In the beginning of this class we identified inertial coordinate systems as those systems where free particles move with constant velocity. The transformations relating different inertial coordinate were shown to be generated by translations, rotations, time translation, and constant velocity translations. These elementary transformations generated the Galilean group, which is the group of transformations between different inertial coordinate systems.

The principle of Galilean relativity asserts that the form of the equations of motion is independent of the origin, orientation, velocity of the origin and the time in an inertial coordinate system. This is equivalent to the statement that the laws of physics are preserved under Galilean transformations.

The problem with this requirement is that Maxwell’s equations for electric and magnetic fields do not satisfy this property because Galilean transformations do not preserve the speed of light, but the Michelson-Morely experiment showed that it is preserved in moving coordinate systems. Since charged particles, which both generate and interact with electric and magnetic fields, are governed by Newton’s laws, there is a problem with the Galilean principle of relativity.

This requires a modification of the Galilean principle of relativity in order to be consistent with Maxwell’s equations. Einstein replaced this with the principle of special relativity, which replaces the Galilean group by another group that leaves the speed of light invariant. The form of Maxwell’s equations is preserved under a transformation group called the Poincaré group, which is the group of transformations that preserve

$$d_{AB}^{2} = |x_{A} - x_{B}|^{2} - c^{2}(t_{A} - t_{B})^{2}$$

$d_{AB}$ is is called the proper distance between two events that occur at position $x$ and time $t$ when $d_{AB}^{2} > 0$. Here $c$ is the speed of light in a vacuum. This ensures that a light signal sent from $A$ to $B$ has the same speed in all of these coordinate systems. Note that $d_{AB}^{2}$ it is not necessarily positive. When $d_{AB}^{2} < 0$ it is useful to replace $c$ it by $\tau_{AB}^{2} = -d_{AB}^{2}$, where in this case $\tau_{AB}$ is called the proper time. While Poincaré transformations were known before Einstein, the important realization is that they do not preserve the distance or time between events. More surprising is which event happens first may depend on the choice of coordinate system.

In order to treat problems in special relativity it is useful to use the following notation. So the coordinates of all events have the same dimensions define

$$x^{0} := ct.$$  

Coordinates of events are labeled by 4-vectors:

$$x^{\mu} := (x^{0}, x^{1}, x^{2}, x^{3}) = (ct, x, y, z).$$
The metric tensor is defined by

\[ \eta_{\mu\nu} = \eta^{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \]

With this definition the equation for \( d_{AB}^2 \) becomes

\[ d_{AB}^2 = \sum_{\mu,\nu=0}^{3} \eta_{\mu\nu}(x_A - x_B)^\mu(x_A - x_B)^\nu. \]

This quantity is invariant. It is also useful to define

\[ x_\mu = \sum_{\nu=0}^{3} \eta_{\mu\nu}x^\nu = (-ct, z, y, z) \]

so the invariant \( d_{AB}^2 \) is

\[ d_{AB}^2 = \sum_{\mu=0}^{3} (x_A - x_B)^\mu(x_A - x_B)_\mu. \]

Since \( \eta \) is its own inverse it follows that

\[ x^\mu = \sum_{\nu=0}^{3} \eta^{\mu\nu}x_\nu \quad \eta^{\mu\nu} = \sum_{\gamma=0}^{3} \eta^{\mu\gamma}\eta_{\gamma\nu} = \delta^\mu_\nu. \]

The quantities \( x^\mu \) are called contravariant components of the 4-vector \( x \) while \( x_\mu \) are called covariant components of \( x \). Both quantities represent the same physical event. The advantage of having both notations is that the sum over the components of an upper and lower index quantity is invariant. It is also convenient to adopt the Einstein summation convention - which means that there is an implied sum from 0 to 3 for any repeated pair of upper and lower Greek indices. In this notation the expression for \( d_{AB}^2 \) becomes

\[ d_{AB}^2 = (x_A - x_B)^\mu(x_A - x_B)_\mu. \]

**Lecture 40.20_b**

**Special relativity:**

The Poincaré group

To understand the structure of Poincaré transformations consider a change of coordinates of the form

\[ x^\mu \rightarrow f^\mu(x). \]

For this transformation to preserve the proper distance between events we require

\[ d_{AB}^2 = \eta_{\mu\nu}(x_A - x_B)^\mu(x_A - x_B)^\nu = \eta_{\mu\nu}(f(x_A) - f(x_B))^\mu(f(x_A) - f(x_B))^\nu \]
If we differentiate with respect to $x_A^\gamma$ and set $x_A = 0$ we get

$$-2\eta_{\gamma\nu}x_B^\nu = 2\eta_{\mu\nu}\frac{\partial f(0)^\mu}{\partial x^\gamma}(f(0) - f(x_B))^\nu$$  \hspace{1cm} (1)

If we differentiate (1) again with respect to $x_B^\delta$ and set $x_B = 0$ we get

$$-2\eta_{\gamma\delta} = -2\eta_{\mu\nu}\frac{\partial f(0)^\mu}{\partial x^\gamma}\frac{\partial f(0)^\nu}{\partial x^\delta}$$ \hspace{1cm} (2)

We define the constant matrix

$$\Lambda^\mu{}_{\gamma} = \frac{\partial f(0)^\mu}{\partial x^\gamma},$$

which is called a Lorentz transformation, and the constant four vector

$$a^\mu = f^\mu(0).$$

We can write equation (2) as

$$\eta_{\gamma\delta} = \Lambda^\mu{}_{\gamma}\eta_{\mu\nu}\Lambda^\nu{}_{\delta}$$

and equation (1) as

$$\eta_{\gamma\nu}x^\nu = \eta_{\mu\nu}\Lambda^\mu{}_{\gamma}(f^\nu(x) - a^\nu).$$

These equations can be expressed in matrix form as:

$$\eta = \Lambda^t\eta \Lambda \quad \text{and} \quad \eta x = \Lambda^t\eta(f(x) - a).$$  \hspace{1cm} (3)

Using

$$\eta^2 = I$$  \hspace{1cm} (4)

the first equation gives

$$\Lambda^{-1} = \eta\Lambda^t\eta$$  \hspace{1cm} (5)

Using (5) in the second equation in (3) gives

$$\Lambda x = \Lambda\eta x = \Lambda\eta\Lambda^t\eta(f(x) - a) = f(x) - a$$

which can be solved for $f^\mu(x)$

$$f(x) = \Lambda x + a.$$  

Putting back the indices this becomes

$$f^\mu(x) = \Lambda^\mu{}_{\nu}x^\nu + a^\mu.$$  \hspace{1cm} (6)

This is the general form of a Poincaré transformation. It shows that the requirement that $d^2_{AB}$ is invariant means that the Taylor expansion of $f^\mu(x)$

$$f^\mu(x) = f^\mu(0) + \frac{\partial f^\mu(0)}{\partial x^\nu}x^\nu$$
terminates after the first derivative terms. Equation (2) implies

$$\text{det}(\Lambda)^2 = 1 \quad (\Lambda^0_0)^2 = 1 + \sum_{i=1}^{3} (\Lambda^i_0)^2.$$  

This breaks the Lorentz transformations into four topologically distinct sets $\text{det}(\Lambda) = 1, \Lambda^0_0 \geq 1$, $\text{det}(\Lambda) = -1, \Lambda^0_0 \geq 1$, $\text{det}(\Lambda) = 1, \Lambda^0_0 \leq -1$, $\text{det}(\Lambda) = -1, \Lambda^0_0 \leq -1$. The first set includes the identity and is a group, the second set involves a space reflection, the third set involves a spacetime reflection and the fourth set involves time reversal. Since the weak interaction is known to not be invariant with respect to space reflection or time reversal, only the component containing the identity is considered to be a symmetry of special relativity.

The Poincaré transformations are generated by space and time translations, rotations and rotationless Lorentz transformations.

The Lorentz transformations that replace the Galilean transformations that change velocity are called Lorentz boosts. They are not uniquely defined, but a useful class are the rotationless Lorentz boosts which have the form

$$\Lambda^{\mu\nu}(v) = \left( \begin{array}{cc} \gamma & \gamma \frac{v}{c} \\ \frac{v}{c} & \gamma \delta_{ij} + (\gamma - 1) \frac{v^i v^j}{c^2} \end{array} \right)$$

where $\gamma = (1 - v^2/c^2)^{-1/2}$. A general boost $\Lambda'(v)$ to the same $v$ is related to the rotationless boost by

$$\Lambda'(v) = \Lambda(v)R(v)$$

where $R(v)$ is a rotation matrix. This follows because rotations leave 0 velocity unchanged.

**Lecture 40.20_c**

**Special relativity:**

**4 vectors**

Four vectors have the property that they transform like (6). The four velocity is defined by

$$v^\mu = \frac{dx^\mu}{d\tau}$$

where $\tau$ is the invariant proper time. Since

$$x^{\mu'} = \Lambda^{\mu}_{\nu} x^\nu + a^\mu,$$

$\Lambda^{\mu}_{\nu}$, $a^\mu$ are constant and $\tau$ is invariant it follows that

$$v^{\mu'} = \frac{dx^{\mu'}}{d\tau'} = \frac{dx^{\mu'}}{d\tau} = \Lambda^{\mu}_{\nu} \frac{dx^\nu}{d\tau} = \Lambda^{\mu}_{\nu} v^\nu$$

which shows that the four-velocity transforms like a 4-vector, except in this case there is no constant term analogous to $a^\mu$. 

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Similarly we can define the four acceleration:

\[ a^\mu = \frac{d^2x^\mu}{d\tau^2} \]

and the four momentum:

\[ p^\mu = mv^\mu \]

The same argument that was used to show that the four-velocity is a four-vector can be used to show that both \( a^\mu \) and \( p^\mu \) are four vectors.

Note that for any four vector that transforms like

\[ b^\mu = \Lambda^\mu_\nu b^\nu \]

it follows that

\[ v^\mu v'_\mu = \Lambda^\mu_\nu v^\nu \eta_{\mu\alpha} \Lambda^\alpha_\beta v^\beta = \eta_{\nu\beta} v^\nu v^\beta = v^\nu v^\nu \]

is invariant. Using the expression for \( \tau^2_{AB} \) for small displacements between \( A \) and \( B \) gives

\[ c^2(d\tau)^2 = c^2(dt)^2 - (dx)^2 \]

which implies

\[ c^2 \left( \frac{d\tau}{dt} \right)^2 = c^2 - v^2. \]

This can be used to solve for \( \frac{d\tau}{dt} \): solving for

\[ \frac{d\tau}{dt} = \frac{1}{\sqrt{1 - v^2/c^2}} = \frac{1}{\gamma}. \]

It follows that

\[ v^\mu = (c \frac{dt}{d\tau}, \frac{dx}{d\tau} \frac{dt}{d\tau}) = (\gamma c, \gamma v) \]

and

\[ p^\mu = (mc \frac{dt}{d\tau}, m \frac{dx}{d\tau} \frac{dt}{d\tau}) = (\gamma mc, \gamma mv) \]

which gives

\[ v^\mu v_\mu = \gamma^2 (-c^2 + v^2) = -c^2 \]

and

\[ p^\mu p_\mu = mv^\mu v_\mu = -m^2c^2 = -m^2c^2\gamma^2 + p^2\gamma^2. \]

Multiplying by \( c^2 \) gives

\[ m^2c^4\gamma^2 = m^2c^4 + p^2c^2\gamma^2. \] (8)

To understand how to reformulate Newton’s second law consider

\[ \frac{md^2a^\mu}{d\tau^2} = \frac{dp^\mu}{d\tau} = m \frac{dv^\mu}{d\tau} = (mc \frac{d\gamma}{d\tau}, mv \frac{d\gamma}{d\tau} + m\gamma \frac{d^2x}{d\tau^2} \left( \frac{dt}{d\tau} \right)) = \]
\[ (mc \frac{d\gamma}{d\tau}, m\mathbf{v} \frac{d\gamma}{d\tau} + m\gamma^2 \frac{d^2\mathbf{x}}{dt^2}) \]  

where

\[ \frac{d\gamma}{d\tau} = \gamma \frac{d\gamma}{dt} = \gamma(-1/2)\gamma^{-3}(-2\mathbf{v} \cdot \frac{d\mathbf{v}}{c^2dt}) = \gamma^{-2} \mathbf{v} \cdot \dot{\mathbf{v}} \]  

If we use (10) in (9) and set \( \mathbf{v} = 0 \) in (9) it becomes

\[ m \frac{d^2a^\mu}{dt^2} \rightarrow (0, m \frac{d^2\mathbf{x}}{dt^2}) \]  

Now we make an important observation - if two four vectors are equal in one coordinate system, then they are equal in all inertial coordinate systems. This follows trivially:

\[ (a^\mu' - b^\mu') = \Lambda^\mu_\nu (a^\nu - b^\nu) = \Lambda^\mu_\nu 0^\nu = 0 \]

This suggests defining a four vector force \( f^\mu \) to be the four vector that has the form

\[ f^\mu \rightarrow (0, \mathbf{F}) \]  

in the frame where the particles velocity \( \mathbf{v} = 0 \). The equation

\[ \frac{dp^\mu}{d\tau} = f^\mu \]  

is a generalization of Newton’s laws that has the same form in all coordinate systems related by Lorentz transformations and agrees with the non-relativistic form in the particle’s instantaneous rest frame.

The general form of this force can be found applying the Lorentz boost (7) to (13)

\[ f^\mu = \Lambda^\mu_\nu \begin{pmatrix} 0 \\ \mathbf{F} \end{pmatrix} = \begin{pmatrix} \gamma & \gamma \frac{\mathbf{v}}{c} \\ \gamma \frac{\mathbf{v}}{c} & \delta_{ij} + (\gamma - 1)\frac{\mathbf{v}_i \cdot \mathbf{v}_j}{c^2} \end{pmatrix} \begin{pmatrix} 0 \\ \mathbf{F} \end{pmatrix} = \begin{pmatrix} \gamma \frac{\mathbf{v} \cdot \mathbf{F}}{c^2} \\ \mathbf{F} + (\gamma - 1)\mathbf{v}(\mathbf{v} \cdot \frac{\mathbf{F}}{c^2}) \end{pmatrix} \]

Note that when the force vanishes \( \frac{dp^\mu}{d\tau} = 0 \) which means that \( p^\mu \) is a conserved quantity. The conserved quantities are \( m\gamma c \) and \( \gamma \mathbf{p} \). The term, \( \gamma \mathbf{p} \), represents that relativistic momentum. Note that it differs for the non-relativistic momentum by the factor \( \gamma \). To interpret the 0 component note

\[ m\gamma c = \frac{mc}{\sqrt{1 - \frac{\mathbf{v}^2}{c^2}}} = mc(1 + \frac{1}{2} \frac{\mathbf{v}^2}{c^2} + \frac{3}{8} \left(\frac{\mathbf{v}^2}{c^2}\right)^2 + \cdots) = \frac{1}{c}(mc^2 + \frac{m}{2} \mathbf{v}^2 + \frac{3m}{8} \left(\frac{\mathbf{v}^2}{c^2}\right)^2 + \cdots) \]
We see that up to the factor of $1/c$, this involves a rest energy term $(mc^2)$ plus the non-relativistic kinetic energy, plus relativistic corrections. This defines the conserved energy of a particle of mass $m$

$$E := \gamma mc^2$$

which leads to

$$p^\mu = (E/c, \gamma m v)$$

$$p^2 c^2 = -E^2 + p_r^2 c^2 = -m^2 c^4$$

or

$$E^2 = m^2 c^4 + p_r^2 c^2$$  \hspace{1cm} (14)

where

$$p_r := \gamma m v$$

is the conserved relativistic momentum.

Note that while $p_r$ is the non-relativistic momentum in the rest frame, $p_r$ is conserved in all inertial frames while $mv$ is not.