Lecture 24

review constraint dynamics

1. \( \det \left( \frac{\partial^2 L}{\partial \dot{q}_i \partial \dot{q}_j} \right) = 0 \)

   same as \( \det \left( \frac{\partial^2 L}{\partial q_i \partial \dot{q}_j} \right) = 0 \)

\[ \text{x} \quad \text{when the determinant is never 0 it is always possible to solve for } \dot{q}_i, (p,q) \]

\[ \text{x} \quad \text{in general} \]

(1) there are still \( N \) \( p_i = \frac{\partial L}{\partial \dot{q}_i} \)

(2) Lagrange's equations still hold

\( \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_i} \right) - \frac{\partial L}{\partial q_i} = 0 \)

\[ \text{x} \quad \text{when the determinant vanishes the } p_i \text{ are no longer independent functions of the } \dot{q}_i \text{'s. This leads to relations of the form} \]

\( \phi_i(\rho,\dot{q}) = 0 \quad i=1 \ldots N \)

These are called primary constraints. They come from \( L \).
if we consider

$$SA = \hat{S} \sum (2 \ p_i \dot{q}_i - L)$$

$$= \hat{S} \sum (s_i \dot{q}_i + p_i \dot{q}_i - \frac{\partial}{\partial q_i} s_i q_i - \frac{\partial}{\partial \dot{q}_i} s_i \dot{q}_i)$$

$$= \hat{S} \sum (s_i \dot{q}_i + p_i s_i q_i - p_i s_i \dot{q}_i - p_i \dot{s}_i)$$

because we still have Lagrange's equation

$$- \hat{S} \sum (s_i \dot{q}_i - p_i \dot{s}_i)$$

This means that the quantity

$$H = \hat{S} \sum p_i \dot{q}_i - L$$

only depends on $p_i \dot{q}_i$.

However, since $\dot{\phi}_i (p_i q_i)$ is always 0, we define

$$\hat{H} = \hat{S} \sum p_i \dot{q}_i - L + \hat{S} \sum c_m \phi_m$$

where $c_m (\tilde{p} q_i)$ can be any function of $p_i q_i$ and $\hat{H}$ has the same value as $H$ provided the constraints are satisfied.
\[ \phi_m = 0 \quad 8\phi_m = 0 \]

\[ 8 \tilde{H} = 8H \]

provided the constraints hold using Lagrange multipliers

\[ \dot{q}_i = \frac{\partial H}{\partial q_i} + \sum_{m} \mu_m \frac{\partial \phi_m}{\partial q_i} \]

\[ \dot{\mu}_m = -\frac{\partial H}{\partial \mu_m} - \sum_{i} \mu_m \frac{\partial \phi_m}{\partial q_i} \]

where here the \( \mu_m \) are chosen to make the coefficients of the dependent variables vanish at this point the Lagrange multipliers are unknown.

we have

\[ \begin{align*}
\frac{dF}{dt} &= \{ F, H \} + 2\sum_{m} \mu_m \phi_m \\
&= \sum_{i} \phi_i H_i + \sum_{m} \{ F, \mu_m \} \phi_m + 2\sum_{m} \mu_m \phi_m
\end{align*} \]

vanish if constraints are applied (often computing bracket)
\[
\frac{d\theta}{dt} = \sum \phi_i H_i + \sum \sigma_i \theta_i \phi_i
\]

to determine \( \theta_i \)

\[
\frac{d\phi_i}{dt} = \sum \phi_k H_j + \sum \sigma_i \theta_i \phi_k \phi_i = 0
\]

these can be considered as equations for the \( \theta_i \)

4 cases

(i) equation makes no sense \( L = 0 \),

   can happen with a bad Lagrangian \( L = \theta \) \rightarrow 

   Lagranges eq give \( L = 0 \)

(ii) if Lagranges equations make sense then

   (a) \( L = 0 \)

   (b) equation independent \( i \)

   \[ \phi_k H_j = \beta \neq 0 \]

   when \( \beta \neq \sum \lambda_i \phi_i \)

   (a new constraint independent of the \( \phi_i \) )

   secondary constraints - require Lagranges equations
(a) The equations are consistent.

*When (b) occurs we call \( \mathcal{K}(\rho q) \) to the set of constraints — and repeat.*

This continues until

\[
\sum_m \phi_m H \frac{\partial H}{\partial \phi_m} u_m = 0
\]

where the \( \phi_m \) now include the secondary constraints.

This is a system of linear equations for the \( u_m \)

\[
M_{mn} = \sum_k \phi_m \frac{\partial \phi_n}{\partial k}
\]

\[
u_m = \sum_k \phi_m \frac{\partial H}{\partial k}
\]

\[
M u = \nu
\]

In general \( M \) will not have an inverse. The general solution has the form

\[
u_m = F_m(\rho q) + \sum_k \lambda_k \nu^k_m(\rho q)
\]

where

\[
\mathcal{M}(\rho q) F(\rho q) = \nu(\rho q)
\]

\[
\mathcal{M}(\rho q) \nu(\rho q) = 0
\]
so the general form of $H$ can be written

$$u_m = F_m + 2 \lambda \epsilon u_m^k$$

$$H = H + 2 F_m \phi_m + \sum \lambda \epsilon u_m^k \phi_m$$

It is useful to make the separation

$$H' = H + 2 \frac{F_m(qp) \phi_m(qp)}{u_m}$$

$$\bar{\Phi}_k = \sum u_m^k(pq) \phi_m(pq)$$

$$\bar{H} = H' + 2 \lambda \epsilon \bar{\Phi}_k$$

where arbitrary

when the constraints are applied

$$\bar{\Phi}_k = 0$$ and $H' = H$

The equations of motion are still

$$\frac{d\bar{H}}{dt} = \{\bar{H} H' \} + 2 \lambda \epsilon \{\bar{\Phi}_k \}$$

In this case

$$\frac{d\bar{\Phi}_k}{dt} = \{\bar{\Phi}_k H' \} + 2 \lambda \epsilon \{\bar{\Phi}_k \}$$

$$\{\bar{\Phi}_k H' \} = 2 u^k \{\phi_m \} (H + 2 F_m \phi_m)$$

$$= 2 u^k \{\phi_m H' \} + 2 F_m \phi_m$$

$$= 0 \text{ because } \bar{H}$$

is a specific sol of this eq
This gives
\[ \sum_k \Phi_k \Phi_e = \sum_{m} \Phi_m \Phi_e \]
This vanishes after applying the constraint because
\[ \sum \Phi_m \Phi_e \Phi_n \]
was constructed to be 0.

This shows that
\[ H = H' + \sum \lambda_e \Phi_e \]
\[ \sum \lambda_e \Phi_e \equiv 0 \quad \text{after applying the constraint} \]
\[ \sum \Phi_e \Phi_e \equiv 0 \]

In general any function \( G(q_0) \)
satisfying
\[ \sum G \Phi_e \equiv 0 \quad \text{for all } a \quad \text{after applying the constraint} \]
is called first class.

This means
\[ \sum G \Phi_e = \sum G_m \Phi_m \]

Note that if \( A, B \) are first class then
\[ \sum A \Phi_n = \sum B \Phi_n \]
\[ \sum A \Phi_n = \sum B \Phi_n \]
\[ \sum A \Phi_n - \sum B \Phi_n \]
\[ = - \sum (\Phi_n A - \Phi_n B) \]
which vanish after applying the constraint.
We get

1. The equations of motion become

\[ \frac{d\mathbf{s}}{dt} = \{ \mathbf{s}, \mathbf{H}' \} + \sum_{r} \lambda_{r} \{ \mathbf{s}, \phi_{r} \} \]

where the \( \lambda_{r} \) are arbitrary.

2. \( D_{\phi_{n}} D_{\phi_{k}} F = D_{\phi_{k}} D_{\phi_{n}} F \)

This holds when the constraints are enforced.

\[ e^{i \pi \phi_{n}} \]

generates transformations that leave \( \mathbf{H}' \) and all of the other \( \phi_{r} \) unchanged.

These are the classical mechanics versions of \( \xi \)-gauge transformations.

This shows that the Hamiltonian is constant on \( K \)-dimensional surfaces where \( K \) is the number of \( \phi_{r} \).

We can get a single valued \( \mathbf{H} \) by fixing a point on the \( K \)-dim surface — the reduces the phase space to \( 2N-2K \) degrees of freedom.
Consider a harmonic oscillator Hamiltonian
\[ H = \frac{1}{2} \left( p^2 + q^2 \right) \]

since
\[ \{ H, H \} = 0 \quad \frac{dH}{dt} = 0 \]

so \( H \) is a constant of motion
\[ E = H = \frac{1}{2} \left( p^2 + q^2 \right) \]

Consider a canonical transformation where the new generalized momentum is \( E = (p=E) \)
\[ S(q,p) \]
\[ p \, dq + q \, dp = S(q,p) = \frac{\partial S}{\partial q} \, dq + \frac{\partial S}{\partial p} \, dp \]
\[ p = \frac{\partial S}{\partial q} \quad q = \frac{\partial S}{\partial p} \]

to find \( S \) let
\[ p^2 = 2E - q^2 \]
\[ p = \sqrt{2E - q^2} = \sqrt{2E - q^2} \]
define
\[ S(q,E) = \int^q \sqrt{2E - q^2} \, dq \]
This is constructed so

\[ p = \frac{\partial S}{\partial q} \]

\[ \omega = \frac{\partial S}{\partial p} = \frac{\partial S}{\partial E} = \frac{2}{\sqrt{2E - q^2}} \int \sqrt{2E - q^2} \, dq \]

\[ = \int \frac{\frac{1}{2} \cdot 2}{\sqrt{2E - q^2}} \, dq \]

Let \( q = \sqrt{2E \sin \theta} \), \( dq = \sqrt{2E \cos \theta} \, d\theta \)

\[ \int \frac{\sin^2 \left( \frac{\theta}{\sqrt{2E}} \right)}{\sqrt{2E \cos \theta} \, d\theta} = \sin \left( \frac{\theta}{\sqrt{2E}} \right) = \theta \]

We see that \( (p, \omega) = (E, \theta) \) \( \theta = \sin \left( \frac{\theta}{\sqrt{2E}} \right) \)

are canonical variables

\[ \dot{\theta} = \frac{\partial H}{\partial p} = \frac{\partial H}{\partial E} = 1 \]

\( Q(t) = t \)

\[ q(t) = \sqrt{2E \sin (t)} \]

\[ p(t) = \sqrt{2E - q^2} = \sqrt{2E} \sqrt{1 - \sin^2 (t)} \]

\[ = \sqrt{2E} \cos (t) \]

The nice lecture in this example is that when one of the canonical variables is a constant of motion we get a simple solution.
Definition

A mechanical system with $2N$ degrees of freedom is called completely integrable if there are $N$ functions $F_i(\mathbf{q}, \mathbf{p})$ satisfying

1. $\{H, F_i\} = 0 \quad i = 1, \ldots, N$
2. $\{F_i, F_j\} = 0 \quad i, j = 1, \ldots, N$
3. $dF_1 \wedge dF_2 \cdots \wedge dF_N \neq 0$ (neven 0)

meaning of these conditions

1. means that each $F_i$ is a constant of motion - i.e. $\frac{dF_i}{dt} = 0 \quad i = 1, \ldots, N$

2. means that $F_i$ is a constant of motion corresponding to the Hamiltonian $F_j$ ($y = i$)

3. means that the functions $F_i$ are independent at every point - i.e. the vectors

$$\left( \frac{\partial F_i}{\partial q_j}, \frac{\partial F_i}{\partial p_i} \right) = v_i$$

are independent
under these conditions the solution of the equations of motion can be reduced to integration

**Step 1** Define the following subsets of 2N dimensional phase space

$$ M_{\bar{s}} = \left\{ (q, p) \mid F_j(q, p) = s_j \right\} $$

where \((\bar{s}_1, \ldots, \bar{s}_N)\) is a constant 2N dimensional vector.

**Step 2** At any point on \( M_{\bar{s}} \) there are \(N\) linearly independent tangent and normal vectors.

**Normal vector**

$$ \bar{n}_i = \left( \frac{\partial F_i}{\partial q_1}, \frac{\partial F_i}{\partial q_2}, \ldots, \frac{\partial F_i}{\partial q_N}, \frac{\partial F_i}{\partial p_1}, \ldots, \frac{\partial F_i}{\partial p_N} \right) $$

To show independence assume there are constants \(\alpha_i\) such that

$$ \sum \alpha_i \bar{n}_i = 0 \quad \text{at some point on } M_{\bar{s}} $$

Note \( dF_i = 2 \alpha_i \bar{n}_i d\bar{s}_i \) \((\bar{s}_i = (q, p))\)

This means

$$ \sum \alpha_i dF_i = \sum \alpha_i \bar{n}_i d\bar{s}_i = 0 $$
If this is true for \( \bar{\mathbf{q}} \neq 0 \), there is at least one component (\( \lambda_i \neq 0 \))

\[
dF_i = -\frac{1}{\lambda} \sum_{j \neq i} \alpha_j dF_j
\]

\[
dF_i \lambda = \lambda \left( -\frac{1}{\lambda_i} \sum_{j \neq i} \alpha_j dF_j \right) \lambda dF_{i-1} \quad \text{and} \quad dF_i = 0
\]

since \( dF_j \) in \( dF_i \) causes this to be - contradicting the third condition

\[
\lambda_i dF_i = 0 \quad i=1
\]

**Tangent Vectors**

To find the tangent vector consider

\[
\dot{q}_i = \frac{\partial F_j}{\partial p_i}; \quad \{ \text{treat } F_i \text{ as a Hamiltonian} \}
\]

\[
\dot{p}_i = -\frac{\partial F_j}{\partial q_i}
\]

\[
\dot{z}_i = \sum_{j,k} J_{ik} \frac{\partial F_j}{\partial z_k}
\]

Consider an initial point \( \bar{z}_0 \) on \( M_3 \). The condition \( \sum F_i F_j \bar{z}_j = 0 \) means that each of the \( F_i \) can conserve quantities with respect to letting any one of them be a Hamiltonian. This means that they leave the value of \( \bar{z} \) unchanged - so the solutions remain on \( M_3 \).
This means that each of the vectors

$$\mathbf{t}_i = \left( \frac{\partial F_i}{\partial q_j} - \frac{\partial F_i}{\partial p_j} \right) = J_{jk} \frac{\partial F_i}{\partial z_k}$$

are tangent to the surface $M$.

Again we assume by contradiction that

$$\sum a_i \mathbf{t}_i = 0 \quad \text{with} \quad a \neq 0$$

$$\sum a_i \mathbf{t}_i = \sum a_i J_{jk} \frac{\partial F_i}{\partial z_k} = 0$$

$$= J_{jk} \left( \sum a_i \frac{\partial F_i}{\partial z_k} \right)$$

since $J$ has an inverse ($-J$) multiply on the left by $(-J)$ to get

$$0 = \sum a_i \frac{\partial F_i}{\partial z_k} = 0$$

This gives $\sum a_j = 0$,

$$\frac{\partial F_j}{\partial z_k} = - \sum a_j \frac{\partial F_i}{\partial z_k}$$

multiplying by $dz_k$

$$dF_j = - \sum a_j \frac{\partial F_i}{\partial z} dF_i$$

which in contradicts $\sum_i dF_i$, to
This proves that any point on \( M^3 \) has \( N \) independent normal vectors and \( N \) independent tangent vectors.

Next we make the additional assumption that in a given set of constants of motion \( \vec{\phi} \), that \( M^3 \) is a bounded set.

(Note that this does not have to happen - it depends on the system.)

Note - obviously the Hamiltonians \( F_i \) provide a means to move continuously on \( M^3 \), but it can happen that \( M^3 \) has disconnected parts.

**Theorem:** Each connected component of \( M^3 \) is an \( N \) torus.
What do we mean by an N torus -
It means we can find periodic coordinates on \( M^4 \) so any point on \( M^4 \) can be expressed as

\[
(\varphi P) = (\vartheta_1, \ldots, \vartheta_0)
\]

\[
\vartheta_i + 2\pi i = \vartheta_i;
\]

Step 1 - Construct the coordinate system
consider Hamilton's equation

\[
\frac{dz_j^i}{dt_j} = \sum_{j=1}^{N} J_{kR} \frac{\partial F_j}{\partial z_j^R}
\]

Each of these differential equations satisfies a local existence theorem

(a) Pick a starting point on \( M^4 \)

\( (q_0, p_0) = \bar{z}_0 \)

(b) Let \((t_1, t_2)\) be the times associated with each of the \( N \) equations

(c) Note

\[
D_{F_1} D_{F_2} Z - D_{F_2} D_{F_1} Z = \\
\{ F_1, \{ F_2, Z \} \} - \{ F_2, \{ F_1, Z \} \} = \\
\{ F_1, \{ F_2, Z \} \} + \{ F_2, \{ Z, F_1 \} \}
\]
\[ \mathbf{z} = \{ F_i, F_j \} = 0 \]

Since \( \{ F_i, F_j \} = 0 \)

This means

\[ \mathbf{-D}_{F_i,t_1} \mathbf{-D}_{F_j,t_2} \mathbf{z} = \mathbf{0} \]

\[ \mathbf{z} = \mathbf{E} \mathbf{E} \mathbf{E} \mathbf{z}_0 \]

we get to the same final point independent of the order.

For sufficiently small \( t_1, t_2 \)

we can label points near \( \mathbf{z}_0 \)

by

\[ \mathbf{z}(t_1, t_2) = \mathbf{E} \mathbf{E} \mathbf{z}_0 \]

For sufficiently small \( \mathbf{t} \), because

the tangent vectors are independent

each \( \mathbf{t} \) corresponds to a different point

\[ \mathbf{d} t_1 \neq \mathbf{d} t_2 \]

\[ \mathbf{-D}_F \mathbf{d} t_1 \mathbf{-D}_F \mathbf{d} t_2 \mathbf{z} = \mathbf{0} \mathbf{z}_0 \]

Since there are \( \mathbf{N} \) independent vectors,

and \( \mathbf{N} \) normals \( \mathbf{M}_F \) is \( \mathbf{N} \) dimensional.

This means that the times \(( t_1, t_2) \)

define local coordinates in a neighborhood of each point on \( \mathbf{M}_F \).
Next we appeal to some math from point set topology.

If $M_{f}$ is a closed bounded set then any collection of open sets that cover $M_{f}$ can be replaced by a finite sub cover.

closed means that is $Z_{n} \to Z$ and $Z_{n} \in M_{f}$, $Z \in M_{f}$

$F_{i}(Z_{n}) = f_{i}$, $\lim_{Z_{n} \to Z} F_{i}(Z_{n}) = f_{i}$

which is true if $F_{i}(Z)$ is continuous.

open means $\sum t_{i} < 6$

This means that we can

we can patch solutions in each of a finite # of open sets to get global solutions on all of $M_{f}$.

This means

$(t_{1}, t_{r})$

can be extended to $-\infty \to \infty$
\((t_1, t_\omega)\) \(\colon \mathbb{R}^n \to M_\delta\)

\(M_\delta\) is bounded, \(\mathbb{R}^n\) is not.
This means that there are multiple times corresponding to the same point in \(M_\delta\).

Since the tangents are independent, they are not in the local region near \(\overline{o}\).

More is a shortest vector satisfying:
\[\overline{o} = \overline{e}_1 = (t_1, t_\omega)\]

Replace
\[\overline{e}_1 \to \frac{2\pi}{1} \overline{e}_1\]

Next we take \(n-1\) vectors \(1\) to \(t\), and use
\[(\Theta_1, t_2, t_\omega, \ldots)\]

Set \(\Theta_1 = 0\) - what remains \(\colon \mathbb{R}^{n-1} \to M_\delta\)
Again there must be a shortest time vector.
\[\overline{e}_2 \to \Theta_1 \frac{2\pi}{1} \overline{e}_2\]

We can keep doing this until we run out of dimensions:

\[M_\delta = (\Theta_1, \Theta_2) \quad \Theta_1 = \Theta_1 + 2\pi\]
This completes the proof that each connected component of $M_\S$ is an $N$ torus.

We generalize what we did in the harmonic oscillator

$$F_i (\tilde{q}, \tilde{p}) = \tilde{s}_i \quad \text{(defines $M_\S$)}$$

we solve for $\tilde{p}$ as a function of $\tilde{q}$

$$\tilde{p}_i = \tilde{p}_i (\tilde{q}, \tilde{p})$$

we replace $\tilde{s}$ by another set of constants

$$I_j (\tilde{q}) \equiv \frac{1}{2\pi} \int_{\tilde{q}} 2 \tilde{s}_i (\tilde{q}, \tilde{p}). dq_i$$

where we integrate the canonical one-form on $M_\S$, around each of the $N$ directions on the torus

$$\tilde{y}_i (h) = (0, 0, 0, 0) \quad \text{(periodic coord on torus)}$$

we use the $I_j (\tilde{q})$ as generalized momenta replacing $\tilde{s}$

$$S (\tilde{q}, \tilde{p}) = \int q_i \tilde{s}_i (\tilde{q}, \tilde{p}). dq_i$$
with this generating function

\[ p_i = \frac{\partial S}{\partial q_i} = p_i(q, \tilde{\theta}) \]

\[ q_i = \frac{\partial S}{\partial q_i} = \int q \frac{\partial}{\partial q_i} \sum p_i(q, \tilde{\theta}(\tilde{x})) \, dq \]

In order to interpret the \( q_i \), consider the change in \( q_i \):

If we integrate around the loop \( \gamma_j \)

\[ \Delta q_i = \frac{2}{\partial q_i} \int_{\gamma_j} \sum p_i(q, \tilde{\theta}(\tilde{x})) \, dq \]

\[ = \frac{2}{\partial q_i} \cdot 2\pi I_j = s_{ij} \cdot 2\pi \]

This shows that \( q_i \) increases by \( 2\pi \) along the \( i \) curve, but does not change along the \( j \) curve. This returns back to the same point - the integral \( x \) returns to the same point so the \( q_i \) are periodic.

The \( q_i \) are simply the independent angles on the torus.
since $I_i(t) = I_i(0)$ we have

$$\dot{q}_i = -\frac{\partial H}{\partial I_i};$$

$$\dot{I}_i = -\frac{\partial H}{\partial \dot{q}_i} = 0$$

the last equation follows because $I_i$ are constants of motion. This means that $H(I_i(t), \dot{q}) = H(I)$ which are constant

$$\dot{q}_i = \frac{\partial H}{\partial I_i}, \quad \Theta_i(t) = \Theta_i(0) + \frac{\partial H}{\partial I_i} t$$

$$\Theta_i(t) + \frac{\partial H}{\partial I_i} t = \int_{q_i(0)}^{q_i(t)} \frac{\partial }{\partial I_i} \Theta_i(q_i, s(I)) dq_i$$

in principle can be solved in $q_i(t)$

$$p_i(t) = p_i(q_i(t), s(I))$$

This means that the motion is a superposition of $N$ periodic motions. It is periodic if the frequencies are rational multiples of each other, otherwise the motion is quasiperiodic.