Lecture 22

Canonical Transformations

\[ \tilde{Z} = (\tilde{q}, \tilde{p}) \]

1. Preserve form of Hamilton equations
2. Preserve \[ \{ \tilde{Z}, \tilde{Z}_i \} = J_{ij} \]
3. \[ M = \frac{\partial}{\partial z_i} \] symplectic
4. Preserve \[ \omega = \sum_i dp_i \wedge dq_i = -\frac{1}{2} \sum_i J_{ij} dz_i \wedge dz_j \]

Construction - Method of generating functions

\( \Theta_1, \Theta_2 \) 1-forms

\[ d\Theta_1 = d\Theta_2 = \omega \]

\[ \Theta_1 - \Theta_2 = dF \]

Example

\[ \Theta_1 = -2 \tilde{q}_i dp_i \]

\[ \Theta_2 = 2 \tilde{p}_i dz_i \]

\[ -\sum q_i dp_i - \sum p_i dz_i = dF(\tilde{q}, \tilde{p}) = 2 \frac{\partial F}{\partial \tilde{p}_i} dp_i + 2 \frac{\partial F}{\partial \tilde{q}_i} dz_i \]

\[ -\tilde{q}_i = \frac{\partial F}{\partial \tilde{p}_i}; \quad -\tilde{p}_i = \frac{\partial F}{\partial \tilde{q}_i} \]

These equations can be solved for

\[ \tilde{q} = \tilde{q}(\tilde{p}, \tilde{\omega}) \]

\[ \tilde{p} = \tilde{p}(\tilde{q}, \tilde{\omega}) \]
the interesting case is

\[ \Theta_1 = \sum p_i dq_i \]
\[ \Theta_2 = -\sum q_i dp_i \]
\[ \Gamma = \sum p_i q_i \] generating function for the identity

\[ \Theta_1 - \Theta_2 = \sum p_i dq_i + \sum q_i dp_i = \sum p_i dq_i + q_i dp_i \]
\[ p_i = p_i \quad q_i = q_i \]

second case let \( F(q,p) \) be a function of \( q, p \) define

\[ q_i' = e^{\Delta F_i} q_i \]
\[ p_i' = e^{\Delta F_i} p_i \]

formally

\[ q_i' = q_i + \sum_{j \neq i} \{ F_{ij} q_j + \frac{1}{2!} \{ F_{ij} F_{ij} q_j \} + \frac{1}{3!} \{ F_{ij} F_{ij} F_{ij} q_j \} + \cdots \} \]
\[ p_i' = p_i + \sum_{j \neq i} \{ F_{ij} p_j + \frac{1}{2!} \{ F_{ij} F_{ij} p_j \} + \frac{1}{3!} \{ F_{ij} F_{ij} F_{ij} p_j \} + \cdots \} \]

while this is a series

\[ q_i(\lambda) = e^{\Delta F_i} q_i \]

\[ \frac{dq_i}{d\lambda} = \sum_{j \neq i} \{ F_{ij} q_j + \frac{1}{2!} \{ F_{ij} F_{ij} q_j \} + \frac{1}{3!} \{ F_{ij} F_{ij} F_{ij} q_j \} + \cdots \} \]
\[ \frac{dp_i}{d\lambda} = \sum_{j \neq i} \{ F_{ij} p_j + \frac{1}{2!} \{ F_{ij} F_{ij} p_j \} + \frac{1}{3!} \{ F_{ij} F_{ij} F_{ij} p_j \} + \cdots \} \]

\[ q_i = q_i(\lambda = 1) \]
we can also get the solution by solving the system of differential equations

\[ e_0(GH) = GH + \sum_{i=1}^{n} \mathcal{E}_i \mathcal{F}_1 G H^i + \sum_{i=2}^{n} \frac{1}{i!} \mathcal{E}_i \mathcal{F}_1 \mathcal{F}_2 G H^{i-2} + \ldots \]

claim

\[ D^n_F (GH) = \sum_{k=0}^{n} \frac{n!}{k! (n-k)!} D^k_G D^{n-k}_F H \]

check by induction

\[ D^n_F D^n_F (GH) = \sum_{k=0}^{n} \frac{n!}{k! (n-k)!} (D^k_F G D^{n-k}_F H + D^k_G D^k_F D^{n-k}_F H) \]

= \sum_{k=0}^{n} \frac{n!}{k! (n-k)!} \left( D^k_F G D^{n-k}_F H + \sum_{k'=0}^{k} \frac{k!}{k'! (n-k')!} \mathcal{F}_1 \mathcal{F}_2 D^{n-k-k'}_F H \right)

let \( k' = k+1 \)

\[ \sum_{k=0}^{n} \frac{n!}{k! (n-k)!} \left( D^k_F G D^{n-k}_F H + \sum_{k'=0}^{k} \frac{k!}{k'! (n-k')!} \mathcal{F}_1 \mathcal{F}_2 D^{n-k-k'}_F H \right) \]

= \sum_{k=0}^{n} \frac{n!}{(k+1)! (n-k)!} \left( D^k_F G D^{n-k}_F H + \sum_{k'=0}^{k} \frac{n!}{k'! (n-k')!} \mathcal{F}_1 \mathcal{F}_2 D^{n-k-k'}_F H \right)

= \sum_{k=0}^{n} \frac{n!}{(k+1)! (n-k)!(n-k-1)!} \left( D^k_F G D^{n-k}_F H + \sum_{k'=0}^{k} \frac{n!}{k'! (n-k')!} \mathcal{F}_1 \mathcal{F}_2 D^{n-k-k'}_F H \right)

= \sum_{k=0}^{n} \frac{n!}{(k+1)! (n-k-1)!} \mathcal{F}_1 \mathcal{F}_2 D^{n-k}_F H + \sum_{k=0}^{n} \frac{n!}{(k+1)! (n-k)!} \mathcal{F}_1 \mathcal{F}_2 D^{n-k}_F H

= \sum_{k=0}^{n} \frac{n!}{(k+1)! (n-k)!} \left( \mathcal{F}_1 \mathcal{F}_2 D^{n-k}_F H + \mathcal{F}_1 \mathcal{F}_2 D^{n-k}_F H \right)

= \sum_{k=0}^{n} \frac{n!}{(k+1)! (n-k)!} \mathcal{F}_1 \mathcal{F}_2 D^{n-k}_F H + \sum_{k=0}^{n} \frac{n!}{(k+1)! (n-k)!} \mathcal{F}_1 \mathcal{F}_2 D^{n-k}_F H

= \sum_{k=0}^{n} \frac{n!}{(k+1)! (n-k)!} \left( \mathcal{F}_1 \mathcal{F}_2 D^{n-k}_F H + \mathcal{F}_1 \mathcal{F}_2 D^{n-k}_F H \right)

= \sum_{k=0}^{n} \frac{n!}{(k+1)! (n-k)!} \mathcal{F}_1 \mathcal{F}_2 D^{n-k}_F H
\[ \sum _{k=0}^{n+1} \frac{n! \cdot k! + n! \cdot (n+1-k)!}{k! \cdot (n+1-k)!} \cdot D_f^k \cdot G \cdot D_p^{n+1-k} \cdot H + D_f^k \cdot G \cdot H + G \cdot D_p^{n+1} \cdot H \]

This completes the induction.

\[ E(G \cdot H) = \sum _{k=0}^{n} \sum _{m=0}^{n} \frac{n!}{k! \cdot (n-k)!} \cdot D_f^k \cdot G \cdot D_p^{n-k} \cdot H \]

Let \( m = n-k \rightarrow k \quad m: 0 \rightarrow n \quad k: 0 \rightarrow n \quad m+k = n \)

\[ = \sum _{k=0}^{n} \sum _{m=0}^{n} \frac{1}{k!} \cdot D_f^k \cdot G \cdot \sum _{m=0}^{n} \frac{1}{m!} \cdot D_p^m \cdot H. \]

Which gives the result

\[ E(D_f \cdot G \cdot H) = (E(G \cdot H))(E(D_f \cdot H)). \]

We also have

\[ \sum _{k=0}^{n} \sum _{m=0}^{n} \frac{1}{k!} \cdot D_f^k \cdot G \cdot \sum _{m=0}^{n} \frac{1}{m!} \cdot D_p^m \cdot H. \]

This satisfies the same type of chain rule as the above.
It follows that

\[ D_F \{ e, H \} = \{ e \} \]  
\[ D_F \{ e \} \subseteq H \]

we are now in a position to show that

\[ Z_i = e_z \]

is a canonical transformation

\[ \{ Z'_i, Z'_j \} = \{ e^{D_F} Z_i, e^{D_F} Z_j \} = \]

\[ e^{D_F} \{ Z_i, Z_j \} = e^{D_F} J_{ij} = \]

\[ J_{ij} + \{ F, J_{ij} \} + \cdots \]

0 because \( J_{ij} \) is a constant

So the method of solving:

\[ \frac{dz_i}{dx} = \{ F, z_i(x) \} = \frac{\partial F}{\partial z_i} J_{ij} \frac{dz_j(x)}{dx} \]

\( z_i(x) \) defines a canonical transformation
an immediate corollary of this result is that

\[ Z_i' = Z_i(t) = e^{-tD_H} Z_i(w) = e^{tD_H} Z_i(0) \]

is a canonical transformation.

\[ p_i' = p_i(H) \]
\[ q_i' = q_i(H) \]

where

\[ \dot{p}_i = -\frac{\partial H}{\partial q_i}; \]
\[ \dot{q}_i = \frac{\partial H}{\partial p_i}; \]

is canonical.

example - harmonic oscillator

\[ H = \frac{p^2}{2m} + \frac{1}{2} k q^2 \]
\[ \dot{p} = -kq \]
\[ \dot{q} = \frac{p}{m} \]
\[ \ddot{p} = -k \dot{q} = -\frac{k}{m} p \]
\[ \ddot{q} = \frac{\dot{p}}{m} = -\frac{k}{m} q \]

\[ q(t) = q(0) \cos \omega t + c \sin (\omega t) \]
\[ p(t) = m \dot{q} = -mw q(t) \sin \omega t + mc \omega \cos \omega t \]
\[ = 0 \]
\[ c = p(0)/\omega m \]
\[ q(t) = q_0 \cos \omega t + \frac{p(0)}{m\omega} \sin \omega t \quad \omega = \sqrt{\frac{k}{m}} \]

\[ p(t) = p(0) \cos \omega t - m\omega q(0) \sin \omega t \]

to check to see if this is canonical, we compute the canonical 2-form

\[ dp(t) \wedge dq(t) = \left( dp(0) \cos \omega t - m\omega dq(0) \sin \omega t \right) \wedge \left( dq(0) \cos \omega t + \frac{dp(0)}{m\omega} \sin \omega t \right) \]

\[ = dp(0) \wedge dq(0) \cos^2 \omega t - dq(0) \wedge dp(0) m\omega \sin^2 \omega t \]

\[ = dp(0) \wedge dq(0) (\cos^2 \omega t + \sin^2 \omega t) \]

\[ = dp(0) \wedge dq(0) \]

This shows that time evolution is a canonical transformation.

Consider

\[ \int_V dp(0) \wedge dq(0) \]

change variables

\[ p = p(t, \dot{q}) \quad q = q(t, \dot{p}) \]

\[ \frac{dp}{dt} = \frac{\partial p}{\partial q} \frac{dq}{dt} = \frac{\partial p}{\partial \dot{q}} \frac{d\dot{q}}{dt} \]

\[ p_0 = e^{\frac{d\dot{q}}{dt}} p \quad q_0 = e^{\frac{dq}{dt}} \]

This shows that the 2-dimensional phase space volume for the oscillator is preserved.
In 2n degrees of freedom:

\[ \sum_{\omega} \sum_{\omega} \ldots \sum_{\omega} \ldots \]

Integrals over all of these 2,4,6- dimensional surfaces are unchanged with respect to time evolution.

Remove \( \omega \) to

\[
\left( dp_1 dq_1 + dp_2 dq_2 \right) \ldots \left( dp_1 dq_1 + dp_2 dq_2 \right) = 2 \left( dp_1 dq_1 + dp_2 dq_2 \right) \]

In general,

\[ L' \omega \]

is up to a constant the volume in phase space. We have shown that this volume is preserved with respect to time evolution.
Application: The Poincaré recurrence theorem

Consider a system of a large number of interacting particles in a box of finite volume.

Assume that the particles have initial conditions:

\[ q_i(0), p_i(0) \quad \ldots \quad q_n(0), p_n(0) \]

Assume that there is no dissipation; pick \( \epsilon_q, \epsilon_p \) arbitrarily small with dimensions of momentum and position.

Consider the following volume in phase space:

\[ V_\epsilon = \{ q_i(0) - \epsilon_q < q_i < q_i + \epsilon_q, \]
\[ p_i(0) - \epsilon_p < p_i < p_i + \epsilon_p \}

Then there is a point in \( V_\epsilon \) that returns to \( V_\epsilon \) in a finite amount of time—no matter how small we make \( \epsilon \).
pick a fixed $\Delta t$ and define

$$V_n = e^{-D_H \Delta t} V_{n-1}$$

1) each $V_n$ has the same volume

$$\left[ \left( \frac{4}{3} \pi e_n^3 \right) \left( \frac{4}{3} \pi e_r^3 \right) \right]^N$$

2) the available phase space has a finite volume $(6N \text{ dim})$

3) if $V_n$ cannot all be disjoint because they have the same volume and the total volume must be less than the volume of the phase space.

there must be a $k$ such that

$$V_n \cap V_{n+k} \neq \emptyset \quad k < \alpha$$

since this is a finite volume and is preserved under time evolution

$$V_0 \cap V_\alpha \neq \emptyset$$

This means that often $t = k\Delta t$ a point (initial cond.) in the original volume returns $t$ within $\epsilon$ of its starting point in a finite amount of time.
Note that this means that every particle arbitrarily close to its own initial position and momentum

while this result holds for any finite system - the time for recurrence is very large

Hamilton Jacobi Theory

Since time evolution is given by a canonical transformation - can we find a generating function for this transformation

properties

1. It must depend on time and know something about the Hamiltonian

2. It should become the identity at time 0

These considerations indicate a generating function of the form

\[ S(q, p, t) \]

where \( q(t) \), \( p(t) \) are solutions of Hamilton equations and \( q, p \) are initial coordinates and momenta
To relate this to H consider Hamilton's principle

\[ SA = S \int \left( \sum p_i \dot{q}_i - H(q, p) \right) dt = 0 \]

\[ S \left( \sum p_i dq_i - H dt \right) dt = 0 \]

The initial conditions do not move
\[ \dot{q} = \dot{p} = 0 \Rightarrow H = 0 \]

\[ SA' = S \left( \sum p_i dq_i - 0 dt \right) - \]

\[ S \left( -\sum q_i dp_i - 0 dt \right) \]

Both will satisfy Hamilton's eq. if the Lagrangian differs by a total time derivative

\[ \sum p_i dq_i - H dt + 2 \dot{q} \cdot dp_i = dF(q, p, t) \]

\[ p_i = \frac{\partial F}{\partial q_i}, \quad q_i = \frac{\partial F}{\partial p_i}, \quad H = -\frac{\partial F}{\partial t} \]

This by using this time dependent generating function, we get an equation involving the Hamiltonian

\[ H(q, p) = -\frac{\partial F}{\partial t} \]

\[ -H(q, \frac{\partial F}{\partial q}) = -\frac{\partial F}{\partial t} (q, p, t) \]
The equation in the box is called the Hamilton-Jacobi equation.

Example - harmonic oscillator

\[ H = \frac{p^2}{2m} + \frac{1}{2}kq^2 \]

\[ \frac{1}{2m} \left( \frac{\partial F}{\partial q} \right)^2 + \frac{1}{2}kq^2 = -\frac{\partial F}{\partial t} \]

Since \( H \) has no explicit time dependence, we try

\[ F(q,p,t) = F_1(q,p) + F_2(p,t) \]

This gives

\[ \frac{1}{2m} \left( \frac{\partial F_1}{\partial q} \right)^2 + \frac{1}{2}kq^2 = -\frac{\partial F_2}{\partial t} \]

Consistency requires that both sides can only be functions of \( p \)

\[ \frac{1}{2m} \left( \frac{\partial F_1}{\partial q} \right)^2 + \frac{1}{2}kq^2 = C(p) \]

\[ -\frac{\partial F_2}{\partial t} = C(p) \]

The second equation has the solution

\[ F_2 = -C_1(p) + C_2(p) \]

The Hamilton equation has the form

\[ \left( \frac{\partial F_1}{\partial q} \right)^2 = 2m(C(p) - \frac{1}{2}kq^2) \]
since \( \frac{\partial F}{\partial q} = p \) this equation has the form

\[ p^2 = 2mc(p) - mkq^2 \]

we define the constant \( \sqrt{p^2 - mkq^2} \) (this gives the constant the same units as \( p^2 \)) with this definition

\[ \frac{\partial F}{\partial q} = \sqrt{p^2 - mkq^2} \]

\[ F_1 = \int \sqrt{p^2 - mkq^2} \, dq \]

\[ F_2 = -c(p) \, t = -\frac{p^2}{2m} \, t \]

\[ F = \int_q^q \sqrt{p^2 - mkq^2} \, dq = \frac{p^2}{2m} + c \]

from this we get

\[ p = \frac{\partial F}{\partial q} = \sqrt{p^2 - mkq^2} \]

\[ Q = \frac{\partial F}{\partial p} = \int_q^q \frac{p}{\sqrt{p^2 - mkq^2}} \, dq = \frac{p}{m} \, t + c \]

\[ = \int_q^q \frac{dq}{\sqrt{1 - \frac{mkq^2}{p^2}}} \]
\[ \begin{align*}
\text{Let } \sin \theta &= u = \sqrt{\frac{q}{l}} \\
\cos \theta \, dq &= \frac{q}{\sqrt{4mkq^2 - P^2}} \\
\int_{q_0}^{q} \frac{dq}{\sqrt{1 - \frac{m^2q^2}{l^2}}} &= \int_0^\phi \frac{\cos \theta \, d\theta}{\cos \theta} = \frac{P}{\sqrt{mk}} \sin^{-1}\left(\sqrt{mk \frac{q}{l}}\right) \\
Q &= \frac{P}{\sqrt{mk}} \sin^{-1}\left(\sqrt{mk \frac{q}{l}}\right) - \frac{P}{m} + \\
\frac{\sqrt{mk}}{P} \left(\omega + \frac{P}{m} t\right) &= \sin^{-1}\left(\sqrt{mk \frac{q}{l}}\right) = \frac{\sqrt{mk}}{P} Q + \sqrt{\frac{P}{m}} t \\
q(t) &= \frac{P}{\sqrt{mk}} \sin\left(\sqrt{\frac{k}{m}} t + \sqrt{\frac{mk}{P}} Q\right) \\
&= \frac{P}{\sqrt{mk}} \sin\left(\sqrt{\frac{k}{m}} t + \phi\right)
\end{align*} \]

Note that the new canonical variables are not the initial momentum and position --- but they are related to the initial conditions.