if it preserves the Hamilton equations:

\[
\frac{dz_i}{dt} = \sum J_{ik} \frac{\partial H}{\partial z_k} = 0
\]

\[
\frac{dz_i'}{dt} = \sum J_{ik} \frac{\partial H}{\partial z_k'}
\]

Note

\[
\frac{dz_i'}{dt} = \sum \frac{\partial z_i'}{\partial z_j} \frac{dz_j}{dt} = \sum \frac{\partial z_i'}{\partial z_j} J_{jk} \frac{\partial H}{\partial z_k} = 2 \frac{\partial z_i'}{\partial z_j} J_{jk} \frac{\partial z_k'}{\partial z_k} \frac{\partial H}{\partial z_k} = \sum J_{ij} \frac{\partial H}{\partial z_j}
\]

In this to be true for all \( \frac{\partial H}{\partial z} = 0 \)

\[
J_{ij} = \sum \frac{\partial z_i'}{\partial z_j} J_{jk} \frac{\partial z_k'}{\partial z_k}
\]

If we define

\[
M_{ij} = \frac{\partial z_i'}{\partial z_j}
\]

this equation becomes

\[
J_{ij} = \sum M_{ik} J_{kj} M_{kj} = \sum M_{ik} J_{kj} M_{kj}^T
\]
In matrix form this becomes

\[
MJM^T = J
\]

If this is true

\[
\{Z_i \dot{Z}_j\} = \sum \frac{\partial F}{\partial \dot{Z}_i} \frac{\partial F}{\partial Z_j} \frac{\partial^2 F}{\partial \dot{Z}_i \partial \dot{Z}_j} \frac{\partial \dot{Z}_m}{\partial \dot{Z}_j} \frac{\partial \dot{Z}_m}{\partial \ddot{Z}_i}
\]

\[
= MJM^T = J = \{Z_i, \dot{Z}_j\}
\]

which shows that canonical transformation satisfy

\[
\{Z_i, \dot{Z}_j\} = \{Z_i, \dot{Z}_j\} = \delta_{ij}
\]

more generally

\[
\{F, G\} = \frac{\partial F}{\partial \dot{Z}_i} \frac{\partial G}{\partial Z_j} \frac{\partial^2 F}{\partial \dot{Z}_i \partial Z_j} \frac{\partial \dot{Z}_m}{\partial \dot{Z}_i} \frac{\partial \dot{Z}_m}{\partial \ddot{Z}_j}
\]

\[
= MJM^T = J
\]

\[
= \{F, G\} = \{F, G\} = \{F, \dot{G}\}
\]

dividual transformation preserve all Poisson brackets

det 2n dimensional real matrices satisfying

\[
MJM^T = J
\]

are called symplectic matrices