Lecture 9

The second variation, when is the action minimal

\[ S^2[A[\tilde{y}_0, \tilde{y}_1]] = \frac{d^2}{d \lambda^2} A[\tilde{y}_0 + \lambda \tilde{y}_1] \bigg|_{\lambda=0} \]

\[ \frac{d}{d \lambda} \int_{t_1}^{t_2} L (\tilde{y}_0 + \lambda \tilde{y}_1, \dot{\tilde{y}}_0 + \lambda \dot{\tilde{y}}_1, t) \, dt = \]

\[ \sum_{m=1}^{n} \int_{t_1}^{t_2} \left( \frac{\partial^2 L}{\partial \tilde{y}_m \partial \tilde{y}_n} \right) \tilde{y}_m(t) \tilde{y}_n(t) \, dt \]

\[ + \sum_{m=1}^{n} \int_{t_1}^{t_2} \left( \frac{\partial^2 L}{\partial \dot{\tilde{y}}_m \partial \dot{\tilde{y}}_n} \right) \dot{y}_m(t) \dot{y}_n(t) \, dt \]

\[ + \sum_{m=1}^{n} \int_{t_1}^{t_2} \left( \frac{\partial^2 L}{\partial \tilde{y}_m \partial \dot{\tilde{y}}_n} \right) \tilde{y}_m(t) \dot{y}_n(t) \, dt \]

The coefficients

\[ A_{mn}(t) = \frac{\partial^2 L}{\partial \tilde{y}_m \partial \tilde{y}_n} \]

\[ B_{mn}(t) = \frac{\partial^2 L}{\partial \dot{\tilde{y}}_m \partial \dot{\tilde{y}}_n} \]

\[ C_{mn}(t) = \frac{\partial^2 L}{\partial \tilde{y}_m \partial \dot{\tilde{y}}_n} \]

are known functions of \( t \); they are evaluated with \( \tilde{y}_0(t) \) that makes the action stationary.
we use these to define a new functional of the $S_\gamma(t)$

$$S[ S_\gamma ] = $$

$$\sum_{mn} \int_{t_1}^{t_2} \left( A_{mn}(t) S_\gamma^m(t) S_\gamma^n(t) + 2 B_{mn}(t) S_\gamma^m(t) \dot{S}_\gamma^n(t) \right) dt$$

This represents the value of the second variation for a given $S_\gamma(t)$.

If this is positive for every $S_\gamma(t)$

then $S_\gamma(t)$ is a local minimum of $A S_\gamma$

This will be true if the $S_\gamma$ that minimizes $S[S_\gamma]$ is positive.

If we call this $S_\gamma_0(t)$ then

$$S[S_\gamma_0; S_\gamma] = 0 \quad \text{for all } S_\gamma(t)$$

Note

1. $S_\gamma(t_1) = S_\gamma(t_2) = 0 \quad S_\gamma(t) = \dot{S}_\gamma(t) = 0$

2. $\int S[S_\gamma] = \eta^2 \int S[S_\gamma] \quad$ this means $S[S_\gamma]$ is homogeneous of degree 2 - which means there is no minimum - it can always be reduced by lowering $\eta$. 
To eliminate this trivial rescaling we require

\[ \int_{t_1}^{t_2} \overrightarrow{\delta y}(t) \cdot \overrightarrow{\delta y}(t) \, dt = 1 \]

Or

\[ \int_{t_1}^{t_2} \left[ \overrightarrow{\delta y}(t) \cdot \overrightarrow{\delta y}(t) - \frac{1}{t_2-t_1} \right] dt = 0 \]

We treat this constraint using the method of Lagrange multipliers:

\[ \mathcal{F} = \int_{t_1}^{t_2} \left( \sum_{mn=1}^{n} \left( A_{mn}(t) \overrightarrow{\delta y}_m(t) \overrightarrow{\delta y}_n(t) + 2 B_{mn}(t) \overrightarrow{\delta y}_m(t) \overrightarrow{\delta y}_n(t) \\
- C_{mn}(t) \overrightarrow{\delta y}_m(t) \overrightarrow{\delta y}_n(t) \right) \\
- \eta (\overrightarrow{\delta y}(t) \cdot \overrightarrow{\delta y}(t) - \frac{1}{t_2-t_1}) \right) dt \]

We look for stationary values of this functional subject to the constraint \( \eta = \text{Lagrange multiplier} \)

\[ \mathcal{F}[\overrightarrow{\delta y}_m; \overrightarrow{\delta y}_n] = \frac{d}{d\lambda} \mathcal{F}[\overrightarrow{\delta y}_m + \lambda \overrightarrow{\delta y}_n] = 0 = \]

\[ \int_{t_1}^{t_2} \sum_{mn=1}^{n} \left( 2 A_{mn}(t) \overrightarrow{\delta y}_m(t) \overrightarrow{\delta y}_n(t) + 2 B_{mn}(t) \overrightarrow{\delta y}_m(t) \overrightarrow{\delta y}_n(t) \\
+ 2 B_{mn}(t) \overrightarrow{\delta y}_m(t) \overrightarrow{\delta y}_n(t) + 2 C_{mn}(t) \overrightarrow{\delta y}_m(t) \overrightarrow{\delta y}_n(t) \right) \\
- \sum_{n=1}^{n} 2 \eta \overrightarrow{\delta y}_n(t) \overrightarrow{\delta y}_n(t) \right) \]
cancelling the factor of 2 integrating by parts to remove the derivatives from $\delta y_n(t)$ gives

\[ 0 = \int_{t_1}^{t_2} \left( \sum_{m} \left( A_{mn}(t) \delta y_m(t) + B_{mn}(t) \delta \dot{y}_m(t) \right) - \frac{d}{dt} \left( C_{mn}(t) \delta \dot{y}_m(t) \right) \right) \]

\[ - \frac{d}{dt} \left( B_{nm}(t) \delta y_m(t) \right) - \frac{d}{dt} \left( C_{mn}(t) \delta \dot{y}_m(t) \right) \]

\[ - \eta \delta y_{n_0}(t) \delta y_n(t) + \]

\[ \sum_{m} B_{nm}(t) \delta y_m(t) \delta y_n(t) \right) \]

\[ \left| \sum_{m} C_{mn}(t) \delta \dot{y}_m(t) \delta y_n(t) \right| \]

The boundary term vanishes because $\delta y_{n}(t_1) = \delta y_{n}(t_2)$ for all $n$. The coefficients of $\delta y_n(t)$ must vanish for all $t_1 < t < t_2$ since the $\delta y_n(t)$ are arbitrary. This gives the differential equation

\[ \sum_{m} \left[ \frac{d}{dt} \left( C_{mn}(t) \delta \dot{y}_m(t) \right) + \frac{d}{dt} \left( B_{nm}(t) \delta y_m(t) \right) \right] \]

\[ - B_{mn}(t) \delta \dot{y}_m(t) - A_{nm}(t) \delta y_m(t) \right) \]

\[ = - \eta \delta y_{n_0}(t) \]

where we have used $A_{mn} = A_{nm}$, $C_{mn} = C_{nm}$.
This is a linear second order differential equation that has the structure of an eigenvalue equation

$$\sum_{mn} \delta_{mn} \dot{y}_{m0} = \eta \delta_{n0}$$

where $\delta_{mn}$ is a linear differential operator.

So far we have not used the constraint --- multiply both sides by $\delta_{n0}(t)$, sum over $n$, and integrate the result from $t_1$ to $t_2$.

This gives

$$\eta \int_{t_1}^{t_2} \sum_{n} \delta_{n0}(t) \dot{y}_{n0}(t) = \eta \cdot 1 =$$

$$\sum_{mn} \int \left[ \frac{-d}{dt} \left( C_{nm}(t) \dot{y}_{m0}(t) \right) - \frac{d}{dt} \left( B_{nm}(t) \dot{y}_{m0}(t) \right) \right] \dot{y}_{n0}(t) dt + \int \left( B_{mn}(t) \dot{y}_{m0}(t) + A_{mn}(t) \dot{y}_{m0}(t) \right) dt$$

Integrating the first, second and third terms separately gives

$$M = \sum_{mn} \int \left( C_{nm}(t) \dot{y}_{m0}(t) \dot{y}_{n0}(t) + B_{mn}(t) \dot{y}_{m0}(t) \dot{y}_{n0}(t) \right)$$

$$+ \left( B_{mn}(t) \dot{y}_{m0}(t) \dot{y}_{n0}(t) + A_{mn}(t) \dot{y}_{m0}(t) \dot{y}_{n0}(t) \right)$$
The middle 2 terms are identical:

\[ \eta = \sum_{mn} \int_{t_1}^{t_2} \left( A_{mn}(t) \dot{y}_{mn}(t) y_{mn}(t) + 2 B_{mn}(t) \dot{y}_{mn}(t) \dot{y}_{mn}(t) + C_{mn}(t) \ddot{y}_{mn}(t) \dot{y}_{mn}(t) \right) dt = \mathcal{S}[\tilde{y}_0] \]

This shows that the Lagrange multiplier \( \eta \) is the value of the functional of the second variation at \( y_0(t) \) for the eigenvector \( \tilde{y}_0 \).

Note that if we integrate by parts a second time we get:

\[ \eta = \sum_{mn} \int \left( A_{mn}(t) \dot{y}_{mn}(t) y_{mn}(t) - \frac{d}{dt} \left( y_{mn}(t) \dot{y}_{mn}(t) \right) y_{mn}(t) \right) dt = \mathcal{S}[\tilde{y}_0] \]

which is completely symmetric with respect to \( mn \):

\[ \int y_{mn} \theta_{nm} y_{nm} = \int (\theta_{nm} y_{nm}) y_{mn} \]
This means that the differential operator

\[ -\frac{d}{dt} (C_{nm}(t) \frac{d}{dt}) - \frac{d}{dt} B_{mn}(t) + B_{mn}(t) \frac{d}{dt} + A_{mn} \]

with boundary conditions is

a Hermitean operator. Linear equations that are Hermitean
on a finite interval and with symmetric boundary conditions
are called Sturm-Liouville problems.

They have the following properties:

1. The eigenvalues are real.
2. The eigenvalues accumulate at \( \infty \).
   (If there are a finite number
   with absolute value less than
   any constant.
3. The eigenfunctions are a
   basis for functions satisfying
   the boundary conditions.
Since any \( \overline{\psi}(t) \) satisfying the boundary conditions can be expanded

\[
\overline{\psi}(t) = \sum_{n=1}^{\infty} a_n \overline{\psi}_n(t)
\]

\[
\overline{\psi}_1(t) = \sum a_n \overline{\psi}_n(t) = \sum a_n \eta_n \phi_n
\]

multiplying by \( \overline{\psi}(t) \) and integrating

\[
\int \overline{\psi} \overline{\psi} = \sum a_n \eta_n \phi_n \phi_n 
= \sum \eta_n a_n^2
\]

Since the \( a_n^2 \) are real and non-negative, this will be positive if all of the eigenvalues are positive.

This analysis is essentially identical to the function of \( n \) variables case discussed previously.
For purposes of illustration consider a particle of mass \( m \) in one dimension, the equations of motion are

\[
m \ddot{x} = -\frac{dV}{dx}
\]

The solution for short time is

\[
x(t) = x(t_1) + \dot{x}(t_1) (t-t_1) - \frac{1}{2m} \frac{dV}{dx} (t_1) (x-x_1)^2 + \ldots
\]

For small \( t-t_1 \) the solution is approximately

\[
x(t) \approx x(t_1) + \dot{x}(t_1) (t-t_1)
\]

We convert this to a boundary value for

\[
x(t_2) = x(t_1) - \dot{x}(t_1) (t_2-t_1) = 0
\]

\[
\dot{x}(t_1) = \frac{x(t_2) - x(t_1)}{t_2-t_1}
\]

\[
x(t) \approx x_1 + \frac{x_2-x_1}{t_2-t_1} (t-t_1)
\]

\[
x(t) \rightarrow x_0 + \frac{x_2-x_1}{t_2-t_1} (t-t_1)
\]
for these short times the potential does not matter. In that case

\[ L \to \frac{1}{2} mx^2 \]

\[ C = \frac{\partial^2 L}{\partial x^2} = m \quad B = \frac{\partial^2 L}{\partial x \partial t} = 0 \quad A = \frac{\partial^2 L}{\partial x^2} = 0 \]

the eigenvalue equation is

\[-\frac{d}{dt} \left( m \frac{d}{dt} \right) s \cdot x_0 = m s \cdot x.\]

we see that

\[ s \cdot x(t) = \sin \left( \sqrt{\frac{m}{\eta}} (t - t_1) \right) \]

to satisfy \[ s \cdot x(t_2) = 0 \]

\[ \sin \left( \sqrt{\frac{\eta}{m}} (t_2 - t_1) \right) = 0 \]

\[ \sqrt{\frac{\eta}{m}} \Delta t = n \pi \]

\[ \eta = m \frac{n^3 \pi^4}{\Delta t^4} > 0 \]

we see in this case all of the \( n^2 \) > 0.

As time is increased the potential term becomes more important.
In that case the eigenvalues move as \( \Delta t \) is increased, while they all start off positive, some of them can change sign. When this happens the curve that makes the action stationary does not make it a minimum (local).

To understand what is going on, we consider the differential equation when the times are adjusted so one of the eigenvalues is 0. The resulting equation

\[
-\frac{\mathbf{S}_n}{m} \frac{d}{dt} \left( C_{mn} \frac{d}{dt} \delta_{Xm} + B_{nm} \delta_{Ym} \right) + \sum B_{mn} \frac{d}{dt} \delta_{Xn} + A_{mn} \delta_{Yn} = 0
\]

This equation (with \( n \) set to 0) is called the Jacobi equation.
Returning to the one dimensional problem

\[ L = \frac{1}{2} m \dot{x}^2 - V(x) \]

In this case

\[ C = \frac{d^2 L}{dx^2} = m \]
\[ A = \frac{d^3 L}{dx^3} = -\frac{d^2 V}{dx^2} \]
\[ B = \frac{d^4 L}{dx^4} = 0 \]

The Jacobi equation is

\[ (-m \frac{d^2}{dt^2} + A(x)) \frac{dx}{dt} = 0 \]

Let \( x(t, v_0) \) be the solution of Lagrange's with initial coordinate \( x_i \) and initial velocity \( v \).

Consider

\[ \frac{dx}{dv} (t, v, v_0) \]

Note this is the derivative of the solution with respect to initial condition.

\[ \frac{dx}{dv} (t, v, v_1) = 0 \]

Since \( x(t, v, v_1) = x_i \) independent of \( v \)
Consider

\[ \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{x}} \right) - \frac{\partial L}{\partial x} = 0. \]

\[ \frac{d}{dv} \left[ \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{x}} \right) - \frac{\partial L}{\partial x} \right] = 0. \]

\[ \frac{d}{dt} \left( \frac{\partial^2 L}{\partial x^2} \frac{dx}{dv} + \frac{\partial^2 L}{\partial x \partial x} \frac{dx}{dv} \right) - \frac{\partial^2 L}{\partial x \partial x} \frac{dx}{dv} - \frac{\partial^2 L}{\partial x^2} \frac{dx}{dv} = 0. \]

Note that

\[ \frac{d}{dt} \left( C \frac{d}{dt} \left( \frac{dx}{dv} \right) + B \frac{dx}{dv} \right) - B \frac{d}{dx} \frac{dx}{dv} - A \frac{dx}{dv} = 0. \]

This is exactly the Jacobi equation.

So we see that

\[ \frac{dx}{dv} = f(t, v, t') \]

is a solution to the Jacobi equation.

This solution does not have to vanish at \( t_F \) since we are solving the initial value problem.

If \( \frac{dx}{dv} (t_i, v, t_F) \) vanishes, \( t_F \) is called a conjugate point.
Men

1. \( \frac{dx}{dv}(t,v,t) \) satisfies boundary conditions on \([t,t_f]\).

2. \( \frac{dx}{dv}(t,v,t_f) = 0 \)

This means that solutions for different values of \( V \) have the same coordinate at \( t_f \).

This means that at \( t=t_f \) the solutions for different values of \( V \) have the same coordinate.

If we think in terms of Fermat's principle, there are many rays that come to a focus at the conjugate point.