Recall that the relativistic equations of motion for a particle of mass $m$ acted on by a force $F$ are

$$\frac{dp^0}{d\tau} = \frac{v\gamma \cdot F}{c}$$

$$\frac{dp}{d\tau} = F + (\gamma - 1)v(v \cdot \frac{F}{v^2})$$

where

$$\gamma = \left(1 - \frac{v^2}{c^2}\right)^{-1/2}.$$ 

Using the definition of the four momentum

$$p^0 = \gamma mc \quad p = \gamma mv$$

these equations become

$$m\frac{d\gamma}{d\tau}c + \gamma m\frac{dm}{d\tau}c = \frac{v\gamma \cdot F}{c}$$

and

$$m\frac{d\gamma}{d\tau}v + \gamma m\frac{dm}{d\tau}v + m\gamma \frac{dv}{d\tau} = F + (\gamma - 1)v(v \cdot \frac{F}{v^2}).$$

The first equation gives

$$m\frac{d\gamma}{d\tau} + \gamma m\frac{dm}{d\tau} = \frac{v\gamma \cdot F}{c^2}.$$ 

Using this in the second equation gives

$$(m\frac{d\gamma}{d\tau} + \gamma m\frac{dm}{d\tau})v + m\gamma \frac{dv}{d\tau} =$$

$$\left(\frac{v\gamma \cdot F}{c^2}\right)v + m\gamma \frac{dv}{d\tau} = F + (\gamma - 1)v(v \cdot \frac{F}{v^2})$$

or

$$m\gamma \frac{dv}{d\tau} = F + (\gamma - 1)v(v \cdot \frac{F}{v^2}) - \frac{v\gamma \cdot F}{c^2}.$$ 

Using

$$\frac{dt}{d\tau} = \gamma,$$

the equation for $v$ becomes

$$m\gamma^2 \frac{dv}{dt} = F + (\gamma - 1)v(v \cdot \frac{F}{v^2}) - \frac{v\gamma \cdot F}{c^2}.$$
dividing by \( m\gamma^2 \) gives

\[
\frac{dv}{dt} = \frac{F}{m\gamma^2} + \frac{\gamma - 1}{m\gamma^2}v(v \cdot \frac{F}{\gamma^2}) - \frac{v\gamma \cdot F}{c^2} \frac{v}{m\gamma^2}
\]

In general \( F \) will depend on the coordinates which requires the additional equation

\[
\frac{dx}{dt} = v
\]

These can be combined to get a single second order differential equation for \( x \).

\[ \text{Lecture 40.20.b} \]

**Special relativity:**

**Dynamics - constant force**

Consider the motion of a particle of mass \( m \) in one dimension. The equation for the velocity parallel to \( F \) is

\[
\frac{dv}{dt} = \frac{F}{m\gamma^2} + \frac{\gamma - 1}{m\gamma^2}F - \frac{\gamma v^2 F}{mc^2\gamma^2} = \frac{\gamma F}{m\gamma^2}(1 - \frac{v^2}{c^2}) = \frac{F}{m\gamma^3}
\]

or

\[
\frac{dv}{dt} = \frac{F}{m\gamma^3}
\]

If \( F \) depends on \( x \) then we have to use

\[
v = \frac{dx}{dt}
\]

\[
m\frac{d^2x}{dt^2} = \frac{F}{\gamma^3}.
\]

When the force is constant the first equation

\[
(1 - \frac{v^2}{c^2})^{-3/2}dv = \frac{F}{m}dt
\]

can be integrated

\[
\int_{v_0}^{v} (1 - \frac{v^2}{c^2})^{-3/2}dv = \frac{F}{m}(t - t_0)
\]

To compute the integral let \( v/c = \sin(\theta) \), \( dv = c\cos(\theta)d\theta \)

\[
\int_{\theta_0}^{\theta} c\cos(\theta)(1 - \sin^2(\theta))^{-3/2}d\theta =
\]
\[ c \int_{\theta_0}^{\theta} \frac{d\theta}{\cos^2(\theta)} = \]
\[ c(\tan(\theta) - \tan(\theta_0)) = \frac{F}{m}(t - t_0) \]
\[ \frac{v}{\sqrt{1 - \frac{v^2}{c^2}}} - \frac{v_0}{\sqrt{1 - \frac{v_0^2}{c^2}}} = \frac{F}{m}(t - t_0) \]

Solving this for \( v \) for the case \( t_0 = 0 \) and \( v_0 = 0 \) gives

\[ v(t) = \frac{Ft}{m} \left( \frac{1}{\sqrt{1 + \left(\frac{Ft}{mc}\right)^2}} \right) \]

In this case we see that in the limit that \( t \to \infty, v \to c \) as expected. In the limit that \( v \to 0 \) we recover the non-relativistic result.

**Lecture 40.20.c**  
**Special relativity:**  
**Dynamics - rocket problem**

For the next example consider a rocket where the in the rocket’s instantaneous rest frame the rocket burns fuel as a constant rate

\[ \frac{dm}{dt} = \kappa = \text{constant} \]

and the exhaust gas has velocity \( v_g \) relative to the rocket. Again we restrict considerations to one dimension. We use relativistic four momentum conservation to determine the force on the rocket in its rest frame by considering the rocket and ejected mass as an isolated system. The initial four momentum is 0 (in the rocket’s instantaneous rest frame). After a proper time \( \Delta \tau \) the rockets mass is reduced by \( \Delta m \), while the mass of the exhaust gas is \( \Delta m \) and it is moving with velocity \( -v_g \) relative to the positive direction of the rocket. Relativistic three momentum conservation gives

\[ 0 = \frac{1}{\sqrt{1 - \left(\frac{\Delta v_r}{c}\right)}}(m_r - \Delta m_r)\Delta v_r - \gamma_g \Delta m_r v_g \]

Keeping terms linear in small quantities

\[ 0 = m_r \Delta v_r - \gamma_g \Delta m_r v_g \]

and dividing by \( \Delta \tau \) gives

\[ 0 = m_r \frac{dv_r}{d\tau} = \gamma_g \frac{dm_r}{d\tau} v_g \]

so the force on the rocket in its rest frame is

\[ F = \gamma_g \frac{dm_r}{d\tau} v_g = \gamma_g \frac{dm}{dt} \gamma v_g \]
In the rocket’s rest frame $\gamma = 1$ so the force on the rocket in its rest frame is
\[ F = \frac{dm}{dt} \gamma v_e = \kappa \gamma v_e \]
Since $\kappa \gamma v_e$ is a constant, this is a constant force problem, but with a time dependent mass.

This represents the force on the rocket in its rest frame. Using the result that we derived for the case of a constant force gives
\[ \gamma^2 dv = \frac{F}{m} \, dt = \frac{\kappa v_e \gamma v_e}{m(t)} \, dt \]
\[ m(t) = m_0 (1 - \frac{\kappa}{m_0} t) \]
In this case the velocity integral is identical to the integral for the constant force problem, while the other side of the equation has a more complicated time dependence which gives the relation
\[ \frac{v}{\sqrt{1 - v^2/c^2}} - \frac{v_0}{\sqrt{1 - v_0^2/c^2}} = \int_{t_0}^{t} \frac{\kappa v_e \gamma v_e}{m(t)} \, dt = \]
\[ v_e \gamma \ln(1 - \frac{\kappa t_0}{m_t}) \]
Solving for $v(t)$ for the case that $t_0 = 0, v_0 = 0$ gives
\[ v(t) = \frac{v_e \gamma \ln(1 - \frac{t}{m_0 t})}{\sqrt{1 + (v_e \gamma/c)(\ln(1 - \frac{t}{m_0 t}))^2}} \]
As $t \to \frac{m_0}{\kappa}$ the speed approaches the speed of light - the problem is that assumes that the rocket is all fuel. If the percentage of the mass of the rocket that is fuel is $p$, then the terminal speed becomes
\[ v_f = \frac{v_e \gamma \ln(\frac{1}{1-p})}{\sqrt{1 + (v_e \gamma/c)(\ln(\frac{1}{1-p}))^2}} \]