In the treatment of the classical three-body problem initially the orbits of the Earth and Jupiter were on a 2 torus with periods $T_e$ and $T_j$. We discussed the case when the ratios of the periods were in a full volume set of irrational multiples of each other, where KAM established stability for sufficiently small perturbations. This did not answer the question of what happens when periods of the orbits are rational or almost rational multiples of each other. It turns out that they become unstable and have a very rich structure. One way to study these is to take a cross section of the original torus and follow the trajectory of the planets and follow where they intersect the cross section. This converts the dynamical problem to a mapping problem.

Mapping problems can have a very rich structure. A simple illustration comes from population dynamics. Let $x$ represent the population of some type of animal. A simple model assume that initially the population grows proportional to the number of members, until the resources (food supply) can no longer support the population growth. A crude model is given by the differential equation

$$\frac{dx}{dt} = a x - b x^2.$$  

The first term on the right represents the growth, while the second term restricts the growth after population reached a certain size.

To convert this to a mapping problem we replace the derivative by a finite difference approximation

$$\frac{x_{n+1} - x_n}{\Delta t} = a x_n - b x_n^2$$

which can be put in the form

$$x_{n+1} = (1 + a \Delta t)x_n - b \Delta t x_n^2 = (1 + a \Delta t)x_n(1 - \frac{b \Delta t}{1 + a \Delta t} x_n).$$

Let

$$y_n = \frac{b \Delta t}{1 + a \Delta t} x_n$$

and multiply both sides of the equation by

$$\frac{b \Delta t}{1 + a \Delta t}$$

to get

$$y_{n+1} = (1 + a \Delta t)y_n(1 - y_n).$$

Finally define

$$c = (1 + a \Delta t)$$
Figure 8.3-3. The VAK, according to Arnold [1963b].

Figure 1: torus
to get

\[ y_{n+1} = cy_n(1 - y_n). \]

This map is called the \textbf{logistical map}. The orbit of the points in this map depend on both the initial condition and the value of the constant \( c \).

Studying this map made Mitch Feigenbaum famous. It has a very rich structure.

We are interested in the case that \( 0 \leq y_n \leq 1 \). For this to be a mapping from \([0, 1]\) to \([0, 1]\) we require

\[ 0 \leq cy(1 - y) \leq 1 \]

for \( 0 \leq y \leq 1 \). The largest value this can take is when

\[ 0 = \frac{d}{dy}(cy(1 - y)) = c - 2cy \quad \rightarrow \quad y = \frac{1}{2} \]

The value at that point is

\[ cy(1 - y) \rightarrow c\frac{1}{2} \cdot \frac{1}{2} = \frac{c}{4}. \]

Thus, in order for this to be a mapping from \([0, 1]\) to \([0, 1]\) we require \( 0 < c \leq 4 \). Later we will consider the case when \( c > 4 \). Note that for \( y < 0 \) and \( y > 1 \) the result in negative, so these points never get mapped into \([0, 1]\).
Fig. 4.7. The graphs of $F^2_\mu(x)$ where $F_\mu(x) = \mu x(1 - x)$ for 
$\mu < 3$, $\mu = 3$, and $\mu > 3$. 

Figure 3: $F(F(y))$
The action of this mapping can be understood graphically (see figure 2). For a given $y \in [0, 1]$ $y_s$ is called a source of $y$ if $F(y_s) = y$. The sources of $y \in [0, 1]$ are the roots $y_s's$ of the quadratic polynomial

$$cy_s(1 - y_s) - y = 0.$$ 

Solving the quadratic equation gives the two roots

$$y_s = \frac{1}{2} \left(1 \pm \sqrt{1 - \frac{4y}{c}}\right).$$ 

Since $y \leq c/4$ this gives 2 sources unless $y = c/4$. There is one on either side of $y = \frac{1}{2}$. This is also apparent from figure 2.

Define $F(y) := cy(1 - y)$.

A point $y_f \in [0, 1]$ is called a fixed point of $F(y)$ if

$$y_f = F(y_f)$$

In population dynamics this corresponds to an initial condition that leads to a stable population.

The logistical map is simple enough that we can solve for the fixed point

$$0 = F(y_f) - y_f = cy_f(1 - y_f) - y_f = y_f(c - 1 - cy_f)$$

$$y_f = 0, y_f = \frac{c - 1}{c} = 1 - \frac{1}{c}$$

which shows that 0 is always a fixed point and there is a second one in $[0, 1]$ at $y_f = 1 - 1/c$ if $c > 1$.

The orbit of $y_0 \in [0, 1]$ is the sequence of points

$$y_0, y_1 = F(y_0), \ldots y_n = F(y_{n-1}), \ldots$$

A period $N$ fixed point is a point $y_0$ satisfying

$$y_N = y_0 \quad y_k \neq y_0 \quad k = 1, \ldots, N - 1$$

Clearly if $y_0$ is a period $N$ fixed point, so are $y_1, \ldots, y_{N-1}$.

Another thing that can be checked is the stability of a fixed point. A fixed point $y_f$ is stable if

$$\left|\frac{dF}{dx}(y_f)\right| < 1$$

it is unstable if

$$\left|\frac{dF}{dx}(y_f)\right| > 1$$

For the logistical map

$$\frac{dF}{dx} = c(1 - 2y)$$

5
For the fixed point \( y_f = 0 \), \( \frac{dF}{dx}(y_f) = c \) which is stable if \( c < 1 \), unstable if \( c > 1 \).
For the fixed point \( y_f = 1 - 1/c \) stability requires

\[-1 < c(1 - 2 + 2/c) = 2 - c < 1\]

or \( 1 < c < 3 \).

To summarize

- \( 0 < c \leq 4 \) for \( F(y) \) to map from \([0, 1]\) to \([0, 1]\)
- \( 0 < c < 1 \) \( F \) has a trivial stable fixed point at 0.
- \( 1 < c < 3 \) \( F \) has a stable non-trivial fixed point at \( y = 1 - \frac{1}{c} \)
- \( 1 < c \leq 4 \) \( F \) has a non-trivial fixed point.

**Lecture 42.20b**

**Period doubling bifurcations**

There are also eventually periodic points. These are sequences where \( y_0y_1 \cdots y_n \), \( y_m \neq y_k \) for \( k \neq m \) with \( y_n \) a fixed point. These points must exist because any fixed point must have 2 sources. One is the fixed point but the other source is not.

When \( 3 < c \leq 4 \) there is still a fixed point but it is no longer stable. If we consider period 2 fixed points

\[ F(F(y)) = y \]

This involves finding roots of the polynomial equation

\[ cF(1 - F) = ccy(1 - y)(1 - cy(1 - y)) = y \]

This is a quadratic polynomial - it has four roots. Two of them are the fixed points of \( F(y) \), \( y = 0 \) and \( y = 1 - 1/c \). These roots can be divided out giving

\[ 0 = y(y - 1 + \frac{1}{c})(c^3y^2 - (c^3 + c^2)y + (c^2 + c)) \]

This has two additional roots at

\[ y = \frac{(c + 1) \pm \sqrt{(c - 3)(c + 1)}}{2c} \]

These roots are imaginary unless \( c > 3 \). When \( c = 3 \)

\[ y = \frac{c + 1}{2c} = \frac{3 + 1}{6} = \frac{2}{3} \]

but this is equal to

\[ y = 1 - 1/c = \frac{2}{3} \]
which is the position of the period one root. The new roots represent period 2 fixed points. They start at the same value of \( c \) where the period 1 fixed point becomes unstable. By computing

\[
\frac{d}{dy} F(F(y)) = \frac{dF}{dy}(F(y)) \frac{dF}{dy}(y) =
\]

\[-4c^3 y^3 + 6c^3 y^2 - 2(c^2 + c^3) + c^2 y =
\]

\[-4c^2(y - \frac{1}{2})(y - \frac{1 + \sqrt{1 - \frac{1}{c}}}{2})(y - \frac{1 - \sqrt{1 - \frac{1}{c}}}{2}) \]

for \( y \) given by (1) it can be shown that the period 2 fixed points are stable for \( c \) slightly greater than 3. What happens is the period 1 fixed point changes stability and at that point two new stable period 2 fixed points are created. This is called a period doubling bifurcation.

The period 2 fixed points do not remain stable as \( c \) is increased. The stability changes at about \( c = 3.449 \) at which point each of the period 2 fixed points becomes unstable and each one creates two new stable period four fixed points. This process keeps repeating itself until \( c = 3.569 \cdots \) which is associated with an infinite number of period doublings.

Lecture 42.20c

Universality - renormalization group

Consider fixed points of

\[ F^{2^n}(y_f, c) = y_f \quad \text{with} \]

\[ |\frac{dF^{2^n}(y)}{dy}(y_f, c)| < 1 \]

These fixed points are called superstable if

\[ |\frac{dF^{2^n}(y)}{dy}(y_f, c)| = 0 \]

For \( n = 1 \)

\[
\frac{dF}{dx} = c(1 - 2y) = 0 \rightarrow y = \frac{1}{2}
\]

for this to be a fixed point we must have

\[
\frac{1}{2} = 1 - \frac{1}{c}
\]

which holds for \( c = 2 \).

In this case if we expand \( F(y) \) about the superstable fixed point the first derivative term vanishes and we get

\[ F(y, 2) = \frac{1}{2} + \frac{1}{2!} \frac{d^2 F(\frac{1}{2}, 2)}{dx^2} (y - \frac{1}{2})^2 \]

(2)
Since $F$ is a degree 2 polynomial this formula is exact. If we let $y = x + \frac{1}{2}$ and define

$$G(x, 2) = \frac{1}{2!} \frac{d^2 F(\frac{1}{2}, x)}{dx^2} (x)^2 - \frac{1}{2}$$

this transforms the superstable fixed point to the origin.

Next we look for superstable periods 2 points. These satisfy

$$F(F(y)) = y$$

and

$$0 = \frac{d}{dy} F(F(y)) = \frac{dF}{dy} (F(y)) \frac{dF}{dy} (y) =$$

$$-4c^2(y - \frac{1}{2})(y - \frac{1 + \sqrt{1 - \frac{2}{c}}}{2})(y - \frac{1 - \sqrt{1 - \frac{2}{c}}}{2})$$

We found the roots of the first equation in the last section - they are

$$y = 0, y = 1 - \frac{1}{c}, y = \frac{c + 1 + \sqrt{(c - 3)(c + 1)}}{2c}, y = \frac{c + 1 - \sqrt{(c - 3)(c + 1)}}{2c}$$

where the first 2 are the period one fixed points and the second two are the period 2 fixed points. The roots of the second equation are

$$y = \frac{1}{2}, y = \frac{1 + \sqrt{1 - \frac{2}{c}}}{2}, y = \frac{1 - \sqrt{1 - \frac{2}{c}}}{2}$$

In this case the polynomial has 3 critical points in \([0, 1]\); (see figure 3) there are two maxima at

$$y^* = 1 \pm \sqrt{1 - \frac{2}{c}}$$

and a minimum at $y = 1/2$. We can choose $c$ so the critical point at $y = \frac{1}{2}$ is a period 2 fixed point by solving

$$\frac{1}{2} = \frac{c + 1 + \pm \sqrt{(c - 3)(c + 1)}}{2c}$$

for $c$. This gives

$$c = 1 \pm \sqrt{5}.$$  

Only the root $c = 1 + \sqrt{5} > 0$. This values of $c$ makes $y = \frac{1}{2}$ a superstable fixed point of period 2.

The important observation is that in the neighborhood of the minimum at $y = \frac{1}{2}$ it has a quadratic structure similar to (2) except it is inverted and the quadratic structure and the derivative is rescaled This suggest that near $\frac{1}{2}$

$$G^2(x, c_1) = F(F(x + \frac{1}{2}, c_1), c_1) - \frac{1}{2}$$
will have a similar quadratic structure to $G(x, \frac{1}{2})$ near the origin, except that it will be inverted and rescaled.

$$G(x) \approx \frac{G''(0)}{2} x^2 \rightarrow \tilde{G}(x) \approx \frac{\tilde{G}''(0)}{2} x^2$$

$$-\alpha \tilde{G}(\frac{y}{-\alpha}) = \frac{\tilde{G}''(0)}{2\alpha} y^2$$

These will approximately equal if

$$\frac{G''(0)}{2} = -\frac{\tilde{G}''(0)}{2\alpha} \alpha$$

This suggest that for $x \approx 0$ that

$$G^2(x, c_1) \approx -\alpha G(\frac{-x}{\alpha}, c_2)$$

where $c_1$ is the $c$ that makes the origin into a superstable fixed point of $G^2$. This can be repeated. Feigenbaum found that

$$G(x) = \lim_{n \to \infty} (-\alpha)^n G_{2^n}(\frac{x}{(-\alpha)^n}, c_n)$$

converges to a universal function $G(x)$ satisfying

$$G(x) = -\alpha G(\frac{-x}{\alpha}) \quad \alpha \approx 2.502 \cdots$$

He also showed that

$$\lim_{n \to \infty} \frac{c_{n+1} - c_n}{c_{n+2} - c_{n+1}} \rightarrow \delta = 4.67 \cdots$$

Universal means that the same function $G(x)$ results from many maps in the neighborhood of superstable fixed points.

**Lecture 42.20d**

**Beyond period doubling**

As $c$ is increased to $c \approx 3.839$ the logistical map has a period three fixed point (see fig 2). There is a general result called the Sarkovskii theorem which implies that if there is a period three fixed point then there must be fixed points of all periods.

The argument is as follows, assume $F(a) = b$, $F(b) = c$, $F(c) = a$. Consider the case $a < b < c$ and let $I_0 = [a, b]$, $I_1 = [b, c]$, it follows that $I_1 \subset F[I_0]$ and $I_0 \cup I_1 \subset F[I_1]$. Let $J$ be the subset of $I_0$ such that maps into $I_1$

$$F[J] = I_1 \rightarrow F^2[J] = F[I_1] \supset I_0 \cup I_1$$

This means that there is a subset of $J$ that (1) leaves $J$ on one application of $F$ and returns to $J$ on a second application. It follows from the intermediate
value theorem that $F$ has a period 2 fixed point in $J$. ($F^2(j) - j$ changes sign as $j$ varies from one endpoint of the subset of $J$ to the other).

Next we show that for any $N$ there is a period $N$ fixed point. The idea of the proof is as follows. Let $J$ in $I_0$ satisfy $F(J) = I_1$. Let $A_1 \in I_1$ satisfy $F(A_1) = J$, let $A_2$ be the inverse image of $A_1$ in $I_1$, · · · let $A_{N-1}$ be the inverse image of $A_{N-2}$ in $I_1$ and finally let $J_N$ be the inverse image of $A_{N-1}$ in $I_0$, which is a subset of $J$. (Recall $F(J)$ covers $I_1$ which covers $A_{N-1}$).

This means $F(J_N) = A_{N-1} \in I_1$

$F^2(J_N) = F(A_{N-1}) = A_{N-2} \in I_1$

$F^3(J_N) = F^2(A_{N-1}) = F(A_{N-2}) = A_{N-3} \in I_1$

\[\vdots\]

$F^{N-1}(J_N) = A_1 \in I_1$

$F^N(J_N) = F(A_1) = J \in I_0$

Again consider $F^N(y) - y$ for $y \in J_N$. Let $j_+$ and $j_-$ be the end points of $J$ and $j_a$ and $j_b$ be in $J_N$ $F^N(j_a) = j_+ < j_a$ and $F^N(j_b) = j_- < j_b$ which means $F^N(j_a) - j_a < 0$, $F^N(j_b) - j_b > 0$. Since $F$ is continuous there must be a point in $J_N$ satisfying $F^N(j) = j$. This must be a period $N$ point since $j \in I_0$ and $F^k(j) \in I_1$ for $k < N$.

The considerations so far correspond to $c \leq 4$. When the $c > 4$ the peak of the parabola at $y = 1/2$ is larger than 1. This means the there is a region centered about $1/2$ where all of the orbits that start in that region move to $y > 1$ and never return to the interval $[0, 1]$. We call $I_1$ the set of points on $[0, 1]$ with $F(y) > 1$. Next let $I_2$ be the set of points satisfying $F[I_2] = I_1$. These points are removed with 2 applications of $F$. This can be continued $I_n$ are the set of points satisfying $F^{n-1}[I_n] = I_1$. These are removed after $n$ applications of $F$. After removing all of these sets the remaining part of $[0, 1]$ is not empty since there are still and infinite number of points that eventually end up at 0. It can be shown that what remains is actually Cantor set, which is an uncountable set that contains no intervals.

This example of the logistical map shows that even a simple mapping problem can have a rich structure that can be use to understand properties of the solution of differential equations.