Lecture 2

Last time we demonstrated that in an inertial coordinate system in the absence of forces the solution of the equations of motion had the form

\[
\vec{r}_i(t) = \vec{r}_i(t_0) + \vec{v}_i(t_0)(t-t_0)
\]

where \(\vec{r}(t_0)\) and \(\vec{v}(t_0)\) are constant vectors. We showed that the equations of motion

\[
M_i \frac{d^2 \vec{r}_i}{dt^2} = 0
\]

were preserved under the following 10 elementary transformations

(3) \(\vec{r}_i'(t) = \vec{r}_i(t) + \vec{r}_0\)

(3) \(\vec{r}_i'(t) = \vec{r}_i(t) + \vec{v}_0 t\)

(3) \(\vec{r}_i'(t) = R \vec{r}_i(t)\) (matrix multiplication)

where \(R\) is a 3x3 matrix representing a rotation.
(1) \( t' = t + t_0 \)

These transformations correspond to (1) changing the origin of the coordinate system by a constant, (2) changing the origin of the coordinate system by a constant velocity, (3) rotating the coordinate axes by a constant rotation (4) changing the clock.

Since each one of these transformations leaves the equations of motion for free particles unchanged, any combination of these transformations leaves the equations of motion unchanged. It is not hard to show that the most general transformation that can be generated by these 10 elementary transformations has the form:
\[ \vec{r}_i(t) \rightarrow \vec{r}_i'(t') = R \vec{r}_i(t) + \vec{V}_0 t + \vec{r}_0. \]
\[ t \rightarrow t' = t + t_0. \]

These transformations are called Galilean transformations.

A general Galilean transformation can be represented by a 5x5 matrix

\[
\begin{pmatrix}
  \vec{r}_i' \\
  t'
\end{pmatrix} = 
\begin{pmatrix}
  R & \vec{V}_0 & \vec{r}_0 \\
  0 & 1 & t_0 \\
  0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
  \vec{r}_i \\
  t
\end{pmatrix}
\]

Each of the 10 elementary transformations corresponds to a special case of this matrix. For example, a translation of the origin is represented by

\[
\begin{pmatrix}
  \vec{r}_i' \\
  0 \\
  0 \\
  0 \\
  0
\end{pmatrix} = 
\begin{pmatrix}
  R & \vec{V}_0 & \vec{r}_0 \\
  0 & 1 & t_0 \\
  0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
  \vec{r}_i \\
  0 \\
  0 \\
  0 \\
  0
\end{pmatrix}
\]
These transformations are called Galilean transformations. Their physical significance is that they are the transformations that relate inertial coordinate systems.

In these systems, if there are no forces particles move with constant velocity. This is the content of Newton's first law of motion.

For homework you will show

1) The "product" of 2 Galilean transformations is another Galilean transformation. (The product can be represented by products of the matrices).

2) Every Galilean transformation has an inverse that is also a Galilean transformation.
3. The identity is a Galilean transformation.

Galilean transformations are associative — i.e.,

\[(G_1 G_2) G_3 = G_1 (G_2 G_3)\]

These properties mean that the set of Galilean transformations form a group under composition. (The four properties above characterize a group.) The group is called the Galilean group.

Some remarks:

1. These are not the only transformations that preserve the equations of motion for a free particle:

- \[\vec{r}_i \rightarrow \vec{r}'_i = \vec{r}_i \]
- \[\vec{r}'_i \rightarrow \vec{r}_{''i} = -\vec{r}_i\]
- \[t \rightarrow t' = -t\]
- \[t' \rightarrow t'' = \gamma t\]

All leave the equations of motion for a free particle unchanged.
The first and fourth simply correspond to change of units when interactions are introduced; the forces have dimensions an must scale accordingly, but will not be invariant under change of units. It also turns out experimentally that the weak interaction is not invariant under space reflection or time reversal.

(2) The matrix $R$ for a rotation is a real $3 \times 3$ orthogonal matrix with unit determinant:

$$R^T R = I$$

ensures that $R$ preserves the length of vectors, i.e., $\det R = 1$ eliminates space reflection.
In the moving car point particles interact. If we expect the laws of physics to be more generally independent of inertial coordinate systems, then there will also be restrictions on the interactions.

Remark - this only applies to the universe as isolated subsystems.
If there is an electric charge at the origin and another charge in the moving car - the force on the charge in the car will depend on how far the car is from the origin. On the other hand if both charges are in the car - they should behave the same as if both charges were at rest in the original coordinate system.
This leads to the principle of Galilean relativity: the equations of motion for an isolated system have the same form in all inertial coordinate systems.

Remark - this sounds a lot like the principle of special relativity. The difference is the set of transformations that relate different inertial coordinate systems.

Experimentally it is known that the Galilean group must be replaced by the Poincaré group - generated by Lorentz transformations and spacetime translations - in this case Galilean symmetry is only approximate.

In what follows we consider the implications of assuming that Galilean symmetry is exact.
Implications of Galilean symmetry

\[ m_i \frac{d^2 \vec{r}_i}{dt^2} = \vec{F}_i(t, \vec{r}_i, \vec{\dot{r}}_i, \vec{\ddot{r}}_i, \ldots, \vec{\dddot{r}}_i) \]

\[ \text{so} \quad \vec{r}'_i = \vec{r}_i + \vec{a} \]

\[ m_i \frac{d^2 \vec{r}'_i}{dt^2} = m_i \frac{d^2 \vec{r}_i}{dt^2} = \vec{F}_i(t, \vec{r}'_i, \vec{\dot{r}}'_i, \vec{\ddot{r}}'_i) \Rightarrow \]

\[ m_i \frac{d^2 \vec{r}_i}{dt^2} = \vec{F}_i(t, \vec{r}_i + \vec{a}, \vec{\dot{r}}_i + \vec{a}, \vec{\ddot{r}}_i, \vec{\dddot{r}}_i) \]

Comparing this to the equation on the top of the page gives

\[ \vec{F}_i(t, \vec{r}_i, \ldots, \vec{\dddot{r}}_i) = \vec{F}_i(t, \vec{r}_i + \vec{a}, \ldots, \vec{\dddot{r}}_i + \vec{a}) \]

For the forces to be independent of \( \vec{a} \), \( \vec{F}_i \) can only depend on coordinate differences

Next let

\[ \vec{r}'_i = \vec{r}_i + \vec{v}_0 \cdot t \quad \vec{r}_i = \vec{r}_i + \vec{v}_0 \]

and assume that \( \vec{F}_i \) only depends on coordinate differences, velocities and time.
\[
    m_i \frac{d^2 \vec{r}_i}{dt^2} = \vec{F}_i(t, \vec{r}_2-\vec{r}_i, \vec{r}_0-\vec{r}_i, \vec{r}_i, \vec{\hat{r}}_i, \vec{\dot{r}}_i)
\]

\[
    m_i \frac{d^2 \vec{r}_i}{dt^2} = m_i \frac{d^2 \vec{p}_i}{dt^2} = \vec{F}_i(t, \vec{r}_2-\vec{r}_i, \vec{r}_0-\vec{r}_i, \vec{r}_i, \vec{\dot{r}}_i + \vec{v}_0, \vec{\dot{p}}_0 + \vec{v}_0]
\]

Comparing these equations we get
\[
    \vec{F}_i(t, \vec{r}_2-\vec{r}_i, \vec{r}_0-\vec{r}_i, \vec{r}_i, \vec{\dot{r}}_i + \vec{v}_0, \vec{\dot{p}}_0 + \vec{v}_0)
\]

This means that the force in a Galilean invariant system can only depend on velocity differences.

For the case of rotations, Galilean invariance implies
\[
    \vec{F}_i(t, \vec{r}_2-\vec{r}_i, \vec{r}_0-\vec{r}_i, \vec{r}_i, \vec{\dot{r}}_i + \vec{v}_0, \vec{\dot{p}}_0 + \vec{v}_0) = \begin{bmatrix} \mathbb{R}^T & \mathbb{R} (\vec{r}_2-\vec{r}_i) & \mathbb{R} (\vec{r}_0-\vec{r}_i) & \mathbb{R} (\vec{r}_i, \vec{\dot{r}}_i) & \mathbb{R} (\vec{\dot{p}}_0 + \vec{v}_0) \end{bmatrix}
\]

This means that the force must transform like a vector made out of the coordinate and velocity differences.
Finally if we shift the time we get

\[ F(t, \vec{r}_2 - \vec{r}_1, \vec{p}_2 - \vec{p}_1, \vec{p}_3 - \vec{p}_2, \ldots, \vec{r}_n - \vec{r}_2) = F(t+t_0, \vec{r}_2 - \vec{r}_1, \vec{p}_n - \vec{p}_2, \vec{p}_2 - \vec{p}_1, \ldots, \vec{r}_n - \vec{r}_2) \]

which means that the force cannot have an explicit time dependence.

To summarize - forces consistent with Galilean symmetry:

1. depend only on coordinate or velocity differences
2. have no explicit time dependence
3. are rotationally covariant:
   \[ F = R^T F(R \vec{r}, R \vec{p}) \]

These considerations apply only to the universe of isolated systems. If a system interacts with an external environment then the forces are not restricted.
So far we have determined how inertial coordinate systems are related; we have given a definition. We do know that in the absence of forces that they are the coordinate systems where particles move with constant velocity. In order to get more insight it is useful to consider a general coordinate change from an inertial coordinate system to a general coordinate system and determine how the equations of motion must be modified.

Let \( \vec{F}_i(t) \) be the cartesian coordinates of the \( i \)-th particle in an inertial coordinate system. We assume

\[
\frac{d^2 \vec{r}_i}{dt^2} = \vec{F}_i
\]

Let \( \vec{r}_i(F_i, t) \) be a set of general coordinates. We consider the
sum of the equations of motion in this coordinate system

\[
\frac{d^2 y_i}{dt^2} = \sum_k \frac{\partial y_i}{\partial r_{i_k}} \frac{dr_i}{dt} + \frac{\partial y_i}{\partial t}
\]

\[
\frac{d^2 y_i}{dt^2} = \sum_k \frac{\partial^2 y_i}{\partial r_{i_k} \partial r_{i_l}} \frac{dr_i}{dt} \frac{dr_l}{dt} + \sum_k \frac{\partial y_i}{\partial r_{i_k}} \frac{dr_i}{dt} + 2 \sum_k \frac{\partial^2 y_i}{\partial r_{i_k} \partial t} \frac{dr_i}{dt} \frac{dt}{dt} + \frac{\partial^2 y_i}{\partial t^2}
\]

Next, multiply by \(m_i\) and use \(m_i \frac{d^2 r_i}{dt^2} = F_i\).

Then this can be expressed as

\[
m_i \frac{d^2 y_i}{dt^2} = \sum_k \frac{\partial y_i}{\partial r_{i_k}} F_i + m_i \sum_k \frac{\partial^2 y_i}{\partial r_{i_k} \partial r_{i_l}} \frac{dr_i}{dt} \frac{dr_l}{dt} + 2m_i \frac{\partial^2 y_i}{\partial r_{i_k} \partial t} \frac{dr_i}{dt} \frac{dt}{dt} + m_i \frac{\partial^2 y_i}{\partial t^2}
\]

We also have

\[
\frac{dr_i}{dt} = \sum_k \frac{\partial r_{i_k}}{\partial y_i} \left( \frac{dy_i}{dt} - \frac{\partial y_i}{\partial t} \right)
\]

Using this in the above equation gives
This is the form of Newton's second law in a non inertial coordinate system.

There are 2 important observations:

1. The original force gets transformed to the new coordinate system.

2. Even where $\bar{F}_i = 0$, there are a number of other "inertial forces" on the right. They have the property that they are all proportional to the inertial mass of the particle — unlike $F_i$, these forces will lead to spontaneous acceleration even when $\bar{F}_i = 0$. 
Thus we see that for an isolated system we are in an inertial coordinate system if there are no forces proportional to the inertial masses.

3. This breaks down when $\vec{F}$ is a gravitational force - because the gravitational and inertial masses are identified. Fixing this led to general relativity - (choosing coordinates locally eliminate the gravitational force).

Many forces in nature are velocity independent and are derivable from a single valued potential function. A set of forces are conservative if they can be expressed in the form

$$\vec{F}_i = -\frac{\partial}{\partial \vec{r}_i} V(\vec{r}_i - \vec{r}_n)$$
where \( V \) is a single valued potential function.

The work done by a force \( \vec{F}_i \) moving the \( i \)-th particle from \( \vec{r}_{iA} \) to \( \vec{r}_{iB} \) along a path \( \vec{y}_i(t) \)

\[
\vec{y}_i(0) = \vec{r}_{iA} \quad \vec{y}_i(1) = \vec{r}_{iB}
\]

is

\[
W_{AB}[\vec{y}_i] = \int_0^1 \vec{F}_i \cdot \frac{d\vec{y}_i}{dt} \, dt
\]

Summing this over all particles gives the total work done by the forces in moving all particles

\[
W_{AB}[\vec{y}] = \sum_{i=1}^n W_{AB}[\vec{y}_i] =
\]

\[
= \sum_{i=1}^n \int_0^1 \vec{F}_i \cdot \frac{d\vec{y}_i}{dt} \, dt = -\sum_{i=1}^n \int_0^1 \frac{\partial V}{\partial \vec{r}_i} \cdot \frac{d\vec{y}_i}{dt} \, dt =
\]

\[
- \int_0^1 \frac{\partial}{\partial t} V(\vec{y}_1(t), \ldots, \vec{y}_n(t)) =
\]

\[
- V(\vec{r}_{iA}, \ldots, \vec{r}_{iB}) + V(\vec{r}_{iA}, \ldots, \vec{r}_{nA})
\]
This shows that for a conservative force the work only depends on the endpoints of the paths and the difference in the potential energy.

If we ask how much work has to be done to move the particle from A to B against the force along the path then the work is the change in potential energy. This result can be equivalently expressed as

$$\oint_{C_i} \mathbf{F} \cdot d\mathbf{r_i} = 0$$

where $\oint_{C_i}$ indicates an integral around closed paths $C_i(0) = C_i(1)$.

Newton's third law - For an isolated system

$$\mathbf{F} = -\sum_{i=1}^{n} \frac{\partial V}{\partial r_i} (r_{2i} - r_i, r_{3i} - r_i) =$$
\[ \overline{F} = \sum_{i=2}^{n} \frac{\partial \mathbf{V}}{\partial \mathbf{x}_i} - \sum_{i=1}^{n} \frac{\partial \mathbf{V}}{\partial \mathbf{x}_i} = 0 \]

This is **Newton's third law**.

In this case we showed that for a conserved translationally invariant system with velocity independent forces is 0.

When applied to a 2 particle system we get the standard form of the third law - the force on particle 1 due to particle 2 is - the force on particle 2 due to particle 1.