Lecture 1: Classical Mechanics

Goal: Determine the motion of systems of particles

Point particles - idealized particles whose internal dimensions can be ignored

Coordinates - we use Cartesian coordinates to describe the position of the particle at time \( t \).

\[
\mathbf{r}(t) = (x(t), y(t), z(t))
\]

The instantaneous velocity and acceleration of the particle are defined by

\[
\mathbf{v}(t) = \frac{d\mathbf{r}}{dt} = \left( \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \right)
\]

\[
\mathbf{a}(t) = \frac{d\mathbf{v}}{dt} = \frac{d^2\mathbf{r}}{dt^2} = \left( \frac{d^2x}{dt^2}, \frac{d^2y}{dt^2}, \frac{d^2z}{dt^2} \right)
\]
Idealized experimental observations

* The Newtonian principle of determinacy.

\( \vec{R}(t) \) can be determined for all time given an initial time \( t_0 \) and \( \vec{R}(t_0) = \vec{R}_0, \quad \vec{V}(t_0) = \vec{V}_0 \)

This is a non-trivial observation - the motion of the particle does not depend on higher derivatives at time \( t_0 \) or on the past history of \( \vec{R}(t) \).

Mathematically this means that there is a vector valued function \( \vec{R}(t, t_0, \vec{R}(t_0), \vec{V}(t_0)) \) with the property

\[ \vec{R}(t) = \vec{R}(t, t_0, \vec{R}(t_0), \vec{V}(t_0)) \] (1)

Differentiate equation (1) twice with respect to \( t \) and set \( t_0 = t \) to get

\[ \vec{a}(t) = \frac{d^2\vec{R}}{dt^2} = \left. \frac{\partial^2 \vec{R}(t, t, \vec{R}(t), \vec{V}(t))}{\partial t^2} \right|_{t=t} \]

(note that on the right the
time derivative must be computed before setting $t_0 = t$.

This shows that $\tilde{F}(t)$ satisfies a second order differential equation

$$\frac{d^2 \tilde{F}}{dt^2}(t) = \bar{G}(t, \tilde{F}(t), \frac{d \tilde{F}}{dt}(t))$$

where

$$\bar{G}(t, r(t), \frac{d r}{dt}(t)) = \frac{\partial^2}{\partial \xi^2} R(t', t, \xi(t), \frac{d \xi}{dt}(t)) \bigg|_{t' = t}$$

For a physical system $\bar{G}(t, \tilde{F}, \tilde{v})$ is not completely arbitrary. Since real particles do not spontaneously disappear, physically acceptable $\bar{G}(t, \tilde{F}(t), \frac{d \tilde{F}}{dt}(t))$ must have solutions $\tilde{F}(t)$ that are defined for all time.

The next goal is to rewrite this differential equation so the left side only depends on properties of the point particle.
and the right side does not depend on the particle.

Consider the following experiment — two different point particles are connected to identical springs.

The observation is that (i) in some restricted set of coordinate systems (called inertial coordinate systems) for the same initial conditions $F(t)$, $\vec{v}(t)$, the acceleration of the 2 particles differ by a multiplicative constant.

This means that in these special coordinate systems the differential equation can be put in the form

$$m \frac{\dd^2 \vec{r}}{dt^2} = \vec{F}(t, \vec{r}(t), \frac{d\vec{r}}{dt}(t))$$

where $m$ distinguishes different particles and $\vec{F}(t, \vec{r}, \vec{v})$ is the same for any particle.
In this equation

\[ m \equiv \text{inertial mass} \]

\[ \vec{F}(t, \vec{r}(t), \vec{v}(t)) \equiv \text{called the force on the particle} \]

The equation

\[ m \frac{d^2 \vec{r}}{dt^2}(t) = \vec{F}(t, \vec{r}(t), \frac{d\vec{r}}{dt}(t)) \]

is called \textit{Newton's second law}.

These observations also apply to systems of \( N \) point particles. In this case, the experimental observation is that the motion of every particle in the system can be determined given the position and velocity of each particle at some initial time.
using the same reasoning leads to

$$m_i \frac{d^2 \bar{r}_i}{dt^2} = \bar{F}_i \left( t, \bar{r}_i(t), \bar{v}_i(t), \bar{v}_N(t) \right)$$

where each particle has a different inertial mass and feels a different force.

This equation is Newton's second law for systems of particles. Everything that is discussed in this class is based on this equation.

Solutions - In general it is not trivial to show that the above equation has solutions that can be extended for all time. However it is not difficult to construct local solutions.
The method is to replace the second order equation by twice as many first order equations:

\[ \frac{d\vec{F}_i}{dt} = \vec{V}_i(t) \]

\[ \frac{d\vec{V}_i}{dt} = \frac{d^2\vec{F}_i}{dt^2} = \frac{1}{m_i} \vec{F}_i\left( \vec{r}_i(t), \vec{r}_i(t') \right) \vec{V}_i(t) \vec{V}_i(t') t' \]

The next step is to replace these equations by the integral equation:

\[ \vec{r}_i(t) = \vec{r}_i(t_0) + \int_{t_0}^{t} \vec{V}_i(t') dt' \]

\[ \vec{V}_i(t) = \vec{V}_i(t_0) + \int_{t_0}^{t} \frac{1}{m_i} \vec{F}_i\left( t', \vec{r}_i(t'), \vec{r}_i(t') \right) \vec{V}_i(t') \vec{V}_i(t') dt' \]

These equations are solved by iteration:

\[ \vec{r}_i^{(n)}(t) = \vec{r}_i(t_0) \]

\[ \vec{V}_i^{(n)}(t) = \vec{V}_i(t_0) \]

\[ \vec{V}_i^{(n)}(t) = \vec{V}_i(t_0) + \int_{t_0}^{t} \frac{1}{m_i} \vec{F}_i\left( t', \vec{r}_i^{(n-1)}(t'), \vec{r}_i^{(n-1)}(t') \right) \vec{V}_i^{(n-1)}(t') \vec{V}_i^{(n-1)}(t') dt' \]
as long as $\bar{F}_i$ is well behaved

$$\bar{V}_i(t) = \lim_{n \to \infty} V_i^{(n)}(t)$$

$$\bar{V}_i(t) = \lim_{n \to \infty} V_i^{(n)}(t)$$

converges to a solution to this equation for sufficiently short time (this is exactly Picard's method for proving the existence of local solutions of differential equations.)

The next set of idealized experiments are used to say something about the structure of the force.

Principle of Galilean Relativity

The statement of this principle is that for an isolated system the form of the equations should be the same in all inertial coordinate systems.
consider the special case of a free point particle in the absence of forces. In this case Newton's second law gives
\[
\frac{d^2 r}{dt^2} = 0
\]
This equation should have the same form in any inertial coordinate system.
The form of this equation is unchanged by

1. Changing the origin of the coordinate system
\[
\vec{r}' = \vec{r} - \vec{c} \quad (\vec{c} = \text{constant vector})
\]
\[
\frac{d^2 \vec{r}'}{dt^2} = \frac{d^2 \vec{r}}{dt^2} = 0
\]

2. Changing the velocity of the origin of the coordinate system
\[
\vec{r}' = \vec{r} - \vec{v}_0 t
\]
\[
\frac{d^2 \vec{r}'}{dt^2} = \frac{d^2 \vec{r}}{dt^2} - \frac{d \vec{v}_0}{dt} = \frac{d^2 \vec{r}}{dt^2} = 0
\]
(3) Resettling clock

\[ t' = t - c \]

\[ \frac{d^2 r}{d t'^2} = \left( \frac{dt}{d t'} \frac{dt'}{dt} \right) \left( \frac{dt}{d t'} \frac{dt'}{dt} \right) F = \left( \frac{dt}{d t'} \right)^2 F = \frac{d^2 r}{d t_i^2} = 0 \]

(4) Rotating the coordinate system,

\[ \vec{r}'(t) = \mathbf{R} \vec{r}(t) \]

\[ \mathbf{R} \text{ constant rotation matrix} \]

\[ \frac{d^2 \vec{r}'}{d t_i^2} = \mathbf{R} \frac{d^2 \vec{r}}{d t_i^2} = 0 \]

Combining these transformations together gives

\[ \vec{r}'(t') = \mathbf{R} \vec{r}(t) + \vec{v}_o t + \vec{r}_0 \]

\[ t' = t + c \]

Transformations generated by

by these elementary transformations have the above form — they also apply to systems of particles,

\[ \vec{r}'_i(t') = \mathbf{R} \vec{r}_i(t) + \vec{v}_o t + \vec{r}_i \]

\[ t' = t + c \]
For homework you will show that $R$ is a 3x3 orthogonal matrix $R^T R = I$.
If $\det R = 1$ it is a rotation; if $\det R = -1$ it includes a space reflection. The relevant property of $R$ is that it preserves length.

A Galilean transformation can be expressed in a 5x5 matrix form

\[
\begin{pmatrix}
\vec{r}' \\
\vec{v}' \\
t'
\end{pmatrix} =
\begin{pmatrix}
R & \vec{V}_0 & \vec{r}_0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
\vec{r} \\
\vec{v} \\
t
\end{pmatrix} =
\begin{pmatrix}
R \vec{r} + \vec{V}_0 t + \vec{r}_0 \\
0 \\
t + c
\end{pmatrix}
\]

For homework you will show

1) products of 2 matrices of the above form give another matrix of the same form

2) each matrix of this form has an inverse matrix of the same form
(3) The identity matrix has this form with 
\[ R = I, \quad V_0 = R_0 = 0 \quad c = 0 \]

(4) Matrix multiplication is associative 
\[ M_1(M_2M_3) = (M_1M_2)M_3 \]

These properties mean that the Galilean transformations form a group, called the Galilean group.

**Galilean Relativity:** The equations of motion have the same form in all inertial coordinate systems - different inertial coordinate systems are related by Galilean transformations.

*Note: Space reflections preserve \[ \frac{d^2\vec{r}}{dt^2} = 0 \] - whether they are included in the transformations relating inertial coordinate systems is optimal - this is because the weak force is not invariant under space reflections*
* Maxwell's equations are not invariant under Galilean transformations. They are invariant under Poincaré transformations that preserve
\[ c^2 \Delta t^2 - (\Delta \vec{r})^2 = c^2 \Delta s^2 \]

Fixing this problem leads to special relativity - where inertial coordinate systems are related by Poincaré transformations. This requires a modification of Newton's second law. (It holds in the particle's instantaneous rest frame.)

For the moment we only consider the consequences of Galilean symmetry question: Assume that Newton's second law has the same form in all inertial coordinate systems (related by Galilean transformations.)
Consider
\[ m_i \frac{d^2 \vec{r}_i}{dt^2} = \vec{F}_i(t, \vec{r}_i, \vec{v}_i, \vec{w}_i, \vec{v}_w) \]

If we let \( \vec{r}_i \to \vec{r}_i + \vec{c} \quad \vec{r}_i = \vec{r}_i' - \vec{c} \)
\[ m_i \frac{d^2 \vec{r}_i'}{dt^2} = \vec{F}_i(t, \vec{r}_i' - \vec{c}, \vec{v}_i - \vec{c}, \vec{v}_w) \]

These will have the same form if \( \vec{F}_i \) only depends on coordinate differences.

If we let \( \vec{r}_i \to \vec{r}_i + \vec{v}_i \cdot t \quad \vec{r}_i = \vec{r}_i' - \vec{v}_i t \quad \vec{v}_i = \vec{v}_i' - \vec{v}_i' \)
\[ m_i \frac{d^2 \vec{r}_i'}{dt^2} = \vec{F}_i(t, \vec{r}_i' - \vec{r}_i, \vec{v}_i - \vec{v}_i, \vec{v}_w' - \vec{v}_w) \]

For this to have the same form as the original equation, the force can only depend on velocity differences.
If we let $t' = t + \tau$

$$m_i \frac{d^2 \vec{r}_i}{dt'^2} = \vec{F}_i \left( t' - c, \vec{r}_{2i} - \vec{r}_i, \vec{r}_{w} - \vec{r}_i, \vec{v}_w - \vec{v}_i, \vec{v}_w - \vec{v}_i \right)$$

This will have the same form as the original equation if $\vec{F}_i$ has no explicit time dependence - i.e. $\vec{F}_i$ only depends on coordinate and velocity differences.

Next we consider $\vec{r}' = \vec{r}_i + R \vec{r}_i, \vec{v}' = \vec{v}_i + R \vec{v}_i$

$$m_i \frac{d^2 \vec{r}_i}{dt'^2} = \vec{F}_i \left( R (\vec{r}_2 - \vec{r}_i), R (\vec{w} - \vec{r}_i), R (\vec{v}_2 - \vec{v}_i), R (\vec{w} - \vec{v}_i) \right)$$

This will have the same form as the original equation if

$$R \vec{F} \left( R (\vec{r}_2 - \vec{r}_i), R (\vec{w} - \vec{r}_i), R (\vec{v}_2 - \vec{v}_i), R (\vec{w} - \vec{v}_i) \right) =$$

$$= \vec{F} \left( R (\vec{r}_2 - \vec{r}_i), R (\vec{w} - \vec{r}_i), R (\vec{v}_2 - \vec{v}_i), R (\vec{w} - \vec{v}_i) \right)$$

This will be true if $\vec{F}_i$ is a linear combination of vectors made out of these coefficients multiplied...
by rotationally invariant scalar quantities

Example

\[
m_1 \frac{\partial^2 \vec{r}_1}{\partial t^2} = -k (\vec{r}_1 - \vec{r}_2)
\]

\[
m_2 \frac{\partial^2 \vec{r}_2}{\partial t^2} = -k (\vec{r}_2 - \vec{r}_1)
\]

\[
\vec{R}' (\vec{L} (\vec{R}_1 - \vec{R}_2)) = -k \vec{R}' \vec{R} (\vec{r}_1 - \vec{r}_2) = -k (\vec{r}_1 - \vec{r}_2)
\]

We see that for isolated systems the Newtonian principle of determinacy and the Galilean principle of relativity limit the allowed forces that

1. are independent of time
2. depend on coordinate differences
3. depend on velocity differences
4. rotate like vectors when the coordinates and velocities are rotated
we still do not know how to identify an inertial coordinate system (it is not enough to say that free particles move with constant velocity because we can always define a moving coordinate system where that is true.

To understand this it is useful to consider the form of Newton's second law in non-inertial coordinate systems:

\[ \mathbf{F}_i = \text{coordinate of } i\text{th particle in an inertial coordinate system} \]

\[ \mathbf{F}'_i = \mathbf{F}'_i (\mathbf{F}, t) = \text{coordinate of } i\text{th particle in a non-inertial coordinate system} \]

\[ \frac{d\mathbf{r}'_i}{dt} = 2 \frac{\partial \mathbf{F}'_i}{\partial r'_j} \frac{dr'_j}{dt} + \frac{\partial \mathbf{F}'_i}{\partial t} \]

\[ \frac{d^2\mathbf{r}'_i}{dt^2} = 2 \frac{\partial^2 \mathbf{F}'_i}{\partial r'_j \partial r'_k} \frac{dr'_j}{dt} \frac{dr'_k}{dt} + 2 \frac{\partial \mathbf{F}'_i}{\partial t} \frac{dr'_j}{dt} + \frac{\partial^2 \mathbf{F}'_i}{\partial t^2} \]
**using Newton's second law in an inertial coordinate system:**

\[
m_i \frac{d^2 \mathbf{r}_i}{dt^2} = \sum_{j=1}^{3} \frac{\partial}{\partial r_{ij}} \mathbf{F}_i + \sum_{k=1}^{3} \frac{\partial}{\partial r_{ik}} \mathbf{F}_i + \sum_{l=1}^{3} \frac{\partial}{\partial r_{il}} \mathbf{F}_i
\]

Note that the first term on the right just involves changing coordinates of the force - it is independent of the inertial mass. The other three terms are all on the force side of the equation - but each are proportional to the inertial mass.

From this it follows that the coordinate system is inertial if there are no mass dependent forces.
The mass dependent forces are not real, they can be transformed away by changing to an *inertial* coordinate system.

There is one interesting exception to this - that *is* the gravitational force - where the acceleration is independent of the mass.

The reason for this is that the gravitational mass is equal to the inertial mass.

This is interesting because the origin of the inertial and gravitational masses are so different.

Reconciling the notion of inertial coordinate systems with gravity led to general relativity.
Examples

\[ \vec{\tau}'(t) = \vec{\tau}(t) + \vec{c} t^3 \]

This transformation corresponds to an accelerated reference frame.

If we look at Newton's second law in this frame

\[ m \frac{d^2 \vec{r}'}{dt^2} = m \frac{d^2 \vec{r}}{dt^2} + 6 \vec{c} t m \]

\[ = F (\vec{r}(t) - ct^3) + 6mc \vec{c} t \]

On the right side of this equation, \( F \) is the force as a function of the coordinates in the primed frame, and an additional force \( F = 6mc \vec{c} t \).

This is an inertial force because it depends on the mass – it is not a real force because it arises because of the choice of coordinates.
Example 2 - Rotating Coordinate System

Consider rotations about the z axis.

\[ \begin{align*}
\hat{x}' &= \hat{x} \cos(\omega t) + \hat{y} \sin(\omega t) \\
\hat{y}' &= \hat{y} \cos(\omega t) - \hat{x} \sin(\omega t)
\end{align*} \]

The axes in each coordinate system are related by

The coordinates of a vector \( \vec{a} \) in both coordinate systems are related by

\[ \begin{align*}
\vec{r} &= x' \hat{x} + y' \hat{y} + z' \hat{z} \\
x' &= \hat{x} \cdot \vec{r} = \hat{x} \cdot \vec{r} \cos(\omega t) + \hat{y} \cdot \vec{r} \sin(\omega t) \\
&= x \cos(\omega t) + y \sin(\omega t) \\
y' &= \hat{y} \cdot \vec{r} = \hat{y} \cdot \vec{r} \cos(\omega t) - \hat{x} \cdot \vec{r} \sin(\omega t) \\
&= y \cos(\omega t) - x \sin(\omega t)
\end{align*} \]
we write these as matrix equations

\[
\begin{pmatrix}
  x' \\
  y' \\
  z'
\end{pmatrix} =
\begin{pmatrix}
  \cos(\omega t) & \sin(\omega t) & 0 \\
  -\sin(\omega t) & \cos(\omega t) & 0 \\
  0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
  x \\
  y \\
  z
\end{pmatrix}
\]

\[
\begin{pmatrix}
  x \\
  y \\
  z
\end{pmatrix} =
\begin{pmatrix}
  \cos(\omega t) & -\sin(\omega t) & 0 \\
  \sin(\omega t) & \cos(\omega t) & 0 \\
  0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
  x' \\
  y' \\
  z'
\end{pmatrix}
\]

these have the form

\[
\begin{pmatrix}
  \ddot{\mathbf{r}}(t) \\
  \mathbf{\ddot{\mathbf{r}}}(t)
\end{pmatrix} =
\begin{pmatrix}
  \ddot{R}(t) & \dddot{R}(t) \\
  \dot{R} & \ddot{R} \\
\end{pmatrix}
\begin{pmatrix}
  \mathbf{\dot{r}}(t) \\
  \mathbf{\dot{r}}(t)
\end{pmatrix}
\]

\[
\frac{d\mathbf{\dot{r}}}{dt} = \frac{dR}{dt} \cdot \mathbf{\dot{r}}(t) + R \frac{d\mathbf{r}}{dt}
\]

\[
\frac{d^2\mathbf{\dot{r}}}{dt^2} = \frac{d^2R}{dt^2} \mathbf{\dot{r}}(t) + 2 \frac{dR}{dt} \frac{d\mathbf{r}}{dt} + R \frac{d^2\mathbf{r}}{dt^2}
\]

we write everything in terms of \( \mathbf{\dot{r}} \)

\[
\mathbf{\ddot{r}} = \mathbf{\ddot{r}} + \mathbf{\dddot{r}} \mathbf{\dddot{r}} + 2 \mathbf{\dddot{r}} \mathbf{\dddot{r}} (R \frac{d\mathbf{r}}{dt} - R \frac{d\mathbf{r}}{dt} \frac{d\mathbf{r}}{dt})
\]

\[
+ R \mathbf{\dddot{r}} (R \mathbf{\dot{r}})
\]

to compute this not:

\[
\frac{d\mathbf{r}}{dt} = \begin{pmatrix}
  -\omega \sin(\omega t) & \omega \cos(\omega t) & 0 \\
  -\omega \cos(\omega t) & -\omega \sin(\omega t) & 0 \\
  0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
  \cos(\omega t) & -\sin(\omega t) & 0 \\
  \sin(\omega t) & \cos(\omega t) & 0 \\
  0 & 0 & 1
\end{pmatrix}
\]
\[
\begin{pmatrix}
0 & \omega & 0 \\
-\omega & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}
\]

\[
\frac{dR}{dt} R^{-1} \begin{pmatrix}
x' \\
y' \\
z'
\end{pmatrix} = \begin{pmatrix}
0 & \omega & 0 \\
-\omega & 0 & 0 \\
0 & 0 & 0
\end{pmatrix} \begin{pmatrix}
x' \\
y' \\
z'
\end{pmatrix} = \begin{pmatrix}
\omega y' \\
-\omega x' \\
0
\end{pmatrix}
= -\omega \hat{Z} \times \vec{F}'
\]

\[
\frac{d^2 R}{dt^2} R^{-1} = -\omega^2 \begin{pmatrix}
\cos \omega t & \sin \omega t & 0 \\
-\sin \omega t & \cos \omega t & 0 \\
0 & 0 & 0
\end{pmatrix} \begin{pmatrix}
\cos \omega t & -\sin \omega t & 0 \\
\sin \omega t & \cos \omega t & 0 \\
0 & 0 & 0
\end{pmatrix}
\]

\[
\frac{d^2 R}{dt^2} R^{-1} = -\omega^2 \begin{pmatrix}
100 \\
010 \\
001
\end{pmatrix}
\]

\[R \vec{F} = \begin{pmatrix}
\cos \omega t & \sin \omega t & 0 \\
-\sin \omega t & \cos \omega t & 0 \\
0 & 0 & 0
\end{pmatrix} \begin{pmatrix}
F_x \\
F_y \\
F_z
\end{pmatrix}
\]

\[R \vec{F} = \begin{pmatrix}
F_x \cos \omega t + F_y \sin \omega t \\
F_y \cos \omega t - F_x \sin \omega t \\
F_z
\end{pmatrix}
\]

Putting everything together, the equations of motion in the rotating coordinate system are

\[
m \frac{d^2 \vec{r}}{dt^2} = \begin{pmatrix}
F_x \cos \omega t + F_y \sin \omega t \\
F_y \cos \omega t - F_x \sin \omega t \\
F_z
\end{pmatrix} - 2m \omega \hat{Z} \times (\vec{v} + \omega \hat{Z} \times \vec{r}) +
\]

\[-m \omega^2 \begin{pmatrix}
x' \\
y' \\
z'
\end{pmatrix}
\]
Note that
\[ 2m \omega \hat{z} \times (-\omega \hat{z} \times \vec{F}') = \]
\[ -2m \omega^2 (\hat{z} (\hat{z} \times \vec{r}) - \vec{r}) = 2m \omega^2 (\vec{x}') \]

This gives
\[ m \frac{d^2 \vec{r}'}{dt^2} = \vec{F} - 2m \omega \hat{z} \times \vec{v}' + m \omega^2 (\vec{x}') \]

(The Coriolis force is only active if the particle is moving in the rotating coordinate system.)

Both of these forces have mass dependence.